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RESEARCH ARTICLE

On classification of groups generated by 3-state automata over a 2-letter alphabet

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Dedicated to V. V. Kirichenko on his 65th birthday and V. I. Sushchansky on his 60th birthday

Abstract. We show that the class of groups generated by 3-state automata over a 2-letter alphabet has no more than 122 members. For each group in the class we provide some basic information, such as short relators, a few initial values of the growth function, a few initial values of the sizes of the quotients by level stabilizers (congruence quotients), and hystogram of the spectrum of the adjacency operator of the Schreier graph of the action on level 9. In most cases we provide more information, such as whether the group is contracting, self-replicating, or (weakly) branch group, and exhibit elements of infinite order (we show that no group in the class is an infinite torsion group). A GAP package, written by Muntyan and Savchuk, was used to perform some necessary calculations. For some of the examples, we establish that they are (virtually) iterated monodromy groups of post-critically finite rational functions, in which cases we describe the functions and the limit spaces. There are exactly 6 finite groups in the class (of order no greater than 16), two free abelian groups (of rank 1 and 2), and only one free nonabelian group (of rank 3). The other examples in the class range from familiar (some virtually abelian groups, lamplighter group, Baumslag-Solitar groups $BS(1, \pm 3)$, and a free product $C_2 * C_2 * C_2$) to enticing (Basilica group and a few other iterated monodromy groups).

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1. Introduction

Automaton groups were formally introduced in the beginning of 1960's [Glu61, Hoř63] but it took a while to realize their importance, utility, and, at the same time, complexity. Among the publications from the first decade of the study of automaton groups let us distinguish [Zar64, Zar65] and the book [GP72].

The first substantial results came only in the 1970's and in the beginning of the 1980's when it was shown in [Ale72, Sus79, Gri80, GS83b] that automaton groups provide examples of finitely generated infinite torsion groups, thus making a contribution to one of the most famous problems in algebra — the General Burnside Problem (more information on all three versions of the Burnside problem can be found in [Adi79, Gol68, Gup89, Kos90, Zel91, GL02]). The methods used to study the properties of the examples from [Ale72, Sus79, Gri80] are very different. The methods used in [Ale72] are typical for the theory of finite automata (in fact the provided proof was incorrect; the first correct proof appears in [Mer83] as a combination of the results from [Gri80] and [Mer83], as well as in the third edition of the book [KM82] and in [KAP85]). The exposition in [Sus79] is based on Kalujnin's tableaux coming from his theory of iterated wreath products of cyclic groups of prime order p. The approach in [Gri80] is based on the ideas of self-similarity and contraction. These ideas are apparent both in the proof of the infiniteness and the torsion property of the group. The self-similarity is apparent from the fact that the set of all states of the automaton is used as a generating set for the group (now it is common to call such groups self-similar). The contraction property here means that the length of the elements contracts by a factor bounded away from 1 when one passes to sections. A principal tool introduced in the beginning of the 1980's was the language of actions on rooted trees suggested by Gupta and Sidki in [GS83b], which helped tremendously in bringing geometric insight to the subject.

A new indication of the importance of automaton groups came when it was shown that some of them provided the first examples of groups of intermediate growth [Gri83, Gri84, Gri85]. This not only answered the question of J.Milnor [Mil68] about existence of such groups, but also answered a number of other questions in and around group theory, including M. Day's problem [Day57] on existence of amenable but not elementary amenable groups. Basically, even to this day, all known examples of groups of intermediate growth and non-elementary amenable groups are based on automaton groups.

Investigations in the last two decades [Gri84, Gri85, GS83b, GS83a, Lys85, Neu86, Sid87a, Sid87b, Gri89, Roz93, Gri98, Gri99, Gri00, BG00a, BG00b, GŻ01, Nek05, GŠ06] show that many automaton groups possess numerous interesting, and sometimes unusual, properties. This includes just infiniteness (the groups constructed in [Gri84, Gri85] as well as in [GS83a] answer a question from [CM82] on new examples of infinite groups with finite quotients), finiteness of width, or more generally polynomial growth of the dimension of the successive quotients in the lower central series [BG00b] (answering a question of E. Zelmanov on classification of groups of finite width), branch properties [Gri84, Neu86, Gri00] (answering some questions of S. Pride and M. Edjvet [Pri80, EP84]), finiteness of the index of maximal subgroups and presence or absence of the congruence property [Per00, Per02] (related to topics in pro-finite groups), existence of groups with exponential but not uniformly exponential growth [Wil04b, Wil04a, Bar03, Nek07b] (providing an answer to a question of M. Gromov), subgroup separability and conjugacy separability [GW00], further examples of amenable groups but not amenable (or even sub-exponentially amenable) groups [GŻ02a, BV05, GNŠ06a], amenability of groups generated by bounded automata [BKN], and so on. The word problem can be solved in contracting self-similar groups by using an extremely effective *branch algorithm* [Gri84, Sav03]. The conjugacy problem can also be solved in many cases [WZ97, Roz98, Leo98, GW00] (in fact we do not know of an example of an automaton group with unsolvable conjugacy problem). In some instances, it is even known that the membership problem is solvable [GW03].

In addition to the formulation of many algebraic properties of groups generated by finite automata, a number of links and applications were discovered during the last decade. This includes asymptotic and spectral properties of the Cayley graphs and Schreier graphs associated to the action on the rooted tree with respect to the set of generators given by the set of states of the automaton. For instance, it is shown in [GŻ01] that the discrete Laplacian on the Cayley graph of the Lamplighter group $\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ has pure point spectrum. This fact was used to answer a question of M. Atiyah on L^2-Betti numbers of closed manifolds [GLSŻ00]. The methods developed in the study of the spectral properties of Schreier graphs of self-similar groups can be used to construct Laplacians on fractal sets and to study their spectral properties (see [GN07, NT08].

A new and fruitful direction, bringing further applications of self-similar groups, was established by the introduction of the notions of iterated monodromy groups and limit spaces by V. Nekrashevych. The theory established a link between contracting self-similar groups and the geometry of Julia sets of expanding maps. An example of an application of self-similar groups to holomorphic dynamics is given by the solution (by L. Bartholdi and V. Nekrashevych in [BN06]) of the "twisted rabbit" problem of J. Hubbard. The book [Nek05] provides a comprehensive introduction to this theory.

In many situations automaton groups serve as renorm groups. For instance this happens in the study of classical fractals, in the study of the behavior of dynamical systems [Oli98], and in combinatorics — for example in Hanoi Towers games on k pegs, $k \geq 3$, as observed by Z. Šunić (see [GŠ06]).

There is interest of computer scientists and logicians in automaton groups, since they may be relevant in the solution of important complexity problems (see [RS] for ideas in this direction). Self-similar groups of intermediate growth are mentioned by Wolfram in [Wol02] as examples of "multiway systems" with complex behavior.

Among the major problems in many areas of mathematics are the classification problems. If the objects are given combinatorially then it is naturally to try to classify them first by complexity and then within each complexity class.

A natural complexity parameter in our situation is the pair (m, n) where m is the number of states of the automaton generating the group and n is the cardinality of the alphabet.

There are 64 invertible 2-state automata acting on a 2-letter alphabet, but there are only six non-isomorphic $(2, 2)$-automaton groups, namely, the trivial group, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, the infinite dihedral group D_∞, and the lamplighter group $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ [GNS00, GŻ01] (more details are given in Theorem 7 below). A classification of semigroups generated by 2-state automata (not necessary invertible) over a 2-letter alphabet is provided by I. Reznikov and V. Sushchanskiĭ [RS02a]. Some examples from this class, including an automaton generating a semigroup of intermediate growth, were studied in the subsequent papers [RS02c, RS02b, BRS06].

It is not known how many pairwise non-isomorphic groups exists for any class (m, n) when either $m > 2$ or $n > 2$. Unfortunately, the number of automata that has to be treated grows super-exponentially with either of the two arguments (there are $m^{mn}(n!)^m$ invertible (m, n)-automata).

Nevertheless, a reasonable task is to consider the problem of classification for small values of m and n and try to classify the $(3, 2)$-automaton groups and $(2, 3)$-automaton groups.

Our research group (with some contribution by Y. Vorobets and M. Vorobets) has been working on the problem of classification of $(3, 2)$-automaton groups for the last four yeas and some of the obtained results are presented in this article. Our research goals moved in three main directions:

1. Search for new interesting groups and an attempt to use them to

solve known problems. An example of such a group is the Basilica group (see automaton [852]). It is the first example of an amenable group (shown in [BV05]) that is not sub-exponentially amenable group (shown in [GŻ02a]).

2. Recognition of already known groups as self-similar groups, and use of the self-similar structure in finding new results and applications for such groups. As examples we can mention the free group of rank 3 (see automaton [2240]), the free product of three copies of $\mathbb{Z}/2\mathbb{Z}$ (see automaton [846]), Baumslag-Solitar groups $BS(1,\pm 3)$ (see automata [870] and [2294]), the Klein bottle group (see automaton [2212]), and the group of orientation preserving automorphisms of the 2-dimensional integer lattice (see automaton [2229]).

3. Understanding of typical phenomena that occur for various classes of automaton groups, formulation and proofs of reasonable conjectures about the structure of self-similar groups.

The results on the class of groups generated by $(3,2)$-automata proven in this article are the following.

Theorem 1. *There are at most 122 non-isomorphic groups generated by* $(3,2)$*-automata.*

The numbers in brackets in the next two theorems are references to the numbers of the corresponding automata (more on this encoding will be said later). Here and thereafter, C_n denotes the cyclic group of order n.

Theorem 2. *There are* 6 *finite groups in the class: the trivial group* $\{1\}$ [1], C_2 [1090], $C_2 \times C_2$ [730], D_4 [847], $C_2 \times C_2 \times C_2$ [802] *and* $D_4 \times C_2$ [748].

Theorem 3. *There are* 6 *abelian groups in the class: the trivial group* $\{1\}$ [1], C_2 [1090], $C_2 \times C_2$ [730], $C_2 \times C_2 \times C_2$ [802], $\mathbb{Z}$ [731] *and* $\mathbb{Z}^2$ [771].

Theorem 4. *The only nonabelian free group in the class is the free group of rank* 3 *generated by the Aleshin-Vorobets-Vorobets automaton* [2240].

Theorem 5. *There are no infinite torsion groups in the class.*

The short list of general results does not give full justice to the work that has been done. Namely, in most individual cases we have provided detailed information for the group in question.

More work and, likely, some new invariants are required to further distinguish the 122 groups that are listed in this paper as potentially non-isomorphic. In some cases one could try to use the rigidity of actions on rooted trees (see [LN02]), since in many cases it is easier to distinguish actions than groups. In the contracting case one could use, for instance, the geometry of the Schreier graphs and limit spaces to distinguish the actions.

Next natural step would be to consider the case of $(2,3)$-automaton groups or 2-generated self-similar groups of binary tree automorphisms defined by recursions in which every section is either trivial, a generator, or an inverse of a generator. The cases $(4,2)$ and $(5,2)$ also seem to be attractive, as there are many remarkable groups in these classes.

Another possible direction is to study more carefully only certain classes of automata (such as the classical linear automata, bounded and polynomially growing automata in the sense of Sidki [Sid00], etc.) and the properties of the corresponding automaton groups.

Many computations used in our work were performed by the package `AutomGrp` for `GAP` system, developed by Y. Muntyan and D. Savchuk [MS08]. The package is not specific to $(3,2)$-automaton groups (in fact, many functions are implemented also for groups of tree automorphisms that are not necessarily generated by automata).

2. Regular rooted trees, automorphisms, and self-similarity

Let X be an alphabet on d ($d \geq 2$) letters. Most often we set $X = \{0, 1, \ldots, d-1\}$. The set of finite words over X, denote by X^*, has the structure of a *regular rooted d-ary tree*, which we also denote by X^*. The empty word $\emptyset$ is the *root* of the tree and every vertex v has d children, namely the words vx, for x in X. The words of length n constitute *level* n in the tree.

The group of tree automorphisms of X^* is denoted by $\mathrm{Aut}(X^*)$. Tree automorphisms are precisely the permutations of the vertices that fix the root and preserve the levels of the tree. Every automorphism f of X^* can be decomposed as

$$f = \alpha_f(f_0, \ldots, f_{d-1}) \tag{1}$$

where f_x, for x in X, are automorphisms of X^* and α_f is a permutation of the set X. The permutation α_f is called the *root permutation* of f and the automorphisms f_x (denoted also by $f|_x$), x in X, are called *sections* of f. The permutation α_f describes the action of f on the first letter of every word, while the automorphism f_x, for x in X, describes the action of f on the tail of the words in the subtree xX^*, consisting of the words in X^* that start with x. Thus the equality (1) describes the action of f through decomposition into two steps. In the first step the d-tuple $(f_0, \dots, f_{d-1})$ acts on the d subtrees hanging below the root, and then the permutation α_f, permutes these d subtrees. Thus we have

$$f(xw) = \alpha_f(x) f_x(w), \tag{2}$$

for x in X and w in X^*. Second level sections of f are defined as the sections of the sections of f, i.e., $f_{xy} = (f_x)_y$, for $x, y \in X$, and more generally, for a word u in X^* and a letter x in X the section of f at ux is defined as $f_{ux} = (f_u)_x$, while the section of f at the root is f itself.

The group $\mathrm{Aut}(X^*)$ decomposes algebraically as

$$\mathrm{Aut}(X^*) = \mathsf{Sym}(X) \ltimes \mathrm{Aut}(X^*)^X = \mathsf{Sym}(X) \wr \mathrm{Aut}(X^*), \tag{3}$$

where $\wr$ is the *permutational wreath product* in which the active group $\mathsf{Sym}(X)$ permutes the coordinates of $\mathrm{Aut}(X^*)^X = (\mathrm{Aut}(X^*), \dots, \mathrm{Aut}(X^*))$. For arbitrary automorphisms f and g in $\mathrm{Aut}(X^*)$ we have

$$\alpha_f(f_0, \dots, f_{d-1})\alpha_g(g_0, \dots, g_{d-1}) = \alpha_f\alpha_g(f_{g(0)}g_0, \dots, f_{g(d-1)}g_{d-1}).$$

For future use we note the following formula regarding the sections of a composition of tree automorphisms. For tree automorphisms f and g and a vertex u in X^*,

$$(fg)_u = f_{g(u)} g_u. \tag{4}$$

The group of tree automorphisms $\mathrm{Aut}(X^*)$ is a pro-finite group. Namely, $\mathrm{Aut}(X^*)$ has the structure of an infinitely iterated wreath product

$$\mathrm{Aut}(X^*) = \mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr \dots))$$

of the finite group $\mathsf{Sym}(X^*)$ (this follows from (3)). This product is the inverse limit of the sequence of finitely iterated wreath products of the form $\mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr (\mathsf{Sym}(X) \wr \cdots \wr \mathsf{Sym}(X)))$. Every subgroup of $\mathrm{Aut}(X^*)$ is residually finite. A canonical sequence of normal subgroups of finite index intersecting trivially is the sequence of level stabilizers. The n-th *level stabilizer* of a group G of tree automorphisms is the subgroup $\mathrm{Stab}_G(n)$ of $\mathrm{Aut}(X^*)$ that consists of all tree automorphisms in G that fix the vertices in the tree X^* up to and including level n.

The *boundary* of the tree X^* is the set X^ω of right infinite words over X (infinite geodesic rays in X^* connecting the root to "infinity"). The boundary has a natural structure of a metric space in which two infinite words are close if they agree on long finite prefixes. More precisely, for two distinct rays ξ and ζ, define the distance to be $d(\xi,\zeta) = 1/2^{|\xi\wedge\zeta|}$, where $|\xi \wedge \zeta|$ denotes the length of the longest common prefix $\xi \wedge \zeta$ of ξ and ζ. The induced topology on X^ω is the Tychonoff product topology (with X discrete), and X^ω is a Cantor set. The group of isometries $\mathrm{Isom}(X^\omega)$ and the group of tree automorphisms $\mathrm{Aut}(X^*)$ are canonically isomorphic. Namely, the action of the automorphism group $\mathrm{Aut}(X^*)$ can be extended to an isometric action on X^ω, simply by declaring that (1) and (2) are valid for right infinite words.

We now turn to the concept of self-similarity. The tree X^* is a highly self-similar object (the subtree uX^* consisting of words with prefix u is canonically isomorphic to the whole tree) and we are interested in groups of tree automorphisms in which this self-similarity structure is reflected.

Definition 1. A group G of tree automorphisms is *self-similar* if, every section of every automorphism in G is an element of G.

Equivalently, self-similarity can be expressed as follows. A group G of tree automorphisms is self-similar if, for every g in G and a letter x in X, there exists a letter y in X and an element h in G such that

$$g(xw) = yh(w),$$

for all words w over X.

Self-replicating groups constitute a special class of self-similar groups. Examples from this class are very common in applications. A self-similar group G is *self-replicating* if, for every vertex u in X^*, the homomorphism $\varphi_u : \mathrm{Stab}_G(u) \to G$ from the stabilizer of the vertex u in G to G, given by $\varphi(g) = g_u$, is surjective.

At the end of the section, let us mention the class of *branch groups*. Branch groups were introduced [Gri00] where it is shown that they constitute one of the three classes of just-infinite groups (infinite groups with no proper, infinite, homomorphic images). If a class of groups $\mathcal{C}$ is closed under homomorphic images and if it contains infinite, finitely generated examples then it contains just-infinite examples (this is because every infinite, finitely generated group has a just-infinite image). Such examples are minimal infinite examples in $\mathcal{C}$. We note that, for example, the group of intermediate growth constructed in [Gri80] is a branch automaton group that is a just-infinite 2-group. i.e., it is an infinite, finitely generated, torsion group that has no proper infinite quotients. The Hanoi Towers group [GŠ07] is a branch group that is not just infinite [GNŠ06b]. The iterated monodromy group $IMG(z^2 + i)$ [GSŠ07] is a branch groups, while $\mathcal{B} = IMG(z^2 - 1)$ is not a branch group, but only weakly branch. More generally, it is shown in [BN07] that the iterated monodromy groups of post-critically finite quadratic maps are branch groups in the pre-periodic case and weakly branch groups in the periodic case (the case refers to the type of post-critical behavior).

We now define regular (weakly) branch groups. A level transitive group $G \leq \mathrm{Aut}(X^*)$ of k-ary tree automorphisms is a *regular branch group* over K if K is a normal subgroup of finite index in G such that $K \times \cdots \times K$ is geometrically contained in K. By definition, the subgroup K has the property that $K \times \cdots \times K$ is geometrically contained in K, denoted by $K \times \cdots \times K \preceq K$, if

$$K \times \cdots \times K \leq \psi(K \cap \mathrm{Stab}_G(1))$$

where ψ is the homomorphism $\psi : \mathrm{Stab}_G(1) \to \mathrm{Aut}(X^*) \times \cdots \times \mathrm{Aut}(X^*)$ given by $\psi(g) = (g_0, g_1, \ldots, g_{k-1})$. If instead of asking for K to have finite index in G we only require that K is nontrivial, we say that G is *regular weakly branch group* over K. Note that if G is level transitive and K is normal in G, in order to show that G is regular (weakly) branch group over K, it is sufficient to show that $K \times 1 \times \cdots \times 1 \preceq K$ (i.e. $K \times 1 \times \cdots \times 1 \leq \psi(K \cap \mathrm{Stab}_G(1))$). More on the class of branch group can be found in [Gri00] and [BGŠ03].

3. Automaton groups

The full group of tree automorphisms $Aut(X^*)$ is self-similar, since the section of every tree automorphism is just another tree automorphism. However, this group is rather large (uncountable). For various reasons, one may be interested in ways to define (construct) finitely generated self-similar groups. Automaton groups constitute a special class of finitely generated self-similar groups. We provide two ways of thinking about automaton groups. One is through finite wreath recursions and the other through finite automata.

Every finite system of recursive relations of the form

$$\begin{cases} s^{(1)} & = & \alpha_1 \left(s_0^{(1)}, s_1^{(1)}, \ldots, s_{d-1}^{(1)} \right), \\ \ldots & & \\ s^{(k)} & = & \alpha_k \left(s_0^{(k)}, s_1^{(k)}, \ldots, s_{d-1}^{(k)} \right), \end{cases} \tag{5}$$

where each symbol $s_j^{(i)}$, $i = 1, \ldots, k$, $j = 0, \ldots, d-1$, is a symbol in the set of symbols $\{s^{(1)}, \ldots, s^{(k)}\}$ and $\alpha_1, \ldots, \alpha_k$ are permutations in $\mathsf{Sym}(X)$, has a unique solution in $\mathrm{Aut}(X^*)$ (in the sense that the above recursive relations represent the decompositions of the tree automorphisms

$s^{(1)}, \dots, s^{(k)}$). Thus, the action of the automorphism defined by the symbol $s^{(i)}$ is given recursively by $s^{(i)}(xw) = \alpha_i(x) s_x^{(i)}(w)$.

The group G generated by the automorphisms $s^{(1)}, \dots, s^{(k)}$ is a finitely generated self-similar group of automorphisms of X^*. This follows since sections of products are products of sections (see (4)) and all sections of the generators of G are generators of G.

When a self-similar group is defined by a system of the form (5), we say that it is defined by a *wreath recursion*. We switch now the point of view from wreath recursions to invertible automata.

Definition 2. A *finite automaton* A is a 4-tuple $\mathcal{A} = (S, X, \pi, \tau)$ where S is a finite set of *states*, X is a finite *alphabet* of cardinality $d \geq 2$, $\pi : S \times X \to X$ is a map, called *output map*, and $\tau : S \times X \to S$ is a map, called *transition map*. If in addition, for each state s in S, the restriction $\pi_s : X \to X$ given by $\pi_s(x) = \pi(s, x)$ is a permutation in $\mathsf{Sym}(X)$, the automaton $\mathcal{A}$ is invertible.

In fact, we will be only concerned with finite invertible automata and, in the rest of the text, we will use the word automaton for such automata.

Each state s of the automaton $\mathcal{A}$ defines a tree automorphism of X^*, which we also denote by s. By definition, the root permutation of the automorphism s (defined by the state s) is the permutation π_s and the section of s at x is $\tau(s, x)$. Therefore

$$s(xw) = \pi_s(x)\tau(s, x)(w) \tag{6}$$

for every state s in S, letter x in X and word w over X.

Definition 3. Given an automaton $\mathcal{A} = (S, X, \pi, \tau)$, the group of tree automorphisms generated by the states of $\mathcal{A}$ is denoted by $G(\mathcal{A})$ and called the *automaton group* defined by $\mathcal{A}$.

The generating set S of the automaton group $G(\mathcal{A})$ generated by the automaton $\mathcal{A} = (S, X, \pi, \tau)$ is called the *standard* generating set of $G(\mathcal{A})$ and plays a distinguished role.

Directed graphs provide convenient representation of automata. The vertices of the graph, called *Moore diagram* of the automaton $\mathcal{A} = (S, X, \pi, \tau)$, are the states in S. Each state s is labeled by the root permutation $\alpha_s = \pi_s$ and, for each pair $(s, x) \in S \times X$, an edge labeled by x connects s to $s_x = \tau(s, x)$. Several examples are presented in Figure 1. The states of the 5-state automaton in the left half of the figure generate the group $\mathcal{G}$ of intermediate growth mentioned in the introduction (σ denotes the permutation exchanging 0 and 1, and 1 denotes the trivial vertex permutation). The top of the three 2-state automata on the

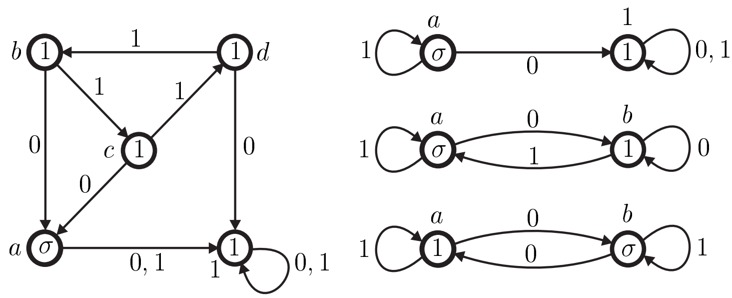


Figure 1: An automaton generating $\mathcal{G}$, the binary adding machine, and two Lamplighter automata

right in Figure 1 is the so called *binary adding machine*, which generates the infinite cyclic group $\mathbb{Z}$. The other two automata both generate the Lamplighter group $L_2 = \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \ltimes (\bigoplus \mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ (see [GNS00]).

The corresponding wreath recursions for the adding machine and for the two automata generating the Lamplighter group are given by

$$\begin{array}{lll} a = \sigma(1, a) & a = \sigma(b, a) & a = \ \ (b, a) \\ 1 = \ \ (1, 1) & b = \ \ (b, a), & b = \sigma(a, b) \end{array} \tag{7}$$

respectively.

The class of *polynomially growing automata* was introduced by Sidki in [Sid00]. Sidki proved in [Sid04] that no group generated by such an automaton contains free subgroups of rank 2. As we already indicated in the introduction, for the subclass of so called bounded automata the corresponding groups are amenable [BKN]. Recall that an automaton $\mathcal{A}$ is called *bounded* if, for every state s of $\mathcal{A}$, the function $f_s(n)$ counting the number of active sections of s at level n is bounded (a state is *active* if its vertex permutation is nontrivial).

There are other classes of automata (and corresponding automaton groups) that deserve special attention. We end the section by mentioning several such classes.

The class of *linear automata* consists of automata in which both the set of states S and the alphabet X have a structure of a vector space (over a finite field) and both the output and the transition function are linear maps (see [GP72] and [Eil76]).

The class of *bi-invertible automata* consists of automata in which both the automaton and its dual are invertible. Some of the automata in our classification are bi-invertible, most notably the Aleshin-Vorobets-Vorobets automaton [2240] generating the free group F_3 of rank 3 and the Bellaterra automaton [846] generating the free product $C_2 * C_2 * C_2$. In fact, both of these have even stronger property of being *fully invertible.* Namely, not only the automaton and its dual are invertible, but also the dual of the inverse automaton is invertible.

Another important class is the class of automata satisfying the *open set* condition. Every automaton in this class contains a *trivial state* (a state defining the trivial tree automorphism) and this state can be reached from any other state.

One may also study automata that are *strongly connected* (i.e. automata for which the corresponding Moore diagrams are strongly connected as directed graphs), automata in which no path contains more than one active state (such as the automaton defining $\mathcal{G}$ in Figure 1), and so on.

4. Schreier graphs

Let G be a group generated by a finite set S and let G act on a set Y. We denote by $\Gamma = \Gamma(G, S, Y)$ the *Schreier graph* of the action of G on Y. The vertices of Γ are the elements of Y. For every pair (s, y) in $S \times Y$ an edge labeled by s connects y to $s(y)$. An *orbital Schreier graph* of the action is the Schreier graph $\Gamma(G, S, y)$ of the action of G on the G-orbit of y, for some y in Y.

Let G be a group of tree automorphisms of X^* generated by a finite set S. The levels X^n, $n \geq 0$, are invariant under the action of G and we can consider the sequence of finite Schreier graphs $\Gamma_n(G, S) = \Gamma(G, S, X^n)$, $n \geq 0$. Let $\xi = x_1x_2x_3\ldots \in X^\omega$ be an infinite ray. Then the pointed Schreier graphs $(\Gamma_n(G, S), x_1x_2\ldots x_n)$ converge in the local topology (see [Gri84] or [GŻ99]) to the pointed orbital Schreier graph $(\Gamma(G, S, \xi), \xi)$.

Schreier graphs may be sometimes used to compute the spectrum of some operators related to the group. For a group of tree automorphisms G generated by a finite symmetric set S there is a natural unitary representation in the space of bounded linear operators $\mathcal{H} = B(L_2(X^\omega))$, given by $\pi_g(f)(x) = f(g^{-1}x)$ (the measure on the boundary X^ω is just the product measure associated to the uniform measure on X). Consider the spectrum of the operator

$$M = \frac{1}{|S|} \sum_{s \in S} \pi_s$$

corresponding to this unitary representation. The spectrum of M for a self-similar group G is approximated by the spectra of the finite dimensional operators induced by the action of G on the levels of the tree (see [BG00a]. Denote by $\mathcal{H}_n$ the subspace of $\mathcal{H} = B(L_2(X^\omega))$ spanned by the characteristic functions f_v, $v \in X^n$, of the cylindrical sets corresponding to the $|X|^n$ vertices on level n. The subspace $\mathcal{H}_n$ is invariant under the action of G and $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Denote by $\pi_g^{(n)}$ the restriction of π_g on $\mathcal{H}_n$. Then, for $n \geq 0$, the operator

$$M_n = \frac{1}{|S|} \sum_{s \in S} \pi_s^{(n)}$$

is finite dimensional. Moreover,

$$sp(M) = \overline{\bigcup_{n \geq 0} sp(M_n)},$$

i.e., the spectra of the operators M_n converge to the spectrum of M.

The table of information provided in Section 8 includes, in each case, the histogram of the spectrum of the operator M_9.

If P is the stabilizer of a point on the boundary X^ω, then one can consider the quasi-regular representation $\rho_{G/P}$ of G in $\ell^2(G/P)$.

Theorem 6 ([BG00a]). *If G is amenable or the Schreier graph G/P (the Schreier graph of the action of G on the cosets of P) is amenable then the spectrum of M and the spectrum of the quasi-regular representation $\rho_{G/P}$ coincide.*

In case the parabolic subgroup P is "small", the last result may be used to compute the spectrum of the Markov operator on the Cayley graph of the group. This approach was successfully used, for instance, to compute the spectrum of the Lamplighter group in [GŻ01] (see also [KSS06]).

5. Contracting groups, limit spaces, and iterated monodromy groups

Definition 4. A group G generated by an automaton over alphabet X is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n (generally depending on g) such that section g_v belongs to $\mathcal{N}$ for all words $v \in X^*$ of length at least n. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the group G.

The above definition makes sense for arbitrary self-similar groups — not necessarily automaton groups and, moreover, not necessarily finitely generated groups. In the case of an automaton group the contracting property may be equivalently stated as follows. An automaton group $G = G(\mathcal{A})$ is contracting if there exist constants κ, C, and N, with

$0 \le \kappa < 1$, such that $|g_v| \le \kappa|g| + C$, for all vertices v of length at least N and $g \in G$ (the length is measured with respect to the standard generating set S consisting of the states of $\mathcal{A}$). The contraction property is a key ingredient in many inductive arguments and algorithms involving the decomposition $g = \alpha_g(g_0, \dots, g_{d-1})$. Indeed, the contraction property implies that, for all sufficiently long elements g, all sections of g at vertices on level at least N are strictly shorter than g.

Contracting groups have rich geometric structure. Each contracting group is the iterated monodromy group of its *limit dynamical system*. This system is an (orbispace) self-covering of the *limit space* of the group. The limit space is a limit of the graphs of the action of G on the levels X^n of the tree X^* and is defined in the following way.

Definition 5. Let G be a contracting group over X. Denote by $X^{-\omega}$ the space of all left-infinite sequences $\dots x_2x_1$ of elements of X with the direct product (Tykhonoff) topology. We say that two sequences $\dots x_2x_1$ and $\dots y_2y_1$ are *asymptotically equivalent* if there exists a sequences $g_k \in G$, assuming a finite set of values, and such that

$$g_k(x_k \dots x_1) = y_k \dots y_1$$

for all $k \ge 1$. The quotient of the space $X^{-\omega}$ by this equivalence relation is called the *limit space* of G.

The following proposition, proved in [Nek05] (Proposition 3.6.4) is a convenient way to compute the asymptotic equivalence.

Proposition 1. *Let a contracting group G be generated by a finite automaton A. Then the asymptotic equivalence is the equivalence relation generated by the set of pairs $(\dots x_2x_1, \dots y_2y_1)$ for which there exists a sequence g_k of states of A such that $g_k(x_k) = y_k$ and $g_k|_{x_k} = g_{k-1}$.*

The limit dynamical system is the map induced by the shift $\dots x_2x_1 \mapsto \dots x_3x_2$. The limit space is a compact metrizable topological space of finite topological dimension (see [Nek05], Theorem 3.6.3). If the group is self-replicating, then the limit space is locally connected and path connected.

The main tool of finding the limit space of a contracting group is realization of the group as the iterated monodromy group of an expanding partial orbispace self-covering. An exposition of the theory of such self-coverings is given in [Nek05]. In particular, if G is the iterated monodromy group of a post-critically finite complex rational function, then the limit space of G is homeomorphic to the Julia set of the function (see Theorems 5.5.3 and 6.4.4 of [Nek05]).

The limit space does not change when we pass from X to X^n in the natural way (we will change then the limit dynamical system to its nth iterate). It also does not change if we post-conjugate the wreath recursion by an element of the wreath product $Symm(X) \ltimes G^X$, i.e., conjugate the group G by an element of the form $\gamma = \pi(g_0\gamma, g_1\gamma)$, where $\pi \in Symm(X)$ and $g_0, g_1 \in G$.

The limit space can be also visualized using its subdivision into *tiles*. This method is especially effective, when the group is generated by bounded automata.

Definition 6. Let G be a contracting group. A *tile* $\mathcal{T}_G$ of G is the quotient of the space $X^{-\omega}$ by the equivalence relation, which identifies two sequences $\ldots x_2x_1$ and $\ldots y_2y_1$ if there exists a sequence $g_k \in G$ assuming a finite number of values and such that

$$g_k(x_k \ldots x_1) = y_k \ldots y_1, \quad g_k|_{x_k \ldots x_1} = 1$$

for all k.

Again, an analog of Proposition 1 is true: the equivalence relation from Definition 6 is generated by the identifications $\ldots x_2x_1 = \ldots y_2y_1$ of sequences for which there exists a sequence $g_k, k = 0, 1, 2, \ldots$ of elements of the nucleus such that $g_k(x_k) = y_k$, $g_k|_{x_k} = g_{k-1}$ and $g_0 = 1$.

Suppose that G satisfies the *open set condition*, i.e., the trivial state can be reached from any other state of the generating automaton. Then the *boundary* of the tile $\mathcal{T}_G$ is the image in $\mathcal{T}_G$ of the set of sequences $\ldots x_2x_1$ such that there exists a sequence $g_k \in G$ assuming a finite number of values and such that $g_k|_{x_k \ldots x_1} \neq 1$. If G is generated by a finite symmetric set S, then it is sufficient to look for the sequence g_k inside S.

The limit space of G is obtained from the tile by some identifications of the points of the boundary. If the group G is generated by bounded automata, then its boundary consists of a finite number of points and it is not hard to identify them (i.e., to identify the sequences encoding them).

For $v \in X^n$ denote by $\mathcal{T}_G v$ the image of the cylindrical set $X^{-\omega}v$ in $\mathcal{T}_G$. It is easy to see that the map $\ldots x_2x_1 \mapsto \ldots x_2x_1v$ induces a homeomorphism of $\mathcal{T}_G$ with $\mathcal{T}_G v$ and that

$$\mathcal{T}_G = \bigcup_{v \in X^n} \mathcal{T}_G v.$$

It is proved in [Nek05] that two pieces $\mathcal{T}_G v_1$ and $\mathcal{T}_G v_2$ intersect if and only if $g(v_1) = v_2$ for an element g of the nucleus of G and that they intersect only along images of the boundary of $\mathcal{T}_G$.

This suggests the following procedure of visualizing the limit space in the case of bounded automata. Identify the points of the boundary of the tile. We get a finite list B of points, represented by a finite list W of infinite sequences (some points may be represented by several sequences). Draw the tile as a graph with $|B|$ "boundary points" (vertices) and identify the boundary points with the points of B labeled by sequences W. Take now $|X|$ copies of this tile, corresponding to different letters of X. Append the corresponding letters $x \in X$ to the ends of the labels $w \in W$ of the boundary points of each of the copy of the tile. Some of the obtained labels will be related by the equivalence relation of Definition 6, i.e., represent the same points of the tile $\mathcal{T}_G$. Glue the corresponding points together. Some of the obtained labels will belong to W. These points will be the new boundary points. In this way we get a new graph with labeled boundary points. Repeat now the procedure several times, rescaling the graph in such a way that the original first order graphs become small. We will get in this way a graph resembling the tile $\mathcal{T}_G$ (see Chapter V in [Bon07] for more details). Making the necessary identifications of its boundary we get an approximation of the limit space of G. More details on this inductive approximation procedure can be found in [Nek05] Section 3.10.

The limit space of a finitely generated contracting self-similar group G can also be viewed as a hyperbolic boundary in the following way. For a given finite generating system S of G define the *self-similarity graph* $\Sigma(G, S)$ as the graph with set of vertices X^* in which two vertices $v_1, v_2 \in X^*$ are connected by an edge if and only if either $v_i = xv_j$, for some $x \in X$ (vertical edges), or $s(v_i) = v_j$ for some $s \in S$ (horizontal edges). In case of a contracting group, the self-similarity graph $\Sigma(G, S)$ is Gromov-hyperbolic and its hyperbolic boundary is homeomorphic to the limit space $\mathcal{J}_G$.

The iterated monodromy group (IMG) construction is dual to the limit space construction. It may be defined for partial self-coverings of orbispaces, but we will only provide the definition in case of topological spaces, since we do not need the more general construction in this text (all iterated monodromy groups that appear later are related to partial self-coverings of the Riemann sphere).

Let $\mathcal{M}$ be a path connected and locally path connected topological space and let $\mathcal{M}_1$ be an open path connected subset of $\mathcal{M}$. Let $f : \mathcal{M}_1 \to \mathcal{M}$ be a d-fold covering. Denote by f^n the n-fold iteration of the map f. Then $f^n : \mathcal{M}_n \to \mathcal{M}$, where $\mathcal{M}_n = f^{-n}(\mathcal{M})$, is a d^n-fold covering.

Fix a base point $t \in \mathcal{M}$ and let T_t be the disjoint union of the sets $f^{-n}(t), n \geq 0$ (formally speaking, these sets may not be disjoint in $\mathcal{M}$). The set of pre-images T_t has a natural structure of a rooted d-ary tree.

The base point t is the root, the vertices in f^{-n} constitute level n and every vertex z in $f^{-n}(t)$ is connected by an edge to $f(z)$ in $f^{-n+1}(t)$, for $n \geq 1$. The fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally, through the monodromy action, on every level $f^{-n}(t)$ and, in fact, acts by automorphisms on T_t.

Definition 7. The *iterated monodromy group* $IMG(f)$ of the covering f is the quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of its action on the tree of pre-images T_t.

6. Classification guide

Every 3-state automaton $\mathcal{A}$ with set of states $S = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ acting on the 2-letter alphabet $X = \{0, 1\}$ is assigned a unique number as follows. Given the wreath recursion

$$\begin{cases} \mathbf{0} = \sigma^{a_{11}}(a_{12}, a_{13}), \\ \mathbf{1} = \sigma^{a_{21}}(a_{22}, a_{23}), \\ \mathbf{2} = \sigma^{a_{31}}(a_{32}, a_{33}), \end{cases}$$

defining the automaton $\mathcal{A}$, where $a_{ij} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ for $j \neq 1$ and $a_{i1} \in \{0, 1\}$, $i = 1, 2, 3$, assign the number

$$\begin{aligned} &\mathrm{Number}(\mathcal{A}) = \\ &\qquad a_{12} + 3a_{13} + 9a_{22} + 27a_{23} + 81a_{32} + \\ &\qquad 243a_{33} + 729(a_{11} + 2a_{21} + 4a_{31}) + 1 \end{aligned}$$

to $\mathcal{A}$. With this agreement every $(3, 2)$-automaton is assigned a unique number in the range from 1 to 5832. The numbering of the automata is induced by the lexicographic ordering of all automata in the class. Each of the automata numbered 1 through 729 generates the trivial group, since all vertex permutations are trivial in this case. Each of the automata numbered 5104 through 5832 generates the cyclic group C_2 of order 2, since both states represent the automorphism that acts by changing all letters in every word over X. Therefore the nontrivial part of the classification is concerned with the automata numbered by 730 through 5103.

Denote by $\mathcal{A}_n$ the automaton numbered by n and by G_n the corresponding group of tree automorphisms. Sometimes we may use just the number to refer to the corresponding automaton or group.

The following three operations on automata do not change the isomorphism class of the group generated by the corresponding automaton (and do not change the action on the tree in essential way):

(i) passing to inverses of all generators,

(ii) permuting the states of the automaton,

(iii) permuting the alphabet letters.

Definition 8. Two automata $\mathcal{A}$ and $\mathcal{B}$ that can be obtained from one another by using a composition of the operations (*i*)–(*iii*), are called *symmetric.*

For instance, the two automata in the lower right part of Figure 1 are symmetric. The wreath recursion for the automaton obtained by permuting both the names of the states and the alphabet letters of the first of these two automata is

$$\begin{aligned} a &= \ \ (b,a) \\ b &= \sigma(b,a) \end{aligned}$$

and this wreath recursion describes exactly the inverses of the tree automorphism defining the second of the two automata.

Additional identifications can be made after automata minimization is applied.

Definition 9. If the minimization of an automaton $\mathcal{A}$ is symmetric to the minimization of an automaton $\mathcal{B}$, we say that the automata $\mathcal{A}$ and $\mathcal{B}$ are *minimally symmetric* and write $\mathcal{A} \sim \mathcal{B}$.

There are 194 classes of $(3,2)$-automata that are pairwise not minimally symmetric. Of these, 10 are minimally symmetric to automata with fewer than 3 states and, as such, are subject of Theorem 7 ([GNS00], see below).

At present, it is known that there are no more than 122 non-isomorphic $(3,2)$-automaton groups. Some information on these groups is given in Section 8.

The proofs of some particular properties of the 194 classes of non-equivalent automata (and in particular, all known isomorphisms) can be found in Section 9. The few general results that hold in the whole class were already mentioned in the introduction.

The table in Section 7 may be used to determine the equivalence and the group isomorphism class for each automaton. Every class is numbered by the smallest number of an automaton in the class. For instance, an entry such as $x \sim y \cong z$ means that the automata with the smallest number in the equivalence and the (known) isomorphism class of x are y and z, respectively. While the equivalence classes are easy to determine the isomorphism class is not. Therefore, there may still be some additional isomorphisms between some of the classes (which would eventually cause changes in the z numbers and consolidation of some of the current isomorphism classes).

If one is interested in some particular $(3, 2)$-automaton $\mathcal{A}$, we recommend the following procedure:

- Use the table in Section 7 to find numbers for the representatives of the equivalence and the isomorphism class of $\mathcal{A}$. Minimizing the automaton and finding the symmetry is a straightforward task, which is not presented here.
- Use Section 8 to find information on the group generated by $\mathcal{A}$ (more precisely, the isomorphic group generated by the chosen representative in the class).
- Use Section 9 to find the proof of the isomorphism and some known properties.

7. Table of equivalence classes (and known isomorphisms)

For explanation of the entries see Section 6.

1 through 729 $\sim 1 \simeq 1$,

$730 \sim 730 \cong 730$	$767 \sim 767 \cong 731$	$804 \sim 804 \cong 731$	$841 \sim 839 \cong 821$
$731 \sim 731 \cong 731$	$768 \sim 768 \cong 731$	$805 \sim 803 \cong 771$	$842 \sim 842 \cong 838$
$732 \sim 731 \cong 731$	$769 \sim 767 \cong 731$	$806 \sim 806 \cong 802$	$843 \sim 843 \cong 843$
$733 \sim 731 \cong 731$	$770 \sim 770 \cong 730$	$807 \sim 807 \cong 771$	$844 \sim 840 \cong 840$
$734 \sim 734 \cong 730$	$771 \sim 771 \cong 771$	$808 \sim 804 \cong 731$	$845 \sim 843 \cong 843$
$735 \sim 734 \cong 730$	$772 \sim 768 \cong 731$	$809 \sim 807 \cong 771$	$846 \sim 846 \cong 846$
$736 \sim 731 \cong 731$	$773 \sim 771 \cong 771$	$810 \sim 810 \cong 802$	$847 \sim 847 \cong 847$
$737 \sim 734 \cong 730$	$774 \sim 774 \cong 730$	$811 \sim 748 \cong 748$	$848 \sim 848 \cong 750$
$738 \sim 734 \cong 730$	$775 \sim 775 \cong 775$	$812 \sim 750 \cong 750$	$849 \sim 849 \cong 849$
$739 \sim 739 \cong 739$	$776 \sim 776 \cong 776$	$813 \sim 749 \cong 749$	$850 \sim 848 \cong 750$
$740 \sim 740 \cong 740$	$777 \sim 777 \cong 777$	$814 \sim 750 \cong 750$	$851 \sim 851 \cong 847$
$741 \sim 741 \cong 741$	$778 \sim 776 \cong 776$	$815 \sim 756 \cong 748$	$852 \sim 852 \cong 852$
$742 \sim 740 \cong 740$	$779 \sim 779 \cong 779$	$816 \sim 753 \cong 753$	$853 \sim 849 \cong 849$
$743 \sim 743 \cong 739$	$780 \sim 780 \cong 780$	$817 \sim 749 \cong 749$	$854 \sim 852 \cong 852$
$744 \sim 744 \cong 744$	$781 \sim 777 \cong 777$	$818 \sim 753 \cong 753$	$855 \sim 855 \cong 847$
$745 \sim 741 \cong 741$	$782 \sim 780 \cong 780$	$819 \sim 752 \cong 752$	$856 \sim 856 \cong 856$
$746 \sim 744 \cong 744$	$783 \sim 783 \cong 775$	$820 \sim 820 \cong 820$	$857 \sim 857 \cong 857$
$747 \sim 747 \cong 739$	$784 \sim 748 \cong 748$	$821 \sim 821 \cong 821$	$858 \sim 858 \cong 858$
$748 \sim 748 \cong 748$	$785 \sim 749 \cong 749$	$822 \sim 821 \cong 821$	$859 \sim 857 \cong 857$
$749 \sim 749 \cong 749$	$786 \sim 750 \cong 750$	$823 \sim 821 \cong 821$	$860 \sim 860 \cong 860$
$750 \sim 750 \cong 750$	$787 \sim 749 \cong 749$	$824 \sim 824 \cong 820$	$861 \sim 861 \cong 861$
$751 \sim 749 \cong 749$	$788 \sim 752 \cong 752$	$825 \sim 824 \cong 820$	$862 \sim 858 \cong 858$
$752 \sim 752 \cong 752$	$789 \sim 753 \cong 753$	$826 \sim 821 \cong 821$	$863 \sim 861 \cong 861$
$753 \sim 753 \cong 753$	$790 \sim 750 \cong 750$	$827 \sim 824 \cong 820$	$864 \sim 864 \cong 864$
$754 \sim 750 \cong 750$	$791 \sim 753 \cong 753$	$828 \sim 824 \cong 820$	$865 \sim 865 \cong 820$
$755 \sim 753 \cong 753$	$792 \sim 756 \cong 748$	$829 \sim 820 \cong 820$	$866 \sim 866 \cong 866$
$756 \sim 756 \cong 748$	$793 \sim 775 \cong 775$	$830 \sim 821 \cong 821$	$867 \sim 866 \cong 866$
$757 \sim 739 \cong 739$	$794 \sim 776 \cong 776$	$831 \sim 821 \cong 821$	$868 \sim 866 \cong 866$
$758 \sim 740 \cong 740$	$795 \sim 777 \cong 777$	$832 \sim 821 \cong 821$	$869 \sim 869 \cong 869$
$759 \sim 741 \cong 741$	$796 \sim 776 \cong 776$	$833 \sim 824 \cong 820$	$870 \sim 870 \cong 870$
$760 \sim 740 \cong 740$	$797 \sim 779 \cong 779$	$834 \sim 824 \cong 820$	$871 \sim 866 \cong 866$
$761 \sim 743 \cong 739$	$798 \sim 780 \cong 780$	$835 \sim 821 \cong 821$	$872 \sim 870 \cong 870$
$762 \sim 744 \cong 744$	$799 \sim 777 \cong 777$	$836 \sim 824 \cong 820$	$873 \sim 869 \cong 869$
$763 \sim 741 \cong 741$	$800 \sim 780 \cong 780$	$837 \sim 824 \cong 820$	$874 \sim 874 \cong 874$
$764 \sim 744 \cong 744$	$801 \sim 783 \cong 775$	$838 \sim 838 \cong 838$	$875 \sim 875 \cong 875$
$765 \sim 747 \cong 739$	$802 \sim 802 \cong 802$	$839 \sim 839 \cong 821$	$876 \sim 876 \cong 876$
$766 \sim 766 \cong 730$	$803 \sim 803 \cong 771$	$840 \sim 840 \cong 840$	$877 \sim 875 \cong 875$

878	$\sim$	878	$\cong$	878	920	$\sim$	920	$\cong$	920	962	$\sim$	960	$\cong$	960	1004	$\sim$	824	$\cong$	820
879	$\sim$	879	$\cong$	879	921	$\sim$	920	$\cong$	920	963	$\sim$	963	$\cong$	963	1005	$\sim$	824	$\cong$	820
880	$\sim$	876	$\cong$	876	922	$\sim$	920	$\cong$	920	964	$\sim$	964	$\cong$	739	1006	$\sim$	821	$\cong$	821
881	$\sim$	879	$\cong$	879	923	$\sim$	923	$\cong$	923	965	$\sim$	965	$\cong$	965	1007	$\sim$	824	$\cong$	820
882	$\sim$	882	$\cong$	882	924	$\sim$	924	$\cong$	870	966	$\sim$	966	$\cong$	966	1008	$\sim$	824	$\cong$	820
883	$\sim$	883	$\cong$	883	925	$\sim$	920	$\cong$	920	967	$\sim$	965	$\cong$	965	1009	$\sim$	847	$\cong$	847
884	$\sim$	884	$\cong$	884	926	$\sim$	924	$\cong$	870	968	$\sim$	968	$\cong$	968	1010	$\sim$	848	$\cong$	750
885	$\sim$	885	$\cong$	885	927	$\sim$	923	$\cong$	923	969	$\sim$	969	$\cong$	969	1011	$\sim$	849	$\cong$	849
886	$\sim$	884	$\cong$	884	928	$\sim$	928	$\cong$	820	970	$\sim$	966	$\cong$	966	1012	$\sim$	848	$\cong$	750
887	$\sim$	887	$\cong$	887	929	$\sim$	929	$\cong$	929	971	$\sim$	969	$\cong$	969	1013	$\sim$	851	$\cong$	847
888	$\sim$	888	$\cong$	888	930	$\sim$	930	$\cong$	821	972	$\sim$	972	$\cong$	739	1014	$\sim$	852	$\cong$	852
889	$\sim$	885	$\cong$	885	931	$\sim$	929	$\cong$	929	973	$\sim$	748	$\cong$	748	1015	$\sim$	849	$\cong$	849
890	$\sim$	888	$\cong$	888	932	$\sim$	932	$\cong$	820	974	$\sim$	750	$\cong$	750	1016	$\sim$	852	$\cong$	852
891	$\sim$	891	$\cong$	891	933	$\sim$	933	$\cong$	849	975	$\sim$	749	$\cong$	749	1017	$\sim$	855	$\cong$	847
892	$\sim$	739	$\cong$	739	934	$\sim$	930	$\cong$	821	976	$\sim$	750	$\cong$	750	1018	$\sim$	874	$\cong$	874
893	$\sim$	741	$\cong$	741	935	$\sim$	933	$\cong$	849	977	$\sim$	756	$\cong$	748	1019	$\sim$	875	$\cong$	875
894	$\sim$	740	$\cong$	740	936	$\sim$	936	$\cong$	820	978	$\sim$	753	$\cong$	753	1020	$\sim$	876	$\cong$	876
895	$\sim$	741	$\cong$	741	937	$\sim$	937	$\cong$	937	979	$\sim$	749	$\cong$	749	1021	$\sim$	875	$\cong$	875
896	$\sim$	747	$\cong$	739	938	$\sim$	938	$\cong$	938	980	$\sim$	753	$\cong$	753	1022	$\sim$	878	$\cong$	878
897	$\sim$	744	$\cong$	744	939	$\sim$	939	$\cong$	939	981	$\sim$	752	$\cong$	752	1023	$\sim$	879	$\cong$	879
898	$\sim$	740	$\cong$	740	940	$\sim$	938	$\cong$	938	982	$\sim$	838	$\cong$	838	1024	$\sim$	876	$\cong$	876
899	$\sim$	744	$\cong$	744	941	$\sim$	941	$\cong$	941	983	$\sim$	839	$\cong$	821	1025	$\sim$	879	$\cong$	879
900	$\sim$	743	$\cong$	739	942	$\sim$	942	$\cong$	942	984	$\sim$	840	$\cong$	840	1026	$\sim$	882	$\cong$	882
901	$\sim$	820	$\cong$	820	943	$\sim$	939	$\cong$	939	985	$\sim$	839	$\cong$	821	1027	$\sim$	820	$\cong$	820
902	$\sim$	821	$\cong$	821	944	$\sim$	942	$\cong$	942	986	$\sim$	842	$\cong$	838	1028	$\sim$	821	$\cong$	821
903	$\sim$	821	$\cong$	821	945	$\sim$	945	$\cong$	941	987	$\sim$	843	$\cong$	843	1029	$\sim$	821	$\cong$	821
904	$\sim$	821	$\cong$	821	946	$\sim$	838	$\cong$	838	988	$\sim$	840	$\cong$	840	1030	$\sim$	821	$\cong$	821
905	$\sim$	824	$\cong$	820	947	$\sim$	840	$\cong$	840	989	$\sim$	843	$\cong$	843	1031	$\sim$	824	$\cong$	820
906	$\sim$	824	$\cong$	820	948	$\sim$	839	$\cong$	821	990	$\sim$	846	$\cong$	846	1032	$\sim$	824	$\cong$	820
907	$\sim$	821	$\cong$	821	949	$\sim$	840	$\cong$	840	991	$\sim$	865	$\cong$	820	1033	$\sim$	821	$\cong$	821
908	$\sim$	824	$\cong$	820	950	$\sim$	846	$\cong$	846	992	$\sim$	866	$\cong$	866	1034	$\sim$	824	$\cong$	820
909	$\sim$	824	$\cong$	820	951	$\sim$	843	$\cong$	843	993	$\sim$	866	$\cong$	866	1035	$\sim$	824	$\cong$	820
910	$\sim$	820	$\cong$	820	952	$\sim$	839	$\cong$	821	994	$\sim$	866	$\cong$	866	1036	$\sim$	856	$\cong$	856
911	$\sim$	821	$\cong$	821	953	$\sim$	843	$\cong$	843	995	$\sim$	869	$\cong$	869	1037	$\sim$	857	$\cong$	857
912	$\sim$	821	$\cong$	821	954	$\sim$	842	$\cong$	838	996	$\sim$	870	$\cong$	870	1038	$\sim$	858	$\cong$	858
913	$\sim$	821	$\cong$	821	955	$\sim$	955	$\cong$	937	997	$\sim$	866	$\cong$	866	1039	$\sim$	857	$\cong$	857
914	$\sim$	824	$\cong$	820	956	$\sim$	956	$\cong$	956	998	$\sim$	870	$\cong$	870	1040	$\sim$	860	$\cong$	860
915	$\sim$	824	$\cong$	820	957	$\sim$	957	$\cong$	957	999	$\sim$	869	$\cong$	869	1041	$\sim$	861	$\cong$	861
916	$\sim$	821	$\cong$	821	958	$\sim$	956	$\cong$	956	1000	$\sim$	820	$\cong$	820	1042	$\sim$	858	$\cong$	858
917	$\sim$	824	$\cong$	820	959	$\sim$	959	$\cong$	959	1001	$\sim$	821	$\cong$	821	1043	$\sim$	861	$\cong$	861
918	$\sim$	824	$\cong$	820	960	$\sim$	960	$\cong$	960	1002	$\sim$	821	$\cong$	821	1044	$\sim$	864	$\cong$	864
919	$\sim$	919	$\cong$	820	961	$\sim$	957	$\cong$	957	1003	$\sim$	821	$\cong$	821	1045	$\sim$	883	$\cong$	883

$1046 \sim 884 \cong 884$	$1088 \sim 969 \cong 969$	$1130 \sim 1094 \cong 1090$	$1172 \sim 1091 \cong 731$
$1047 \sim 885 \cong 885$	$1089 \sim 968 \cong 968$	$1131 \sim 1094 \cong 1090$	$1173 \sim 1091 \cong 731$
$1048 \sim 884 \cong 884$	$1090 \sim 1090 \cong 1090$	$1132 \sim 1091 \cong 731$	$1174 \sim 1091 \cong 731$
$1049 \sim 887 \cong 887$	$1091 \sim 1091 \cong 731$	$1133 \sim 1094 \cong 1090$	$1175 \sim 1094 \cong 1090$
$1050 \sim 888 \cong 888$	$1092 \sim 1091 \cong 731$	$1134 \sim 1094 \cong 1090$	$1176 \sim 1094 \cong 1090$
$1051 \sim 885 \cong 885$	$1093 \sim 1091 \cong 731$	$1135 \sim 775 \cong 775$	$1177 \sim 1091 \cong 731$
$1052 \sim 888 \cong 888$	$1094 \sim 1094 \cong 1090$	$1136 \sim 777 \cong 777$	$1178 \sim 1094 \cong 1090$
$1053 \sim 891 \cong 891$	$1095 \sim 1094 \cong 1090$	$1137 \sim 776 \cong 776$	$1179 \sim 1094 \cong 1090$
$1054 \sim 802 \cong 802$	$1096 \sim 1091 \cong 731$	$1138 \sim 777 \cong 777$	$1180 \sim 1090 \cong 1090$
$1055 \sim 804 \cong 731$	$1097 \sim 1094 \cong 1090$	$1139 \sim 783 \cong 775$	$1181 \sim 1091 \cong 731$
$1056 \sim 803 \cong 771$	$1098 \sim 1094 \cong 1090$	$1140 \sim 780 \cong 780$	$1182 \sim 1091 \cong 731$
$1057 \sim 804 \cong 731$	$1099 \sim 1090 \cong 1090$	$1141 \sim 776 \cong 776$	$1183 \sim 1091 \cong 731$
$1058 \sim 810 \cong 802$	$1100 \sim 1091 \cong 731$	$1142 \sim 780 \cong 780$	$1184 \sim 1094 \cong 1090$
$1059 \sim 807 \cong 771$	$1101 \sim 1091 \cong 731$	$1143 \sim 779 \cong 779$	$1185 \sim 1094 \cong 1090$
$1060 \sim 803 \cong 771$	$1102 \sim 1091 \cong 731$	$1144 \sim 955 \cong 937$	$1186 \sim 1091 \cong 731$
$1061 \sim 807 \cong 771$	$1103 \sim 1094 \cong 1090$	$1145 \sim 957 \cong 957$	$1187 \sim 1094 \cong 1090$
$1062 \sim 806 \cong 802$	$1104 \sim 1094 \cong 1090$	$1146 \sim 956 \cong 956$	$1188 \sim 1094 \cong 1090$
$1063 \sim 964 \cong 739$	$1105 \sim 1091 \cong 731$	$1147 \sim 957 \cong 957$	$1189 \sim 856 \cong 856$
$1064 \sim 966 \cong 966$	$1106 \sim 1094 \cong 1090$	$1148 \sim 963 \cong 963$	$1190 \sim 858 \cong 858$
$1065 \sim 965 \cong 965$	$1107 \sim 1094 \cong 1090$	$1149 \sim 960 \cong 960$	$1191 \sim 857 \cong 857$
$1066 \sim 966 \cong 966$	$1108 \sim 883 \cong 883$	$1150 \sim 956 \cong 956$	$1192 \sim 858 \cong 858$
$1067 \sim 972 \cong 739$	$1109 \sim 885 \cong 885$	$1151 \sim 960 \cong 960$	$1193 \sim 864 \cong 864$
$1068 \sim 969 \cong 969$	$1110 \sim 884 \cong 884$	$1152 \sim 959 \cong 959$	$1194 \sim 861 \cong 861$
$1069 \sim 965 \cong 965$	$1111 \sim 885 \cong 885$	$1153 \sim 874 \cong 874$	$1195 \sim 857 \cong 857$
$1070 \sim 969 \cong 969$	$1112 \sim 891 \cong 891$	$1154 \sim 876 \cong 876$	$1196 \sim 861 \cong 861$
$1071 \sim 968 \cong 968$	$1113 \sim 888 \cong 888$	$1155 \sim 875 \cong 875$	$1197 \sim 860 \cong 860$
$1072 \sim 883 \cong 883$	$1114 \sim 884 \cong 884$	$1156 \sim 876 \cong 876$	$1198 \sim 1090 \cong 1090$
$1073 \sim 885 \cong 885$	$1115 \sim 888 \cong 888$	$1157 \sim 882 \cong 882$	$1199 \sim 1091 \cong 731$
$1074 \sim 884 \cong 884$	$1116 \sim 887 \cong 887$	$1158 \sim 879 \cong 879$	$1200 \sim 1091 \cong 731$
$1075 \sim 885 \cong 885$	$1117 \sim 1090 \cong 1090$	$1159 \sim 875 \cong 875$	$1201 \sim 1091 \cong 731$
$1076 \sim 891 \cong 891$	$1118 \sim 1091 \cong 731$	$1160 \sim 879 \cong 879$	$1202 \sim 1094 \cong 1090$
$1077 \sim 888 \cong 888$	$1119 \sim 1091 \cong 731$	$1161 \sim 878 \cong 878$	$1203 \sim 1094 \cong 1090$
$1078 \sim 884 \cong 884$	$1120 \sim 1091 \cong 731$	$1162 \sim 937 \cong 937$	$1204 \sim 1091 \cong 731$
$1079 \sim 888 \cong 888$	$1121 \sim 1094 \cong 1090$	$1163 \sim 939 \cong 939$	$1205 \sim 1094 \cong 1090$
$1080 \sim 887 \cong 887$	$1122 \sim 1094 \cong 1090$	$1164 \sim 938 \cong 938$	$1206 \sim 1094 \cong 1090$
$1081 \sim 964 \cong 739$	$1123 \sim 1091 \cong 731$	$1165 \sim 939 \cong 939$	$1207 \sim 1090 \cong 1090$
$1082 \sim 966 \cong 966$	$1124 \sim 1094 \cong 1090$	$1166 \sim 945 \cong 941$	$1208 \sim 1091 \cong 731$
$1083 \sim 965 \cong 965$	$1125 \sim 1094 \cong 1090$	$1167 \sim 942 \cong 942$	$1209 \sim 1091 \cong 731$
$1084 \sim 966 \cong 966$	$1126 \sim 1090 \cong 1090$	$1168 \sim 938 \cong 938$	$1210 \sim 1091 \cong 731$
$1085 \sim 972 \cong 739$	$1127 \sim 1091 \cong 731$	$1169 \sim 942 \cong 942$	$1211 \sim 1094 \cong 1090$
$1086 \sim 969 \cong 969$	$1128 \sim 1091 \cong 731$	$1170 \sim 941 \cong 941$	$1212 \sim 1094 \cong 1090$
$1087 \sim 965 \cong 965$	$1129 \sim 1091 \cong 731$	$1171 \sim 1090 \cong 1090$	$1213 \sim 1091 \cong 731$

$1214 \sim 1094 \cong 1090$	$1256 \sim 932 \cong 820$	$1298 \sim 777 \cong 777$	$1340 \sim 1094 \cong 1090$
$1215 \sim 1094 \cong 1090$	$1257 \sim 933 \cong 849$	$1299 \sim 776 \cong 776$	$1341 \sim 1094 \cong 1090$
$1216 \sim 739 \cong 739$	$1258 \sim 930 \cong 821$	$1300 \sim 777 \cong 777$	$1342 \sim 1090 \cong 1090$
$1217 \sim 741 \cong 741$	$1259 \sim 933 \cong 849$	$1301 \sim 783 \cong 775$	$1343 \sim 1091 \cong 731$
$1218 \sim 740 \cong 740$	$1260 \sim 936 \cong 820$	$1302 \sim 780 \cong 780$	$1344 \sim 1091 \cong 731$
$1219 \sim 741 \cong 741$	$1261 \sim 955 \cong 937$	$1303 \sim 776 \cong 776$	$1345 \sim 1091 \cong 731$
$1220 \sim 747 \cong 739$	$1262 \sim 956 \cong 956$	$1304 \sim 780 \cong 780$	$1346 \sim 1094 \cong 1090$
$1221 \sim 744 \cong 744$	$1263 \sim 957 \cong 957$	$1305 \sim 779 \cong 779$	$1347 \sim 1094 \cong 1090$
$1222 \sim 740 \cong 740$	$1264 \sim 956 \cong 956$	$1306 \sim 937 \cong 937$	$1348 \sim 1091 \cong 731$
$1223 \sim 744 \cong 744$	$1265 \sim 959 \cong 959$	$1307 \sim 939 \cong 939$	$1349 \sim 1094 \cong 1090$
$1224 \sim 743 \cong 739$	$1266 \sim 960 \cong 960$	$1308 \sim 938 \cong 938$	$1350 \sim 1094 \cong 1090$
$1225 \sim 919 \cong 820$	$1267 \sim 957 \cong 957$	$1309 \sim 939 \cong 939$	$1351 \sim 874 \cong 874$
$1226 \sim 920 \cong 920$	$1268 \sim 960 \cong 960$	$1310 \sim 945 \cong 941$	$1352 \sim 876 \cong 876$
$1227 \sim 920 \cong 920$	$1269 \sim 963 \cong 963$	$1311 \sim 942 \cong 942$	$1353 \sim 875 \cong 875$
$1228 \sim 920 \cong 920$	$1270 \sim 820 \cong 820$	$1312 \sim 938 \cong 938$	$1354 \sim 876 \cong 876$
$1229 \sim 923 \cong 923$	$1271 \sim 821 \cong 821$	$1313 \sim 942 \cong 942$	$1355 \sim 882 \cong 882$
$1230 \sim 924 \cong 870$	$1272 \sim 821 \cong 821$	$1314 \sim 941 \cong 941$	$1356 \sim 879 \cong 879$
$1231 \sim 920 \cong 920$	$1273 \sim 821 \cong 821$	$1315 \sim 856 \cong 856$	$1357 \sim 875 \cong 875$
$1232 \sim 924 \cong 870$	$1274 \sim 824 \cong 820$	$1316 \sim 858 \cong 858$	$1358 \sim 879 \cong 879$
$1233 \sim 923 \cong 923$	$1275 \sim 824 \cong 820$	$1317 \sim 857 \cong 857$	$1359 \sim 878 \cong 878$
$1234 \sim 838 \cong 838$	$1276 \sim 821 \cong 821$	$1318 \sim 858 \cong 858$	$1360 \sim 1090 \cong 1090$
$1235 \sim 840 \cong 840$	$1277 \sim 824 \cong 820$	$1319 \sim 864 \cong 864$	$1361 \sim 1091 \cong 731$
$1236 \sim 839 \cong 821$	$1278 \sim 824 \cong 820$	$1320 \sim 861 \cong 861$	$1362 \sim 1091 \cong 731$
$1237 \sim 840 \cong 840$	$1279 \sim 937 \cong 937$	$1321 \sim 857 \cong 857$	$1363 \sim 1091 \cong 731$
$1238 \sim 846 \cong 846$	$1280 \sim 938 \cong 938$	$1322 \sim 861 \cong 861$	$1364 \sim 1094 \cong 1090$
$1239 \sim 843 \cong 843$	$1281 \sim 939 \cong 939$	$1323 \sim 860 \cong 860$	$1365 \sim 1094 \cong 1090$
$1240 \sim 839 \cong 821$	$1282 \sim 938 \cong 938$	$1324 \sim 955 \cong 937$	$1366 \sim 1091 \cong 731$
$1241 \sim 843 \cong 843$	$1283 \sim 941 \cong 941$	$1325 \sim 957 \cong 957$	$1367 \sim 1094 \cong 1090$
$1242 \sim 842 \cong 838$	$1284 \sim 942 \cong 942$	$1326 \sim 956 \cong 956$	$1368 \sim 1094 \cong 1090$
$1243 \sim 820 \cong 820$	$1285 \sim 939 \cong 939$	$1327 \sim 957 \cong 957$	$1369 \sim 1090 \cong 1090$
$1244 \sim 821 \cong 821$	$1286 \sim 942 \cong 942$	$1328 \sim 963 \cong 963$	$1370 \sim 1091 \cong 731$
$1245 \sim 821 \cong 821$	$1287 \sim 945 \cong 941$	$1329 \sim 960 \cong 960$	$1371 \sim 1091 \cong 731$
$1246 \sim 821 \cong 821$	$1288 \sim 964 \cong 739$	$1330 \sim 956 \cong 956$	$1372 \sim 1091 \cong 731$
$1247 \sim 824 \cong 820$	$1289 \sim 965 \cong 965$	$1331 \sim 960 \cong 960$	$1373 \sim 1094 \cong 1090$
$1248 \sim 824 \cong 820$	$1290 \sim 966 \cong 966$	$1332 \sim 959 \cong 959$	$1374 \sim 1094 \cong 1090$
$1249 \sim 821 \cong 821$	$1291 \sim 965 \cong 965$	$1333 \sim 1090 \cong 1090$	$1375 \sim 1091 \cong 731$
$1250 \sim 824 \cong 820$	$1292 \sim 968 \cong 968$	$1334 \sim 1091 \cong 731$	$1376 \sim 1094 \cong 1090$
$1251 \sim 824 \cong 820$	$1293 \sim 969 \cong 969$	$1335 \sim 1091 \cong 731$	$1377 \sim 1094 \cong 1090$
$1252 \sim 928 \cong 820$	$1294 \sim 966 \cong 966$	$1336 \sim 1091 \cong 731$	$1378 \sim 766 \cong 730$
$1253 \sim 929 \cong 929$	$1295 \sim 969 \cong 969$	$1337 \sim 1094 \cong 1090$	$1379 \sim 768 \cong 731$
$1254 \sim 930 \cong 821$	$1296 \sim 972 \cong 739$	$1338 \sim 1094 \cong 1090$	$1380 \sim 767 \cong 731$
$1255 \sim 929 \cong 929$	$1297 \sim 775 \cong 775$	$1339 \sim 1091 \cong 731$	$1381 \sim 768 \cong 731$

$1382 \sim 774 \cong 730$	$1424 \sim 1091 \cong 731$	$1466 \sim 891 \cong 891$	$1508 \sim 803 \cong 771$
$1383 \sim 771 \cong 771$	$1425 \sim 1091 \cong 731$	$1467 \sim 1094 \cong 1090$	$1509 \sim 884 \cong 884$
$1384 \sim 767 \cong 731$	$1426 \sim 1091 \cong 731$	$1468 \sim 1091 \cong 731$	$1510 \sim 1091 \cong 731$
$1385 \sim 771 \cong 771$	$1427 \sim 1094 \cong 1090$	$1469 \sim 966 \cong 966$	$1511 \sim 884 \cong 884$
$1386 \sim 770 \cong 730$	$1428 \sim 1094 \cong 1090$	$1470 \sim 1091 \cong 731$	$1512 \sim 1091 \cong 731$
$1387 \sim 928 \cong 820$	$1429 \sim 1091 \cong 731$	$1471 \sim 966 \cong 966$	$1513 \sim 1094 \cong 1090$
$1388 \sim 930 \cong 821$	$1430 \sim 1094 \cong 1090$	$1472 \sim 804 \cong 731$	$1514 \sim 969 \cong 969$
$1389 \sim 929 \cong 929$	$1431 \sim 1094 \cong 1090$	$1473 \sim 885 \cong 885$	$1515 \sim 1094 \cong 1090$
$1390 \sim 930 \cong 821$	$1432 \sim 847 \cong 847$	$1474 \sim 1091 \cong 731$	$1516 \sim 969 \cong 969$
$1391 \sim 936 \cong 820$	$1433 \sim 849 \cong 849$	$1475 \sim 885 \cong 885$	$1517 \sim 807 \cong 771$
$1392 \sim 933 \cong 849$	$1434 \sim 848 \cong 750$	$1476 \sim 1091 \cong 731$	$1518 \sim 888 \cong 888$
$1393 \sim 929 \cong 929$	$1435 \sim 849 \cong 849$	$1477 \sim 1094 \cong 1090$	$1519 \sim 1094 \cong 1090$
$1394 \sim 933 \cong 849$	$1436 \sim 855 \cong 847$	$1478 \sim 969 \cong 969$	$1520 \sim 888 \cong 888$
$1395 \sim 932 \cong 820$	$1437 \sim 852 \cong 852$	$1479 \sim 1094 \cong 1090$	$1521 \sim 1094 \cong 1090$
$1396 \sim 847 \cong 847$	$1438 \sim 848 \cong 750$	$1480 \sim 969 \cong 969$	$1522 \sim 1091 \cong 731$
$1397 \sim 849 \cong 849$	$1439 \sim 852 \cong 852$	$1481 \sim 807 \cong 771$	$1523 \sim 965 \cong 965$
$1398 \sim 848 \cong 750$	$1440 \sim 851 \cong 847$	$1482 \sim 888 \cong 888$	$1524 \sim 1091 \cong 731$
$1399 \sim 849 \cong 849$	$1441 \sim 1090 \cong 1090$	$1483 \sim 1094 \cong 1090$	$1525 \sim 965 \cong 965$
$1400 \sim 855 \cong 847$	$1442 \sim 1091 \cong 731$	$1484 \sim 888 \cong 888$	$1526 \sim 803 \cong 771$
$1401 \sim 852 \cong 852$	$1443 \sim 1091 \cong 731$	$1485 \sim 1094 \cong 1090$	$1527 \sim 884 \cong 884$
$1402 \sim 848 \cong 750$	$1444 \sim 1091 \cong 731$	$1486 \sim 1091 \cong 731$	$1528 \sim 1091 \cong 731$
$1403 \sim 852 \cong 852$	$1445 \sim 1094 \cong 1090$	$1487 \sim 966 \cong 966$	$1529 \sim 884 \cong 884$
$1404 \sim 851 \cong 847$	$1446 \sim 1094 \cong 1090$	$1488 \sim 1091 \cong 731$	$1530 \sim 1091 \cong 731$
$1405 \sim 928 \cong 820$	$1447 \sim 1091 \cong 731$	$1489 \sim 966 \cong 966$	$1531 \sim 1094 \cong 1090$
$1406 \sim 930 \cong 821$	$1448 \sim 1094 \cong 1090$	$1490 \sim 804 \cong 731$	$1532 \sim 968 \cong 968$
$1407 \sim 929 \cong 929$	$1449 \sim 1094 \cong 1090$	$1491 \sim 885 \cong 885$	$1533 \sim 1094 \cong 1090$
$1408 \sim 930 \cong 821$	$1450 \sim 1090 \cong 1090$	$1492 \sim 1091 \cong 731$	$1534 \sim 968 \cong 968$
$1409 \sim 936 \cong 820$	$1451 \sim 1091 \cong 731$	$1493 \sim 885 \cong 885$	$1535 \sim 806 \cong 802$
$1410 \sim 933 \cong 849$	$1452 \sim 1091 \cong 731$	$1494 \sim 1091 \cong 731$	$1536 \sim 887 \cong 887$
$1411 \sim 929 \cong 929$	$1453 \sim 1091 \cong 731$	$1495 \sim 1090 \cong 1090$	$1537 \sim 1094 \cong 1090$
$1412 \sim 933 \cong 849$	$1454 \sim 1094 \cong 1090$	$1496 \sim 964 \cong 739$	$1538 \sim 887 \cong 887$
$1413 \sim 932 \cong 820$	$1455 \sim 1094 \cong 1090$	$1497 \sim 1090 \cong 1090$	$1539 \sim 1094 \cong 1090$
$1414 \sim 1090 \cong 1090$	$1456 \sim 1091 \cong 731$	$1498 \sim 964 \cong 739$	$1540 \sim 851 \cong 847$
$1415 \sim 1091 \cong 731$	$1457 \sim 1094 \cong 1090$	$1499 \sim 802 \cong 802$	$1541 \sim 824 \cong 820$
$1416 \sim 1091 \cong 731$	$1458 \sim 1094 \cong 1090$	$1500 \sim 883 \cong 883$	$1542 \sim 878 \cong 878$
$1417 \sim 1091 \cong 731$	$1459 \sim 1094 \cong 1090$	$1501 \sim 1090 \cong 1090$	$1543 \sim 842 \cong 838$
$1418 \sim 1094 \cong 1090$	$1460 \sim 972 \cong 739$	$1502 \sim 883 \cong 883$	$1544 \sim 756 \cong 748$
$1419 \sim 1094 \cong 1090$	$1461 \sim 1094 \cong 1090$	$1503 \sim 1090 \cong 1090$	$1545 \sim 869 \cong 869$
$1420 \sim 1091 \cong 731$	$1462 \sim 972 \cong 739$	$1504 \sim 1091 \cong 731$	$1546 \sim 860 \cong 860$
$1421 \sim 1094 \cong 1090$	$1463 \sim 810 \cong 802$	$1505 \sim 965 \cong 965$	$1547 \sim 824 \cong 820$
$1422 \sim 1094 \cong 1090$	$1464 \sim 891 \cong 891$	$1506 \sim 1091 \cong 731$	$1548 \sim 887 \cong 887$
$1423 \sim 1090 \cong 1090$	$1465 \sim 1094 \cong 1090$	$1507 \sim 965 \cong 965$	$1549 \sim 848 \cong 750$

$1550 \sim 821 \cong 821$	$1592 \sim 821 \cong 821$	$1634 \sim 777 \cong 777$	$1676 \sim 942 \cong 942$
$1551 \sim 875 \cong 875$	$1593 \sim 885 \cong 885$	$1635 \sim 876 \cong 876$	$1677 \sim 1094 \cong 1090$
$1552 \sim 839 \cong 821$	$1594 \sim 852 \cong 852$	$1636 \sim 1091 \cong 731$	$1678 \sim 960 \cong 960$
$1553 \sim 750 \cong 750$	$1595 \sim 824 \cong 820$	$1637 \sim 858 \cong 858$	$1679 \sim 780 \cong 780$
$1554 \sim 866 \cong 866$	$1596 \sim 879 \cong 879$	$1638 \sim 1091 \cong 731$	$1680 \sim 879 \cong 879$
$1555 \sim 857 \cong 857$	$1597 \sim 843 \cong 843$	$1639 \sim 1094 \cong 1090$	$1681 \sim 1094 \cong 1090$
$1556 \sim 821 \cong 821$	$1598 \sim 753 \cong 753$	$1640 \sim 942 \cong 942$	$1682 \sim 861 \cong 861$
$1557 \sim 884 \cong 884$	$1599 \sim 870 \cong 870$	$1641 \sim 1094 \cong 1090$	$1683 \sim 1094 \cong 1090$
$1558 \sim 852 \cong 852$	$1600 \sim 861 \cong 861$	$1642 \sim 960 \cong 960$	$1684 \sim 1091 \cong 731$
$1559 \sim 824 \cong 820$	$1601 \sim 824 \cong 820$	$1643 \sim 780 \cong 780$	$1685 \sim 938 \cong 938$
$1560 \sim 879 \cong 879$	$1602 \sim 888 \cong 888$	$1644 \sim 879 \cong 879$	$1686 \sim 1091 \cong 731$
$1561 \sim 843 \cong 843$	$1603 \sim 849 \cong 849$	$1645 \sim 1094 \cong 1090$	$1687 \sim 956 \cong 956$
$1562 \sim 753 \cong 753$	$1604 \sim 821 \cong 821$	$1646 \sim 861 \cong 861$	$1688 \sim 776 \cong 776$
$1563 \sim 870 \cong 870$	$1605 \sim 876 \cong 876$	$1647 \sim 1094 \cong 1090$	$1689 \sim 875 \cong 875$
$1564 \sim 861 \cong 861$	$1606 \sim 840 \cong 840$	$1648 \sim 1091 \cong 731$	$1690 \sim 1091 \cong 731$
$1565 \sim 824 \cong 820$	$1607 \sim 749 \cong 749$	$1649 \sim 939 \cong 939$	$1691 \sim 857 \cong 857$
$1566 \sim 888 \cong 888$	$1608 \sim 866 \cong 866$	$1650 \sim 1091 \cong 731$	$1692 \sim 1091 \cong 731$
$1567 \sim 848 \cong 750$	$1609 \sim 858 \cong 858$	$1651 \sim 957 \cong 957$	$1693 \sim 1094 \cong 1090$
$1568 \sim 821 \cong 821$	$1610 \sim 821 \cong 821$	$1652 \sim 777 \cong 777$	$1694 \sim 941 \cong 941$
$1569 \sim 875 \cong 875$	$1611 \sim 885 \cong 885$	$1653 \sim 876 \cong 876$	$1695 \sim 1094 \cong 1090$
$1570 \sim 839 \cong 821$	$1612 \sim 855 \cong 847$	$1654 \sim 1091 \cong 731$	$1696 \sim 959 \cong 959$
$1571 \sim 750 \cong 750$	$1613 \sim 824 \cong 820$	$1655 \sim 858 \cong 858$	$1697 \sim 779 \cong 779$
$1572 \sim 866 \cong 866$	$1614 \sim 882 \cong 882$	$1656 \sim 1091 \cong 731$	$1698 \sim 878 \cong 878$
$1573 \sim 857 \cong 857$	$1615 \sim 846 \cong 846$	$1657 \sim 1090 \cong 1090$	$1699 \sim 1094 \cong 1090$
$1574 \sim 821 \cong 821$	$1616 \sim 752 \cong 752$	$1658 \sim 937 \cong 937$	$1700 \sim 860 \cong 860$
$1575 \sim 884 \cong 884$	$1617 \sim 869 \cong 869$	$1659 \sim 1090 \cong 1090$	$1701 \sim 1094 \cong 1090$
$1576 \sim 847 \cong 847$	$1618 \sim 864 \cong 864$	$1660 \sim 955 \cong 937$	$1702 \sim 851 \cong 847$
$1577 \sim 820 \cong 820$	$1619 \sim 824 \cong 820$	$1661 \sim 775 \cong 775$	$1703 \sim 842 \cong 838$
$1578 \sim 874 \cong 874$	$1620 \sim 891 \cong 891$	$1662 \sim 874 \cong 874$	$1704 \sim 860 \cong 860$
$1579 \sim 838 \cong 838$	$1621 \sim 1094 \cong 1090$	$1663 \sim 1090 \cong 1090$	$1705 \sim 824 \cong 820$
$1580 \sim 748 \cong 748$	$1622 \sim 945 \cong 941$	$1664 \sim 856 \cong 856$	$1706 \sim 756 \cong 748$
$1581 \sim 865 \cong 820$	$1623 \sim 1094 \cong 1090$	$1665 \sim 1090 \cong 1090$	$1707 \sim 824 \cong 820$
$1582 \sim 856 \cong 856$	$1624 \sim 963 \cong 963$	$1666 \sim 1091 \cong 731$	$1708 \sim 878 \cong 878$
$1583 \sim 820 \cong 820$	$1625 \sim 783 \cong 775$	$1667 \sim 938 \cong 938$	$1709 \sim 869 \cong 869$
$1584 \sim 883 \cong 883$	$1626 \sim 882 \cong 882$	$1668 \sim 1091 \cong 731$	$1710 \sim 887 \cong 887$
$1585 \sim 849 \cong 849$	$1627 \sim 1094 \cong 1090$	$1669 \sim 956 \cong 956$	$1711 \sim 848 \cong 750$
$1586 \sim 821 \cong 821$	$1628 \sim 864 \cong 864$	$1670 \sim 776 \cong 776$	$1712 \sim 839 \cong 821$
$1587 \sim 876 \cong 876$	$1629 \sim 1094 \cong 1090$	$1671 \sim 875 \cong 875$	$1713 \sim 857 \cong 857$
$1588 \sim 840 \cong 840$	$1630 \sim 1091 \cong 731$	$1672 \sim 1091 \cong 731$	$1714 \sim 821 \cong 821$
$1589 \sim 749 \cong 749$	$1631 \sim 939 \cong 939$	$1673 \sim 857 \cong 857$	$1715 \sim 750 \cong 750$
$1590 \sim 866 \cong 866$	$1632 \sim 1091 \cong 731$	$1674 \sim 1091 \cong 731$	$1716 \sim 821 \cong 821$
$1591 \sim 858 \cong 858$	$1633 \sim 957 \cong 957$	$1675 \sim 1094 \cong 1090$	$1717 \sim 875 \cong 875$

$1718 \sim 866 \cong 866$	$1760 \sim 753 \cong 753$	$1802 \sim 744 \cong 744$	$1844 \sim 753 \cong 753$
$1719 \sim 884 \cong 884$	$1761 \sim 824 \cong 820$	$1803 \sim 780 \cong 780$	$1845 \sim 807 \cong 771$
$1720 \sim 852 \cong 852$	$1762 \sim 879 \cong 879$	$1804 \sim 744 \cong 744$	$1846 \sim 768 \cong 731$
$1721 \sim 843 \cong 843$	$1763 \sim 870 \cong 870$	$1805 \sim 734 \cong 730$	$1847 \sim 741 \cong 741$
$1722 \sim 861 \cong 861$	$1764 \sim 888 \cong 888$	$1806 \sim 753 \cong 753$	$1848 \sim 777 \cong 777$
$1723 \sim 824 \cong 820$	$1765 \sim 849 \cong 849$	$1807 \sim 780 \cong 780$	$1849 \sim 741 \cong 741$
$1724 \sim 753 \cong 753$	$1766 \sim 840 \cong 840$	$1808 \sim 753 \cong 753$	$1850 \sim 731 \cong 731$
$1725 \sim 824 \cong 820$	$1767 \sim 858 \cong 858$	$1809 \sim 807 \cong 771$	$1851 \sim 750 \cong 750$
$1726 \sim 879 \cong 879$	$1768 \sim 821 \cong 821$	$1810 \sim 767 \cong 731$	$1852 \sim 777 \cong 777$
$1727 \sim 870 \cong 870$	$1769 \sim 749 \cong 749$	$1811 \sim 740 \cong 740$	$1853 \sim 750 \cong 750$
$1728 \sim 888 \cong 888$	$1770 \sim 821 \cong 821$	$1812 \sim 776 \cong 776$	$1854 \sim 804 \cong 731$
$1729 \sim 848 \cong 750$	$1771 \sim 876 \cong 876$	$1813 \sim 740 \cong 740$	$1855 \sim 774 \cong 730$
$1730 \sim 839 \cong 821$	$1772 \sim 866 \cong 866$	$1814 \sim 731 \cong 731$	$1856 \sim 747 \cong 739$
$1731 \sim 857 \cong 857$	$1773 \sim 885 \cong 885$	$1815 \sim 749 \cong 749$	$1857 \sim 783 \cong 775$
$1732 \sim 821 \cong 821$	$1774 \sim 855 \cong 847$	$1816 \sim 776 \cong 776$	$1858 \sim 747 \cong 739$
$1733 \sim 750 \cong 750$	$1775 \sim 846 \cong 846$	$1817 \sim 749 \cong 749$	$1859 \sim 734 \cong 730$
$1734 \sim 821 \cong 821$	$1776 \sim 864 \cong 864$	$1818 \sim 803 \cong 771$	$1860 \sim 756 \cong 748$
$1735 \sim 875 \cong 875$	$1777 \sim 824 \cong 820$	$1819 \sim 766 \cong 730$	$1861 \sim 783 \cong 775$
$1736 \sim 866 \cong 866$	$1778 \sim 752 \cong 752$	$1820 \sim 739 \cong 739$	$1862 \sim 756 \cong 748$
$1737 \sim 884 \cong 884$	$1779 \sim 824 \cong 820$	$1821 \sim 775 \cong 775$	$1863 \sim 810 \cong 802$
$1738 \sim 847 \cong 847$	$1780 \sim 882 \cong 882$	$1822 \sim 739 \cong 739$	$1864 \sim 932 \cong 820$
$1739 \sim 838 \cong 838$	$1781 \sim 869 \cong 869$	$1823 \sim 730 \cong 730$	$1865 \sim 923 \cong 923$
$1740 \sim 856 \cong 856$	$1782 \sim 891 \cong 891$	$1824 \sim 748 \cong 748$	$1866 \sim 941 \cong 941$
$1741 \sim 820 \cong 820$	$1783 \sim 770 \cong 730$	$1825 \sim 775 \cong 775$	$1867 \sim 824 \cong 820$
$1742 \sim 748 \cong 748$	$1784 \sim 743 \cong 739$	$1826 \sim 748 \cong 748$	$1868 \sim 747 \cong 739$
$1743 \sim 820 \cong 820$	$1785 \sim 779 \cong 779$	$1827 \sim 802 \cong 802$	$1869 \sim 824 \cong 820$
$1744 \sim 874 \cong 874$	$1786 \sim 743 \cong 739$	$1828 \sim 768 \cong 731$	$1870 \sim 959 \cong 959$
$1745 \sim 865 \cong 820$	$1787 \sim 734 \cong 730$	$1829 \sim 741 \cong 741$	$1871 \sim 846 \cong 846$
$1746 \sim 883 \cong 883$	$1788 \sim 752 \cong 752$	$1830 \sim 777 \cong 777$	$1872 \sim 968 \cong 968$
$1747 \sim 849 \cong 849$	$1789 \sim 779 \cong 779$	$1831 \sim 741 \cong 741$	$1873 \sim 929 \cong 929$
$1748 \sim 840 \cong 840$	$1790 \sim 752 \cong 752$	$1832 \sim 731 \cong 731$	$1874 \sim 920 \cong 920$
$1749 \sim 858 \cong 858$	$1791 \sim 806 \cong 802$	$1833 \sim 750 \cong 750$	$1875 \sim 938 \cong 938$
$1750 \sim 821 \cong 821$	$1792 \sim 767 \cong 731$	$1834 \sim 777 \cong 777$	$1876 \sim 821 \cong 821$
$1751 \sim 749 \cong 749$	$1793 \sim 740 \cong 740$	$1835 \sim 750 \cong 750$	$1877 \sim 741 \cong 741$
$1752 \sim 821 \cong 821$	$1794 \sim 776 \cong 776$	$1836 \sim 804 \cong 731$	$1878 \sim 821 \cong 821$
$1753 \sim 876 \cong 876$	$1795 \sim 740 \cong 740$	$1837 \sim 771 \cong 771$	$1879 \sim 956 \cong 956$
$1754 \sim 866 \cong 866$	$1796 \sim 731 \cong 731$	$1838 \sim 744 \cong 744$	$1880 \sim 840 \cong 840$
$1755 \sim 885 \cong 885$	$1797 \sim 749 \cong 749$	$1839 \sim 780 \cong 780$	$1881 \sim 965 \cong 965$
$1756 \sim 852 \cong 852$	$1798 \sim 776 \cong 776$	$1840 \sim 744 \cong 744$	$1882 \sim 933 \cong 849$
$1757 \sim 843 \cong 843$	$1799 \sim 749 \cong 749$	$1841 \sim 734 \cong 730$	$1883 \sim 924 \cong 870$
$1758 \sim 861 \cong 861$	$1800 \sim 803 \cong 771$	$1842 \sim 753 \cong 753$	$1884 \sim 942 \cong 942$
$1759 \sim 824 \cong 820$	$1801 \sim 771 \cong 771$	$1843 \sim 780 \cong 780$	$1885 \sim 824 \cong 820$

1886	$\sim$	744	$\cong$	744	1928	$\sim$	920	$\cong$	920	1970	$\sim$	879	$\cong$	879	2012	$\sim$	776	$\cong$	776
1887	$\sim$	824	$\cong$	820	1929	$\sim$	939	$\cong$	939	1971	$\sim$	1094	$\cong$	1090	2013	$\sim$	857	$\cong$	857
1888	$\sim$	960	$\cong$	960	1930	$\sim$	821	$\cong$	821	1972	$\sim$	1091	$\cong$	731	2014	$\sim$	1091	$\cong$	731
1889	$\sim$	843	$\cong$	843	1931	$\sim$	740	$\cong$	740	1973	$\sim$	957	$\cong$	957	2015	$\sim$	875	$\cong$	875
1890	$\sim$	969	$\cong$	969	1932	$\sim$	821	$\cong$	821	1974	$\sim$	1091	$\cong$	731	2016	$\sim$	1091	$\cong$	731
1891	$\sim$	929	$\cong$	929	1933	$\sim$	957	$\cong$	957	1975	$\sim$	939	$\cong$	939	2017	$\sim$	1094	$\cong$	1090
1892	$\sim$	920	$\cong$	920	1934	$\sim$	839	$\cong$	821	1976	$\sim$	777	$\cong$	777	2018	$\sim$	959	$\cong$	959
1893	$\sim$	938	$\cong$	938	1935	$\sim$	966	$\cong$	966	1977	$\sim$	858	$\cong$	858	2019	$\sim$	1094	$\cong$	1090
1894	$\sim$	821	$\cong$	821	1936	$\sim$	936	$\cong$	820	1978	$\sim$	1091	$\cong$	731	2020	$\sim$	941	$\cong$	941
1895	$\sim$	741	$\cong$	741	1937	$\sim$	923	$\cong$	923	1979	$\sim$	876	$\cong$	876	2021	$\sim$	779	$\cong$	779
1896	$\sim$	821	$\cong$	821	1938	$\sim$	945	$\cong$	941	1980	$\sim$	1091	$\cong$	731	2022	$\sim$	860	$\cong$	860
1897	$\sim$	956	$\cong$	956	1939	$\sim$	824	$\cong$	820	1981	$\sim$	1090	$\cong$	1090	2023	$\sim$	1094	$\cong$	1090
1898	$\sim$	840	$\cong$	840	1940	$\sim$	743	$\cong$	739	1982	$\sim$	955	$\cong$	937	2024	$\sim$	878	$\cong$	878
1899	$\sim$	965	$\cong$	965	1941	$\sim$	824	$\cong$	820	1983	$\sim$	1090	$\cong$	1090	2025	$\sim$	1094	$\cong$	1090
1900	$\sim$	928	$\cong$	820	1942	$\sim$	963	$\cong$	963	1984	$\sim$	937	$\cong$	937	2026	$\sim$	932	$\cong$	820
1901	$\sim$	919	$\cong$	820	1943	$\sim$	842	$\cong$	838	1985	$\sim$	775	$\cong$	775	2027	$\sim$	824	$\cong$	820
1902	$\sim$	937	$\cong$	937	1944	$\sim$	972	$\cong$	739	1986	$\sim$	856	$\cong$	856	2028	$\sim$	959	$\cong$	959
1903	$\sim$	820	$\cong$	820	1945	$\sim$	1094	$\cong$	1090	1987	$\sim$	1090	$\cong$	1090	2029	$\sim$	923	$\cong$	923
1904	$\sim$	739	$\cong$	739	1946	$\sim$	963	$\cong$	963	1988	$\sim$	874	$\cong$	874	2030	$\sim$	747	$\cong$	739
1905	$\sim$	820	$\cong$	820	1947	$\sim$	1094	$\cong$	1090	1989	$\sim$	1090	$\cong$	1090	2031	$\sim$	846	$\cong$	846
1906	$\sim$	955	$\cong$	937	1948	$\sim$	945	$\cong$	941	1990	$\sim$	1091	$\cong$	731	2032	$\sim$	941	$\cong$	941
1907	$\sim$	838	$\cong$	838	1949	$\sim$	783	$\cong$	775	1991	$\sim$	956	$\cong$	956	2033	$\sim$	824	$\cong$	820
1908	$\sim$	964	$\cong$	739	1950	$\sim$	864	$\cong$	864	1992	$\sim$	1091	$\cong$	731	2034	$\sim$	968	$\cong$	968
1909	$\sim$	930	$\cong$	821	1951	$\sim$	1094	$\cong$	1090	1993	$\sim$	938	$\cong$	938	2035	$\sim$	929	$\cong$	929
1910	$\sim$	920	$\cong$	920	1952	$\sim$	882	$\cong$	882	1994	$\sim$	776	$\cong$	776	2036	$\sim$	821	$\cong$	821
1911	$\sim$	939	$\cong$	939	1953	$\sim$	1094	$\cong$	1090	1995	$\sim$	857	$\cong$	857	2037	$\sim$	956	$\cong$	956
1912	$\sim$	821	$\cong$	821	1954	$\sim$	1091	$\cong$	731	1996	$\sim$	1091	$\cong$	731	2038	$\sim$	920	$\cong$	920
1913	$\sim$	740	$\cong$	740	1955	$\sim$	957	$\cong$	957	1997	$\sim$	875	$\cong$	875	2039	$\sim$	741	$\cong$	741
1914	$\sim$	821	$\cong$	821	1956	$\sim$	1091	$\cong$	731	1998	$\sim$	1091	$\cong$	731	2040	$\sim$	840	$\cong$	840
1915	$\sim$	957	$\cong$	957	1957	$\sim$	939	$\cong$	939	1999	$\sim$	1094	$\cong$	1090	2041	$\sim$	938	$\cong$	938
1916	$\sim$	839	$\cong$	821	1958	$\sim$	777	$\cong$	777	2000	$\sim$	960	$\cong$	960	2042	$\sim$	821	$\cong$	821
1917	$\sim$	966	$\cong$	966	1959	$\sim$	858	$\cong$	858	2001	$\sim$	1094	$\cong$	1090	2043	$\sim$	965	$\cong$	965
1918	$\sim$	933	$\cong$	849	1960	$\sim$	1091	$\cong$	731	2002	$\sim$	942	$\cong$	942	2044	$\sim$	933	$\cong$	849
1919	$\sim$	924	$\cong$	870	1961	$\sim$	876	$\cong$	876	2003	$\sim$	780	$\cong$	780	2045	$\sim$	824	$\cong$	820
1920	$\sim$	942	$\cong$	942	1962	$\sim$	1091	$\cong$	731	2004	$\sim$	861	$\cong$	861	2046	$\sim$	960	$\cong$	960
1921	$\sim$	824	$\cong$	820	1963	$\sim$	1094	$\cong$	1090	2005	$\sim$	1094	$\cong$	1090	2047	$\sim$	924	$\cong$	870
1922	$\sim$	744	$\cong$	744	1964	$\sim$	960	$\cong$	960	2006	$\sim$	879	$\cong$	879	2048	$\sim$	744	$\cong$	744
1923	$\sim$	824	$\cong$	820	1965	$\sim$	1094	$\cong$	1090	2007	$\sim$	1094	$\cong$	1090	2049	$\sim$	843	$\cong$	843
1924	$\sim$	960	$\cong$	960	1966	$\sim$	942	$\cong$	942	2008	$\sim$	1091	$\cong$	731	2050	$\sim$	942	$\cong$	942
1925	$\sim$	843	$\cong$	843	1967	$\sim$	780	$\cong$	780	2009	$\sim$	956	$\cong$	956	2051	$\sim$	824	$\cong$	820
1926	$\sim$	969	$\cong$	969	1968	$\sim$	861	$\cong$	861	2010	$\sim$	1091	$\cong$	731	2052	$\sim$	969	$\cong$	969
1927	$\sim$	930	$\cong$	821	1969	$\sim$	1094	$\cong$	1090	2011	$\sim$	938	$\cong$	938	2053	$\sim$	929	$\cong$	929

2054 $\sim$ 821 $\cong$ 821	2096 $\sim$ 821 $\cong$ 821	2138 $\sim$ 768 $\cong$ 731	2180 $\sim$ 932 $\cong$ 820
2055 $\sim$ 956 $\cong$ 956	2097 $\sim$ 966 $\cong$ 966	2139 $\sim$ 849 $\cong$ 849	2181 $\sim$ 1094 $\cong$ 1090
2056 $\sim$ 920 $\cong$ 920	2098 $\sim$ 936 $\cong$ 820	2140 $\sim$ 1091 $\cong$ 731	2182 $\sim$ 932 $\cong$ 820
2057 $\sim$ 741 $\cong$ 741	2099 $\sim$ 824 $\cong$ 820	2141 $\sim$ 849 $\cong$ 849	2183 $\sim$ 770 $\cong$ 730
2058 $\sim$ 840 $\cong$ 840	2100 $\sim$ 963 $\cong$ 963	2142 $\sim$ 1091 $\cong$ 731	2184 $\sim$ 851 $\cong$ 847
2059 $\sim$ 938 $\cong$ 938	2101 $\sim$ 923 $\cong$ 923	2143 $\sim$ 1090 $\cong$ 1090	2185 $\sim$ 1094 $\cong$ 1090
2060 $\sim$ 821 $\cong$ 821	2102 $\sim$ 743 $\cong$ 739	2144 $\sim$ 928 $\cong$ 820	2186 $\sim$ 851 $\cong$ 847
2061 $\sim$ 965 $\cong$ 965	2103 $\sim$ 842 $\cong$ 838	2145 $\sim$ 1090 $\cong$ 1090	2187 $\sim$ 1094 $\cong$ 1090
2062 $\sim$ 928 $\cong$ 820	2104 $\sim$ 945 $\cong$ 941	2146 $\sim$ 928 $\cong$ 820	2188 $\sim$ 730 $\cong$ 730
2063 $\sim$ 820 $\cong$ 820	2105 $\sim$ 824 $\cong$ 820	2147 $\sim$ 766 $\cong$ 730	2189 $\sim$ 730 $\cong$ 730
2064 $\sim$ 955 $\cong$ 937	2106 $\sim$ 972 $\cong$ 739	2148 $\sim$ 847 $\cong$ 847	2190 $\sim$ 2190 $\cong$ 750
2065 $\sim$ 919 $\cong$ 820	2107 $\sim$ 1094 $\cong$ 1090	2149 $\sim$ 1090 $\cong$ 1090	2191 $\sim$ 730 $\cong$ 730
2066 $\sim$ 739 $\cong$ 739	2108 $\sim$ 936 $\cong$ 820	2150 $\sim$ 847 $\cong$ 847	2192 $\sim$ 730 $\cong$ 730
2067 $\sim$ 838 $\cong$ 838	2109 $\sim$ 1094 $\cong$ 1090	2151 $\sim$ 1090 $\cong$ 1090	2193 $\sim$ 2193 $\cong$ 2193
2068 $\sim$ 937 $\cong$ 937	2110 $\sim$ 936 $\cong$ 820	2152 $\sim$ 1091 $\cong$ 731	2194 $\sim$ 2190 $\cong$ 750
2069 $\sim$ 820 $\cong$ 820	2111 $\sim$ 774 $\cong$ 730	2153 $\sim$ 929 $\cong$ 929	2195 $\sim$ 2193 $\cong$ 2193
2070 $\sim$ 964 $\cong$ 739	2112 $\sim$ 855 $\cong$ 847	2154 $\sim$ 1091 $\cong$ 731	2196 $\sim$ 2196 $\cong$ 802
2071 $\sim$ 930 $\cong$ 821	2113 $\sim$ 1094 $\cong$ 1090	2155 $\sim$ 929 $\cong$ 929	2197 $\sim$ 730 $\cong$ 730
2072 $\sim$ 821 $\cong$ 821	2114 $\sim$ 855 $\cong$ 847	2156 $\sim$ 767 $\cong$ 731	2198 $\sim$ 730 $\cong$ 730
2073 $\sim$ 957 $\cong$ 957	2115 $\sim$ 1094 $\cong$ 1090	2157 $\sim$ 848 $\cong$ 750	2199 $\sim$ 2199 $\cong$ 2199
2074 $\sim$ 920 $\cong$ 920	2116 $\sim$ 1091 $\cong$ 731	2158 $\sim$ 1091 $\cong$ 731	2200 $\sim$ 730 $\cong$ 730
2075 $\sim$ 740 $\cong$ 740	2117 $\sim$ 930 $\cong$ 821	2159 $\sim$ 848 $\cong$ 750	2201 $\sim$ 730 $\cong$ 730
2076 $\sim$ 839 $\cong$ 821	2118 $\sim$ 1091 $\cong$ 731	2160 $\sim$ 1091 $\cong$ 731	2202 $\sim$ 2202 $\cong$ 2202
2077 $\sim$ 939 $\cong$ 939	2119 $\sim$ 930 $\cong$ 821	2161 $\sim$ 1094 $\cong$ 1090	2203 $\sim$ 2203 $\cong$ 2203
2078 $\sim$ 821 $\cong$ 821	2120 $\sim$ 768 $\cong$ 731	2162 $\sim$ 933 $\cong$ 849	2204 $\sim$ 2204 $\cong$ 2204
2079 $\sim$ 966 $\cong$ 966	2121 $\sim$ 849 $\cong$ 849	2163 $\sim$ 1094 $\cong$ 1090	2205 $\sim$ 2205 $\cong$ 775
2080 $\sim$ 933 $\cong$ 849	2122 $\sim$ 1091 $\cong$ 731	2164 $\sim$ 933 $\cong$ 849	2206 $\sim$ 2206 $\cong$ 748
2081 $\sim$ 824 $\cong$ 820	2123 $\sim$ 849 $\cong$ 849	2165 $\sim$ 771 $\cong$ 771	2207 $\sim$ 2207 $\cong$ 2207
2082 $\sim$ 960 $\cong$ 960	2124 $\sim$ 1091 $\cong$ 731	2166 $\sim$ 852 $\cong$ 852	2208 $\sim$ 731 $\cong$ 731
2083 $\sim$ 924 $\cong$ 870	2125 $\sim$ 1094 $\cong$ 1090	2167 $\sim$ 1094 $\cong$ 1090	2209 $\sim$ 2209 $\cong$ 2209
2084 $\sim$ 744 $\cong$ 744	2126 $\sim$ 933 $\cong$ 849	2168 $\sim$ 852 $\cong$ 852	2210 $\sim$ 2210 $\cong$ 2210
2085 $\sim$ 843 $\cong$ 843	2127 $\sim$ 1094 $\cong$ 1090	2169 $\sim$ 1094 $\cong$ 1090	2211 $\sim$ 731 $\cong$ 731
2086 $\sim$ 942 $\cong$ 942	2128 $\sim$ 933 $\cong$ 849	2170 $\sim$ 1091 $\cong$ 731	2212 $\sim$ 2212 $\cong$ 2212
2087 $\sim$ 824 $\cong$ 820	2129 $\sim$ 771 $\cong$ 771	2171 $\sim$ 929 $\cong$ 929	2213 $\sim$ 2213 $\cong$ 2213
2088 $\sim$ 969 $\cong$ 969	2130 $\sim$ 852 $\cong$ 852	2172 $\sim$ 1091 $\cong$ 731	2214 $\sim$ 2214 $\cong$ 748
2089 $\sim$ 930 $\cong$ 821	2131 $\sim$ 1094 $\cong$ 1090	2173 $\sim$ 929 $\cong$ 929	2215 $\sim$ 730 $\cong$ 730
2090 $\sim$ 821 $\cong$ 821	2132 $\sim$ 852 $\cong$ 852	2174 $\sim$ 767 $\cong$ 731	2216 $\sim$ 730 $\cong$ 730
2091 $\sim$ 957 $\cong$ 957	2133 $\sim$ 1094 $\cong$ 1090	2175 $\sim$ 848 $\cong$ 750	2217 $\sim$ 2203 $\cong$ 2203
2092 $\sim$ 920 $\cong$ 920	2134 $\sim$ 1091 $\cong$ 731	2176 $\sim$ 1091 $\cong$ 731	2218 $\sim$ 730 $\cong$ 730
2093 $\sim$ 740 $\cong$ 740	2135 $\sim$ 930 $\cong$ 821	2177 $\sim$ 848 $\cong$ 750	2219 $\sim$ 730 $\cong$ 730
2094 $\sim$ 839 $\cong$ 821	2136 $\sim$ 1091 $\cong$ 731	2178 $\sim$ 1091 $\cong$ 731	2220 $\sim$ 2204 $\cong$ 2204
2095 $\sim$ 939 $\cong$ 939	2137 $\sim$ 930 $\cong$ 821	2179 $\sim$ 1094 $\cong$ 1090	2221 $\sim$ 2199 $\cong$ 2199

2222 ∼ 2202 ≅ 2202	2264 ∼ 2264 ≅ 730	2306 ∼ 730 ≅ 730	2348 ∼ 2295 ≅ 2295
2223 ∼ 2205 ≅ 775	2265 ∼ 2265 ≅ 2265	2307 ∼ 2307 ≅ 2307	2349 ∼ 734 ≅ 730
2224 ∼ 730 ≅ 730	2266 ∼ 2262 ≅ 750	2308 ∼ 730 ≅ 730	2350 ∼ 820 ≅ 820
2225 ∼ 730 ≅ 730	2267 ∼ 2265 ≅ 2265	2309 ∼ 730 ≅ 730	2351 ∼ 820 ≅ 820
2226 ∼ 2226 ≅ 820	2268 ∼ 734 ≅ 730	2310 ∼ 2287 ≅ 2287	2352 ∼ 2352 ≅ 740
2227 ∼ 730 ≅ 730	2269 ∼ 730 ≅ 730	2311 ∼ 2307 ≅ 2307	2353 ∼ 820 ≅ 820
2228 ∼ 730 ≅ 730	2270 ∼ 730 ≅ 730	2312 ∼ 2287 ≅ 2287	2354 ∼ 820 ≅ 820
2229 ∼ 2229 ≅ 2229	2271 ∼ 2271 ≅ 2271	2313 ∼ 2313 ≅ 2277	2355 ∼ 2355 ≅ 2355
2230 ∼ 2226 ≅ 820	2272 ∼ 730 ≅ 730	2314 ∼ 2307 ≅ 2307	2356 ∼ 2352 ≅ 740
2231 ∼ 2229 ≅ 2229	2273 ∼ 730 ≅ 730	2315 ∼ 2284 ≅ 2284	2357 ∼ 2355 ≅ 2355
2232 ∼ 2232 ≅ 730	2274 ∼ 2274 ≅ 2274	2316 ∼ 731 ≅ 731	2358 ∼ 2358 ≅ 820
2233 ∼ 2233 ≅ 2233	2275 ∼ 2271 ≅ 2271	2317 ∼ 2280 ≅ 2280	2359 ∼ 820 ≅ 820
2234 ∼ 2234 ≅ 2234	2276 ∼ 2274 ≅ 2274	2318 ∼ 2271 ≅ 2271	2360 ∼ 820 ≅ 820
2235 ∼ 731 ≅ 731	2277 ∼ 2277 ≅ 2277	2319 ∼ 731 ≅ 731	2361 ∼ 2361 ≅ 2361
2236 ∼ 2236 ≅ 2236	2278 ∼ 730 ≅ 730	2320 ∼ 2320 ≅ 2294	2362 ∼ 820 ≅ 820
2237 ∼ 2237 ≅ 2237	2279 ∼ 730 ≅ 730	2321 ∼ 2293 ≅ 2293	2363 ∼ 820 ≅ 820
2238 ∼ 731 ≅ 731	2280 ∼ 2280 ≅ 2280	2322 ∼ 2322 ≅ 2322	2364 ∼ 2364 ≅ 2364
2239 ∼ 2239 ≅ 2239	2281 ∼ 730 ≅ 730	2323 ∼ 2287 ≅ 2287	2365 ∼ 2365 ≅ 2365
2240 ∼ 2240 ≅ 2240	2282 ∼ 730 ≅ 730	2324 ∼ 2283 ≅ 2283	2366 ∼ 2366 ≅ 2366
2241 ∼ 2241 ≅ 739	2283 ∼ 2283 ≅ 2283	2325 ∼ 2293 ≅ 2293	2367 ∼ 2367 ≅ 2367
2242 ∼ 2206 ≅ 748	2284 ∼ 2284 ≅ 2284	2326 ∼ 2285 ≅ 2285	2368 ∼ 2368 ≅ 739
2243 ∼ 2209 ≅ 2209	2285 ∼ 2285 ≅ 2285	2327 ∼ 2274 ≅ 2274	2369 ∼ 2369 ≅ 2369
2244 ∼ 2212 ≅ 2212	2286 ∼ 2286 ≅ 2286	2328 ∼ 2294 ≅ 2294	2370 ∼ 821 ≅ 821
2245 ∼ 2207 ≅ 2207	2287 ∼ 2287 ≅ 2287	2329 ∼ 731 ≅ 731	2371 ∼ 2371 ≅ 2371
2246 ∼ 2210 ≅ 2210	2288 ∼ 2285 ≅ 2285	2330 ∼ 731 ≅ 731	2372 ∼ 2372 ≅ 2372
2247 ∼ 2213 ≅ 2213	2289 ∼ 731 ≅ 731	2331 ∼ 2295 ≅ 2295	2373 ∼ 821 ≅ 821
2248 ∼ 731 ≅ 731	2290 ∼ 2283 ≅ 2283	2332 ∼ 2307 ≅ 2307	2374 ∼ 2374 ≅ 821
2249 ∼ 731 ≅ 731	2291 ∼ 2274 ≅ 2274	2333 ∼ 2280 ≅ 2280	2375 ∼ 2375 ≅ 2375
2250 ∼ 2214 ≅ 748	2292 ∼ 731 ≅ 731	2334 ∼ 2320 ≅ 2294	2376 ∼ 2376 ≅ 739
2251 ∼ 2233 ≅ 2233	2293 ∼ 2293 ≅ 2293	2335 ∼ 2284 ≅ 2284	2377 ∼ 820 ≅ 820
2252 ∼ 2236 ≅ 2236	2294 ∼ 2294 ≅ 2294	2336 ∼ 2271 ≅ 2271	2378 ∼ 820 ≅ 820
2253 ∼ 2239 ≅ 2239	2295 ∼ 2295 ≅ 2295	2337 ∼ 2293 ≅ 2293	2379 ∼ 2365 ≅ 2365
2254 ∼ 2234 ≅ 2234	2296 ∼ 730 ≅ 730	2338 ∼ 731 ≅ 731	2380 ∼ 820 ≅ 820
2255 ∼ 2237 ≅ 2237	2297 ∼ 730 ≅ 730	2339 ∼ 731 ≅ 731	2381 ∼ 820 ≅ 820
2256 ∼ 2240 ≅ 2240	2298 ∼ 2284 ≅ 2284	2340 ∼ 2322 ≅ 2322	2382 ∼ 2366 ≅ 2366
2257 ∼ 731 ≅ 731	2299 ∼ 730 ≅ 730	2341 ∼ 2313 ≅ 2277	2383 ∼ 2361 ≅ 2361
2258 ∼ 731 ≅ 731	2300 ∼ 730 ≅ 730	2342 ∼ 2286 ≅ 2286	2384 ∼ 2364 ≅ 2364
2259 ∼ 2241 ≅ 739	2301 ∼ 2285 ≅ 2285	2343 ∼ 2322 ≅ 2322	2385 ∼ 2367 ≅ 2367
2260 ∼ 2260 ≅ 802	2302 ∼ 2280 ≅ 2280	2344 ∼ 2286 ≅ 2286	2386 ∼ 820 ≅ 820
2261 ∼ 2261 ≅ 2261	2303 ∼ 2283 ≅ 2283	2345 ∼ 2277 ≅ 2277	2387 ∼ 820 ≅ 820
2262 ∼ 2262 ≅ 750	2304 ∼ 2286 ≅ 2286	2346 ∼ 2295 ≅ 2295	2388 ∼ 2388 ≅ 821
2263 ∼ 2261 ≅ 2261	2305 ∼ 730 ≅ 730	2347 ∼ 2322 ≅ 2322	2389 ∼ 820 ≅ 820

$2390 \sim 820 \cong 820$	$2432 \sim 730 \cong 730$	$2474 \sim 2287 \cong 2287$	$2516 \sim 730 \cong 730$
$2391 \sim 2391 \cong 2391$	$2433 \sim 2271 \cong 2271$	$2475 \sim 2313 \cong 2277$	$2517 \sim 2210 \cong 2210$
$2392 \sim 2388 \cong 821$	$2434 \sim 730 \cong 730$	$2476 \sim 2307 \cong 2307$	$2518 \sim 2237 \cong 2237$
$2393 \sim 2391 \cong 2391$	$2435 \sim 730 \cong 730$	$2477 \sim 2284 \cong 2284$	$2519 \sim 2210 \cong 2210$
$2394 \sim 2394 \cong 820$	$2436 \sim 2274 \cong 2274$	$2478 \sim 731 \cong 731$	$2520 \sim 2264 \cong 730$
$2395 \sim 2395 \cong 2395$	$2437 \sim 2271 \cong 2271$	$2479 \sim 2280 \cong 2280$	$2521 \sim 730 \cong 730$
$2396 \sim 2396 \cong 2396$	$2438 \sim 2274 \cong 2274$	$2480 \sim 2271 \cong 2271$	$2522 \sim 730 \cong 730$
$2397 \sim 821 \cong 821$	$2439 \sim 2277 \cong 2277$	$2481 \sim 731 \cong 731$	$2523 \sim 2236 \cong 2236$
$2398 \sim 2398 \cong 2398$	$2440 \sim 730 \cong 730$	$2482 \sim 2320 \cong 2294$	$2524 \sim 730 \cong 730$
$2399 \sim 2399 \cong 2399$	$2441 \sim 730 \cong 730$	$2483 \sim 2293 \cong 2293$	$2525 \sim 730 \cong 730$
$2400 \sim 821 \cong 821$	$2442 \sim 2280 \cong 2280$	$2484 \sim 2322 \cong 2322$	$2526 \sim 2209 \cong 2209$
$2401 \sim 2401 \cong 2401$	$2443 \sim 730 \cong 730$	$2485 \sim 2287 \cong 2287$	$2527 \sim 2234 \cong 2234$
$2402 \sim 2402 \cong 2402$	$2444 \sim 730 \cong 730$	$2486 \sim 2283 \cong 2283$	$2528 \sim 2207 \cong 2207$
$2403 \sim 2403 \cong 2287$	$2445 \sim 2283 \cong 2283$	$2487 \sim 2293 \cong 2293$	$2529 \sim 2261 \cong 2261$
$2404 \sim 2368 \cong 739$	$2446 \sim 2284 \cong 2284$	$2488 \sim 2285 \cong 2285$	$2530 \sim 2229 \cong 2229$
$2405 \sim 2371 \cong 2371$	$2447 \sim 2285 \cong 2285$	$2489 \sim 2274 \cong 2274$	$2531 \sim 2204 \cong 2204$
$2406 \sim 2374 \cong 821$	$2448 \sim 2286 \cong 2286$	$2490 \sim 2294 \cong 2294$	$2532 \sim 731 \cong 731$
$2407 \sim 2369 \cong 2369$	$2449 \sim 2287 \cong 2287$	$2491 \sim 731 \cong 731$	$2533 \sim 2202 \cong 2202$
$2408 \sim 2372 \cong 2372$	$2450 \sim 2285 \cong 2285$	$2492 \sim 731 \cong 731$	$2534 \sim 2193 \cong 2193$
$2409 \sim 2375 \cong 2375$	$2451 \sim 731 \cong 731$	$2493 \sim 2295 \cong 2295$	$2535 \sim 731 \cong 731$
$2410 \sim 821 \cong 821$	$2452 \sim 2283 \cong 2283$	$2494 \sim 2307 \cong 2307$	$2536 \sim 2240 \cong 2240$
$2411 \sim 821 \cong 821$	$2453 \sim 2274 \cong 2274$	$2495 \sim 2280 \cong 2280$	$2537 \sim 2213 \cong 2213$
$2412 \sim 2376 \cong 739$	$2454 \sim 731 \cong 731$	$2496 \sim 2320 \cong 2294$	$2538 \sim 2265 \cong 2265$
$2413 \sim 2395 \cong 2395$	$2455 \sim 2293 \cong 2293$	$2497 \sim 2284 \cong 2284$	$2539 \sim 730 \cong 730$
$2414 \sim 2398 \cong 2398$	$2456 \sim 2294 \cong 2294$	$2498 \sim 2271 \cong 2271$	$2540 \sim 730 \cong 730$
$2415 \sim 2401 \cong 2401$	$2457 \sim 2295 \cong 2295$	$2499 \sim 2293 \cong 2293$	$2541 \sim 2234 \cong 2234$
$2416 \sim 2396 \cong 2396$	$2458 \sim 730 \cong 730$	$2500 \sim 731 \cong 731$	$2542 \sim 730 \cong 730$
$2417 \sim 2399 \cong 2399$	$2459 \sim 730 \cong 730$	$2501 \sim 731 \cong 731$	$2543 \sim 730 \cong 730$
$2418 \sim 2402 \cong 2402$	$2460 \sim 2284 \cong 2284$	$2502 \sim 2322 \cong 2322$	$2544 \sim 2207 \cong 2207$
$2419 \sim 821 \cong 821$	$2461 \sim 730 \cong 730$	$2503 \sim 2313 \cong 2277$	$2545 \sim 2236 \cong 2236$
$2420 \sim 821 \cong 821$	$2462 \sim 730 \cong 730$	$2504 \sim 2286 \cong 2286$	$2546 \sim 2209 \cong 2209$
$2421 \sim 2403 \cong 2287$	$2463 \sim 2285 \cong 2285$	$2505 \sim 2322 \cong 2322$	$2547 \sim 2261 \cong 2261$
$2422 \sim 2422 \cong 820$	$2464 \sim 2280 \cong 2280$	$2506 \sim 2286 \cong 2286$	$2548 \sim 730 \cong 730$
$2423 \sim 2423 \cong 2423$	$2465 \sim 2283 \cong 2283$	$2507 \sim 2277 \cong 2277$	$2549 \sim 730 \cong 730$
$2424 \sim 2424 \cong 966$	$2466 \sim 2286 \cong 2286$	$2508 \sim 2295 \cong 2295$	$2550 \sim 2233 \cong 2233$
$2425 \sim 2423 \cong 2423$	$2467 \sim 730 \cong 730$	$2509 \sim 2322 \cong 2322$	$2551 \sim 730 \cong 730$
$2426 \sim 2426 \cong 2277$	$2468 \sim 730 \cong 730$	$2510 \sim 2295 \cong 2295$	$2552 \sim 730 \cong 730$
$2427 \sim 2427 \cong 2427$	$2469 \sim 2307 \cong 2307$	$2511 \sim 734 \cong 730$	$2553 \sim 2206 \cong 748$
$2428 \sim 2424 \cong 966$	$2470 \sim 730 \cong 730$	$2512 \sim 730 \cong 730$	$2554 \sim 2233 \cong 2233$
$2429 \sim 2427 \cong 2427$	$2471 \sim 730 \cong 730$	$2513 \sim 730 \cong 730$	$2555 \sim 2206 \cong 748$
$2430 \sim 824 \cong 820$	$2472 \sim 2287 \cong 2287$	$2514 \sim 2237 \cong 2237$	$2556 \sim 2260 \cong 802$
$2431 \sim 730 \cong 730$	$2473 \sim 2307 \cong 2307$	$2515 \sim 730 \cong 730$	$2557 \sim 2226 \cong 820$

2558 ∼ 2203 ≅ 2203	2600 ∼ 2372 ≅ 2372	2642 ∼ 2352 ≅ 740	2684 ∼ 820 ≅ 820
2559 ∼ 731 ≅ 731	2601 ∼ 2426 ≅ 2277	2643 ∼ 821 ≅ 821	2685 ∼ 2361 ≅ 2361
2560 ∼ 2199 ≅ 2199	2602 ∼ 820 ≅ 820	2644 ∼ 2401 ≅ 2401	2686 ∼ 820 ≅ 820
2561 ∼ 2190 ≅ 750	2603 ∼ 820 ≅ 820	2645 ∼ 2374 ≅ 821	2687 ∼ 820 ≅ 820
2562 ∼ 731 ≅ 731	2604 ∼ 2398 ≅ 2398	2646 ∼ 2424 ≅ 966	2688 ∼ 2364 ≅ 2364
2563 ∼ 2239 ≅ 2239	2605 ∼ 820 ≅ 820	2647 ∼ 2391 ≅ 2391	2689 ∼ 2365 ≅ 2365
2564 ∼ 2212 ≅ 2212	2606 ∼ 820 ≅ 820	2648 ∼ 2364 ≅ 2364	2690 ∼ 2366 ≅ 2366
2565 ∼ 2262 ≅ 750	2607 ∼ 2371 ≅ 2371	2649 ∼ 2402 ≅ 2402	2691 ∼ 2367 ≅ 2367
2566 ∼ 2229 ≅ 2229	2608 ∼ 2396 ≅ 2396	2650 ∼ 2366 ≅ 2366	2692 ∼ 2368 ≅ 739
2567 ∼ 2202 ≅ 2202	2609 ∼ 2369 ≅ 2369	2651 ∼ 2355 ≅ 2355	2693 ∼ 2369 ≅ 2369
2568 ∼ 2240 ≅ 2240	2610 ∼ 2423 ≅ 2423	2652 ∼ 2375 ≅ 2375	2694 ∼ 821 ≅ 821
2569 ∼ 2204 ≅ 2204	2611 ∼ 2391 ≅ 2391	2653 ∼ 821 ≅ 821	2695 ∼ 2371 ≅ 2371
2570 ∼ 2193 ≅ 2193	2612 ∼ 2366 ≅ 2366	2654 ∼ 821 ≅ 821	2696 ∼ 2372 ≅ 2372
2571 ∼ 2213 ≅ 2213	2613 ∼ 821 ≅ 821	2655 ∼ 2427 ≅ 2427	2697 ∼ 821 ≅ 821
2572 ∼ 731 ≅ 731	2614 ∼ 2364 ≅ 2364	2656 ∼ 2388 ≅ 821	2698 ∼ 2374 ≅ 821
2573 ∼ 731 ≅ 731	2615 ∼ 2355 ≅ 2355	2657 ∼ 2361 ≅ 2361	2699 ∼ 2375 ≅ 2375
2574 ∼ 2265 ≅ 2265	2616 ∼ 821 ≅ 821	2658 ∼ 2401 ≅ 2401	2700 ∼ 2376 ≅ 739
2575 ∼ 2226 ≅ 820	2617 ∼ 2402 ≅ 2402	2659 ∼ 2365 ≅ 2365	2701 ∼ 820 ≅ 820
2576 ∼ 2199 ≅ 2199	2618 ∼ 2375 ≅ 2375	2660 ∼ 2352 ≅ 740	2702 ∼ 820 ≅ 820
2577 ∼ 2239 ≅ 2239	2619 ∼ 2427 ≅ 2427	2661 ∼ 2374 ≅ 821	2703 ∼ 2365 ≅ 2365
2578 ∼ 2203 ≅ 2203	2620 ∼ 820 ≅ 820	2662 ∼ 821 ≅ 821	2704 ∼ 820 ≅ 820
2579 ∼ 2190 ≅ 750	2621 ∼ 820 ≅ 820	2663 ∼ 821 ≅ 821	2705 ∼ 820 ≅ 820
2580 ∼ 2212 ≅ 2212	2622 ∼ 2396 ≅ 2396	2664 ∼ 2424 ≅ 966	2706 ∼ 2366 ≅ 2366
2581 ∼ 731 ≅ 731	2623 ∼ 820 ≅ 820	2665 ∼ 2394 ≅ 820	2707 ∼ 2361 ≅ 2361
2582 ∼ 731 ≅ 731	2624 ∼ 820 ≅ 820	2666 ∼ 2367 ≅ 2367	2708 ∼ 2364 ≅ 2364
2583 ∼ 2262 ≅ 750	2625 ∼ 2369 ≅ 2369	2667 ∼ 2403 ≅ 2287	2709 ∼ 2367 ≅ 2367
2584 ∼ 2232 ≅ 730	2626 ∼ 2398 ≅ 2398	2668 ∼ 2367 ≅ 2367	2710 ∼ 820 ≅ 820
2585 ∼ 2205 ≅ 775	2627 ∼ 2371 ≅ 2371	2669 ∼ 2358 ≅ 820	2711 ∼ 820 ≅ 820
2586 ∼ 2241 ≅ 739	2628 ∼ 2423 ≅ 2423	2670 ∼ 2376 ≅ 739	2712 ∼ 2388 ≅ 821
2587 ∼ 2205 ≅ 775	2629 ∼ 820 ≅ 820	2671 ∼ 2403 ≅ 2287	2713 ∼ 820 ≅ 820
2588 ∼ 2196 ≅ 802	2630 ∼ 820 ≅ 820	2672 ∼ 2376 ≅ 739	2714 ∼ 820 ≅ 820
2589 ∼ 2214 ≅ 748	2631 ∼ 2395 ≅ 2395	2673 ∼ 824 ≅ 820	2715 ∼ 2391 ≅ 2391
2590 ∼ 2241 ≅ 739	2632 ∼ 820 ≅ 820	2674 ∼ 820 ≅ 820	2716 ∼ 2388 ≅ 821
2591 ∼ 2214 ≅ 748	2633 ∼ 820 ≅ 820	2675 ∼ 820 ≅ 820	2717 ∼ 2391 ≅ 2391
2592 ∼ 734 ≅ 730	2634 ∼ 2368 ≅ 739	2676 ∼ 2352 ≅ 740	2718 ∼ 2394 ≅ 820
2593 ∼ 820 ≅ 820	2635 ∼ 2395 ≅ 2395	2677 ∼ 820 ≅ 820	2719 ∼ 2395 ≅ 2395
2594 ∼ 820 ≅ 820	2636 ∼ 2368 ≅ 739	2678 ∼ 820 ≅ 820	2720 ∼ 2396 ≅ 2396
2595 ∼ 2399 ≅ 2399	2637 ∼ 2422 ≅ 820	2679 ∼ 2355 ≅ 2355	2721 ∼ 821 ≅ 821
2596 ∼ 820 ≅ 820	2638 ∼ 2388 ≅ 821	2680 ∼ 2352 ≅ 740	2722 ∼ 2398 ≅ 2398
2597 ∼ 820 ≅ 820	2639 ∼ 2365 ≅ 2365	2681 ∼ 2355 ≅ 2355	2723 ∼ 2399 ≅ 2399
2598 ∼ 2372 ≅ 2372	2640 ∼ 821 ≅ 821	2682 ∼ 2358 ≅ 820	2724 ∼ 821 ≅ 821
2599 ∼ 2399 ≅ 2399	2641 ∼ 2361 ≅ 2361	2683 ∼ 820 ≅ 820	2725 ∼ 2401 ≅ 2401

$2726 \sim 2402 \cong 2402$	$2768 \sim 820 \cong 820$	$2810 \sim 2364 \cong 2364$	$2852 \sim 2852 \cong 849$
$2727 \sim 2403 \cong 2287$	$2769 \sim 2371 \cong 2371$	$2811 \sim 2402 \cong 2402$	$2853 \sim 2853 \cong 2853$
$2728 \sim 2368 \cong 739$	$2770 \sim 2396 \cong 2396$	$2812 \sim 2366 \cong 2366$	$2854 \sim 2854 \cong 847$
$2729 \sim 2371 \cong 2371$	$2771 \sim 2369 \cong 2369$	$2813 \sim 2355 \cong 2355$	$2855 \sim 2852 \cong 849$
$2730 \sim 2374 \cong 821$	$2772 \sim 2423 \cong 2423$	$2814 \sim 2375 \cong 2375$	$2856 \sim 1091 \cong 731$
$2731 \sim 2369 \cong 2369$	$2773 \sim 2391 \cong 2391$	$2815 \sim 821 \cong 821$	$2857 \sim 2850 \cong 2850$
$2732 \sim 2372 \cong 2372$	$2774 \sim 2366 \cong 2366$	$2816 \sim 821 \cong 821$	$2858 \sim 2841 \cong 2841$
$2733 \sim 2375 \cong 2375$	$2775 \sim 821 \cong 821$	$2817 \sim 2427 \cong 2427$	$2859 \sim 1091 \cong 731$
$2734 \sim 821 \cong 821$	$2776 \sim 2364 \cong 2364$	$2818 \sim 2388 \cong 821$	$2860 \sim 2860 \cong 2212$
$2735 \sim 821 \cong 821$	$2777 \sim 2355 \cong 2355$	$2819 \sim 2361 \cong 2361$	$2861 \sim 2861 \cong 731$
$2736 \sim 2376 \cong 739$	$2778 \sim 821 \cong 821$	$2820 \sim 2401 \cong 2401$	$2862 \sim 2862 \cong 847$
$2737 \sim 2395 \cong 2395$	$2779 \sim 2402 \cong 2402$	$2821 \sim 2365 \cong 2365$	$2863 \sim 1090 \cong 1090$
$2738 \sim 2398 \cong 2398$	$2780 \sim 2375 \cong 2375$	$2822 \sim 2352 \cong 740$	$2864 \sim 1090 \cong 1090$
$2739 \sim 2401 \cong 2401$	$2781 \sim 2427 \cong 2427$	$2823 \sim 2374 \cong 821$	$2865 \sim 2851 \cong 929$
$2740 \sim 2396 \cong 2396$	$2782 \sim 820 \cong 820$	$2824 \sim 821 \cong 821$	$2866 \sim 1090 \cong 1090$
$2741 \sim 2399 \cong 2399$	$2783 \sim 820 \cong 820$	$2825 \sim 821 \cong 821$	$2867 \sim 1090 \cong 1090$
$2742 \sim 2402 \cong 2402$	$2784 \sim 2396 \cong 2396$	$2826 \sim 2424 \cong 966$	$2868 \sim 2852 \cong 849$
$2743 \sim 821 \cong 821$	$2785 \sim 820 \cong 820$	$2827 \sim 2394 \cong 820$	$2869 \sim 2847 \cong 929$
$2744 \sim 821 \cong 821$	$2786 \sim 820 \cong 820$	$2828 \sim 2367 \cong 2367$	$2870 \sim 2850 \cong 2850$
$2745 \sim 2403 \cong 2287$	$2787 \sim 2369 \cong 2369$	$2829 \sim 2403 \cong 2287$	$2871 \sim 2853 \cong 2853$
$2746 \sim 2422 \cong 820$	$2788 \sim 2398 \cong 2398$	$2830 \sim 2367 \cong 2367$	$2872 \sim 1090 \cong 1090$
$2747 \sim 2423 \cong 2423$	$2789 \sim 2371 \cong 2371$	$2831 \sim 2358 \cong 820$	$2873 \sim 1090 \cong 1090$
$2748 \sim 2424 \cong 966$	$2790 \sim 2423 \cong 2423$	$2832 \sim 2376 \cong 739$	$2874 \sim 2874 \cong 820$
$2749 \sim 2423 \cong 2423$	$2791 \sim 820 \cong 820$	$2833 \sim 2403 \cong 2287$	$2875 \sim 1090 \cong 1090$
$2750 \sim 2426 \cong 2277$	$2792 \sim 820 \cong 820$	$2834 \sim 2376 \cong 739$	$2876 \sim 1090 \cong 1090$
$2751 \sim 2427 \cong 2427$	$2793 \sim 2395 \cong 2395$	$2835 \sim 824 \cong 820$	$2877 \sim 2854 \cong 847$
$2752 \sim 2424 \cong 966$	$2794 \sim 820 \cong 820$	$2836 \sim 1090 \cong 1090$	$2878 \sim 2874 \cong 820$
$2753 \sim 2427 \cong 2427$	$2795 \sim 820 \cong 820$	$2837 \sim 1090 \cong 1090$	$2879 \sim 2854 \cong 847$
$2754 \sim 824 \cong 820$	$2796 \sim 2368 \cong 739$	$2838 \sim 2838 \cong 750$	$2880 \sim 2880 \cong 730$
$2755 \sim 820 \cong 820$	$2797 \sim 2395 \cong 2395$	$2839 \sim 1090 \cong 1090$	$2881 \sim 2874 \cong 820$
$2756 \sim 820 \cong 820$	$2798 \sim 2368 \cong 739$	$2840 \sim 1090 \cong 1090$	$2882 \sim 2851 \cong 929$
$2757 \sim 2399 \cong 2399$	$2799 \sim 2422 \cong 820$	$2841 \sim 2841 \cong 2841$	$2883 \sim 1091 \cong 731$
$2758 \sim 820 \cong 820$	$2800 \sim 2388 \cong 821$	$2842 \sim 2838 \cong 750$	$2884 \sim 2847 \cong 929$
$2759 \sim 820 \cong 820$	$2801 \sim 2365 \cong 2365$	$2843 \sim 2841 \cong 2841$	$2885 \sim 2838 \cong 750$
$2760 \sim 2372 \cong 2372$	$2802 \sim 821 \cong 821$	$2844 \sim 2844 \cong 730$	$2886 \sim 1091 \cong 731$
$2761 \sim 2399 \cong 2399$	$2803 \sim 2361 \cong 2361$	$2845 \sim 1090 \cong 1090$	$2887 \sim 2887 \cong 731$
$2762 \sim 2372 \cong 2372$	$2804 \sim 2352 \cong 740$	$2846 \sim 1090 \cong 1090$	$2888 \sim 2860 \cong 2212$
$2763 \sim 2426 \cong 2277$	$2805 \sim 821 \cong 821$	$2847 \sim 2847 \cong 929$	$2889 \sim 2889 \cong 750$
$2764 \sim 820 \cong 820$	$2806 \sim 2401 \cong 2401$	$2848 \sim 1090 \cong 1090$	$2890 \sim 2854 \cong 847$
$2765 \sim 820 \cong 820$	$2807 \sim 2374 \cong 821$	$2849 \sim 1090 \cong 1090$	$2891 \sim 2850 \cong 2850$
$2766 \sim 2398 \cong 2398$	$2808 \sim 2424 \cong 966$	$2850 \sim 2850 \cong 2850$	$2892 \sim 2860 \cong 2212$
$2767 \sim 820 \cong 820$	$2809 \sim 2391 \cong 2391$	$2851 \sim 2851 \cong 929$	$2893 \sim 2852 \cong 849$

2894 ∼ 2841 ≅ 2841	2936 ∼ 878 ≅ 878	2978 ∼ 824 ≅ 820	3020 ∼ 888 ≅ 888
2895 ∼ 2861 ≅ 731	2937 ∼ 824 ≅ 820	2979 ∼ 756 ≅ 748	3021 ∼ 824 ≅ 820
2896 ∼ 1091 ≅ 731	2938 ∼ 860 ≅ 860	2980 ∼ 932 ≅ 820	3022 ∼ 843 ≅ 843
2897 ∼ 1091 ≅ 731	2939 ∼ 887 ≅ 887	2981 ∼ 941 ≅ 941	3023 ∼ 870 ≅ 870
2898 ∼ 2862 ≅ 847	2940 ∼ 824 ≅ 820	2982 ∼ 923 ≅ 923	3024 ∼ 753 ≅ 753
2899 ∼ 2874 ≅ 820	2941 ∼ 842 ≅ 838	2983 ∼ 959 ≅ 959	3025 ∼ 1094 ≅ 1090
2900 ∼ 2847 ≅ 929	2942 ∼ 869 ≅ 869	2984 ∼ 968 ≅ 968	3026 ∼ 1094 ≅ 1090
2901 ∼ 2887 ≅ 731	2943 ∼ 756 ≅ 748	2985 ∼ 846 ≅ 846	3027 ∼ 960 ≅ 960
2902 ∼ 2851 ≅ 929	2944 ∼ 1094 ≅ 1090	2986 ∼ 824 ≅ 820	3028 ∼ 1094 ≅ 1090
2903 ∼ 2838 ≅ 750	2945 ∼ 1094 ≅ 1090	2987 ∼ 824 ≅ 820	3029 ∼ 1094 ≅ 1090
2904 ∼ 2860 ≅ 2212	2946 ∼ 963 ≅ 963	2988 ∼ 747 ≅ 739	3030 ∼ 879 ≅ 879
2905 ∼ 1091 ≅ 731	2947 ∼ 1094 ≅ 1090	2989 ∼ 770 ≅ 730	3031 ∼ 942 ≅ 942
2906 ∼ 1091 ≅ 731	2948 ∼ 1094 ≅ 1090	2990 ∼ 779 ≅ 779	3032 ∼ 861 ≅ 861
2907 ∼ 2889 ≅ 750	2949 ∼ 882 ≅ 882	2991 ∼ 743 ≅ 739	3033 ∼ 780 ≅ 780
2908 ∼ 2880 ≅ 730	2950 ∼ 945 ≅ 941	2992 ∼ 779 ≅ 779	3034 ∼ 1094 ≅ 1090
2909 ∼ 2853 ≅ 2853	2951 ∼ 864 ≅ 864	2993 ∼ 806 ≅ 802	3035 ∼ 1094 ≅ 1090
2910 ∼ 2889 ≅ 750	2952 ∼ 783 ≅ 775	2994 ∼ 752 ≅ 752	3036 ∼ 933 ≅ 849
2911 ∼ 2853 ≅ 2853	2953 ∼ 1094 ≅ 1090	2995 ∼ 743 ≅ 739	3037 ∼ 1094 ≅ 1090
2912 ∼ 2844 ≅ 730	2954 ∼ 1094 ≅ 1090	2996 ∼ 752 ≅ 752	3038 ∼ 1094 ≅ 1090
2913 ∼ 2862 ≅ 847	2955 ∼ 936 ≅ 820	2997 ∼ 734 ≅ 730	3039 ∼ 852 ≅ 852
2914 ∼ 2889 ≅ 750	2956 ∼ 1094 ≅ 1090	2998 ∼ 1094 ≅ 1090	3040 ∼ 933 ≅ 849
2915 ∼ 2862 ≅ 847	2957 ∼ 1094 ≅ 1090	2999 ∼ 1094 ≅ 1090	3041 ∼ 852 ≅ 852
2916 ∼ 1094 ≅ 1090	2958 ∼ 855 ≅ 847	3000 ∼ 969 ≅ 969	3042 ∼ 771 ≅ 771
2917 ∼ 1094 ≅ 1090	2959 ∼ 936 ≅ 820	3001 ∼ 1094 ≅ 1090	3043 ∼ 933 ≅ 849
2918 ∼ 1094 ≅ 1090	2960 ∼ 855 ≅ 847	3002 ∼ 1094 ≅ 1090	3044 ∼ 960 ≅ 960
2919 ∼ 972 ≅ 739	2961 ∼ 774 ≅ 730	3003 ∼ 888 ≅ 888	3045 ∼ 824 ≅ 820
2920 ∼ 1094 ≅ 1090	2962 ∼ 932 ≅ 820	3004 ∼ 969 ≅ 969	3046 ∼ 942 ≅ 942
2921 ∼ 1094 ≅ 1090	2963 ∼ 959 ≅ 959	3005 ∼ 888 ≅ 888	3047 ∼ 969 ≅ 969
2922 ∼ 891 ≅ 891	2964 ∼ 824 ≅ 820	3006 ∼ 807 ≅ 771	3048 ∼ 824 ≅ 820
2923 ∼ 972 ≅ 739	2965 ∼ 941 ≅ 941	3007 ∼ 1094 ≅ 1090	3049 ∼ 924 ≅ 870
2924 ∼ 891 ≅ 891	2966 ∼ 968 ≅ 968	3008 ∼ 1094 ≅ 1090	3050 ∼ 843 ≅ 843
2925 ∼ 810 ≅ 802	2967 ∼ 824 ≅ 820	3009 ∼ 942 ≅ 942	3051 ∼ 744 ≅ 744
2926 ∼ 1094 ≅ 1090	2968 ∼ 923 ≅ 923	3010 ∼ 1094 ≅ 1090	3052 ∼ 852 ≅ 852
2927 ∼ 1094 ≅ 1090	2969 ∼ 846 ≅ 846	3011 ∼ 1094 ≅ 1090	3053 ∼ 861 ≅ 861
2928 ∼ 945 ≅ 941	2970 ∼ 747 ≅ 739	3012 ∼ 861 ≅ 861	3054 ∼ 843 ≅ 843
2929 ∼ 1094 ≅ 1090	2971 ∼ 851 ≅ 847	3013 ∼ 960 ≅ 960	3055 ∼ 879 ≅ 879
2930 ∼ 1094 ≅ 1090	2972 ∼ 860 ≅ 860	3014 ∼ 879 ≅ 879	3056 ∼ 888 ≅ 888
2931 ∼ 864 ≅ 864	2973 ∼ 842 ≅ 838	3015 ∼ 780 ≅ 780	3057 ∼ 870 ≅ 870
2932 ∼ 963 ≅ 963	2974 ∼ 878 ≅ 878	3016 ∼ 852 ≅ 852	3058 ∼ 824 ≅ 820
2933 ∼ 882 ≅ 882	2975 ∼ 887 ≅ 887	3017 ∼ 879 ≅ 879	3059 ∼ 824 ≅ 820
2934 ∼ 783 ≅ 775	2976 ∼ 869 ≅ 869	3018 ∼ 824 ≅ 820	3060 ∼ 753 ≅ 753
2935 ∼ 851 ≅ 847	2977 ∼ 824 ≅ 820	3019 ∼ 861 ≅ 861	3061 ∼ 933 ≅ 849

3062 ∼ 942 ≅ 942	3104 ∼ 866 ≅ 866	3146 ∼ 965 ≅ 965	3188 ∼ 1094 ≅ 1090
3063 ∼ 924 ≅ 870	3105 ∼ 750 ≅ 750	3147 ∼ 840 ≅ 840	3189 ∼ 960 ≅ 960
3064 ∼ 960 ≅ 960	3106 ∼ 1091 ≅ 731	3148 ∼ 821 ≅ 821	3190 ∼ 1094 ≅ 1090
3065 ∼ 969 ≅ 969	3107 ∼ 1091 ≅ 731	3149 ∼ 821 ≅ 821	3191 ∼ 1094 ≅ 1090
3066 ∼ 843 ≅ 843	3108 ∼ 957 ≅ 957	3150 ∼ 741 ≅ 741	3192 ∼ 879 ≅ 879
3067 ∼ 824 ≅ 820	3109 ∼ 1091 ≅ 731	3151 ∼ 767 ≅ 731	3193 ∼ 942 ≅ 942
3068 ∼ 824 ≅ 820	3110 ∼ 1091 ≅ 731	3152 ∼ 776 ≅ 776	3194 ∼ 861 ≅ 861
3069 ∼ 744 ≅ 744	3111 ∼ 876 ≅ 876	3153 ∼ 740 ≅ 740	3195 ∼ 780 ≅ 780
3070 ∼ 771 ≅ 771	3112 ∼ 939 ≅ 939	3154 ∼ 776 ≅ 776	3196 ∼ 1094 ≅ 1090
3071 ∼ 780 ≅ 780	3113 ∼ 858 ≅ 858	3155 ∼ 803 ≅ 771	3197 ∼ 1094 ≅ 1090
3072 ∼ 744 ≅ 744	3114 ∼ 777 ≅ 777	3156 ∼ 749 ≅ 749	3198 ∼ 933 ≅ 849
3073 ∼ 780 ≅ 780	3115 ∼ 1091 ≅ 731	3157 ∼ 740 ≅ 740	3199 ∼ 1094 ≅ 1090
3074 ∼ 807 ≅ 771	3116 ∼ 1091 ≅ 731	3158 ∼ 749 ≅ 749	3200 ∼ 1094 ≅ 1090
3075 ∼ 753 ≅ 753	3117 ∼ 930 ≅ 821	3159 ∼ 731 ≅ 731	3201 ∼ 852 ≅ 852
3076 ∼ 744 ≅ 744	3118 ∼ 1091 ≅ 731	3160 ∼ 1094 ≅ 1090	3202 ∼ 933 ≅ 849
3077 ∼ 753 ≅ 753	3119 ∼ 1091 ≅ 731	3161 ∼ 1094 ≅ 1090	3203 ∼ 852 ≅ 852
3078 ∼ 734 ≅ 730	3120 ∼ 849 ≅ 849	3162 ∼ 969 ≅ 969	3204 ∼ 771 ≅ 771
3079 ∼ 1091 ≅ 731	3121 ∼ 930 ≅ 821	3163 ∼ 1094 ≅ 1090	3205 ∼ 933 ≅ 849
3080 ∼ 1091 ≅ 731	3122 ∼ 849 ≅ 849	3164 ∼ 1094 ≅ 1090	3206 ∼ 960 ≅ 960
3081 ∼ 966 ≅ 966	3123 ∼ 768 ≅ 731	3165 ∼ 888 ≅ 888	3207 ∼ 824 ≅ 820
3082 ∼ 1091 ≅ 731	3124 ∼ 929 ≅ 929	3166 ∼ 969 ≅ 969	3208 ∼ 942 ≅ 942
3083 ∼ 1091 ≅ 731	3125 ∼ 956 ≅ 956	3167 ∼ 888 ≅ 888	3209 ∼ 969 ≅ 969
3084 ∼ 885 ≅ 885	3126 ∼ 821 ≅ 821	3168 ∼ 807 ≅ 771	3210 ∼ 824 ≅ 820
3085 ∼ 966 ≅ 966	3127 ∼ 938 ≅ 938	3169 ∼ 1094 ≅ 1090	3211 ∼ 924 ≅ 870
3086 ∼ 885 ≅ 885	3128 ∼ 965 ≅ 965	3170 ∼ 1094 ≅ 1090	3212 ∼ 843 ≅ 843
3087 ∼ 804 ≅ 731	3129 ∼ 821 ≅ 821	3171 ∼ 942 ≅ 942	3213 ∼ 744 ≅ 744
3088 ∼ 1091 ≅ 731	3130 ∼ 920 ≅ 920	3172 ∼ 1094 ≅ 1090	3214 ∼ 852 ≅ 852
3089 ∼ 1091 ≅ 731	3131 ∼ 840 ≅ 840	3173 ∼ 1094 ≅ 1090	3215 ∼ 861 ≅ 861
3090 ∼ 939 ≅ 939	3132 ∼ 741 ≅ 741	3174 ∼ 861 ≅ 861	3216 ∼ 843 ≅ 843
3091 ∼ 1091 ≅ 731	3133 ∼ 848 ≅ 750	3175 ∼ 960 ≅ 960	3217 ∼ 879 ≅ 879
3092 ∼ 1091 ≅ 731	3134 ∼ 857 ≅ 857	3176 ∼ 879 ≅ 879	3218 ∼ 888 ≅ 888
3093 ∼ 858 ≅ 858	3135 ∼ 839 ≅ 821	3177 ∼ 780 ≅ 780	3219 ∼ 870 ≅ 870
3094 ∼ 957 ≅ 957	3136 ∼ 875 ≅ 875	3178 ∼ 852 ≅ 852	3220 ∼ 824 ≅ 820
3095 ∼ 876 ≅ 876	3137 ∼ 884 ≅ 884	3179 ∼ 879 ≅ 879	3221 ∼ 824 ≅ 820
3096 ∼ 777 ≅ 777	3138 ∼ 866 ≅ 866	3180 ∼ 824 ≅ 820	3222 ∼ 753 ≅ 753
3097 ∼ 848 ≅ 750	3139 ∼ 821 ≅ 821	3181 ∼ 861 ≅ 861	3223 ∼ 933 ≅ 849
3098 ∼ 875 ≅ 875	3140 ∼ 821 ≅ 821	3182 ∼ 888 ≅ 888	3224 ∼ 942 ≅ 942
3099 ∼ 821 ≅ 821	3141 ∼ 750 ≅ 750	3183 ∼ 824 ≅ 820	3225 ∼ 924 ≅ 870
3100 ∼ 857 ≅ 857	3142 ∼ 929 ≅ 929	3184 ∼ 843 ≅ 843	3226 ∼ 960 ≅ 960
3101 ∼ 884 ≅ 884	3143 ∼ 938 ≅ 938	3185 ∼ 870 ≅ 870	3227 ∼ 969 ≅ 969
3102 ∼ 821 ≅ 821	3144 ∼ 920 ≅ 920	3186 ∼ 753 ≅ 753	3228 ∼ 843 ≅ 843
3103 ∼ 839 ≅ 821	3145 ∼ 956 ≅ 956	3187 ∼ 1094 ≅ 1090	3229 ∼ 824 ≅ 820

$3230 \sim 824 \cong 820$	$3272 \sim 1094 \cong 1090$	$3314 \sim 783 \cong 775$	$3356 \sim 857 \cong 857$
$3231 \sim 744 \cong 744$	$3273 \sim 878 \cong 878$	$3315 \sim 747 \cong 739$	$3357 \sim 776 \cong 776$
$3232 \sim 771 \cong 771$	$3274 \sim 941 \cong 941$	$3316 \sim 783 \cong 775$	$3358 \sim 1091 \cong 731$
$3233 \sim 780 \cong 780$	$3275 \sim 860 \cong 860$	$3317 \sim 810 \cong 802$	$3359 \sim 1091 \cong 731$
$3234 \sim 744 \cong 744$	$3276 \sim 779 \cong 779$	$3318 \sim 756 \cong 748$	$3360 \sim 929 \cong 929$
$3235 \sim 780 \cong 780$	$3277 \sim 1094 \cong 1090$	$3319 \sim 747 \cong 739$	$3361 \sim 1091 \cong 731$
$3236 \sim 807 \cong 771$	$3278 \sim 1094 \cong 1090$	$3320 \sim 756 \cong 748$	$3362 \sim 1091 \cong 731$
$3237 \sim 753 \cong 753$	$3279 \sim 932 \cong 820$	$3321 \sim 734 \cong 730$	$3363 \sim 848 \cong 750$
$3238 \sim 744 \cong 744$	$3280 \sim 1094 \cong 1090$	$3322 \sim 1091 \cong 731$	$3364 \sim 929 \cong 929$
$3239 \sim 753 \cong 753$	$3281 \sim 1094 \cong 1090$	$3323 \sim 1091 \cong 731$	$3365 \sim 848 \cong 750$
$3240 \sim 734 \cong 730$	$3282 \sim 851 \cong 847$	$3324 \sim 965 \cong 965$	$3366 \sim 767 \cong 731$
$3241 \sim 1094 \cong 1090$	$3283 \sim 932 \cong 820$	$3325 \sim 1091 \cong 731$	$3367 \sim 930 \cong 821$
$3242 \sim 1094 \cong 1090$	$3284 \sim 851 \cong 847$	$3326 \sim 1091 \cong 731$	$3368 \sim 957 \cong 957$
$3243 \sim 968 \cong 968$	$3285 \sim 770 \cong 730$	$3327 \sim 884 \cong 884$	$3369 \sim 821 \cong 821$
$3244 \sim 1094 \cong 1090$	$3286 \sim 936 \cong 820$	$3328 \sim 965 \cong 965$	$3370 \sim 939 \cong 939$
$3245 \sim 1094 \cong 1090$	$3287 \sim 963 \cong 963$	$3329 \sim 884 \cong 884$	$3371 \sim 966 \cong 966$
$3246 \sim 887 \cong 887$	$3288 \sim 824 \cong 820$	$3330 \sim 803 \cong 771$	$3372 \sim 821 \cong 821$
$3247 \sim 968 \cong 968$	$3289 \sim 945 \cong 941$	$3331 \sim 1091 \cong 731$	$3373 \sim 920 \cong 920$
$3248 \sim 887 \cong 887$	$3290 \sim 972 \cong 739$	$3332 \sim 1091 \cong 731$	$3374 \sim 839 \cong 821$
$3249 \sim 806 \cong 802$	$3291 \sim 824 \cong 820$	$3333 \sim 938 \cong 938$	$3375 \sim 740 \cong 740$
$3250 \sim 1094 \cong 1090$	$3292 \sim 923 \cong 923$	$3334 \sim 1091 \cong 731$	$3376 \sim 849 \cong 849$
$3251 \sim 1094 \cong 1090$	$3293 \sim 842 \cong 838$	$3335 \sim 1091 \cong 731$	$3377 \sim 858 \cong 858$
$3252 \sim 941 \cong 941$	$3294 \sim 743 \cong 739$	$3336 \sim 857 \cong 857$	$3378 \sim 840 \cong 840$
$3253 \sim 1094 \cong 1090$	$3295 \sim 855 \cong 847$	$3337 \sim 956 \cong 956$	$3379 \sim 876 \cong 876$
$3254 \sim 1094 \cong 1090$	$3296 \sim 864 \cong 864$	$3338 \sim 875 \cong 875$	$3380 \sim 885 \cong 885$
$3255 \sim 860 \cong 860$	$3297 \sim 846 \cong 846$	$3339 \sim 776 \cong 776$	$3381 \sim 866 \cong 866$
$3256 \sim 959 \cong 959$	$3298 \sim 882 \cong 882$	$3340 \sim 849 \cong 849$	$3382 \sim 821 \cong 821$
$3257 \sim 878 \cong 878$	$3299 \sim 891 \cong 891$	$3341 \sim 876 \cong 876$	$3383 \sim 821 \cong 821$
$3258 \sim 779 \cong 779$	$3300 \sim 869 \cong 869$	$3342 \sim 821 \cong 821$	$3384 \sim 749 \cong 749$
$3259 \sim 855 \cong 847$	$3301 \sim 824 \cong 820$	$3343 \sim 858 \cong 858$	$3385 \sim 930 \cong 821$
$3260 \sim 882 \cong 882$	$3302 \sim 824 \cong 820$	$3344 \sim 885 \cong 885$	$3386 \sim 939 \cong 939$
$3261 \sim 824 \cong 820$	$3303 \sim 752 \cong 752$	$3345 \sim 821 \cong 821$	$3387 \sim 920 \cong 920$
$3262 \sim 864 \cong 864$	$3304 \sim 936 \cong 820$	$3346 \sim 840 \cong 840$	$3388 \sim 957 \cong 957$
$3263 \sim 891 \cong 891$	$3305 \sim 945 \cong 941$	$3347 \sim 866 \cong 866$	$3389 \sim 966 \cong 966$
$3264 \sim 824 \cong 820$	$3306 \sim 923 \cong 923$	$3348 \sim 749 \cong 749$	$3390 \sim 839 \cong 821$
$3265 \sim 846 \cong 846$	$3307 \sim 963 \cong 963$	$3349 \sim 1091 \cong 731$	$3391 \sim 821 \cong 821$
$3266 \sim 869 \cong 869$	$3308 \sim 972 \cong 739$	$3350 \sim 1091 \cong 731$	$3392 \sim 821 \cong 821$
$3267 \sim 752 \cong 752$	$3309 \sim 842 \cong 838$	$3351 \sim 956 \cong 956$	$3393 \sim 740 \cong 740$
$3268 \sim 1094 \cong 1090$	$3310 \sim 824 \cong 820$	$3352 \sim 1091 \cong 731$	$3394 \sim 768 \cong 731$
$3269 \sim 1094 \cong 1090$	$3311 \sim 824 \cong 820$	$3353 \sim 1091 \cong 731$	$3395 \sim 777 \cong 777$
$3270 \sim 959 \cong 959$	$3312 \sim 743 \cong 739$	$3354 \sim 875 \cong 875$	$3396 \sim 741 \cong 741$
$3271 \sim 1094 \cong 1090$	$3313 \sim 774 \cong 730$	$3355 \sim 938 \cong 938$	$3397 \sim 777 \cong 777$

3398 ∼ 804 ≅ 731	3440 ∼ 1091 ≅ 731	3482 ∼ 749 ≅ 749	3524 ∼ 1091 ≅ 731
3399 ∼ 750 ≅ 750	3441 ∼ 930 ≅ 821	3483 ∼ 731 ≅ 731	3525 ∼ 848 ≅ 750
3400 ∼ 741 ≅ 741	3442 ∼ 1091 ≅ 731	3484 ∼ 1091 ≅ 731	3526 ∼ 929 ≅ 929
3401 ∼ 750 ≅ 750	3443 ∼ 1091 ≅ 731	3485 ∼ 1091 ≅ 731	3527 ∼ 848 ≅ 750
3402 ∼ 731 ≅ 731	3444 ∼ 849 ≅ 849	3486 ∼ 965 ≅ 965	3528 ∼ 767 ≅ 731
3403 ∼ 1091 ≅ 731	3445 ∼ 930 ≅ 821	3487 ∼ 1091 ≅ 731	3529 ∼ 930 ≅ 821
3404 ∼ 1091 ≅ 731	3446 ∼ 849 ≅ 849	3488 ∼ 1091 ≅ 731	3530 ∼ 957 ≅ 957
3405 ∼ 966 ≅ 966	3447 ∼ 768 ≅ 731	3489 ∼ 884 ≅ 884	3531 ∼ 821 ≅ 821
3406 ∼ 1091 ≅ 731	3448 ∼ 929 ≅ 929	3490 ∼ 965 ≅ 965	3532 ∼ 939 ≅ 939
3407 ∼ 1091 ≅ 731	3449 ∼ 956 ≅ 956	3491 ∼ 884 ≅ 884	3533 ∼ 966 ≅ 966
3408 ∼ 885 ≅ 885	3450 ∼ 821 ≅ 821	3492 ∼ 803 ≅ 771	3534 ∼ 821 ≅ 821
3409 ∼ 966 ≅ 966	3451 ∼ 938 ≅ 938	3493 ∼ 1091 ≅ 731	3535 ∼ 920 ≅ 920
3410 ∼ 885 ≅ 885	3452 ∼ 965 ≅ 965	3494 ∼ 1091 ≅ 731	3536 ∼ 839 ≅ 821
3411 ∼ 804 ≅ 731	3453 ∼ 821 ≅ 821	3495 ∼ 938 ≅ 938	3537 ∼ 740 ≅ 740
3412 ∼ 1091 ≅ 731	3454 ∼ 920 ≅ 920	3496 ∼ 1091 ≅ 731	3538 ∼ 849 ≅ 849
3413 ∼ 1091 ≅ 731	3455 ∼ 840 ≅ 840	3497 ∼ 1091 ≅ 731	3539 ∼ 858 ≅ 858
3414 ∼ 939 ≅ 939	3456 ∼ 741 ≅ 741	3498 ∼ 857 ≅ 857	3540 ∼ 840 ≅ 840
3415 ∼ 1091 ≅ 731	3457 ∼ 848 ≅ 750	3499 ∼ 956 ≅ 956	3541 ∼ 876 ≅ 876
3416 ∼ 1091 ≅ 731	3458 ∼ 857 ≅ 857	3500 ∼ 875 ≅ 875	3542 ∼ 885 ≅ 885
3417 ∼ 858 ≅ 858	3459 ∼ 839 ≅ 821	3501 ∼ 776 ≅ 776	3543 ∼ 866 ≅ 866
3418 ∼ 957 ≅ 957	3460 ∼ 875 ≅ 875	3502 ∼ 849 ≅ 849	3544 ∼ 821 ≅ 821
3419 ∼ 876 ≅ 876	3461 ∼ 884 ≅ 884	3503 ∼ 876 ≅ 876	3545 ∼ 821 ≅ 821
3420 ∼ 777 ≅ 777	3462 ∼ 866 ≅ 866	3504 ∼ 821 ≅ 821	3546 ∼ 749 ≅ 749
3421 ∼ 848 ≅ 750	3463 ∼ 821 ≅ 821	3505 ∼ 858 ≅ 858	3547 ∼ 930 ≅ 821
3422 ∼ 875 ≅ 875	3464 ∼ 821 ≅ 821	3506 ∼ 885 ≅ 885	3548 ∼ 939 ≅ 939
3423 ∼ 821 ≅ 821	3465 ∼ 750 ≅ 750	3507 ∼ 821 ≅ 821	3549 ∼ 920 ≅ 920
3424 ∼ 857 ≅ 857	3466 ∼ 929 ≅ 929	3508 ∼ 840 ≅ 840	3550 ∼ 957 ≅ 957
3425 ∼ 884 ≅ 884	3467 ∼ 938 ≅ 938	3509 ∼ 866 ≅ 866	3551 ∼ 966 ≅ 966
3426 ∼ 821 ≅ 821	3468 ∼ 920 ≅ 920	3510 ∼ 749 ≅ 749	3552 ∼ 839 ≅ 821
3427 ∼ 839 ≅ 821	3469 ∼ 956 ≅ 956	3511 ∼ 1091 ≅ 731	3553 ∼ 821 ≅ 821
3428 ∼ 866 ≅ 866	3470 ∼ 965 ≅ 965	3512 ∼ 1091 ≅ 731	3554 ∼ 821 ≅ 821
3429 ∼ 750 ≅ 750	3471 ∼ 840 ≅ 840	3513 ∼ 956 ≅ 956	3555 ∼ 740 ≅ 740
3430 ∼ 1091 ≅ 731	3472 ∼ 821 ≅ 821	3514 ∼ 1091 ≅ 731	3556 ∼ 768 ≅ 731
3431 ∼ 1091 ≅ 731	3473 ∼ 821 ≅ 821	3515 ∼ 1091 ≅ 731	3557 ∼ 777 ≅ 777
3432 ∼ 957 ≅ 957	3474 ∼ 741 ≅ 741	3516 ∼ 875 ≅ 875	3558 ∼ 741 ≅ 741
3433 ∼ 1091 ≅ 731	3475 ∼ 767 ≅ 731	3517 ∼ 938 ≅ 938	3559 ∼ 777 ≅ 777
3434 ∼ 1091 ≅ 731	3476 ∼ 776 ≅ 776	3518 ∼ 857 ≅ 857	3560 ∼ 804 ≅ 731
3435 ∼ 876 ≅ 876	3477 ∼ 740 ≅ 740	3519 ∼ 776 ≅ 776	3561 ∼ 750 ≅ 750
3436 ∼ 939 ≅ 939	3478 ∼ 776 ≅ 776	3520 ∼ 1091 ≅ 731	3562 ∼ 741 ≅ 741
3437 ∼ 858 ≅ 858	3479 ∼ 803 ≅ 771	3521 ∼ 1091 ≅ 731	3563 ∼ 750 ≅ 750
3438 ∼ 777 ≅ 777	3480 ∼ 749 ≅ 749	3522 ∼ 929 ≅ 929	3564 ∼ 731 ≅ 731
3439 ∼ 1091 ≅ 731	3481 ∼ 740 ≅ 740	3523 ∼ 1091 ≅ 731	3565 ∼ 1090 ≅ 1090

3566	∼	1090	≅	1090	3608	∼	847	≅	847	3650	∼	2196	≅	802	3692	∼	2399	≅	2399
3567	∼	964	≅	739	3609	∼	766	≅	730	3651	∼	2193	≅	2193	3693	∼	820	≅	820
3568	∼	1090	≅	1090	3610	∼	928	≅	820	3652	∼	730	≅	730	3694	∼	2399	≅	2399
3569	∼	1090	≅	1090	3611	∼	955	≅	937	3653	∼	2193	≅	2193	3695	∼	2426	≅	2277
3570	∼	883	≅	883	3612	∼	820	≅	820	3654	∼	730	≅	730	3696	∼	2372	≅	2372
3571	∼	964	≅	739	3613	∼	937	≅	937	3655	∼	820	≅	820	3697	∼	820	≅	820
3572	∼	883	≅	883	3614	∼	964	≅	739	3656	∼	2352	≅	740	3698	∼	2372	≅	2372
3573	∼	802	≅	802	3615	∼	820	≅	820	3657	∼	820	≅	820	3699	∼	820	≅	820
3574	∼	1090	≅	1090	3616	∼	919	≅	820	3658	∼	2352	≅	740	3700	∼	730	≅	730
3575	∼	1090	≅	1090	3617	∼	838	≅	838	3659	∼	2358	≅	820	3701	∼	2271	≅	2271
3576	∼	937	≅	937	3618	∼	739	≅	739	3660	∼	2355	≅	2355	3702	∼	730	≅	730
3577	∼	1090	≅	1090	3619	∼	847	≅	847	3661	∼	820	≅	820	3703	∼	2271	≅	2271
3578	∼	1090	≅	1090	3620	∼	856	≅	856	3662	∼	2355	≅	2355	3704	∼	2277	≅	2277
3579	∼	856	≅	856	3621	∼	838	≅	838	3663	∼	820	≅	820	3705	∼	2274	≅	2274
3580	∼	955	≅	937	3622	∼	874	≅	874	3664	∼	730	≅	730	3706	∼	730	≅	730
3581	∼	874	≅	874	3623	∼	883	≅	883	3665	∼	2271	≅	2271	3707	∼	2274	≅	2274
3582	∼	775	≅	775	3624	∼	865	≅	820	3666	∼	730	≅	730	3708	∼	730	≅	730
3583	∼	847	≅	847	3625	∼	820	≅	820	3667	∼	2271	≅	2271	3709	∼	820	≅	820
3584	∼	874	≅	874	3626	∼	820	≅	820	3668	∼	2277	≅	2277	3710	∼	2399	≅	2399
3585	∼	820	≅	820	3627	∼	748	≅	748	3669	∼	2274	≅	2274	3711	∼	820	≅	820
3586	∼	856	≅	856	3628	∼	928	≅	820	3670	∼	730	≅	730	3712	∼	2399	≅	2399
3587	∼	883	≅	883	3629	∼	937	≅	937	3671	∼	2274	≅	2274	3713	∼	2426	≅	2277
3588	∼	820	≅	820	3630	∼	919	≅	820	3672	∼	730	≅	730	3714	∼	2372	≅	2372
3589	∼	838	≅	838	3631	∼	955	≅	937	3673	∼	820	≅	820	3715	∼	820	≅	820
3590	∼	865	≅	820	3632	∼	964	≅	739	3674	∼	2352	≅	740	3716	∼	2372	≅	2372
3591	∼	748	≅	748	3633	∼	838	≅	838	3675	∼	820	≅	820	3717	∼	820	≅	820
3592	∼	1090	≅	1090	3634	∼	820	≅	820	3676	∼	2352	≅	740	3718	∼	730	≅	730
3593	∼	1090	≅	1090	3635	∼	820	≅	820	3677	∼	2358	≅	820	3719	∼	2237	≅	2237
3594	∼	955	≅	937	3636	∼	739	≅	739	3678	∼	2355	≅	2355	3720	∼	730	≅	730
3595	∼	1090	≅	1090	3637	∼	766	≅	730	3679	∼	820	≅	820	3721	∼	2237	≅	2237
3596	∼	1090	≅	1090	3638	∼	775	≅	775	3680	∼	2355	≅	2355	3722	∼	2264	≅	730
3597	∼	874	≅	874	3639	∼	739	≅	739	3681	∼	820	≅	820	3723	∼	2210	≅	2210
3598	∼	937	≅	937	3640	∼	775	≅	775	3682	∼	1090	≅	1090	3724	∼	730	≅	730
3599	∼	856	≅	856	3641	∼	802	≅	802	3683	∼	2838	≅	750	3725	∼	2210	≅	2210
3600	∼	775	≅	775	3642	∼	748	≅	748	3684	∼	1090	≅	1090	3726	∼	730	≅	730
3601	∼	1090	≅	1090	3643	∼	739	≅	739	3685	∼	2838	≅	750	3727	∼	2206	≅	748
3602	∼	1090	≅	1090	3644	∼	748	≅	748	3686	∼	2844	≅	730	3728	∼	731	≅	731
3603	∼	928	≅	820	3645	∼	730	≅	730	3687	∼	2841	≅	2841	3729	∼	2207	≅	2207
3604	∼	1090	≅	1090	3646	∼	730	≅	730	3688	∼	1090	≅	1090	3730	∼	2212	≅	2212
3605	∼	1090	≅	1090	3647	∼	2190	≅	750	3689	∼	2841	≅	2841	3731	∼	2214	≅	748
3606	∼	847	≅	847	3648	∼	730	≅	730	3690	∼	1090	≅	1090	3732	∼	2213	≅	2213
3607	∼	928	≅	820	3649	∼	2190	≅	750	3691	∼	820	≅	820	3733	∼	2209	≅	2209

3734 $\sim$ 731 $\cong$ 731	3776 $\sim$ 2427 $\cong$ 2427	3818 $\sim$ 2361 $\cong$ 2361	3860 $\sim$ 2371 $\cong$ 2371
3735 $\sim$ 2210 $\cong$ 2210	3777 $\sim$ 2375 $\cong$ 2375	3819 $\sim$ 820 $\cong$ 820	3861 $\sim$ 820 $\cong$ 820
3736 $\sim$ 2368 $\cong$ 739	3778 $\sim$ 2364 $\cong$ 2364	3820 $\sim$ 2365 $\cong$ 2365	3862 $\sim$ 730 $\cong$ 730
3737 $\sim$ 821 $\cong$ 821	3779 $\sim$ 821 $\cong$ 821	3821 $\sim$ 2367 $\cong$ 2367	3863 $\sim$ 2280 $\cong$ 2280
3738 $\sim$ 2369 $\cong$ 2369	3780 $\sim$ 2355 $\cong$ 2355	3822 $\sim$ 2366 $\cong$ 2366	3864 $\sim$ 730 $\cong$ 730
3739 $\sim$ 2374 $\cong$ 821	3781 $\sim$ 2287 $\cong$ 2287	3823 $\sim$ 820 $\cong$ 820	3865 $\sim$ 2284 $\cong$ 2284
3740 $\sim$ 2376 $\cong$ 739	3782 $\sim$ 731 $\cong$ 731	3824 $\sim$ 2364 $\cong$ 2364	3866 $\sim$ 2286 $\cong$ 2286
3741 $\sim$ 2375 $\cong$ 2375	3783 $\sim$ 2285 $\cong$ 2285	3825 $\sim$ 820 $\cong$ 820	3867 $\sim$ 2285 $\cong$ 2285
3742 $\sim$ 2371 $\cong$ 2371	3784 $\sim$ 2293 $\cong$ 2293	3826 $\sim$ 730 $\cong$ 730	3868 $\sim$ 730 $\cong$ 730
3743 $\sim$ 821 $\cong$ 821	3785 $\sim$ 2295 $\cong$ 2295	3827 $\sim$ 2280 $\cong$ 2280	3869 $\sim$ 2283 $\cong$ 2283
3744 $\sim$ 2372 $\cong$ 2372	3786 $\sim$ 2294 $\cong$ 2294	3828 $\sim$ 730 $\cong$ 730	3870 $\sim$ 730 $\cong$ 730
3745 $\sim$ 2287 $\cong$ 2287	3787 $\sim$ 2283 $\cong$ 2283	3829 $\sim$ 2284 $\cong$ 2284	3871 $\sim$ 820 $\cong$ 820
3746 $\sim$ 731 $\cong$ 731	3788 $\sim$ 731 $\cong$ 731	3830 $\sim$ 2286 $\cong$ 2286	3872 $\sim$ 2398 $\cong$ 2398
3747 $\sim$ 2285 $\cong$ 2285	3789 $\sim$ 2274 $\cong$ 2274	3831 $\sim$ 2285 $\cong$ 2285	3873 $\sim$ 820 $\cong$ 820
3748 $\sim$ 2293 $\cong$ 2293	3790 $\sim$ 2391 $\cong$ 2391	3832 $\sim$ 730 $\cong$ 730	3874 $\sim$ 2396 $\cong$ 2396
3749 $\sim$ 2295 $\cong$ 2295	3791 $\sim$ 821 $\cong$ 821	3833 $\sim$ 2283 $\cong$ 2283	3875 $\sim$ 2423 $\cong$ 2423
3750 $\sim$ 2294 $\cong$ 2294	3792 $\sim$ 2366 $\cong$ 2366	3834 $\sim$ 730 $\cong$ 730	3876 $\sim$ 2369 $\cong$ 2369
3751 $\sim$ 2283 $\cong$ 2283	3793 $\sim$ 2402 $\cong$ 2402	3835 $\sim$ 820 $\cong$ 820	3877 $\sim$ 820 $\cong$ 820
3752 $\sim$ 731 $\cong$ 731	3794 $\sim$ 2427 $\cong$ 2427	3836 $\sim$ 2361 $\cong$ 2361	3878 $\sim$ 2371 $\cong$ 2371
3753 $\sim$ 2274 $\cong$ 2274	3795 $\sim$ 2375 $\cong$ 2375	3837 $\sim$ 820 $\cong$ 820	3879 $\sim$ 820 $\cong$ 820
3754 $\sim$ 2368 $\cong$ 739	3796 $\sim$ 2364 $\cong$ 2364	3838 $\sim$ 2365 $\cong$ 2365	3880 $\sim$ 730 $\cong$ 730
3755 $\sim$ 821 $\cong$ 821	3797 $\sim$ 821 $\cong$ 821	3839 $\sim$ 2367 $\cong$ 2367	3881 $\sim$ 2236 $\cong$ 2236
3756 $\sim$ 2369 $\cong$ 2369	3798 $\sim$ 2355 $\cong$ 2355	3840 $\sim$ 2366 $\cong$ 2366	3882 $\sim$ 730 $\cong$ 730
3757 $\sim$ 2374 $\cong$ 821	3799 $\sim$ 2229 $\cong$ 2229	3841 $\sim$ 820 $\cong$ 820	3883 $\sim$ 2234 $\cong$ 2234
3758 $\sim$ 2376 $\cong$ 739	3800 $\sim$ 731 $\cong$ 731	3842 $\sim$ 2364 $\cong$ 2364	3884 $\sim$ 2261 $\cong$ 2261
3759 $\sim$ 2375 $\cong$ 2375	3801 $\sim$ 2204 $\cong$ 2204	3843 $\sim$ 820 $\cong$ 820	3885 $\sim$ 2207 $\cong$ 2207
3760 $\sim$ 2371 $\cong$ 2371	3802 $\sim$ 2240 $\cong$ 2240	3844 $\sim$ 1090 $\cong$ 1090	3886 $\sim$ 730 $\cong$ 730
3761 $\sim$ 821 $\cong$ 821	3803 $\sim$ 2265 $\cong$ 2265	3845 $\sim$ 2847 $\cong$ 929	3887 $\sim$ 2209 $\cong$ 2209
3762 $\sim$ 2372 $\cong$ 2372	3804 $\sim$ 2213 $\cong$ 2213	3846 $\sim$ 1090 $\cong$ 1090	3888 $\sim$ 730 $\cong$ 730
3763 $\sim$ 2854 $\cong$ 847	3805 $\sim$ 2202 $\cong$ 2202	3847 $\sim$ 2851 $\cong$ 929	3889 $\sim$ 2206 $\cong$ 748
3764 $\sim$ 1091 $\cong$ 731	3806 $\sim$ 731 $\cong$ 731	3848 $\sim$ 2853 $\cong$ 2853	3890 $\sim$ 2212 $\cong$ 2212
3765 $\sim$ 2852 $\cong$ 849	3807 $\sim$ 2193 $\cong$ 2193	3849 $\sim$ 2852 $\cong$ 849	3891 $\sim$ 2209 $\cong$ 2209
3766 $\sim$ 2860 $\cong$ 2212	3808 $\sim$ 730 $\cong$ 730	3850 $\sim$ 1090 $\cong$ 1090	3892 $\sim$ 731 $\cong$ 731
3767 $\sim$ 2862 $\cong$ 847	3809 $\sim$ 2199 $\cong$ 2199	3851 $\sim$ 2850 $\cong$ 2850	3893 $\sim$ 2214 $\cong$ 748
3768 $\sim$ 2861 $\cong$ 731	3810 $\sim$ 730 $\cong$ 730	3852 $\sim$ 1090 $\cong$ 1090	3894 $\sim$ 731 $\cong$ 731
3769 $\sim$ 2850 $\cong$ 2850	3811 $\sim$ 2203 $\cong$ 2203	3853 $\sim$ 820 $\cong$ 820	3895 $\sim$ 2207 $\cong$ 2207
3770 $\sim$ 1091 $\cong$ 731	3812 $\sim$ 2205 $\cong$ 775	3854 $\sim$ 2398 $\cong$ 2398	3896 $\sim$ 2213 $\cong$ 2213
3771 $\sim$ 2841 $\cong$ 2841	3813 $\sim$ 2204 $\cong$ 2204	3855 $\sim$ 820 $\cong$ 820	3897 $\sim$ 2210 $\cong$ 2210
3772 $\sim$ 2391 $\cong$ 2391	3814 $\sim$ 730 $\cong$ 730	3856 $\sim$ 2396 $\cong$ 2396	3898 $\sim$ 2368 $\cong$ 739
3773 $\sim$ 821 $\cong$ 821	3815 $\sim$ 2202 $\cong$ 2202	3857 $\sim$ 2423 $\cong$ 2423	3899 $\sim$ 2374 $\cong$ 821
3774 $\sim$ 2366 $\cong$ 2366	3816 $\sim$ 730 $\cong$ 730	3858 $\sim$ 2369 $\cong$ 2369	3900 $\sim$ 2371 $\cong$ 2371
3775 $\sim$ 2402 $\cong$ 2402	3817 $\sim$ 820 $\cong$ 820	3859 $\sim$ 820 $\cong$ 820	3901 $\sim$ 821 $\cong$ 821

3902 ∼ 2376 ≅ 739	3944 ∼ 2293 ≅ 2293	3986 ∼ 2427 ≅ 2427	4028 ∼ 734 ≅ 730
3903 ∼ 821 ≅ 821	3945 ∼ 2283 ≅ 2283	3987 ∼ 2426 ≅ 2277	4029 ∼ 2295 ≅ 2295
3904 ∼ 2369 ≅ 2369	3946 ∼ 731 ≅ 731	3988 ∼ 2313 ≅ 2277	4030 ∼ 2286 ≅ 2286
3905 ∼ 2375 ≅ 2375	3947 ∼ 2295 ≅ 2295	3989 ∼ 2322 ≅ 2322	4031 ∼ 2295 ≅ 2295
3906 ∼ 2372 ≅ 2372	3948 ∼ 731 ≅ 731	3990 ∼ 2286 ≅ 2286	4032 ∼ 2277 ≅ 2277
3907 ∼ 2287 ≅ 2287	3949 ∼ 2285 ≅ 2285	3991 ∼ 2322 ≅ 2322	4033 ∼ 2394 ≅ 820
3908 ∼ 2293 ≅ 2293	3950 ∼ 2294 ≅ 2294	3992 ∼ 734 ≅ 730	4034 ∼ 2403 ≅ 2287
3909 ∼ 2283 ≅ 2283	3951 ∼ 2274 ≅ 2274	3993 ∼ 2295 ≅ 2295	4035 ∼ 2367 ≅ 2367
3910 ∼ 731 ≅ 731	3952 ∼ 2391 ≅ 2391	3994 ∼ 2286 ≅ 2286	4036 ∼ 2403 ≅ 2287
3911 ∼ 2295 ≅ 2295	3953 ∼ 2402 ≅ 2402	3995 ∼ 2295 ≅ 2295	4037 ∼ 824 ≅ 820
3912 ∼ 731 ≅ 731	3954 ∼ 2364 ≅ 2364	3996 ∼ 2277 ≅ 2277	4038 ∼ 2376 ≅ 739
3913 ∼ 2285 ≅ 2285	3955 ∼ 821 ≅ 821	3997 ∼ 2422 ≅ 820	4039 ∼ 2367 ≅ 2367
3914 ∼ 2294 ≅ 2294	3956 ∼ 2427 ≅ 2427	3998 ∼ 2424 ≅ 966	4040 ∼ 2376 ≅ 739
3915 ∼ 2274 ≅ 2274	3957 ∼ 821 ≅ 821	3999 ∼ 2423 ≅ 2423	4041 ∼ 2358 ≅ 820
3916 ∼ 2368 ≅ 739	3958 ∼ 2366 ≅ 2366	4000 ∼ 2424 ≅ 966	4042 ∼ 2232 ≅ 730
3917 ∼ 2374 ≅ 821	3959 ∼ 2375 ≅ 2375	4001 ∼ 824 ≅ 820	4043 ∼ 2241 ≅ 739
3918 ∼ 2371 ≅ 2371	3960 ∼ 2355 ≅ 2355	4002 ∼ 2427 ≅ 2427	4044 ∼ 2205 ≅ 775
3919 ∼ 821 ≅ 821	3961 ∼ 2229 ≅ 2229	4003 ∼ 2423 ≅ 2423	4045 ∼ 2241 ≅ 739
3920 ∼ 2376 ≅ 739	3962 ∼ 2240 ≅ 2240	4004 ∼ 2427 ≅ 2427	4046 ∼ 734 ≅ 730
3921 ∼ 821 ≅ 821	3963 ∼ 2202 ≅ 2202	4005 ∼ 2426 ≅ 2277	4047 ∼ 2214 ≅ 748
3922 ∼ 2369 ≅ 2369	3964 ∼ 731 ≅ 731	4006 ∼ 2880 ≅ 730	4048 ∼ 2205 ≅ 775
3923 ∼ 2375 ≅ 2375	3965 ∼ 2265 ≅ 2265	4007 ∼ 2889 ≅ 750	4049 ∼ 2214 ≅ 748
3924 ∼ 2372 ≅ 2372	3966 ∼ 731 ≅ 731	4008 ∼ 2853 ≅ 2853	4050 ∼ 2196 ≅ 802
3925 ∼ 2854 ≅ 847	3967 ∼ 2204 ≅ 2204	4009 ∼ 2889 ≅ 750	4051 ∼ 2233 ≅ 2233
3926 ∼ 2860 ≅ 2212	3968 ∼ 2213 ≅ 2213	4010 ∼ 1094 ≅ 1090	4052 ∼ 2239 ≅ 2239
3927 ∼ 2850 ≅ 2850	3969 ∼ 2193 ≅ 2193	4011 ∼ 2862 ≅ 847	4053 ∼ 2236 ≅ 2236
3928 ∼ 1091 ≅ 731	3970 ∼ 2260 ≅ 802	4012 ∼ 2853 ≅ 2853	4054 ∼ 731 ≅ 731
3929 ∼ 2862 ≅ 847	3971 ∼ 2262 ≅ 750	4013 ∼ 2862 ≅ 847	4055 ∼ 2241 ≅ 739
3930 ∼ 1091 ≅ 731	3972 ∼ 2261 ≅ 2261	4014 ∼ 2844 ≅ 730	4056 ∼ 731 ≅ 731
3931 ∼ 2852 ≅ 849	3973 ∼ 2262 ≅ 750	4015 ∼ 2394 ≅ 820	4057 ∼ 2234 ≅ 2234
3932 ∼ 2861 ≅ 731	3974 ∼ 734 ≅ 730	4016 ∼ 2403 ≅ 2287	4058 ∼ 2240 ≅ 2240
3933 ∼ 2841 ≅ 2841	3975 ∼ 2265 ≅ 2265	4017 ∼ 2367 ≅ 2367	4059 ∼ 2237 ≅ 2237
3934 ∼ 2391 ≅ 2391	3976 ∼ 2261 ≅ 2261	4018 ∼ 2403 ≅ 2287	4060 ∼ 2395 ≅ 2395
3935 ∼ 2402 ≅ 2402	3977 ∼ 2265 ≅ 2265	4019 ∼ 824 ≅ 820	4061 ∼ 2401 ≅ 2401
3936 ∼ 2364 ≅ 2364	3978 ∼ 2264 ≅ 730	4020 ∼ 2376 ≅ 739	4062 ∼ 2398 ≅ 2398
3937 ∼ 821 ≅ 821	3979 ∼ 2422 ≅ 820	4021 ∼ 2367 ≅ 2367	4063 ∼ 821 ≅ 821
3938 ∼ 2427 ≅ 2427	3980 ∼ 2424 ≅ 966	4022 ∼ 2376 ≅ 739	4064 ∼ 2403 ≅ 2287
3939 ∼ 821 ≅ 821	3981 ∼ 2423 ≅ 2423	4023 ∼ 2358 ≅ 820	4065 ∼ 821 ≅ 821
3940 ∼ 2366 ≅ 2366	3982 ∼ 2424 ≅ 966	4024 ∼ 2313 ≅ 2277	4066 ∼ 2396 ≅ 2396
3941 ∼ 2375 ≅ 2375	3983 ∼ 824 ≅ 820	4025 ∼ 2322 ≅ 2322	4067 ∼ 2402 ≅ 2402
3942 ∼ 2355 ≅ 2355	3984 ∼ 2427 ≅ 2427	4026 ∼ 2286 ≅ 2286	4068 ∼ 2399 ≅ 2399
3943 ∼ 2287 ≅ 2287	3985 ∼ 2423 ≅ 2423	4027 ∼ 2322 ≅ 2322	4069 ∼ 2307 ≅ 2307

$4070 \sim 2320 \cong 2294$	$4112 \sim 2293 \cong 2293$	$4154 \sim 2286 \cong 2286$	$4196 \sim 2396 \cong 2396$
$4071 \sim 2280 \cong 2280$	$4113 \sim 2271 \cong 2271$	$4155 \sim 2283 \cong 2283$	$4197 \sim 820 \cong 820$
$4072 \sim 731 \cong 731$	$4114 \sim 2388 \cong 821$	$4156 \sim 730 \cong 730$	$4198 \sim 2398 \cong 2398$
$4073 \sim 2322 \cong 2322$	$4115 \sim 2401 \cong 2401$	$4157 \sim 2285 \cong 2285$	$4199 \sim 2423 \cong 2423$
$4074 \sim 731 \cong 731$	$4116 \sim 2361 \cong 2361$	$4158 \sim 730 \cong 730$	$4200 \sim 2371 \cong 2371$
$4075 \sim 2284 \cong 2284$	$4117 \sim 821 \cong 821$	$4159 \sim 820 \cong 820$	$4201 \sim 820 \cong 820$
$4076 \sim 2293 \cong 2293$	$4118 \sim 2424 \cong 966$	$4160 \sim 2365 \cong 2365$	$4202 \sim 2369 \cong 2369$
$4077 \sim 2271 \cong 2271$	$4119 \sim 821 \cong 821$	$4161 \sim 820 \cong 820$	$4203 \sim 820 \cong 820$
$4078 \sim 2395 \cong 2395$	$4120 \sim 2365 \cong 2365$	$4162 \sim 2361 \cong 2361$	$4204 \sim 730 \cong 730$
$4079 \sim 2401 \cong 2401$	$4121 \sim 2374 \cong 821$	$4163 \sim 2367 \cong 2367$	$4205 \sim 2234 \cong 2234$
$4080 \sim 2398 \cong 2398$	$4122 \sim 2352 \cong 740$	$4164 \sim 2364 \cong 2364$	$4206 \sim 730 \cong 730$
$4081 \sim 821 \cong 821$	$4123 \sim 2226 \cong 820$	$4165 \sim 820 \cong 820$	$4207 \sim 2236 \cong 2236$
$4082 \sim 2403 \cong 2287$	$4124 \sim 2239 \cong 2239$	$4166 \sim 2366 \cong 2366$	$4208 \sim 2261 \cong 2261$
$4083 \sim 821 \cong 821$	$4125 \sim 2199 \cong 2199$	$4167 \sim 820 \cong 820$	$4209 \sim 2209 \cong 2209$
$4084 \sim 2396 \cong 2396$	$4126 \sim 731 \cong 731$	$4168 \sim 1090 \cong 1090$	$4210 \sim 730 \cong 730$
$4085 \sim 2402 \cong 2402$	$4127 \sim 2262 \cong 750$	$4169 \sim 2851 \cong 929$	$4211 \sim 2207 \cong 2207$
$4086 \sim 2399 \cong 2399$	$4128 \sim 731 \cong 731$	$4170 \sim 1090 \cong 1090$	$4212 \sim 730 \cong 730$
$4087 \sim 2874 \cong 820$	$4129 \sim 2203 \cong 2203$	$4171 \sim 2847 \cong 929$	$4213 \sim 2233 \cong 2233$
$4088 \sim 2887 \cong 731$	$4130 \sim 2212 \cong 2212$	$4172 \sim 2853 \cong 2853$	$4214 \sim 731 \cong 731$
$4089 \sim 2847 \cong 929$	$4131 \sim 2190 \cong 750$	$4173 \sim 2850 \cong 2850$	$4215 \sim 2234 \cong 2234$
$4090 \sim 1091 \cong 731$	$4132 \sim 730 \cong 730$	$4174 \sim 1090 \cong 1090$	$4216 \sim 2239 \cong 2239$
$4091 \sim 2889 \cong 750$	$4133 \sim 2203 \cong 2203$	$4175 \sim 2852 \cong 849$	$4217 \sim 2241 \cong 739$
$4092 \sim 1091 \cong 731$	$4134 \sim 730 \cong 730$	$4176 \sim 1090 \cong 1090$	$4218 \sim 2240 \cong 2240$
$4093 \sim 2851 \cong 929$	$4135 \sim 2199 \cong 2199$	$4177 \sim 820 \cong 820$	$4219 \sim 2236 \cong 2236$
$4094 \sim 2860 \cong 2212$	$4136 \sim 2205 \cong 775$	$4178 \sim 2396 \cong 2396$	$4220 \sim 731 \cong 731$
$4095 \sim 2838 \cong 750$	$4137 \sim 2202 \cong 2202$	$4179 \sim 820 \cong 820$	$4221 \sim 2237 \cong 2237$
$4096 \sim 2388 \cong 821$	$4138 \sim 730 \cong 730$	$4180 \sim 2398 \cong 2398$	$4222 \sim 2395 \cong 2395$
$4097 \sim 2401 \cong 2401$	$4139 \sim 2204 \cong 2204$	$4181 \sim 2423 \cong 2423$	$4223 \sim 821 \cong 821$
$4098 \sim 2361 \cong 2361$	$4140 \sim 730 \cong 730$	$4182 \sim 2371 \cong 2371$	$4224 \sim 2396 \cong 2396$
$4099 \sim 821 \cong 821$	$4141 \sim 820 \cong 820$	$4183 \sim 820 \cong 820$	$4225 \sim 2401 \cong 2401$
$4100 \sim 2424 \cong 966$	$4142 \sim 2365 \cong 2365$	$4184 \sim 2369 \cong 2369$	$4226 \sim 2403 \cong 2287$
$4101 \sim 821 \cong 821$	$4143 \sim 820 \cong 820$	$4185 \sim 820 \cong 820$	$4227 \sim 2402 \cong 2402$
$4102 \sim 2365 \cong 2365$	$4144 \sim 2361 \cong 2361$	$4186 \sim 730 \cong 730$	$4228 \sim 2398 \cong 2398$
$4103 \sim 2374 \cong 821$	$4145 \sim 2367 \cong 2367$	$4187 \sim 2284 \cong 2284$	$4229 \sim 821 \cong 821$
$4104 \sim 2352 \cong 740$	$4146 \sim 2364 \cong 2364$	$4188 \sim 730 \cong 730$	$4230 \sim 2399 \cong 2399$
$4105 \sim 2307 \cong 2307$	$4147 \sim 820 \cong 820$	$4189 \sim 2280 \cong 2280$	$4231 \sim 2307 \cong 2307$
$4106 \sim 2320 \cong 2294$	$4148 \sim 2366 \cong 2366$	$4190 \sim 2286 \cong 2286$	$4232 \sim 731 \cong 731$
$4107 \sim 2280 \cong 2280$	$4149 \sim 820 \cong 820$	$4191 \sim 2283 \cong 2283$	$4233 \sim 2284 \cong 2284$
$4108 \sim 731 \cong 731$	$4150 \sim 730 \cong 730$	$4192 \sim 730 \cong 730$	$4234 \sim 2320 \cong 2294$
$4109 \sim 2322 \cong 2322$	$4151 \sim 2284 \cong 2284$	$4193 \sim 2285 \cong 2285$	$4235 \sim 2322 \cong 2322$
$4110 \sim 731 \cong 731$	$4152 \sim 730 \cong 730$	$4194 \sim 730 \cong 730$	$4236 \sim 2293 \cong 2293$
$4111 \sim 2284 \cong 2284$	$4153 \sim 2280 \cong 2280$	$4195 \sim 820 \cong 820$	$4237 \sim 2280 \cong 2280$

$4238 \sim 731 \cong 731$	$4280 \sim 2424 \cong 966$	$4322 \sim 2388 \cong 821$	$4364 \sim 2368 \cong 739$
$4239 \sim 2271 \cong 2271$	$4281 \sim 2374 \cong 821$	$4323 \sim 820 \cong 820$	$4365 \sim 820 \cong 820$
$4240 \sim 2395 \cong 2395$	$4282 \sim 2361 \cong 2361$	$4324 \sim 2388 \cong 821$	$4366 \sim 730 \cong 730$
$4241 \sim 821 \cong 821$	$4283 \sim 821 \cong 821$	$4325 \sim 2394 \cong 820$	$4367 \sim 2233 \cong 2233$
$4242 \sim 2396 \cong 2396$	$4284 \sim 2352 \cong 740$	$4326 \sim 2391 \cong 2391$	$4368 \sim 730 \cong 730$
$4243 \sim 2401 \cong 2401$	$4285 \sim 2226 \cong 820$	$4327 \sim 820 \cong 820$	$4369 \sim 2233 \cong 2233$
$4244 \sim 2403 \cong 2287$	$4286 \sim 731 \cong 731$	$4328 \sim 2391 \cong 2391$	$4370 \sim 2260 \cong 802$
$4245 \sim 2402 \cong 2402$	$4287 \sim 2203 \cong 2203$	$4329 \sim 820 \cong 820$	$4371 \sim 2206 \cong 748$
$4246 \sim 2398 \cong 2398$	$4288 \sim 2239 \cong 2239$	$4330 \sim 1090 \cong 1090$	$4372 \sim 730 \cong 730$
$4247 \sim 821 \cong 821$	$4289 \sim 2262 \cong 750$	$4331 \sim 2874 \cong 820$	$4373 \sim 2206 \cong 748$
$4248 \sim 2399 \cong 2399$	$4290 \sim 2212 \cong 2212$	$4332 \sim 1090 \cong 1090$	$4374 \sim 730 \cong 730$
$4249 \sim 2874 \cong 820$	$4291 \sim 2199 \cong 2199$	$4333 \sim 2874 \cong 820$	$4375 \sim 1094 \cong 1090$
$4250 \sim 1091 \cong 731$	$4292 \sim 731 \cong 731$	$4334 \sim 2880 \cong 730$	$4376 \sim 824 \cong 820$
$4251 \sim 2851 \cong 929$	$4293 \sim 2190 \cong 750$	$4335 \sim 2854 \cong 847$	$4377 \sim 824 \cong 820$
$4252 \sim 2887 \cong 731$	$4294 \sim 730 \cong 730$	$4336 \sim 1090 \cong 1090$	$4378 \sim 824 \cong 820$
$4253 \sim 2889 \cong 750$	$4295 \sim 2226 \cong 820$	$4337 \sim 2854 \cong 847$	$4379 \sim 734 \cong 730$
$4254 \sim 2860 \cong 2212$	$4296 \sim 730 \cong 730$	$4338 \sim 1090 \cong 1090$	$4380 \sim 734 \cong 730$
$4255 \sim 2847 \cong 929$	$4297 \sim 2226 \cong 820$	$4339 \sim 820 \cong 820$	$4381 \sim 824 \cong 820$
$4256 \sim 1091 \cong 731$	$4298 \sim 2232 \cong 730$	$4340 \sim 2395 \cong 2395$	$4382 \sim 734 \cong 730$
$4257 \sim 2838 \cong 750$	$4299 \sim 2229 \cong 2229$	$4341 \sim 820 \cong 820$	$4383 \sim 734 \cong 730$
$4258 \sim 2388 \cong 821$	$4300 \sim 730 \cong 730$	$4342 \sim 2395 \cong 2395$	$4384 \sim 2889 \cong 750$
$4259 \sim 821 \cong 821$	$4301 \sim 2229 \cong 2229$	$4343 \sim 2422 \cong 820$	$4385 \sim 2424 \cong 966$
$4260 \sim 2365 \cong 2365$	$4302 \sim 730 \cong 730$	$4344 \sim 2368 \cong 739$	$4386 \sim 2403 \cong 2287$
$4261 \sim 2401 \cong 2401$	$4303 \sim 820 \cong 820$	$4345 \sim 820 \cong 820$	$4387 \sim 2424 \cong 966$
$4262 \sim 2424 \cong 966$	$4304 \sim 2388 \cong 821$	$4346 \sim 2368 \cong 739$	$4388 \sim 2262 \cong 750$
$4263 \sim 2374 \cong 821$	$4305 \sim 820 \cong 820$	$4347 \sim 820 \cong 820$	$4389 \sim 2322 \cong 2322$
$4264 \sim 2361 \cong 2361$	$4306 \sim 2388 \cong 821$	$4348 \sim 730 \cong 730$	$4390 \sim 2403 \cong 2287$
$4265 \sim 821 \cong 821$	$4307 \sim 2394 \cong 820$	$4349 \sim 2307 \cong 2307$	$4391 \sim 2322 \cong 2322$
$4266 \sim 2352 \cong 740$	$4308 \sim 2391 \cong 2391$	$4350 \sim 730 \cong 730$	$4392 \sim 2241 \cong 739$
$4267 \sim 2307 \cong 2307$	$4309 \sim 820 \cong 820$	$4351 \sim 2307 \cong 2307$	$4393 \sim 2862 \cong 847$
$4268 \sim 731 \cong 731$	$4310 \sim 2391 \cong 2391$	$4352 \sim 2313 \cong 2277$	$4394 \sim 2427 \cong 2427$
$4269 \sim 2284 \cong 2284$	$4311 \sim 820 \cong 820$	$4353 \sim 2287 \cong 2287$	$4395 \sim 2376 \cong 739$
$4270 \sim 2320 \cong 2294$	$4312 \sim 730 \cong 730$	$4354 \sim 730 \cong 730$	$4396 \sim 2427 \cong 2427$
$4271 \sim 2322 \cong 2322$	$4313 \sim 2307 \cong 2307$	$4355 \sim 2287 \cong 2287$	$4397 \sim 2265 \cong 2265$
$4272 \sim 2293 \cong 2293$	$4314 \sim 730 \cong 730$	$4356 \sim 730 \cong 730$	$4398 \sim 2295 \cong 2295$
$4273 \sim 2280 \cong 2280$	$4315 \sim 2307 \cong 2307$	$4357 \sim 820 \cong 820$	$4399 \sim 2376 \cong 739$
$4274 \sim 731 \cong 731$	$4316 \sim 2313 \cong 2277$	$4358 \sim 2395 \cong 2395$	$4400 \sim 2295 \cong 2295$
$4275 \sim 2271 \cong 2271$	$4317 \sim 2287 \cong 2287$	$4359 \sim 820 \cong 820$	$4401 \sim 2214 \cong 748$
$4276 \sim 2388 \cong 821$	$4318 \sim 730 \cong 730$	$4360 \sim 2395 \cong 2395$	$4402 \sim 2889 \cong 750$
$4277 \sim 821 \cong 821$	$4319 \sim 2287 \cong 2287$	$4361 \sim 2422 \cong 820$	$4403 \sim 2424 \cong 966$
$4278 \sim 2365 \cong 2365$	$4320 \sim 730 \cong 730$	$4362 \sim 2368 \cong 739$	$4404 \sim 2403 \cong 2287$
$4279 \sim 2401 \cong 2401$	$4321 \sim 820 \cong 820$	$4363 \sim 820 \cong 820$	$4405 \sim 2424 \cong 966$

$4406 \sim 2262 \cong 750$	$4448 \sim 2426 \cong 2277$	$4490 \sim 2293 \cong 2293$	$4532 \sim 2210 \cong 2210$
$4407 \sim 2322 \cong 2322$	$4449 \sim 2358 \cong 820$	$4491 \sim 2240 \cong 2240$	$4533 \sim 2274 \cong 2274$
$4408 \sim 2403 \cong 2287$	$4450 \sim 2426 \cong 2277$	$4492 \sim 2854 \cong 847$	$4534 \sim 2355 \cong 2355$
$4409 \sim 2322 \cong 2322$	$4451 \sim 2264 \cong 730$	$4493 \sim 2368 \cong 739$	$4535 \sim 2274 \cong 2274$
$4410 \sim 2241 \cong 739$	$4452 \sim 2277 \cong 2277$	$4494 \sim 2391 \cong 2391$	$4536 \sim 2193 \cong 2193$
$4411 \sim 2880 \cong 730$	$4453 \sim 2358 \cong 820$	$4495 \sim 2368 \cong 739$	$4537 \sim 2889 \cong 750$
$4412 \sim 2422 \cong 820$	$4454 \sim 2277 \cong 2277$	$4496 \sim 2206 \cong 748$	$4538 \sim 2403 \cong 2287$
$4413 \sim 2394 \cong 820$	$4455 \sim 2196 \cong 802$	$4497 \sim 2287 \cong 2287$	$4539 \sim 2424 \cong 966$
$4414 \sim 2422 \cong 820$	$4456 \sim 2862 \cong 847$	$4498 \sim 2391 \cong 2391$	$4540 \sim 2403 \cong 2287$
$4415 \sim 2260 \cong 802$	$4457 \sim 2376 \cong 739$	$4499 \sim 2287 \cong 2287$	$4541 \sim 2241 \cong 739$
$4416 \sim 2313 \cong 2277$	$4458 \sim 2427 \cong 2427$	$4500 \sim 2229 \cong 2229$	$4542 \sim 2322 \cong 2322$
$4417 \sim 2394 \cong 820$	$4459 \sim 2376 \cong 739$	$4501 \sim 2850 \cong 2850$	$4543 \sim 2424 \cong 966$
$4418 \sim 2313 \cong 2277$	$4460 \sim 2214 \cong 748$	$4502 \sim 2371 \cong 2371$	$4544 \sim 2322 \cong 2322$
$4419 \sim 2232 \cong 730$	$4461 \sim 2295 \cong 2295$	$4503 \sim 2364 \cong 2364$	$4545 \sim 2262 \cong 750$
$4420 \sim 2853 \cong 2853$	$4462 \sim 2427 \cong 2427$	$4504 \sim 2371 \cong 2371$	$4546 \sim 1091 \cong 731$
$4421 \sim 2423 \cong 2423$	$4463 \sim 2295 \cong 2295$	$4505 \sim 2209 \cong 2209$	$4547 \sim 821 \cong 821$
$4422 \sim 2367 \cong 2367$	$4464 \sim 2265 \cong 2265$	$4506 \sim 2283 \cong 2283$	$4548 \sim 821 \cong 821$
$4423 \sim 2423 \cong 2423$	$4465 \sim 1091 \cong 731$	$4507 \sim 2364 \cong 2364$	$4549 \sim 821 \cong 821$
$4424 \sim 2261 \cong 2261$	$4466 \sim 821 \cong 821$	$4508 \sim 2283 \cong 2283$	$4550 \sim 731 \cong 731$
$4425 \sim 2286 \cong 2286$	$4467 \sim 821 \cong 821$	$4509 \sim 2202 \cong 2202$	$4551 \sim 731 \cong 731$
$4426 \sim 2367 \cong 2367$	$4468 \sim 821 \cong 821$	$4510 \sim 2861 \cong 731$	$4552 \sim 821 \cong 821$
$4427 \sim 2286 \cong 2286$	$4469 \sim 731 \cong 731$	$4511 \sim 2375 \cong 2375$	$4553 \sim 731 \cong 731$
$4428 \sim 2205 \cong 775$	$4470 \sim 731 \cong 731$	$4512 \sim 2375 \cong 2375$	$4554 \sim 731 \cong 731$
$4429 \sim 2862 \cong 847$	$4471 \sim 821 \cong 821$	$4513 \sim 2375 \cong 2375$	$4555 \sim 1091 \cong 731$
$4430 \sim 2427 \cong 2427$	$4472 \sim 731 \cong 731$	$4514 \sim 2213 \cong 2213$	$4556 \sim 821 \cong 821$
$4431 \sim 2376 \cong 739$	$4473 \sim 731 \cong 731$	$4515 \sim 2294 \cong 2294$	$4557 \sim 821 \cong 821$
$4432 \sim 2427 \cong 2427$	$4474 \sim 1091 \cong 731$	$4516 \sim 2375 \cong 2375$	$4558 \sim 821 \cong 821$
$4433 \sim 2265 \cong 2265$	$4475 \sim 821 \cong 821$	$4517 \sim 2294 \cong 2294$	$4559 \sim 731 \cong 731$
$4434 \sim 2295 \cong 2295$	$4476 \sim 821 \cong 821$	$4518 \sim 2213 \cong 2213$	$4560 \sim 731 \cong 731$
$4435 \sim 2376 \cong 739$	$4477 \sim 821 \cong 821$	$4519 \sim 2852 \cong 849$	$4561 \sim 821 \cong 821$
$4436 \sim 2295 \cong 2295$	$4478 \sim 731 \cong 731$	$4520 \sim 2369 \cong 2369$	$4562 \sim 731 \cong 731$
$4437 \sim 2214 \cong 748$	$4479 \sim 731 \cong 731$	$4521 \sim 2366 \cong 2366$	$4563 \sim 731 \cong 731$
$4438 \sim 2853 \cong 2853$	$4480 \sim 821 \cong 821$	$4522 \sim 2369 \cong 2369$	$4564 \sim 2887 \cong 731$
$4439 \sim 2423 \cong 2423$	$4481 \sim 731 \cong 731$	$4523 \sim 2207 \cong 2207$	$4565 \sim 2401 \cong 2401$
$4440 \sim 2367 \cong 2367$	$4482 \sim 731 \cong 731$	$4524 \sim 2285 \cong 2285$	$4566 \sim 2401 \cong 2401$
$4441 \sim 2423 \cong 2423$	$4483 \sim 2860 \cong 2212$	$4525 \sim 2366 \cong 2366$	$4567 \sim 2401 \cong 2401$
$4442 \sim 2261 \cong 2261$	$4484 \sim 2374 \cong 821$	$4526 \sim 2285 \cong 2285$	$4568 \sim 2239 \cong 2239$
$4443 \sim 2286 \cong 2286$	$4485 \sim 2402 \cong 2402$	$4527 \sim 2204 \cong 2204$	$4569 \sim 2320 \cong 2294$
$4444 \sim 2367 \cong 2367$	$4486 \sim 2374 \cong 821$	$4528 \sim 2841 \cong 2841$	$4570 \sim 2401 \cong 2401$
$4445 \sim 2286 \cong 2286$	$4487 \sim 2212 \cong 2212$	$4529 \sim 2372 \cong 2372$	$4571 \sim 2320 \cong 2294$
$4446 \sim 2205 \cong 775$	$4488 \sim 2293 \cong 2293$	$4530 \sim 2355 \cong 2355$	$4572 \sim 2239 \cong 2239$
$4447 \sim 2844 \cong 730$	$4489 \sim 2402 \cong 2402$	$4531 \sim 2372 \cong 2372$	$4573 \sim 2874 \cong 820$

$4574 \sim 2395 \cong 2395$	$4616 \sim 2271 \cong 2271$	$4658 \sim 2206 \cong 748$	$4700 \sim 2358 \cong 820$
$4575 \sim 2388 \cong 821$	$4617 \sim 2190 \cong 750$	$4659 \sim 2287 \cong 2287$	$4701 \sim 2426 \cong 2277$
$4576 \sim 2395 \cong 2395$	$4618 \sim 2862 \cong 847$	$4660 \sim 2391 \cong 2391$	$4702 \sim 2358 \cong 820$
$4577 \sim 2233 \cong 2233$	$4619 \sim 2376 \cong 739$	$4661 \sim 2287 \cong 2287$	$4703 \sim 2196 \cong 802$
$4578 \sim 2307 \cong 2307$	$4620 \sim 2427 \cong 2427$	$4662 \sim 2229 \cong 2229$	$4704 \sim 2277 \cong 2277$
$4579 \sim 2388 \cong 821$	$4621 \sim 2376 \cong 739$	$4663 \sim 2852 \cong 849$	$4705 \sim 2426 \cong 2277$
$4580 \sim 2307 \cong 2307$	$4622 \sim 2214 \cong 748$	$4664 \sim 2369 \cong 2369$	$4706 \sim 2277 \cong 2277$
$4581 \sim 2226 \cong 820$	$4623 \sim 2295 \cong 2295$	$4665 \sim 2366 \cong 2366$	$4707 \sim 2264 \cong 730$
$4582 \sim 2847 \cong 929$	$4624 \sim 2427 \cong 2427$	$4666 \sim 2369 \cong 2369$	$4708 \sim 2838 \cong 750$
$4583 \sim 2398 \cong 2398$	$4625 \sim 2295 \cong 2295$	$4667 \sim 2207 \cong 2207$	$4709 \sim 2352 \cong 740$
$4584 \sim 2361 \cong 2361$	$4626 \sim 2265 \cong 2265$	$4668 \sim 2285 \cong 2285$	$4710 \sim 2399 \cong 2399$
$4585 \sim 2398 \cong 2398$	$4627 \sim 2860 \cong 2212$	$4669 \sim 2366 \cong 2366$	$4711 \sim 2352 \cong 740$
$4586 \sim 2236 \cong 2236$	$4628 \sim 2374 \cong 821$	$4670 \sim 2285 \cong 2285$	$4712 \sim 2190 \cong 750$
$4587 \sim 2280 \cong 2280$	$4629 \sim 2402 \cong 2402$	$4671 \sim 2204 \cong 2204$	$4713 \sim 2271 \cong 2271$
$4588 \sim 2361 \cong 2361$	$4630 \sim 2374 \cong 821$	$4672 \sim 1091 \cong 731$	$4714 \sim 2399 \cong 2399$
$4589 \sim 2280 \cong 2280$	$4631 \sim 2212 \cong 2212$	$4673 \sim 821 \cong 821$	$4715 \sim 2271 \cong 2271$
$4590 \sim 2199 \cong 2199$	$4632 \sim 2293 \cong 2293$	$4674 \sim 821 \cong 821$	$4716 \sim 2237 \cong 2237$
$4591 \sim 2860 \cong 2212$	$4633 \sim 2402 \cong 2402$	$4675 \sim 821 \cong 821$	$4717 \sim 2841 \cong 2841$
$4592 \sim 2402 \cong 2402$	$4634 \sim 2293 \cong 2293$	$4676 \sim 731 \cong 731$	$4718 \sim 2355 \cong 2355$
$4593 \sim 2374 \cong 821$	$4635 \sim 2240 \cong 2240$	$4677 \sim 731 \cong 731$	$4719 \sim 2372 \cong 2372$
$4594 \sim 2402 \cong 2402$	$4636 \sim 2861 \cong 731$	$4678 \sim 821 \cong 821$	$4720 \sim 2355 \cong 2355$
$4595 \sim 2240 \cong 2240$	$4637 \sim 2375 \cong 2375$	$4679 \sim 731 \cong 731$	$4721 \sim 2193 \cong 2193$
$4596 \sim 2293 \cong 2293$	$4638 \sim 2375 \cong 2375$	$4680 \sim 731 \cong 731$	$4722 \sim 2274 \cong 2274$
$4597 \sim 2374 \cong 821$	$4639 \sim 2375 \cong 2375$	$4681 \sim 2850 \cong 2850$	$4723 \sim 2372 \cong 2372$
$4598 \sim 2293 \cong 2293$	$4640 \sim 2213 \cong 2213$	$4682 \sim 2371 \cong 2371$	$4724 \sim 2274 \cong 2274$
$4599 \sim 2212 \cong 2212$	$4641 \sim 2294 \cong 2294$	$4683 \sim 2364 \cong 2364$	$4725 \sim 2210 \cong 2210$
$4600 \sim 2851 \cong 929$	$4642 \sim 2375 \cong 2375$	$4684 \sim 2371 \cong 2371$	$4726 \sim 2838 \cong 750$
$4601 \sim 2396 \cong 2396$	$4643 \sim 2294 \cong 2294$	$4685 \sim 2209 \cong 2209$	$4727 \sim 2352 \cong 740$
$4602 \sim 2365 \cong 2365$	$4644 \sim 2213 \cong 2213$	$4686 \sim 2283 \cong 2283$	$4728 \sim 2399 \cong 2399$
$4603 \sim 2396 \cong 2396$	$4645 \sim 1091 \cong 731$	$4687 \sim 2364 \cong 2364$	$4729 \sim 2352 \cong 740$
$4604 \sim 2234 \cong 2234$	$4646 \sim 821 \cong 821$	$4688 \sim 2283 \cong 2283$	$4730 \sim 2190 \cong 750$
$4605 \sim 2284 \cong 2284$	$4647 \sim 821 \cong 821$	$4689 \sim 2202 \cong 2202$	$4731 \sim 2271 \cong 2271$
$4606 \sim 2365 \cong 2365$	$4648 \sim 821 \cong 821$	$4690 \sim 2841 \cong 2841$	$4732 \sim 2399 \cong 2399$
$4607 \sim 2284 \cong 2284$	$4649 \sim 731 \cong 731$	$4691 \sim 2372 \cong 2372$	$4733 \sim 2271 \cong 2271$
$4608 \sim 2203 \cong 2203$	$4650 \sim 731 \cong 731$	$4692 \sim 2355 \cong 2355$	$4734 \sim 2237 \cong 2237$
$4609 \sim 2838 \cong 750$	$4651 \sim 821 \cong 821$	$4693 \sim 2372 \cong 2372$	$4735 \sim 1090 \cong 1090$
$4610 \sim 2399 \cong 2399$	$4652 \sim 731 \cong 731$	$4694 \sim 2210 \cong 2210$	$4736 \sim 820 \cong 820$
$4611 \sim 2352 \cong 740$	$4653 \sim 731 \cong 731$	$4695 \sim 2274 \cong 2274$	$4737 \sim 820 \cong 820$
$4612 \sim 2399 \cong 2399$	$4654 \sim 2854 \cong 847$	$4696 \sim 2355 \cong 2355$	$4738 \sim 820 \cong 820$
$4613 \sim 2237 \cong 2237$	$4655 \sim 2368 \cong 739$	$4697 \sim 2274 \cong 2274$	$4739 \sim 730 \cong 730$
$4614 \sim 2271 \cong 2271$	$4656 \sim 2391 \cong 2391$	$4698 \sim 2193 \cong 2193$	$4740 \sim 730 \cong 730$
$4615 \sim 2352 \cong 740$	$4657 \sim 2368 \cong 739$	$4699 \sim 2844 \cong 730$	$4741 \sim 820 \cong 820$

$4742 \sim 730 \cong 730$	$4784 \sim 2205 \cong 775$	$4826 \sim 820 \cong 820$	$4868 \sim 2322 \cong 2322$
$4743 \sim 730 \cong 730$	$4785 \sim 2286 \cong 2286$	$4827 \sim 820 \cong 820$	$4869 \sim 2262 \cong 750$
$4744 \sim 1090 \cong 1090$	$4786 \sim 2423 \cong 2423$	$4828 \sim 820 \cong 820$	$4870 \sim 2887 \cong 731$
$4745 \sim 820 \cong 820$	$4787 \sim 2286 \cong 2286$	$4829 \sim 730 \cong 730$	$4871 \sim 2401 \cong 2401$
$4746 \sim 820 \cong 820$	$4788 \sim 2261 \cong 2261$	$4830 \sim 730 \cong 730$	$4872 \sim 2401 \cong 2401$
$4747 \sim 820 \cong 820$	$4789 \sim 2851 \cong 929$	$4831 \sim 820 \cong 820$	$4873 \sim 2401 \cong 2401$
$4748 \sim 730 \cong 730$	$4790 \sim 2365 \cong 2365$	$4832 \sim 730 \cong 730$	$4874 \sim 2239 \cong 2239$
$4749 \sim 730 \cong 730$	$4791 \sim 2396 \cong 2396$	$4833 \sim 730 \cong 730$	$4875 \sim 2320 \cong 2294$
$4750 \sim 820 \cong 820$	$4792 \sim 2365 \cong 2365$	$4834 \sim 2850 \cong 2850$	$4876 \sim 2401 \cong 2401$
$4751 \sim 730 \cong 730$	$4793 \sim 2203 \cong 2203$	$4835 \sim 2364 \cong 2364$	$4877 \sim 2320 \cong 2294$
$4752 \sim 730 \cong 730$	$4794 \sim 2284 \cong 2284$	$4836 \sim 2371 \cong 2371$	$4878 \sim 2239 \cong 2239$
$4753 \sim 2841 \cong 2841$	$4795 \sim 2396 \cong 2396$	$4837 \sim 2364 \cong 2364$	$4879 \sim 2860 \cong 2212$
$4754 \sim 2355 \cong 2355$	$4796 \sim 2284 \cong 2284$	$4838 \sim 2202 \cong 2202$	$4880 \sim 2402 \cong 2402$
$4755 \sim 2372 \cong 2372$	$4797 \sim 2234 \cong 2234$	$4839 \sim 2283 \cong 2283$	$4881 \sim 2374 \cong 821$
$4756 \sim 2355 \cong 2355$	$4798 \sim 2852 \cong 849$	$4840 \sim 2371 \cong 2371$	$4882 \sim 2402 \cong 2402$
$4757 \sim 2193 \cong 2193$	$4799 \sim 2366 \cong 2366$	$4841 \sim 2283 \cong 2283$	$4883 \sim 2240 \cong 2240$
$4758 \sim 2274 \cong 2274$	$4800 \sim 2369 \cong 2369$	$4842 \sim 2209 \cong 2209$	$4884 \sim 2293 \cong 2293$
$4759 \sim 2372 \cong 2372$	$4801 \sim 2366 \cong 2366$	$4843 \sim 1090 \cong 1090$	$4885 \sim 2374 \cong 821$
$4760 \sim 2274 \cong 2274$	$4802 \sim 2204 \cong 2204$	$4844 \sim 820 \cong 820$	$4886 \sim 2293 \cong 2293$
$4761 \sim 2210 \cong 2210$	$4803 \sim 2285 \cong 2285$	$4845 \sim 820 \cong 820$	$4887 \sim 2212 \cong 2212$
$4762 \sim 1090 \cong 1090$	$4804 \sim 2369 \cong 2369$	$4846 \sim 820 \cong 820$	$4888 \sim 1091 \cong 731$
$4763 \sim 820 \cong 820$	$4805 \sim 2285 \cong 2285$	$4847 \sim 730 \cong 730$	$4889 \sim 821 \cong 821$
$4764 \sim 820 \cong 820$	$4806 \sim 2207 \cong 2207$	$4848 \sim 730 \cong 730$	$4890 \sim 821 \cong 821$
$4765 \sim 820 \cong 820$	$4807 \sim 2847 \cong 929$	$4849 \sim 820 \cong 820$	$4891 \sim 821 \cong 821$
$4766 \sim 730 \cong 730$	$4808 \sim 2361 \cong 2361$	$4850 \sim 730 \cong 730$	$4892 \sim 731 \cong 731$
$4767 \sim 730 \cong 730$	$4809 \sim 2398 \cong 2398$	$4851 \sim 730 \cong 730$	$4893 \sim 731 \cong 731$
$4768 \sim 820 \cong 820$	$4810 \sim 2361 \cong 2361$	$4852 \sim 1090 \cong 1090$	$4894 \sim 821 \cong 821$
$4769 \sim 730 \cong 730$	$4811 \sim 2199 \cong 2199$	$4853 \sim 820 \cong 820$	$4895 \sim 731 \cong 731$
$4770 \sim 730 \cong 730$	$4812 \sim 2280 \cong 2280$	$4854 \sim 820 \cong 820$	$4896 \sim 731 \cong 731$
$4771 \sim 1090 \cong 1090$	$4813 \sim 2398 \cong 2398$	$4855 \sim 820 \cong 820$	$4897 \sim 2874 \cong 820$
$4772 \sim 820 \cong 820$	$4814 \sim 2280 \cong 2280$	$4856 \sim 730 \cong 730$	$4898 \sim 2395 \cong 2395$
$4773 \sim 820 \cong 820$	$4815 \sim 2236 \cong 2236$	$4857 \sim 730 \cong 730$	$4899 \sim 2388 \cong 821$
$4774 \sim 820 \cong 820$	$4816 \sim 1090 \cong 1090$	$4858 \sim 820 \cong 820$	$4900 \sim 2395 \cong 2395$
$4775 \sim 730 \cong 730$	$4817 \sim 820 \cong 820$	$4859 \sim 730 \cong 730$	$4901 \sim 2233 \cong 2233$
$4776 \sim 730 \cong 730$	$4818 \sim 820 \cong 820$	$4860 \sim 730 \cong 730$	$4902 \sim 2307 \cong 2307$
$4777 \sim 820 \cong 820$	$4819 \sim 820 \cong 820$	$4861 \sim 2889 \cong 750$	$4903 \sim 2388 \cong 821$
$4778 \sim 730 \cong 730$	$4820 \sim 730 \cong 730$	$4862 \sim 2403 \cong 2287$	$4904 \sim 2307 \cong 2307$
$4779 \sim 730 \cong 730$	$4821 \sim 730 \cong 730$	$4863 \sim 2424 \cong 966$	$4905 \sim 2226 \cong 820$
$4780 \sim 2853 \cong 2853$	$4822 \sim 820 \cong 820$	$4864 \sim 2403 \cong 2287$	$4906 \sim 2851 \cong 929$
$4781 \sim 2367 \cong 2367$	$4823 \sim 730 \cong 730$	$4865 \sim 2241 \cong 739$	$4907 \sim 2396 \cong 2396$
$4782 \sim 2423 \cong 2423$	$4824 \sim 730 \cong 730$	$4866 \sim 2322 \cong 2322$	$4908 \sim 2365 \cong 2365$
$4783 \sim 2367 \cong 2367$	$4825 \sim 1090 \cong 1090$	$4867 \sim 2424 \cong 966$	$4909 \sim 2396 \cong 2396$

4910 ∼ 2234 ≅ 2234	4952 ∼ 2361 ≅ 2361	4994 ∼ 730 ≅ 730	5036 ∼ 2226 ≅ 820
4911 ∼ 2284 ≅ 2284	4953 ∼ 2398 ≅ 2398	4995 ∼ 730 ≅ 730	5037 ∼ 2307 ≅ 2307
4912 ∼ 2365 ≅ 2365	4954 ∼ 2361 ≅ 2361	4996 ∼ 2852 ≅ 849	5038 ∼ 2395 ≅ 2395
4913 ∼ 2284 ≅ 2284	4955 ∼ 2199 ≅ 2199	4997 ∼ 2366 ≅ 2366	5039 ∼ 2307 ≅ 2307
4914 ∼ 2203 ≅ 2203	4956 ∼ 2280 ≅ 2280	4998 ∼ 2369 ≅ 2369	5040 ∼ 2233 ≅ 2233
4915 ∼ 1091 ≅ 731	4957 ∼ 2398 ≅ 2398	4999 ∼ 2366 ≅ 2366	5041 ∼ 2854 ≅ 847
4916 ∼ 821 ≅ 821	4958 ∼ 2280 ≅ 2280	5000 ∼ 2204 ≅ 2204	5042 ∼ 2391 ≅ 2391
4917 ∼ 821 ≅ 821	4959 ∼ 2236 ≅ 2236	5001 ∼ 2285 ≅ 2285	5043 ∼ 2368 ≅ 739
4918 ∼ 821 ≅ 821	4960 ∼ 2850 ≅ 2850	5002 ∼ 2369 ≅ 2369	5044 ∼ 2391 ≅ 2391
4919 ∼ 731 ≅ 731	4961 ∼ 2364 ≅ 2364	5003 ∼ 2285 ≅ 2285	5045 ∼ 2229 ≅ 2229
4920 ∼ 731 ≅ 731	4962 ∼ 2371 ≅ 2371	5004 ∼ 2207 ≅ 2207	5046 ∼ 2287 ≅ 2287
4921 ∼ 821 ≅ 821	4963 ∼ 2364 ≅ 2364	5005 ∼ 1090 ≅ 1090	5047 ∼ 2368 ≅ 739
4922 ∼ 731 ≅ 731	4964 ∼ 2202 ≅ 2202	5006 ∼ 820 ≅ 820	5048 ∼ 2287 ≅ 2287
4923 ∼ 731 ≅ 731	4965 ∼ 2283 ≅ 2283	5007 ∼ 820 ≅ 820	5049 ∼ 2206 ≅ 748
4924 ∼ 2847 ≅ 929	4966 ∼ 2371 ≅ 2371	5008 ∼ 820 ≅ 820	5050 ∼ 2874 ≅ 820
4925 ∼ 2398 ≅ 2398	4967 ∼ 2283 ≅ 2283	5009 ∼ 730 ≅ 730	5051 ∼ 2388 ≅ 821
4926 ∼ 2361 ≅ 2361	4968 ∼ 2209 ≅ 2209	5010 ∼ 730 ≅ 730	5052 ∼ 2395 ≅ 2395
4927 ∼ 2398 ≅ 2398	4969 ∼ 2851 ≅ 929	5011 ∼ 820 ≅ 820	5053 ∼ 2388 ≅ 821
4928 ∼ 2236 ≅ 2236	4970 ∼ 2365 ≅ 2365	5012 ∼ 730 ≅ 730	5054 ∼ 2226 ≅ 820
4929 ∼ 2280 ≅ 2280	4971 ∼ 2396 ≅ 2396	5013 ∼ 730 ≅ 730	5055 ∼ 2307 ≅ 2307
4930 ∼ 2361 ≅ 2361	4972 ∼ 2365 ≅ 2365	5014 ∼ 1090 ≅ 1090	5056 ∼ 2395 ≅ 2395
4931 ∼ 2280 ≅ 2280	4973 ∼ 2203 ≅ 2203	5015 ∼ 820 ≅ 820	5057 ∼ 2307 ≅ 2307
4932 ∼ 2199 ≅ 2199	4974 ∼ 2284 ≅ 2284	5016 ∼ 820 ≅ 820	5058 ∼ 2233 ≅ 2233
4933 ∼ 2838 ≅ 750	4975 ∼ 2396 ≅ 2396	5017 ∼ 820 ≅ 820	5059 ∼ 1090 ≅ 1090
4934 ∼ 2399 ≅ 2399	4976 ∼ 2284 ≅ 2284	5018 ∼ 730 ≅ 730	5060 ∼ 820 ≅ 820
4935 ∼ 2352 ≅ 740	4977 ∼ 2234 ≅ 2234	5019 ∼ 730 ≅ 730	5061 ∼ 820 ≅ 820
4936 ∼ 2399 ≅ 2399	4978 ∼ 1090 ≅ 1090	5020 ∼ 820 ≅ 820	5062 ∼ 820 ≅ 820
4937 ∼ 2237 ≅ 2237	4979 ∼ 820 ≅ 820	5021 ∼ 730 ≅ 730	5063 ∼ 730 ≅ 730
4938 ∼ 2271 ≅ 2271	4980 ∼ 820 ≅ 820	5022 ∼ 730 ≅ 730	5064 ∼ 730 ≅ 730
4939 ∼ 2352 ≅ 740	4981 ∼ 820 ≅ 820	5023 ∼ 2880 ≅ 730	5065 ∼ 820 ≅ 820
4940 ∼ 2271 ≅ 2271	4982 ∼ 730 ≅ 730	5024 ∼ 2394 ≅ 820	5066 ∼ 730 ≅ 730
4941 ∼ 2190 ≅ 750	4983 ∼ 730 ≅ 730	5025 ∼ 2422 ≅ 820	5067 ∼ 730 ≅ 730
4942 ∼ 2853 ≅ 2853	4984 ∼ 820 ≅ 820	5026 ∼ 2394 ≅ 820	5068 ∼ 1090 ≅ 1090
4943 ∼ 2367 ≅ 2367	4985 ∼ 730 ≅ 730	5027 ∼ 2232 ≅ 730	5069 ∼ 820 ≅ 820
4944 ∼ 2423 ≅ 2423	4986 ∼ 730 ≅ 730	5028 ∼ 2313 ≅ 2277	5070 ∼ 820 ≅ 820
4945 ∼ 2367 ≅ 2367	4987 ∼ 1090 ≅ 1090	5029 ∼ 2422 ≅ 820	5071 ∼ 820 ≅ 820
4946 ∼ 2205 ≅ 775	4988 ∼ 820 ≅ 820	5030 ∼ 2313 ≅ 2277	5072 ∼ 730 ≅ 730
4947 ∼ 2286 ≅ 2286	4989 ∼ 820 ≅ 820	5031 ∼ 2260 ≅ 802	5073 ∼ 730 ≅ 730
4948 ∼ 2423 ≅ 2423	4990 ∼ 820 ≅ 820	5032 ∼ 2874 ≅ 820	5074 ∼ 820 ≅ 820
4949 ∼ 2286 ≅ 2286	4991 ∼ 730 ≅ 730	5033 ∼ 2388 ≅ 821	5075 ∼ 730 ≅ 730
4950 ∼ 2261 ≅ 2261	4992 ∼ 730 ≅ 730	5034 ∼ 2395 ≅ 2395	5076 ∼ 730 ≅ 730
4951 ∼ 2847 ≅ 929	4993 ∼ 820 ≅ 820	5035 ∼ 2388 ≅ 821	5077 ∼ 2854 ≅ 847

$5078 \sim 2391 \cong 2391$	$5085 \sim 2206 \cong 748$	$5092 \sim 820 \cong 820$	$5099 \sim 730 \cong 730$
$5079 \sim 2368 \cong 739$	$5086 \sim 1090 \cong 1090$	$5093 \sim 730 \cong 730$	$5100 \sim 730 \cong 730$
$5080 \sim 2391 \cong 2391$	$5087 \sim 820 \cong 820$	$5094 \sim 730 \cong 730$	$5101 \sim 820 \cong 820$
$5081 \sim 2229 \cong 2229$	$5088 \sim 820 \cong 820$	$5095 \sim 1090 \cong 1090$	$5102 \sim 730 \cong 730$
$5082 \sim 2287 \cong 2287$	$5089 \sim 820 \cong 820$	$5096 \sim 820 \cong 820$	$5103 \sim 730 \cong 730$
$5083 \sim 2368 \cong 739$	$5090 \sim 730 \cong 730$	$5097 \sim 820 \cong 820$	
$5084 \sim 2287 \cong 2287$	$5091 \sim 730 \cong 730$	$5098 \sim 820 \cong 820$	

5104 through $5832 \sim 1090 \simeq 1090$.

8. Group information

We use the following notation:

- Rels - a list of some relators in the group. In most cases these are the first few relators in the length-lexicographic order, but in some cases (more precisely, for the automata numbered by 744, 753, 776, 840, 843, 858, 885, 888, 956, 965, 2209, 2210, 2213, 2234, 2261, 2274, 2293, 2355, 2364, 2396, 2402, 2423) there could be some shorter relators. In most cases the given list does not give a presentation of the group (exception are the finite and abelian groups, and the automata numbered by 820, 846, 870, 2212, 2240, 2294).

- SF - these numbers represent the size of the factors $G/\operatorname{Stab}_G(n)$, for $n \geq 0$.

- Gr - these numbers represent the first few values of the growth function $\gamma_G(n)$, for $n \geq 0$, with respect to the generating system a, b, c ($\gamma_G(n)$ counts the number of elements of length at most n in G).

Automaton number 1

$a = (a, a)$ Group: *Trivial Group*

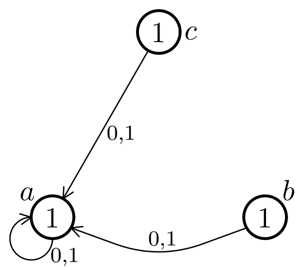
$b = (a, a)$ Contracting: *yes*

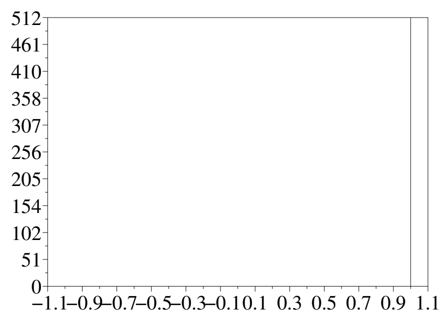
$c = (a, a)$ Self-replicating: *yes*

Rels: a, b, c

SF: 2^0,2^0,2^0,2^0,2^0,2^0,2^0,2^0,2^0

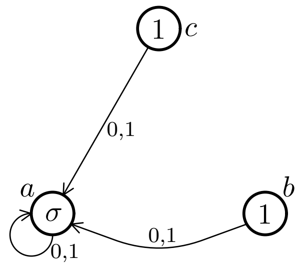
Gr: 1,1,1,1,1,1,1,1,1,1,1

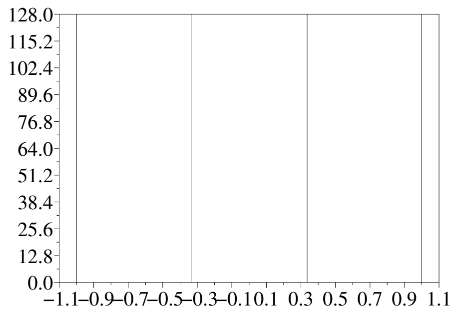




Automaton number 730

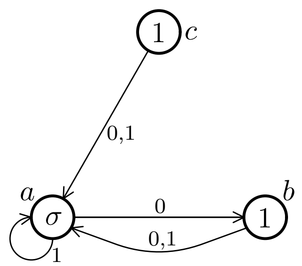
$a = \sigma(a, a)$ Group: *Klein Group*
$b = (a, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: $b^{-1}c$, a^2, b^2, $abab$
SF: 2^0,2^1,2^2,2^2,2^2,2^2,2^2,2^2,2^2
Gr: 1,3,4,4,4,4,4,4,4,4,4

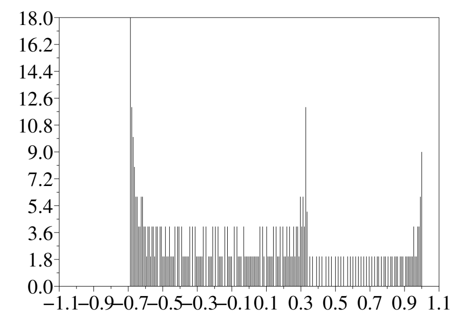




Automaton number 731

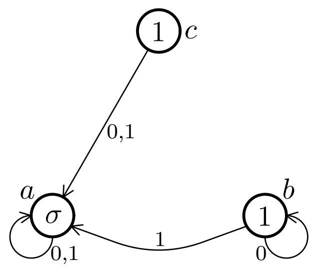
$a = \sigma(b, a)$ Group: $\mathbb{Z}$
$b = (a, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*
Rels: $b^{-1}c$, ba^2
SF: 2^0,2^1,2^2,2^3,2^4,2^5,2^6,2^7,2^8
Gr: 1,5,9,13,17,21,25,29,33,37,41

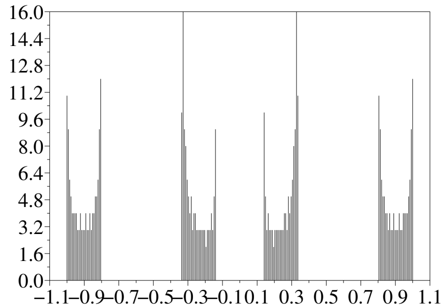




Automaton number 739

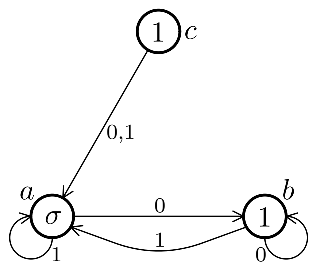
$a = \sigma(a, a)$ Group: $C_2 \ltimes \left(\mathbb{Z} \wr C_2\right)$
$b = (b, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $(ac)^2$, $(acbab)^2$
SF: 2^0,2^1,2^3,2^6,2^8,2^{10},2^{12},2^{14},2^{16}
Gr: 1,4,9,17,30,47,68,93,122,155,192

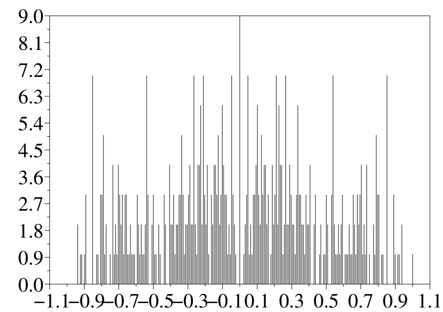




Automaton number 740

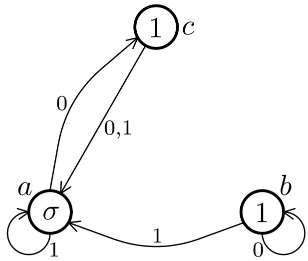
$a = \sigma(b, a)$ Group:
$b = (b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *no*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $a^{-1}c^{-1}ac^{-1}b^2$, $[a, b]^2$
SF: 2^0,2^1,2^3,2^6,2^9,2^{11},2^{14},2^{16},2^{18}
Gr: 1,7,33,135,495,1725

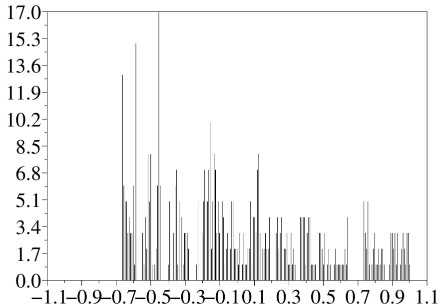




Automaton number 741

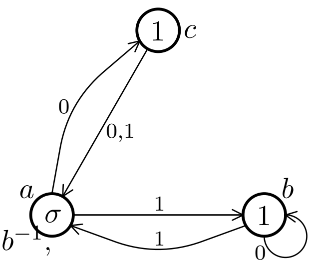
$a = \sigma(c, a)$ Group:
$b = (b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: ca^2, $b^{-1}a^{-3}b^{-1}ababa$,
$b^{-1}a^{-6}b^{-1}a^{-2}ba^{-2}ba^{-2}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,29,115,441,1643

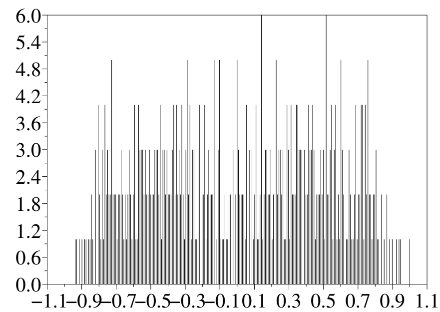




Automaton number 744

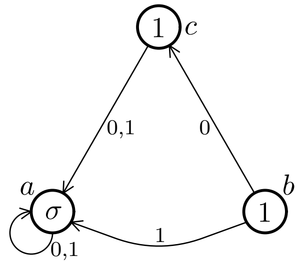
$a = \sigma(c, b)$ Group:
$b = (b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels:
$[a^2ca^{-1}bc^{-1}b^{-1}a^{-1}, aca^{-1}bc^{-1}b^{-1}]$,
$abcb^{-1}ac^{-1}a^{-2}bcb^{-1}ab^{-1}aca^{-1}bc^{-1}a^{-1}bc^{-1}b^{-1}$,
$abcb^{-1}ab^{-1}a^{-2}bcb^{-1}ac^{-1}aba^{-1}bc^{-1}b^{-1}ca^{-1}bc^{-1}b^{-1}$,
$abcb^{-1}ab^{-1}a^{-2}bcb^{-1}ab^{-1}a\cdot$
$ba^{-1}bc^{-1}a^{-1}bc^{-1}b^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4687

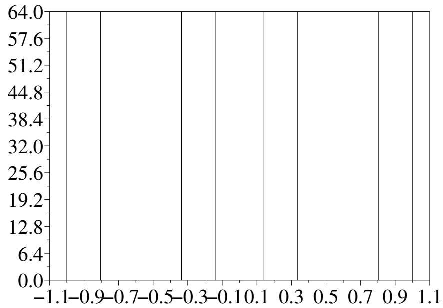




Automaton number 748

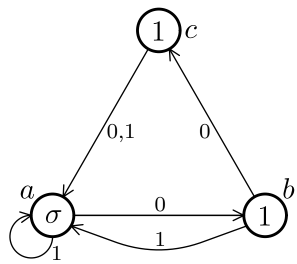
$a = \sigma(a, a)$ Group: $D_4 \times C_2$
$b = (c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $acac$, $bcbc$, $abababab$
SF: 2^0,2^1,2^3,2^4,2^4,2^4,2^4,2^4,2^4
Gr: 1,4,8,12,15,16,16,16,16,16,16

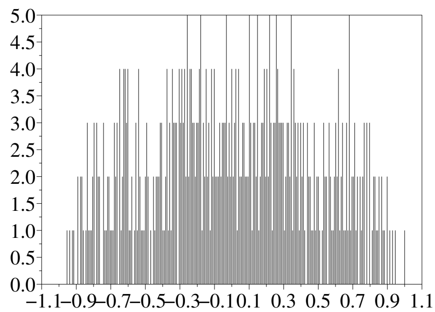




Automaton number 749

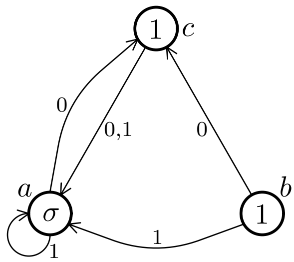
$a = \sigma(b, a)$ Group:
$b = (c, a)$ Contracting: *n/a*
$c = (a, a)$ Self-replicating: *yes*
Rels: $a^{-1}c^{-1}bab^{-1}a^{-1}cb^{-1}ab$,
$a^{-1}c^{-1}bac^{-1}a^{-1}cb^{-1}ac$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4667

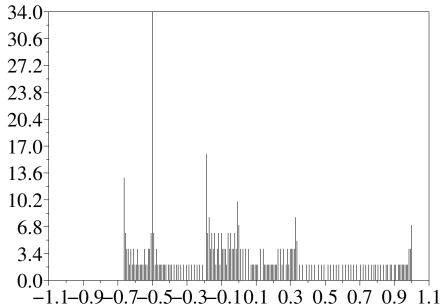




Automaton number 750

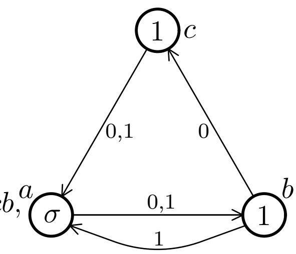
$a = \sigma(c, a)$ Group: $C_2 \wr \mathbb{Z}$
$b = (c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: ca^2, $(a^{-1}b)^2$, $[b, c]$
SF: 2^0,2^1,2^3,2^5,2^7,2^9,2^{11},2^{13},2^{15}
Gr: 1,7,23,49,87,137,199,273,359

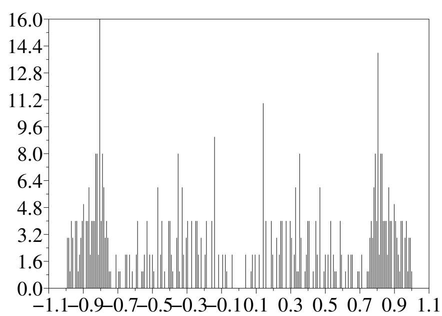




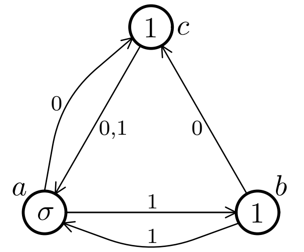
Automaton number 752

$a = \sigma(b, b)$ Group: virtually $\mathbb{Z}^3$
$b = (c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $(acbab)^2$, $(acacb)^2$, $(abc)^2(acb)^2$, $acbcbabacbcbab$, $abcbacbabcbacb$, $acbcacbacbcacb$, $acacbcbacacbcb$, $abc(bca)^2cbcbacb$, $a(cb)^3aba(cb)^3ab$, $abcbcbacbabcbcbacb$, $acbcbcacbacbcbcacb$
SF: 2^0,2^1,2^3,2^5,2^7,2^8,2^{10},2^{11},2^{13}
Gr: 1,4,10,22,46,84,140,217,319,448





Automaton number 753

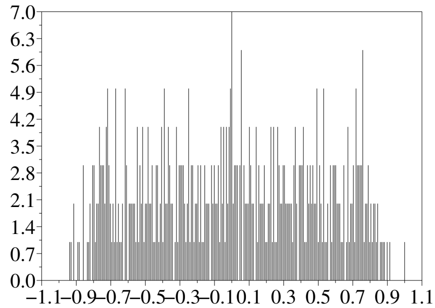


$a = \sigma(c, b)$ Group:
$b = (c, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*

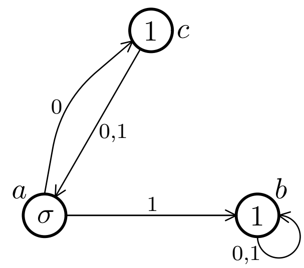
Rels: $aba^{-1}b^{-1}ab^{-1}ca^{-1}ba^{-1}b^{-1}ab^{-1}cac^{-1}b\cdot$
$a^{-1}bab^{-1}a^{-1}c^{-1}ba^{-1}bab^{-1}$,
$aba^{-1}b^{-1}ab^{-1}ca^{-1}c^{-1}ba^{-1}c^{-1}bab^{-1}ca\cdot$
$c^{-1}ba^{-1}bab^{-1}a^{-1}c^{-1}ba^{-1}b^{-1}cab^{-1}c$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,37,187,937,4687



Automaton number 771



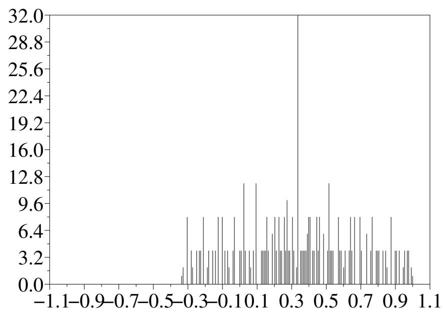
$a = \sigma(c, b)$ Group: $\mathbb{Z}^2$
$b = (b, b)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*

Rels: b, $a^{-1}c^{-1}ac$

SF: 2^0,2^1,2^2,2^3,2^4,2^5,2^6,2^7,2^8

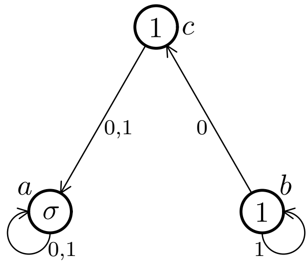
Gr: 1,5,13,25,41,61,85,113,145,181,221

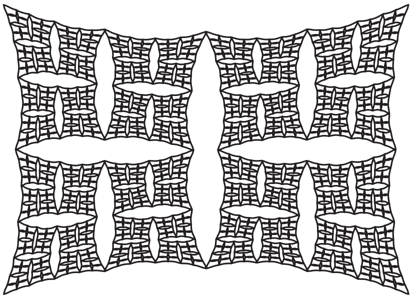
Limit space: 2-dimensional torus T_2

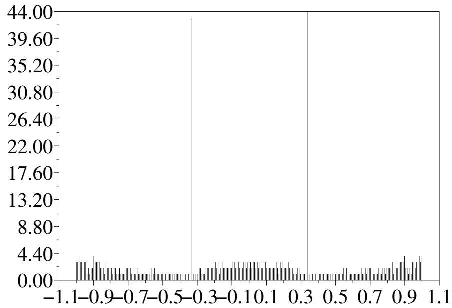


Automaton number 775

$a = \sigma(a,a)$ Group: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = (c,b)$ Contracting: *yes*
$c = (a,a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acac$, $acbcbabcbcabcbabcb$
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,9,17,30,51,85,140,229,367,579
Limit space:

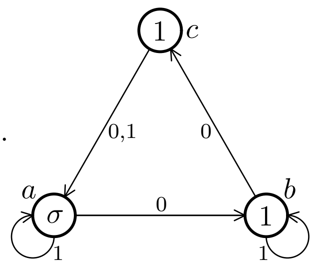


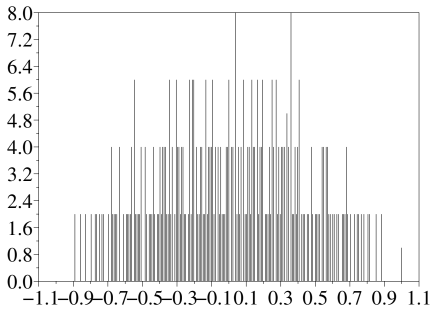




Automaton number 776

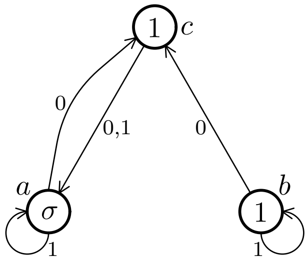
$a = \sigma(b,a)$ Group:
$b = (c,b)$ Contracting: *no*
$c = (a,a)$ Self-replicating: *yes*
Rels: $aba^{-1}b^{-1}a^2c^{-1}ab^{-1}a^{-1}bcb^{-1}ac^{-1}a^{-1}ba^{-1}\cdot$
$b^{-1}a^2c^{-1}ab^{-1}a^{-1}bcb^{-1}ac^{-1}aca^{-1}bc^{-1}b^{-1}ab\cdot$
$a^{-1}ca^{-2}bab^{-1}a^{-1}ca^{-1}bc^{-1}b^{-1}aba^{-1}ca^{-2}bab^{-1}$,
$aba^{-1}b^{-1}a^2c^{-1}ab^{-1}a^{-1}bcb^{-1}ac^{-1}a^{-1}cba^{-1}\cdot$
$b^{-1}a^2c^{-1}ab^{-1}a^{-1}bc^{-1}b^{-1}aba^{-1}ca^{-2}\cdot$
$bab^{-1}aca^{-1}bc^{-1}b^{-1}aba^{-1}ca^{-2}bab^{-1}\cdot$
$a^{-1}ba^{-1}b^{-1}a^2c^{-1}ab^{-1}a^{-1}bcb^{-1}\cdot$
$aba^{-1}ca^{-2}bab^{-1}c^{-1}$
SF: 2^0,2^1,2^2,2^4,2^7,2^{13},2^{24},2^{46},2^{89}
Gr: 1,7,37,187,937,4687

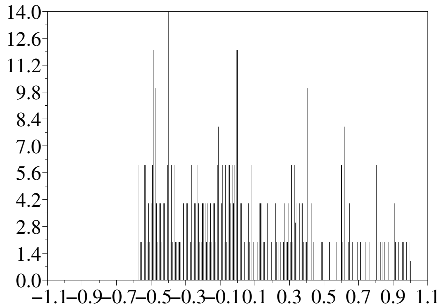




Automaton number 777

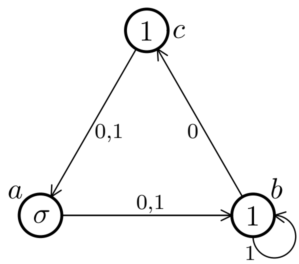
$a = \sigma(c, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: ca^2, $b^{-1}a^5b^{-1}a^{-1}ba^{-3}ba^{-1}$
SF: 2^0,2^1,2^2,2^4,2^7,2^{13},2^{24},2^{46},2^{89}
Gr: 1,7,29,115,441,1695

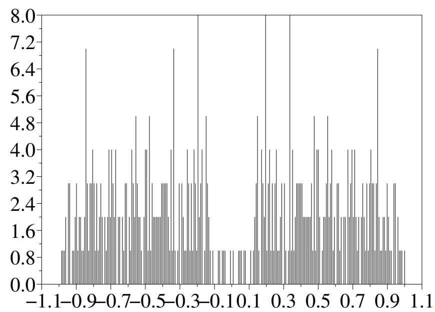




Automaton number 779

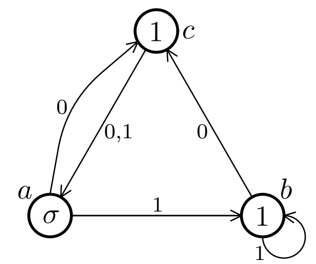
$a = \sigma(b, b)$ Group:
$b = (c, b)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acabcabcbabacabcabcbab$,
$acbcbacacabcbcabcbabcb$
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,10,22,46,94,190,382,766,1534,3070,6120

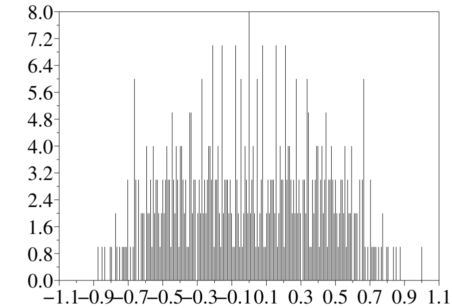




Automaton number 780

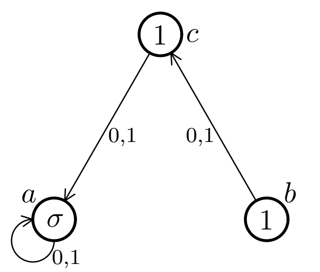
$a = \sigma(c, b)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $[ba^{-1}, c]$
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{27},2^{49}
Gr: 1,7,35,159,705,3107

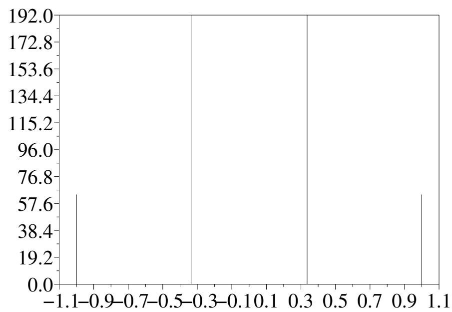




Automaton number 802

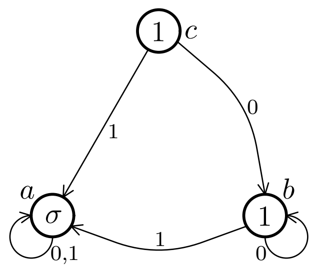
$a = \sigma(a, a)$ Group: $C_2 \times C_2 \times C_2$
$b = (c, c)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $[a, b]$, $[a, c]$, $[b, c]$
SF: 2^0,2^1,2^2,2^3,2^3,2^3,2^3,2^3,2^3
Gr: 1,4,7,8,8,8,8,8,8,8,8

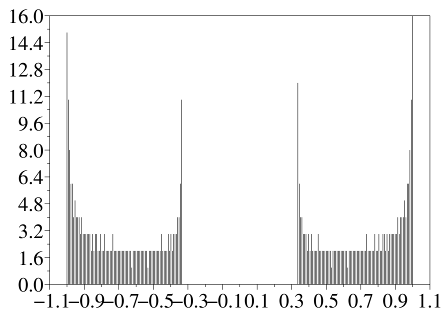




Automaton number 820

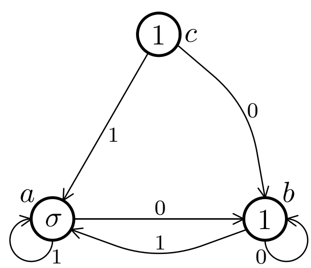
$a = \sigma(a, a)$ Group: D_∞
$b = (b, a)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: $b^{-1}c$, a^2, b^2
SF: 2^0,2^1,2^3,2^4,2^5,2^6,2^7,2^8,2^9
Gr: 1,3,5,7,9,11,13,15,17,19,21

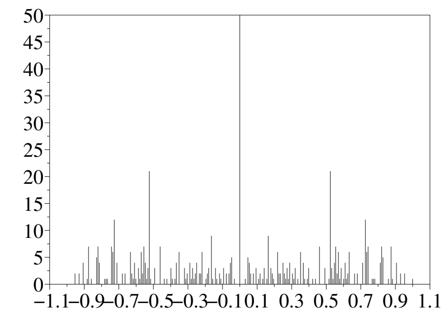




Automaton number 821

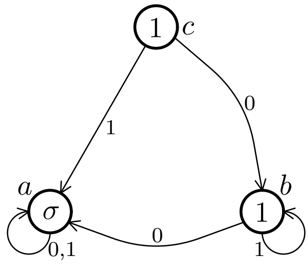
$a = \sigma(b, a)$ Group: *Lamplighter group* $\mathbb{Z} \wr C_2$
$b = (b, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $b^{-1}c$, $(a^{-1}b)^2$, $[a, b]^2$,
$a^{-3}baba^{-2}b^{-1}a^2b$
SF: 2^0,2^1,2^3,2^5,2^6,2^8,2^9,2^{10},2^{11}
Gr: 1,5,15,39,92,208,452,964,2016

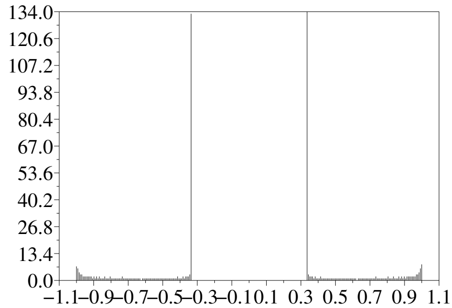




Automaton number 838

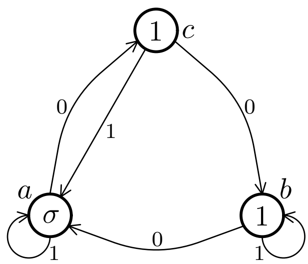
$a = \sigma(a, a)$ Group: $C_2 \ltimes \langle s, t \mid s^2 = t^2 \rangle$
$b = (a, b)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $abcacb$
SF: 2^0,2^1,2^3,2^5,2^7,2^9,2^{11},2^{13},2^{15}
Gr: 1,4,10,19,31,46,64,85,109,136

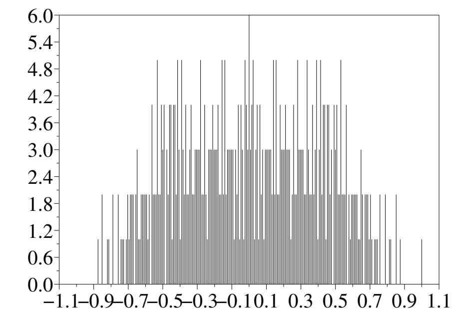




Automaton number 840

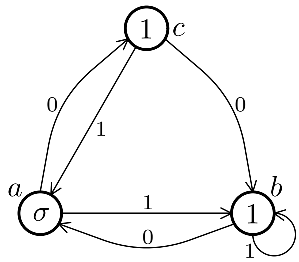
$a = \sigma(c, a)$ Group:
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $abac^{-1}a^{-2}bac^{-1}aca^{-1}b^{-1}ca^{-1}b^{-1}$,
$abac^{-1}a^{-2}cac^{-1}b^{-1}caca^{-1}b^{-1}c^{-1}bca^{-1}c^{-1}$,
$acac^{-1}b^{-1}ca^{-2}bac^{-1}ac^{-1}bca^{-2}b^{-1}$,
$acac^{-1}b^{-1}ca^{-2}cac^{-1}b^{-1}cac^{-1}bca^{-1}c^{-2}bca^{-1}c^{-1}$
SF: 2^0,2^1,2^3,2^5,2^7,2^{10},2^{15},2^{25},2^{41}
Gr: 1,7,37,187,937,4687





Automaton number 843

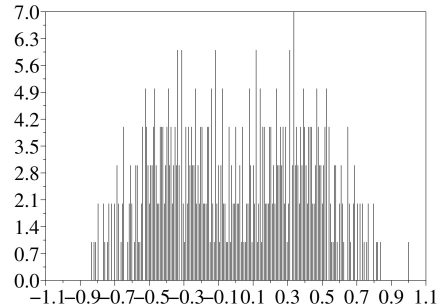
$a = \sigma(c, b)$ Group:
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*



Rels: $acab^{-1}a^{-2}cab^{-1}aba^{-1}c^{-1}ba^{-1}c^{-1}$,
$acab^{-1}a^{-2}cb^{-1}ab^{-1}caba^{-1}c^{-2}ba^{-1}bc^{-1}$,
$acb^{-1}ab^{-1}ca^{-2}cab^{-1}ac^{-1}ba^{-1}bc^{-1}ba^{-1}c^{-1}$,
$acb^{-1}ab^{-1}ca^{-2}cb^{-1}ab^{-1}cac^{-1}ba^{-1}bc^{-2}ba^{-1}bc^{-1}$

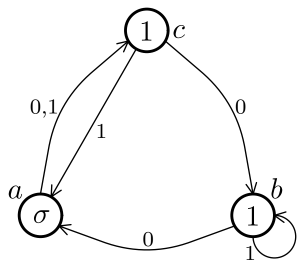
SF: 2^0,2^1,2^3,2^5,2^8,2^{14},2^{24},2^{43},2^{81}

Gr: 1,7,37,187,937,4687



Automaton number 846

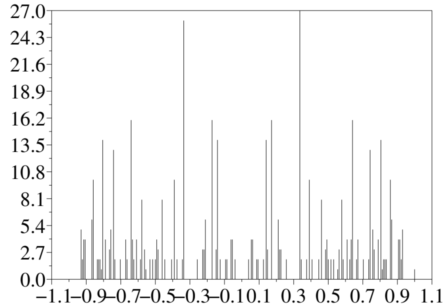
$a = \sigma(c, c)$ Group: $C_2 * C_2 * C_2$
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *no*



Rels: a^2, b^2, c^2

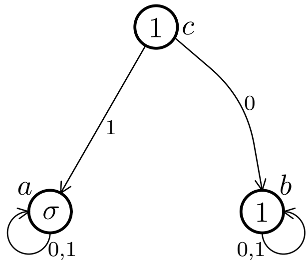
SF: 2^0,2^1,2^3,2^5,2^7,2^{10},2^{13},2^{16},2^{19}

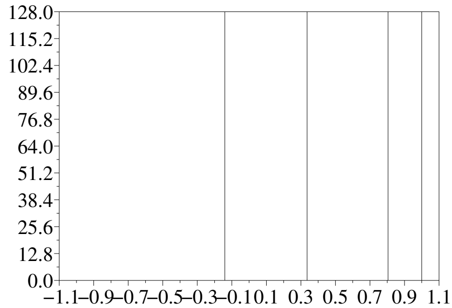
Gr: 1,4,10,22,46,94,190,382,766,1534



Automaton number 847

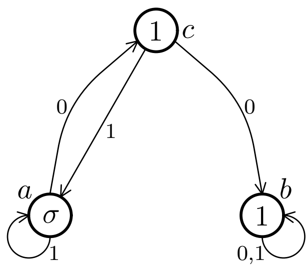
$a = \sigma(a, a)$ Group: D_4
$b = (b, b)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *no*
Rels: b, a^2, c^2, $acacacac$
SF: 2^0,2^1,2^3,2^3,2^3,2^3,2^3,2^3,2^3
Gr: 1,3,5,7,8,8,8,8,8,8,8

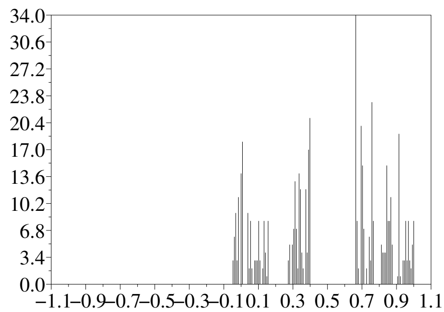




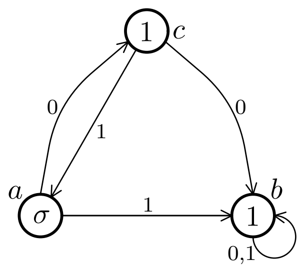
Automaton number 849

$a = \sigma(c, a)$ Group:
$b = (b, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: b, $[ac^{-1}a^{-1}, c]$,
$[a^2, c^{-1}] \cdot [c, a^{-2}]$, $[a^{-1}, c^{-2}] \cdot [a^{-1}, c^2]$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,5,17,53,153,421,1125,2945,7589

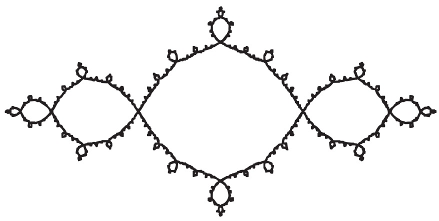


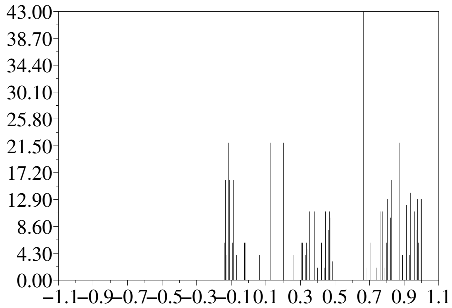


Automaton number 852

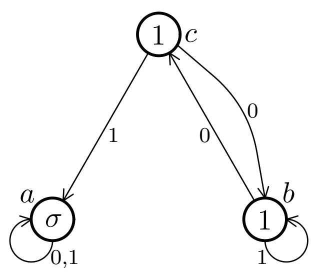


$a = \sigma(c, b)$ Group: $IMG(z^2 - 1)$
$b = (b, b)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: b, $[ac^{-1}a^{-1}, c]$,
$[c, a^2] \cdot [c, a^{-2}]$, $[a^{-1}, c^{-2}] \cdot [a^{-1}, c^2]$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,5,17,53,153,421,1125,2945,7545
Limit space:

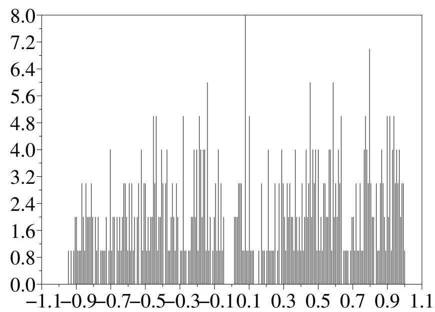




Automaton number 856

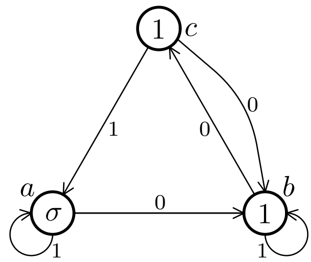


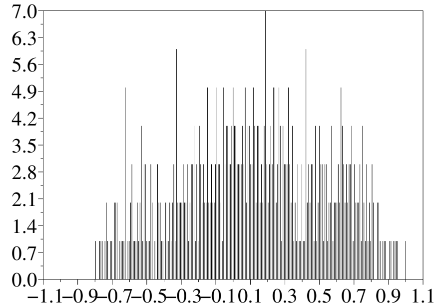
$a = \sigma(a, a)$ Group: $C_2 \ltimes G_{2850}$
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acbcacbcabcacacacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,190,382,766,
1525,3025,5998,11890,23532



Automaton number 857

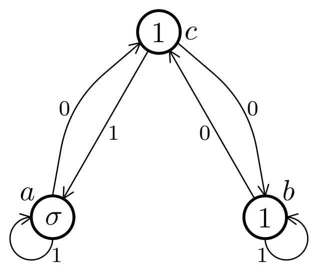
$a = \sigma(b, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}c)^2$, $(a^{-1}b)^4$, $(a^{-1}b^{-1}ac)^2$, $(b^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,758,3460

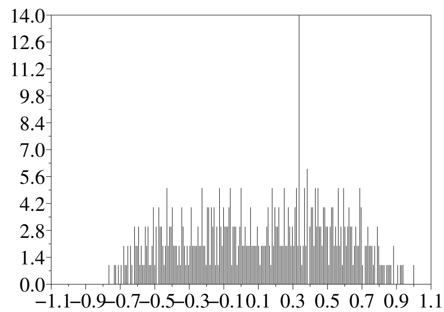




Automaton number 858

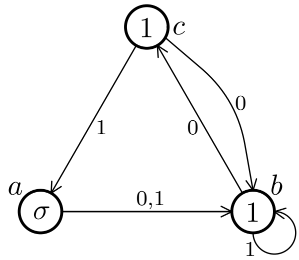
$a = \sigma(c, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $abca^{-1}c^{-1}ab^{-1}a^2c^{-1}b^{-1}a^{-1}bca^{-1}c^{-1}a\cdot b^{-1}a^2c^{-1}b^{-1}abca^{-2}ba^{-1}cac^{-1}b^{-1}a^{-1}\cdot bca^{-2}ba^{-1}cac^{-1}b^{-1}$,
$abca^{-1}c^{-1}ab^{-1}a^2c^{-1}b^{-1}a^{-1}cba^{-1}b^{-1}ab^{-1}a\cdot bca^{-2}ba^{-1}cac^{-1}b^{-1}a^{-1}ba^{-1}bab^{-1}c^{-1}$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{90},2^{176}
Gr: 1,7,37,187,937,4687

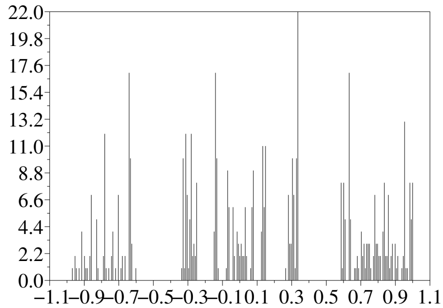




Automaton number 860

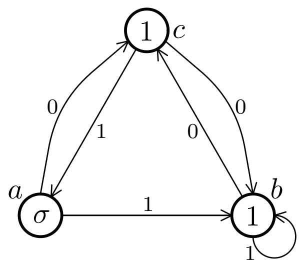
$a = \sigma(b, b)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acbacacabcabab$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,190,375,731,1422,2762,5350

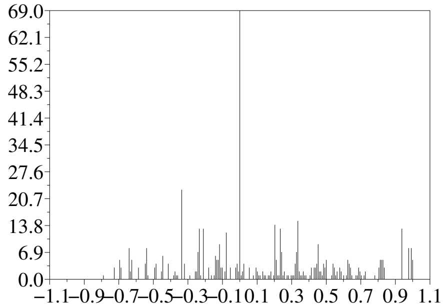




Automaton number 861

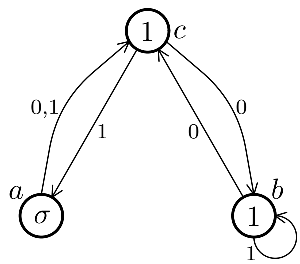
$a = \sigma(c, b)$ Group:
$b = (c, b)$ Contracting: *n/a*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[a, b]^2$, $[b, c]^2$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,599,2485

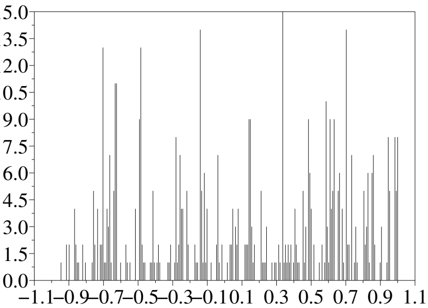




Automaton number 864

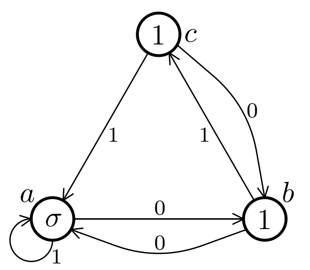
$a = \sigma(c, c)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcbabcbacbabab$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,190,382,766,1525,
3025,5998,11890

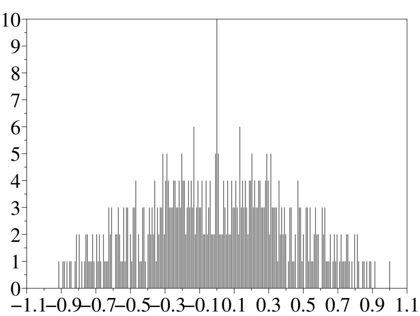




Automaton number 866

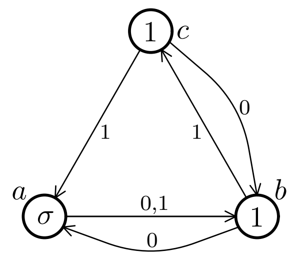
$a = \sigma(b, a)$ Group:
$b = (a, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(ca^{-1})^2$, $ba^{-2}cab^{-1}ab^{-1}c^{-1}aba^{-1}$,
$cab^{-1}cb^{-1}a^{-1}cbc^{-1}ba^{-2}$
SF: 2^0,2^1,2^3,2^5,2^9,2^{15},2^{26},2^{48},2^{92}
Gr: 1,7,35,165,769,3575

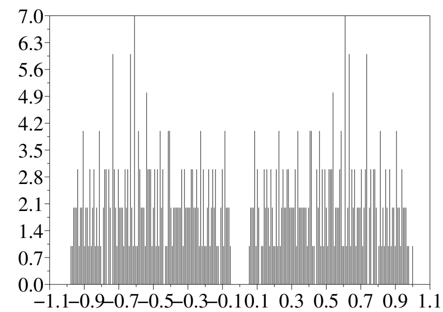




Automaton number 869

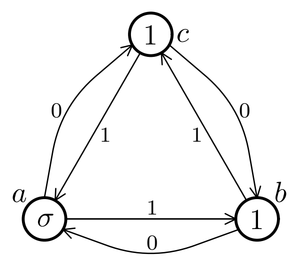
$a = \sigma(b, b)$ Group:
$b = (a, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acbcacbcabcacacacb$
SF: 2^0,2^1,2^3,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,10,22,46,94,190,382,766,1525

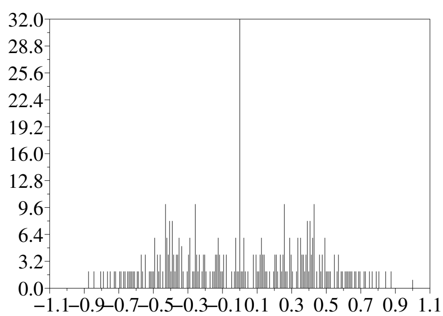




Automaton number 870

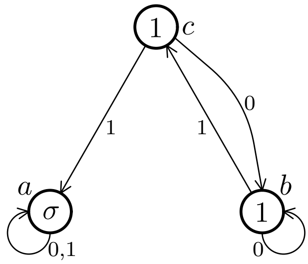
$a = \sigma(c, b)$ Group: $BS(1, 3)$
$b = (a, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $a^{-1}ca^{-1}b$, $(b^{-1}a)^{b^{-1}}(b^{-1}a)^{-3}$
SF: 2^0,2^1,2^3,2^4,2^6,2^8,2^{10},2^{12},2^{14}
Gr: 1,7,33,127,433,1415

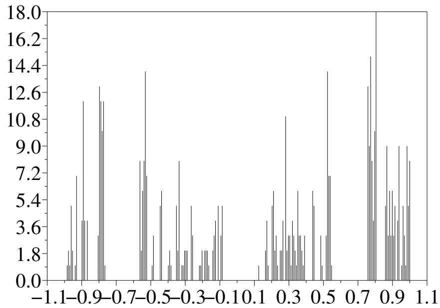




Automaton number 874

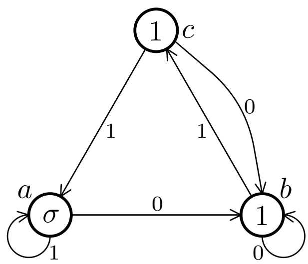
$a = \sigma(a, a)$ Group: $C_2 \ltimes G_{2852}$
$b = (b, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcacbacb$,
$abcbcabcacbcbacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,184,352,664,1244,2320,4288

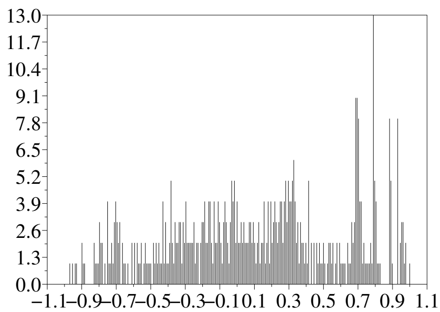




Automaton number 875

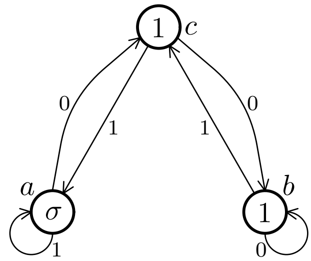
$a = \sigma(b, a)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}c)^2$, $(b^{-1}c)^2$, $(a^{-1}b)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,607,2563

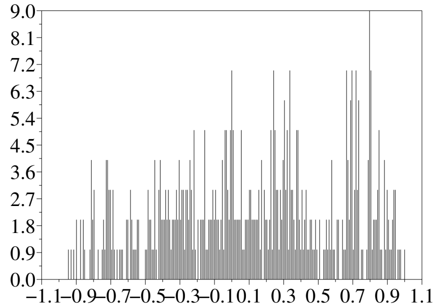




Automaton number 876

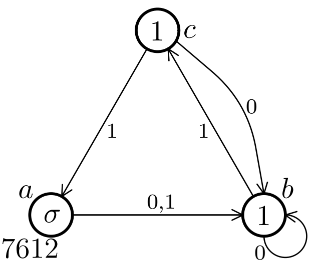
$a = \sigma(c, a)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $a^{-2}bcb^{-2}a^2c^{-1}b$, $a^{-2}cb^{-1}a^2c^{-2}bc$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,937,4667

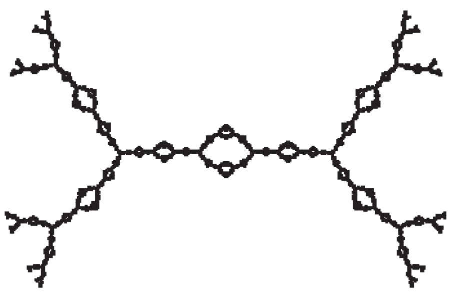


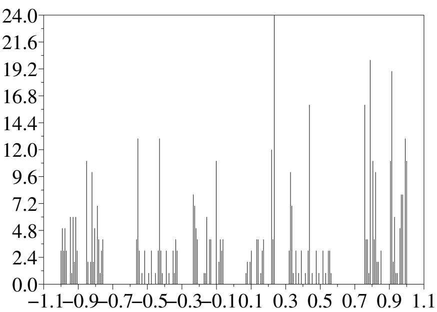


Automaton number 878

$a = \sigma(b, b)$ Group: $C_2 \ltimes IMG(1 - \frac{1}{z^2})$
$b = (b, c)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcacbacb$,
$abcbcabcacbcbacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,184,352,664,1244,2296,4198,7612
Limit space:

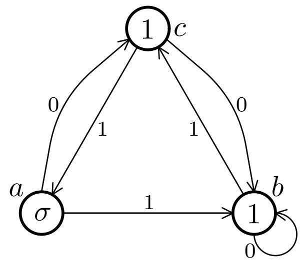


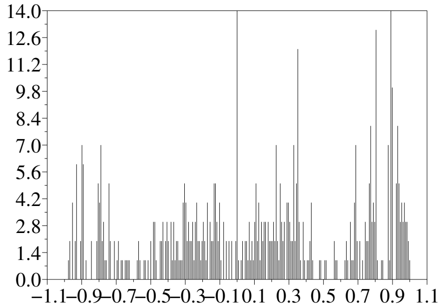




Automaton number 879

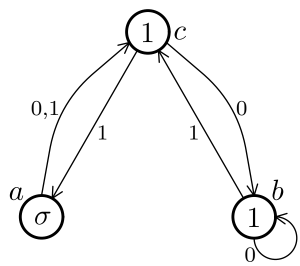
$a = \sigma(c, b)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $a^{-1}ca^{-1}cb^{-1}ac^{-1}ac^{-1}b$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,769,3567

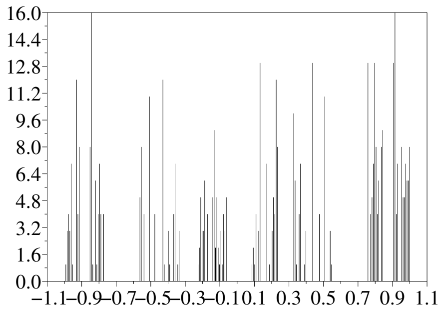




Automaton number 882

$a = \sigma(c, c)$ Group:
$b = (b, c)$ Contracting: *n/a*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcacbacb$,
$abcbcabcacbcbacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,184,352,664,1244





Automaton number 883

$a = \sigma(a,a)$ Group: $C_2 \ltimes G_{2841}$

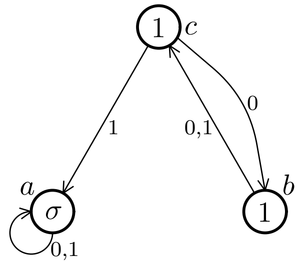
$b = (c,c)$ Contracting: *no*

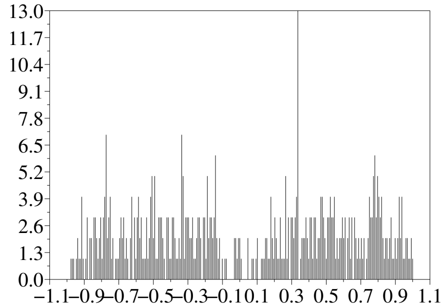
$c = (b,a)$ Self-replicating: *yes*

Rels: a^2, b^2, c^2, $acbcbacbcacbcabcbcabab$,
$acbacbcacabacbacbcacab$

SF: 2^0,2^1,2^3,2^6,2^9,2^{14},2^{24},2^{43},2^{80}

Gr: 1,4,10,22,46,94,190,382,766,1534,3070,6120





Automaton number 884

$a = \sigma(b,a)$ Group:

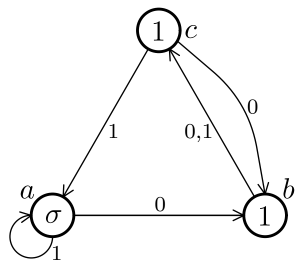
$b = (c,c)$ Contracting: *no*

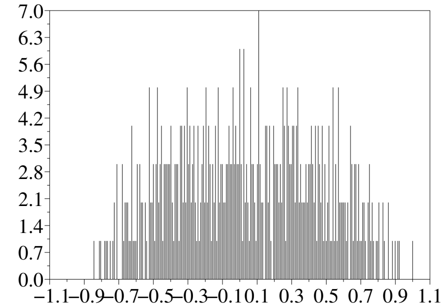
$c = (b,a)$ Self-replicating: *yes*

Rels: $(a^{-1}c)^2$, $(b^{-1}c)^2$, $[b, ac^{-1}]$

SF: 2^0,2^1,2^3,2^6,2^9,2^{15},2^{27},2^{49},2^{93}

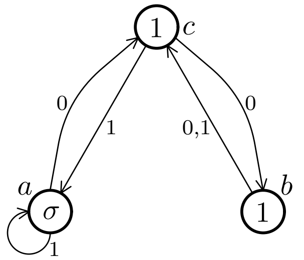
Gr: 1,7,33,135,529,2051



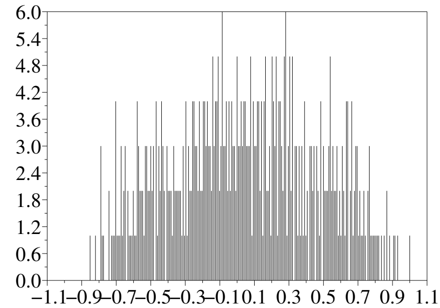


Automaton number 885

$a = \sigma(c, a)$ Group:
$b = (c, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

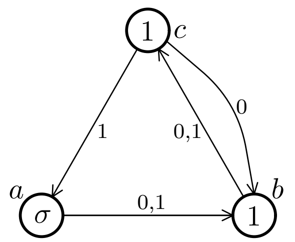


Rels: $acba^{-1}b^{-1}ac^{-1}a^{-1}cba^{-1}b^{-1}ac^{-1}aca^{-1}\cdot$
$bab^{-1}c^{-1}a^{-1}ca^{-1}bab^{-1}c^{-1}$,
$acba^{-1}b^{-1}ac^{-1}a^{-1}ca^{-1}c^{-1}b^{-1}a^{3}c^{-1}aca^{-1}b\cdot$
$ab^{-1}c^{-1}a^{-1}ca^{-3}bcac^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4687

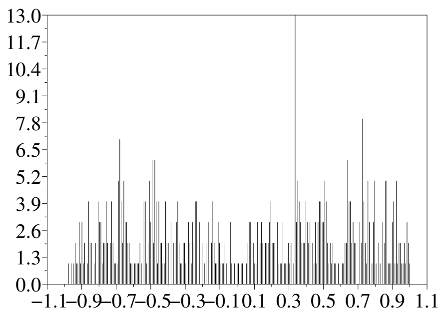


Automaton number 887

$a = \sigma(b, b)$ Group:
$b = (c, c)$ Contracting: *n/a*
$c = (b, a)$ Self-replicating: *yes*

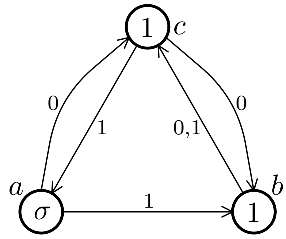


Rels: a^2, b^2, c^2, *babacbcbacbcacbcabcbca*,
bacacbcabcabacacbcabca
SF: 2^0,2^1,2^3,2^6,2^9,2^{14},2^{24},2^{43},2^{80}
Gr: 1,4,10,22,46,94,190,382,766,1534,3070,6120

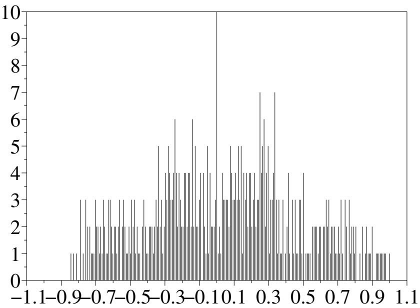


Automaton number 888

$a = \sigma(c, b)$ Group:
$b = (c, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

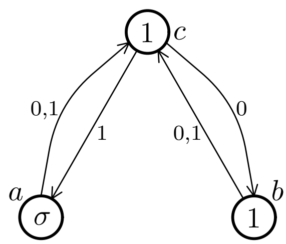


Rels: $aca^{-1}ba^{-2}ca^{-1}bab^{-1}ac^{-1}b^{-1}ac^{-1}$,
$aca^{-1}ba^{-3}bab^{-1}a^2b^{-1}ac^{-1}a^{-1}ba^{-1}b^{-1}a$,
$bab^{-1}a^{-1}ca^{-1}b^2a^{-1}b^{-1}ab^{-1}ac^{-1}$,
$bab^{-1}a^{-2}bab^{-1}aba^{-2}b^{-1}a$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4687

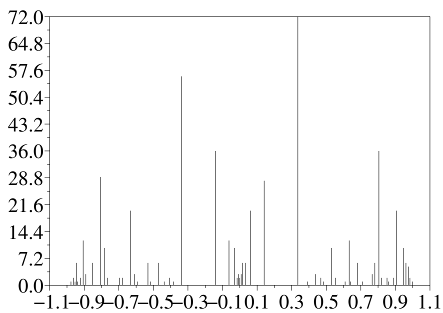


Automaton number 891

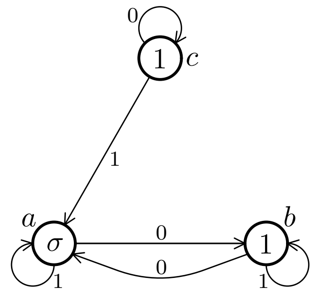
$a = \sigma(c, c)$ Group: $C_2 \ltimes$ *Lampighter*
$b = (c, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*



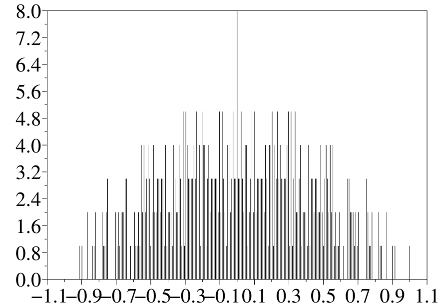
Rels: a^2, b^2, c^2, $abab$, $(acb)^4$,
$[acaca, bcacb]$, $[acaca, bcbcb]$
SF: 2^0,2^1,2^3,2^6,2^7,2^9,2^{10},2^{11},2^{12}
Gr: 1,4,9,17,30,51,82,128,198,304



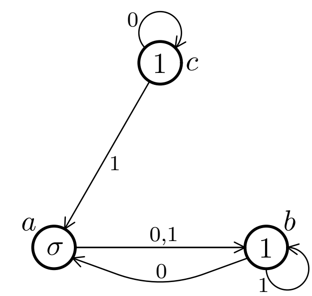
Automaton number 920



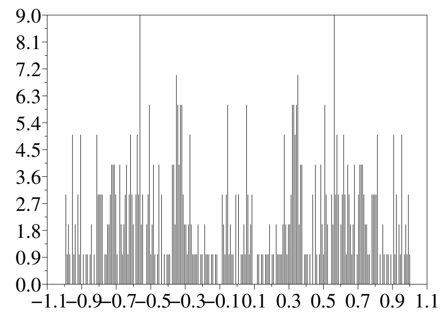
$a = \sigma(b, a)$ Group:
$b = (a, b)$ Contracting: *n/a*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $[a, b]^2$, $(a^{-1}c^{-1}ab)^2$
SF: 2^0,2^1,2^3,2^5,2^9,2^{15},2^{26},2^{48},2^{92}
Gr: 1,7,35,165,757,3447



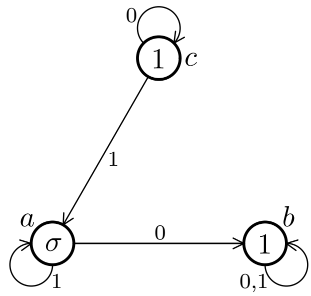
Automaton number 923



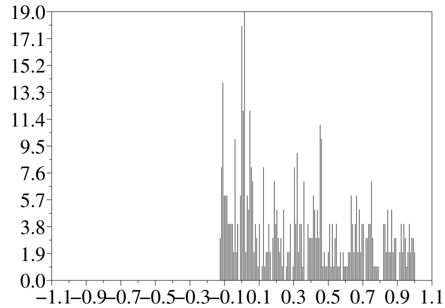
$a = \sigma(b, b)$ Group:
$b = (a, b)$ Contracting: *yes*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcbabcbacbabab$
SF: 2^0,2^1,2^3,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,10,22,46,94,190,382,766,
1525,3025,5998,11890



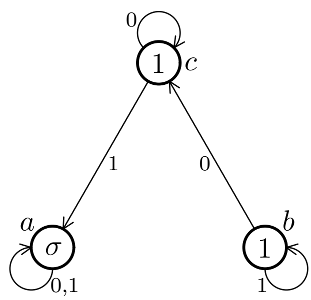
Automaton number 929



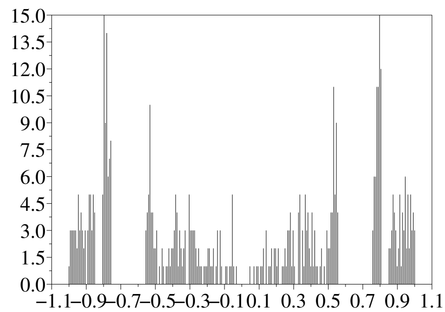
$a = \sigma(b, a)$ Group:
$b = (b, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: b, $a^{-3}cac^{-1}ac^{-1}ac$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,5,17,53,161,475,1387



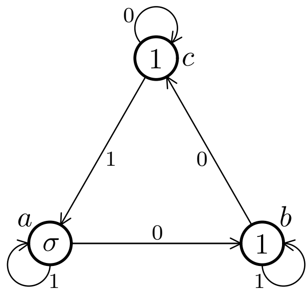
Automaton number 937



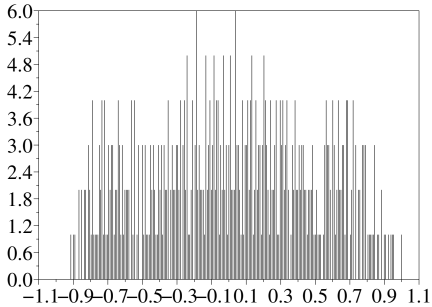
$a = \sigma(a, a)$ Group: $C_2 \ltimes G_{929}$
$b = (c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcacbacb$,
$abcbcabcacbcbacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,184,352,664,1244



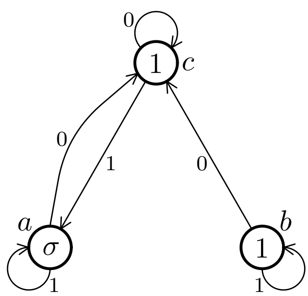
Automaton number 938



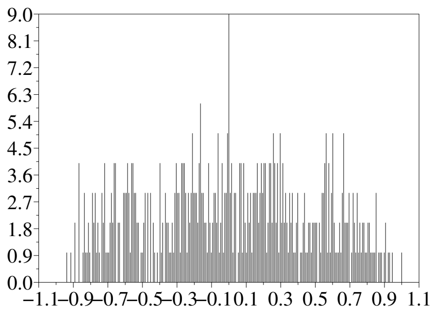
$a = \sigma(b, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $a^{-2}bcb^{-2}a^2c^{-1}b$, $a^{-2}cb^{-1}a^2c^{-2}bc$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,937,4667



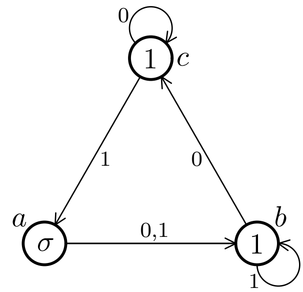
Automaton number 939



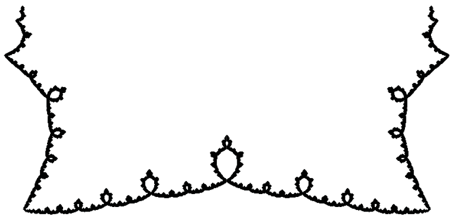
$a = \sigma(c, a)$ Group:
$b = (c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}c)^2$, $(a^{-2}cb)^2$, $[a, c]^2$,
$[ca^{-1}, ba^{-1}b]$, $a^{-1}b^{-1}abc^{-1}a^{-1}bca^{-1}b$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,757,3427

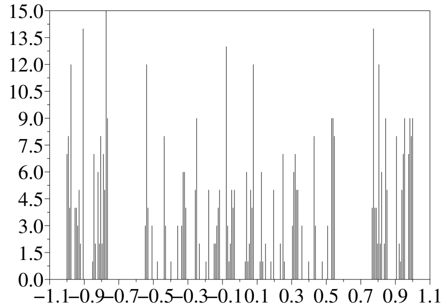


Automaton number 941

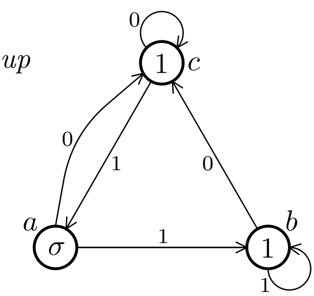


$a = \sigma(b, b)$ Group: $C_2 \ltimes IMG(z^2 - 1)$
$b = (c, b)$ Contracting: *yes*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcacbacb$,
$abcbcabcacbcbacb$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,184,352,664,1244
Limit space:

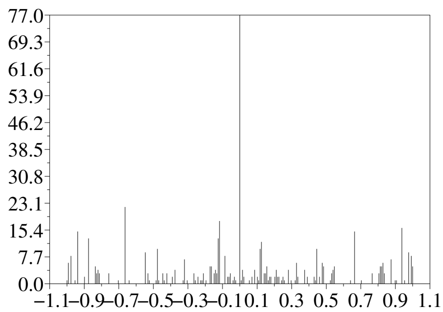




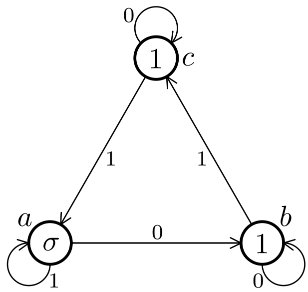
Automaton number 942



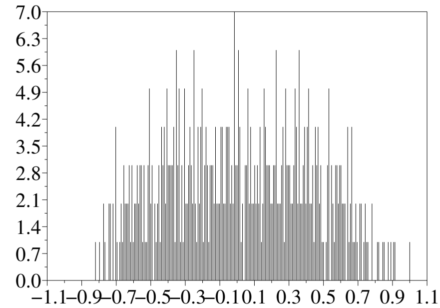
$a = \sigma(c, b)$ Group: *Contains Lamplighter group*
$b = (c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[a, b]^2$, $[b, c]^2$,
$(a^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,597,2465



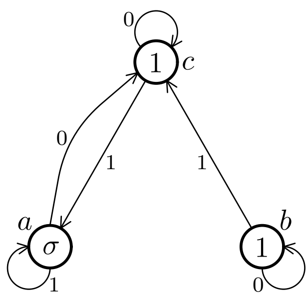
Automaton number 956



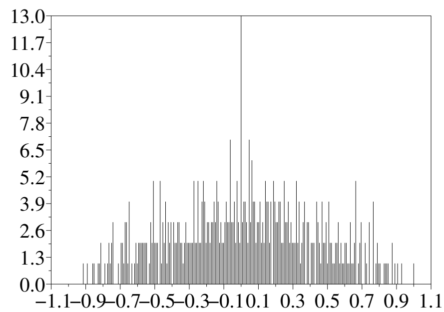
$a = \sigma(b, a)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $acba^{-1}b^{-1}ab^{-1}a^{-1}cba^{-1}b^{-1}ab^{-1}aba^{-1}\cdot bab^{-1}c^{-1}a^{-1}ba^{-1}bab^{-1}c^{-1}$, $acba^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}ca^{-1}caba^{-1}bab^{-1}c^{-1}\cdot a^{-2}bc^{-1}baba^{-1}bab^{-1}c^{-1}a^{-1}b^{-1}cb^{-1}a^{2}cb\cdot a^{-1}b^{-1}ab^{-1}a^{-1}c^{-1}ac^{-1}b$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{90},2^{176}
Gr: 1,7,37,187,937,4687



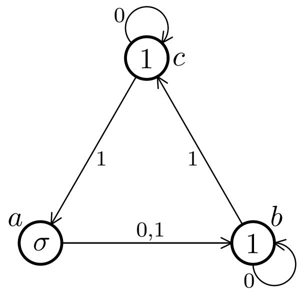
Automaton number 957



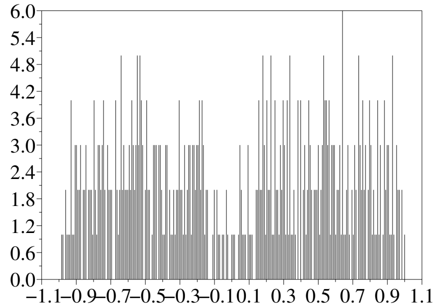
$a = \sigma(c, a)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}c)^2$, $(b^{-1}c)^2$, $[a, c]^2$, $[b, c]^2$, $(a^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,599,2485



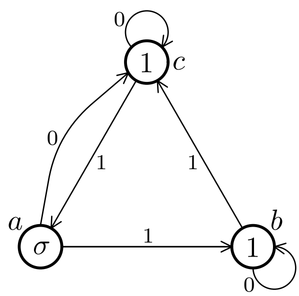
Automaton number 959



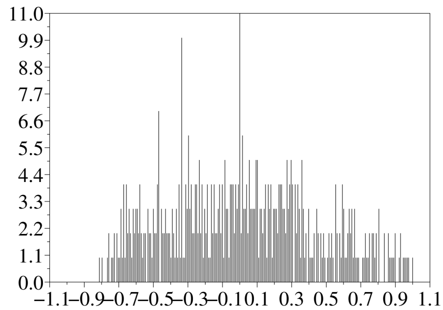
$a = \sigma(b, b)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $abcabcbabcbacbabab$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,190,382,766,1525



Automaton number 960

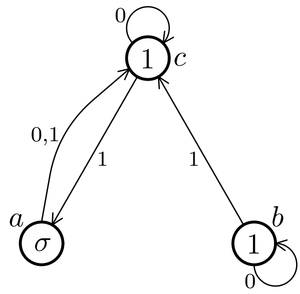


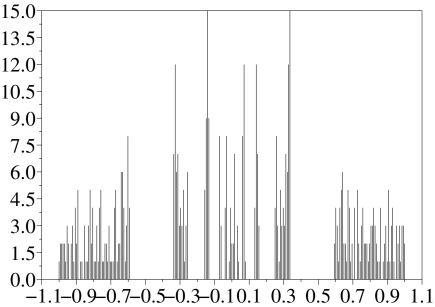
$a = \sigma(c, b)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(a^{-2}bc)^2$, $(a^{-1}c)^4$, $(b^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,758,3460



Automaton number 963

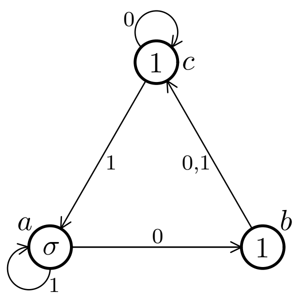
$a = \sigma(c, c)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $acbacacabcabab$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,4,10,22,46,94,190,375,731,
1422,2762,5350,10322

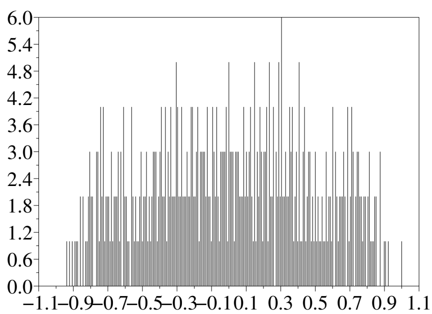




Automaton number 965

$a = \sigma(b, a)$ Group:
$b = (c, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $acb^{-1}a^{-1}cb^{-1}abc^{-1}a^{-1}bc^{-1}$,
$acb^{-1}a^{-1}cac^{-1}b^{-1}cbc^{-2}bca^{-1}c^{-1}$,
$acac^{-1}b^{-1}ca^{-2}cb^{-1}a^2c^{-1}bca^{-1}c^{-1}a^{-1}bc^{-1}$,
$acac^{-1}b^{-1}ca^{-2}cac^{-1}b^{-1}cac^{-1}bca^{-1}c^{-2}bca^{-1}c^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4687

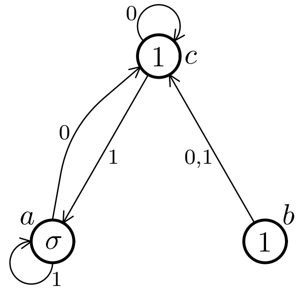


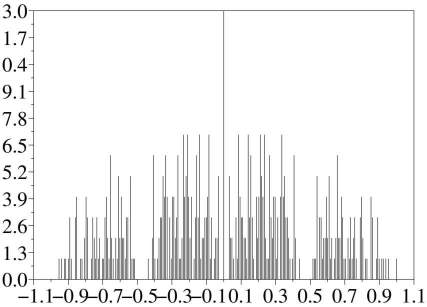


Automaton number 966

$a = \sigma(c, a)$ Group:
$b = (c, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *no*

Rels: $(a^{-1}c)^2$, $(b^{-1}c)^2$, $[ca^{-1}, b]$,
$[a, b]^2$, $(a^{-2}b^2)^2$, $(a^{-1}b)^4$, $[[c^{-1}, a^{-1}], cb^{-1}]$
SF: 2^0,2^1,2^3,2^6,2^9,2^{11},2^{14},2^{16},2^{18}
Gr: 1,7,33,135,495,1725

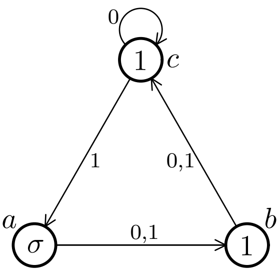


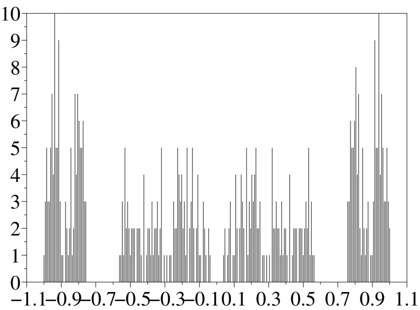


Automaton number 968

$a = \sigma(b, b)$ Group: Virtually $\mathbb{Z}^5$
$b = (c, c)$ Contracting: *yes*
$c = (c, a)$ Self-replicating: *no*

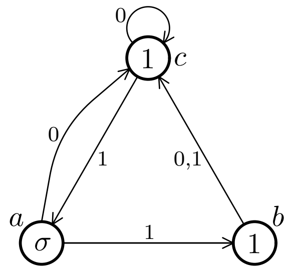
Rels: a^2, b^2, c^2, $(abc)^2(acb)^2$,
$(cbcbaba)^2$, $(cacbcba)^2$,
$(cabacbaba)^2$, $((ac)^4b)^2$
SF: 2^0,2^1,2^3,2^6,2^9,2^{13},2^{17},2^{21},2^{25}
Gr: 1,4,10,22,46,94,184,338,600,1022

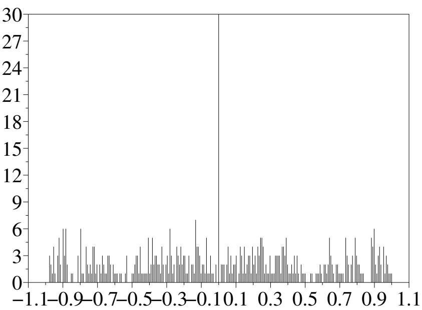




Automaton number 969

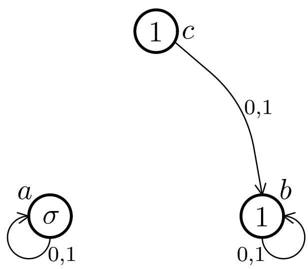
$a = \sigma(c, b)$ Group:
$b = (c, c)$ Contracting: *n/a*
$c = (c, a)$ Self-replicating: *yes*
Rels: $a^{-1}c^{-1}bab^{-1}a^{-1}cb^{-1}ab$,
$a^{-1}c^{-1}bac^{-1}a^{-1}cb^{-1}ac$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,937,4667

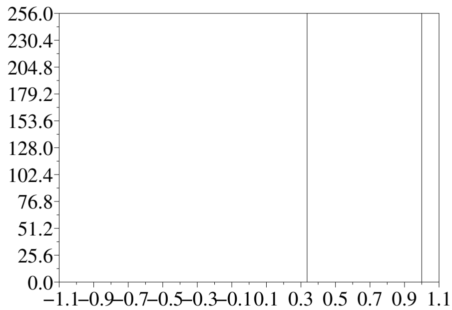




Automaton number 1090

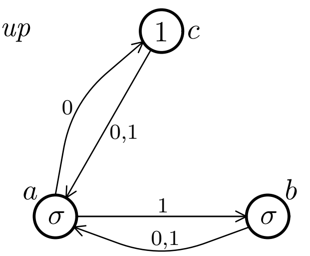
$a = \sigma(a, a)$ Group: C_2
$b = (b, b)$ Contracting: *yes*
$c = (b, b)$ Self-replicating: *no*
Rels: b, c, a^2
SF: 2^0,2^1,2^1,2^1,2^1,2^1,2^1,2^1,2^1
Gr: 1,2,2,2,2,2,2,2,2,2,2

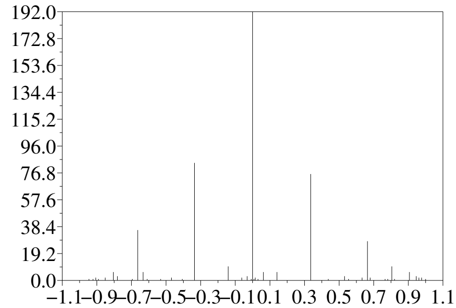




Automaton number 2193

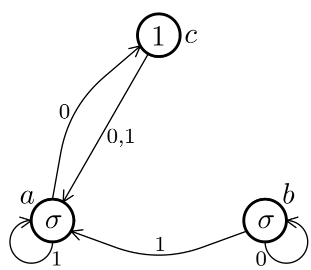
$a = \sigma(c, b)$ Group: *Contains Lamplighter group*
$b = \sigma(a, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $[b, c]$, b^2c^2, a^4, b^4,
$(a^2b)^2$, $(abc)^2$, $(a^2c)^2$
SF: 2^0,2^1,2^3,2^6,2^7,2^9,2^{10},2^{11},2^{12}
Gr: 1,7,27,65,120,204,328,
512,792,1216

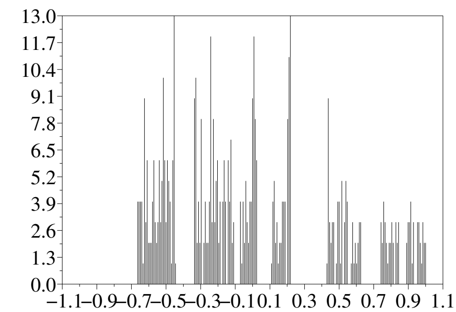




Automaton number 2199

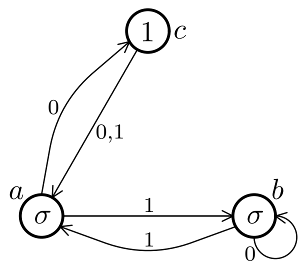
$a = \sigma(c, a)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: ca^2, $[a^{-1}b, ab^{-1}]$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,29,115,417,1505

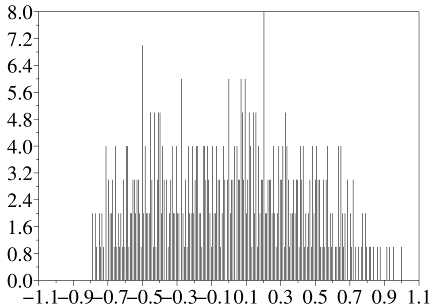




Automaton number 2202

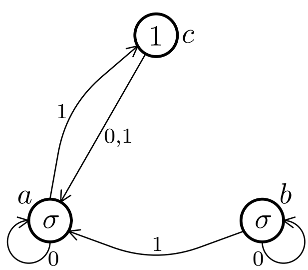
$a = \sigma(c, b)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: cab^2a
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,177,833,3909

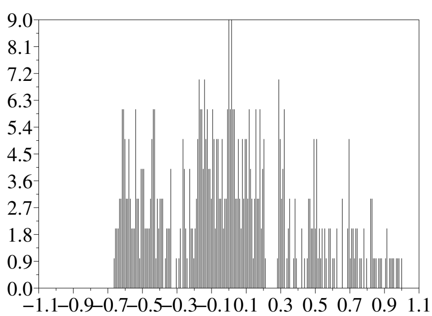




Automaton number 2203

$a = \sigma(a, c)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: ca^2, $[c^{-2}ab, bc^{-2}a]$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,29,115,441,1695





Automaton number 2204

$a = \sigma(b, c)$ Group:

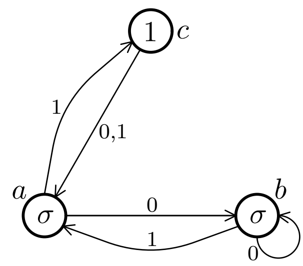
$b = \sigma(b, a)$ Contracting: *no*

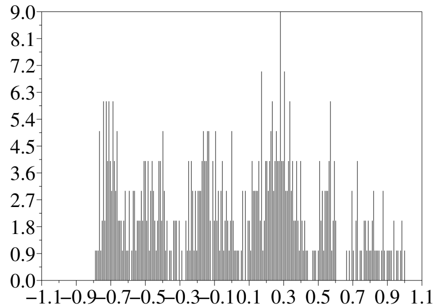
$c = (a, a)$ Self-replicating: *yes*

Rels: $bcba^2$, $[b^{-1}a, ba^{-1}]$,
$a^{-1}ba^2ba^{-2}b^{-2}aba^2b^{-1}a^{-2}$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,37,177,825,3781





Automaton number 2207

$a = \sigma(b, a)$ Group:

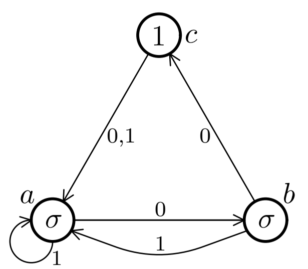
$b = \sigma(c, a)$ Contracting: *no*

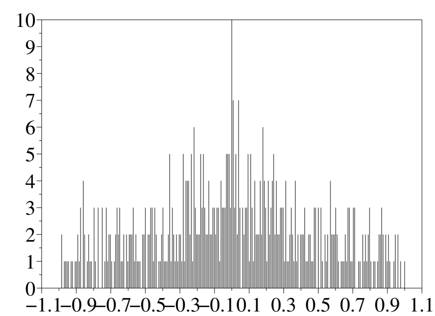
$c = (a, a)$ Self-replicating: *yes*

Rels: $[b^{-1}a, ba^{-1}]$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,37,187,929,4599





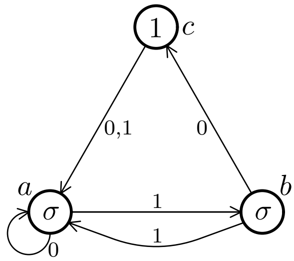
Automaton number 2209

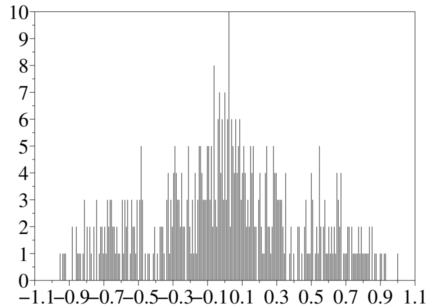
$a = \sigma(a, b)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*

Rels: $aca^{-2}c^{-1}acac^{-1}a^{-2}cac^{-1}$,
$aca^{-2}b^{-1}a^{-1}cacac^{-1}a^{-2}c^{-1}abac^{-1}$,
$aca^{-1}b^{-1}a^{-1}c^2a^{-1}c^{-1}ac^{-1}abac^{-1}a^{-2}cac^{-1}$,
$aca^{-1}b^{-1}a^{-1}c^2a^{-1}b^{-1}a^{-1}cac^{-1}\cdot$
$abac^{-1}a^{-2}c^{-1}abac^{-1}$,
$bca^{-1}c^{-1}ab^{-1}ca^{-1}c^{-1}aba^{-1}ca\cdot$
$c^{-1}b^{-1}a^{-1}cac^{-1}$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,37,187,937,4687





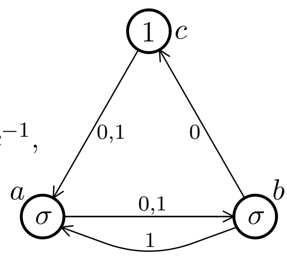
Automaton number 2210

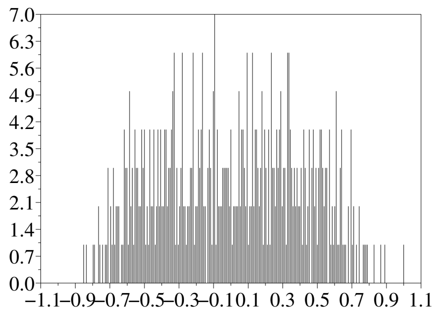
$a = \sigma(b, b)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*

Rels: $acbc^{-1}b^{-1}a^{-1}cbc^{-1}b^{-1}abcb^{-1}c^{-1}a^{-1}bcb^{-1}c^{-1}$,
$bcbc^{-1}b^{-2}cbc^{-1}bcb^{-2}c^{-1}$,
$bcbc^{-1}b^{-2}ca^{-1}b^{-1}cabcb^{-1}c^{-1}a^{-1}c^{-1}bac^{-1}$,
$bca^{-1}b^{-1}cab^{-2}cbc^{-1}ba^{-1}c^{-1}bab^{-1}c^{-1}$,
$bca^{-1}b^{-1}cab^{-2}ca^{-1}b^{-1}caba^{-1}c^{-1}\cdot$
$bac^{-1}a^{-1}c^{-1}bac^{-1}$

SF: 2^0,2^1,2^3,2^5,2^8,2^{13},2^{23},2^{42},2^{79}

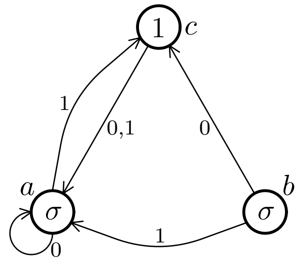
Gr: 1,7,37,187,937,4687

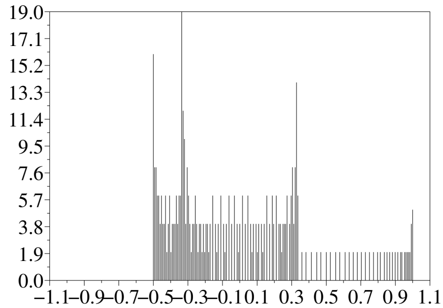




Automaton number 2212

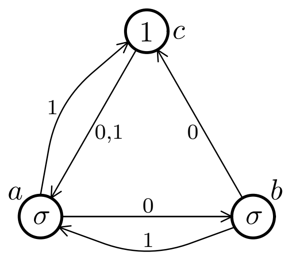
$a = \sigma(a, c)$ Group: *Klein bottle group*
$b = \sigma(c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: ca^2, cb^2
SF: 2^0,2^1,2^2,2^4,2^6,2^8,2^{10},2^{12},2^{14}
Gr: 1,7,19,37,61,91,127,169,217,271,331

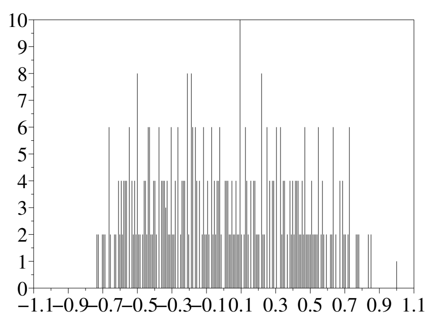




Automaton number 2213

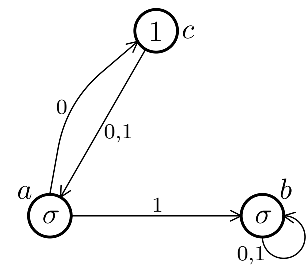
$a = \sigma(b, c)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $bcbc^{-1}b^{-2}cbc^{-1}bcb^{-2}c^{-1}$,
$acbc^{-1}b^{-1}a^{-1}cbc^{-1}b^{-1}abcb^{-1}c^{-1}\cdot$
$a^{-1}bcb^{-1}c^{-1}$,
$acbc^{-1}b^{-1}a^{-1}ba^{-1}c^{-1}b^2c^{-1}abcb^{-1}c^{-1}a^{-1}\cdot$
$cb^{-2}cab^{-1}$,
$aba^{-1}c^{-1}b^2c^{-1}a^{-1}cbc^{-1}b^{-1}\cdot$
$acb^{-2}cab^{-1}a^{-1}bcb^{-1}c^{-1}$,
SF: 2^0,2^1,2^2,2^3,2^5,2^8,2^{14},2^{25},2^{47}
Gr: 1,7,37,187,937,4687

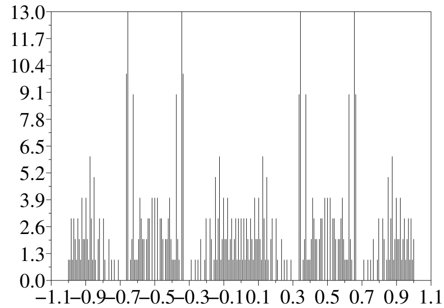




Automaton number 2229

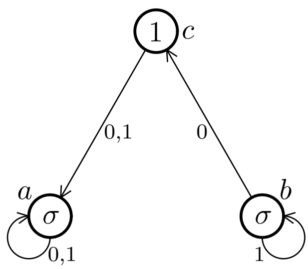
$a = \sigma(c, b)$ Group: $C_4 \ltimes \mathbb{Z}^2$
$b = \sigma(b, b)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: b^2, $(ab)^2$, $(bc)^2$, a^4, c^4, $[a, c]^2$, $(a^{-1}c)^4$, $(ac)^4$
SF: 2^0,2^1,2^3,2^6,2^9,2^{11},2^{13},2^{15},2^{17}
Gr: 1,6,20,54,128,270,510,886,1452

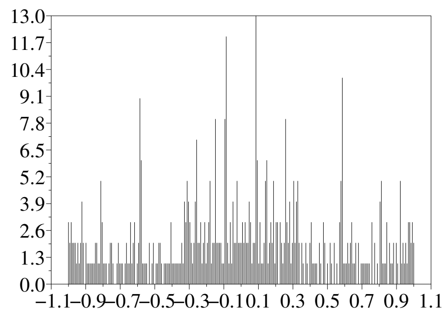




Automaton number 2233

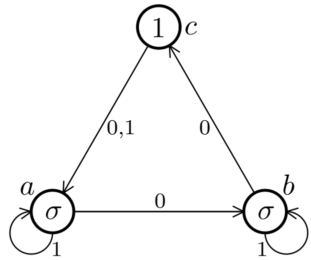
$a = \sigma(a, a)$ Group:
$b = \sigma(c, b)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *yes*
Rels: a^2, c^2, $abab$, $acac$, $cb^2acbcbcab^2cabcba$
SF: 2^0,2^1,2^3,2^6,2^9,2^{15},2^{26},2^{48},2^{91}
Gr: 1,5,14,32,68,140,284,565,1106





Automaton number 2234

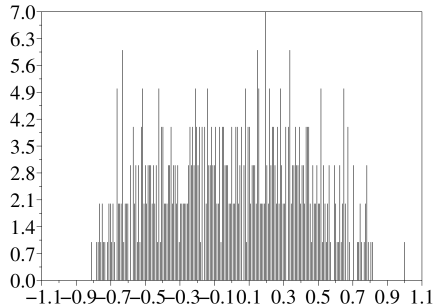
$a = \sigma(b, a)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*



Rels: $ac^{-1}a^2c^{-1}ab^{-1}a^{-1}c^{-1}a^2c^{-1}ab^{-1}ab\cdot$
$a^{-1}ca^{-2}ca^{-1}ba^{-1}ca^{-2}c$,
$ac^{-1}a^2c^{-1}ab^{-1}a^{-1}cbac^{-1}ab^{-1}a^{-1}c^{-1}aba^{-1}\cdot$
$ca^{-1}b^{-1}aba^{-1}ca^{-2}ca^{-1}bac^{-1}ab^{-1}a^{-1}ca\cdot$
$ba^{-1}ca^{-1}b^{-1}c^{-1}$

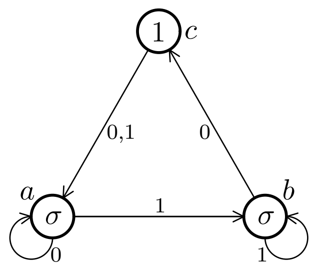
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,37,187,937,4687



Automaton number 2236

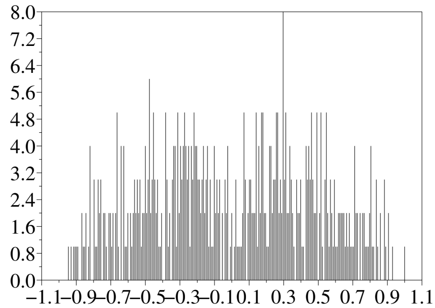
$a = \sigma(a, b)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*



Rels: $[b^{-1}a, ba^{-1}]$, $a^{-1}c^{-1}acb^{-1}ac^{-1}a^{-1}cb$,
$a^{-1}cac^{-1}b^{-1}aca^{-1}c^{-1}b$

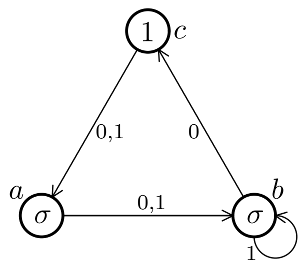
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

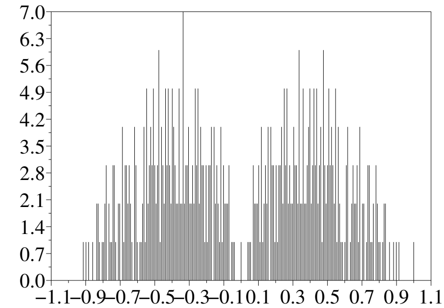
Gr: 1,7,37,187,929,4579



Automaton number 2237

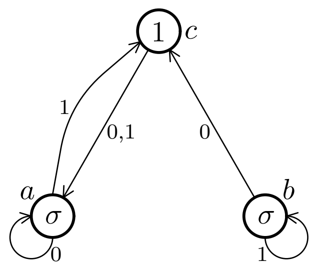
$a = \sigma(b, b)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *no*
Rels: $[b^{-1}a, ba^{-1}]$, $[c^{-1}a, ca^{-1}]$
SF: 2^0,2^1,2^3,2^6,2^9,2^{15},2^{26},2^{45},2^{81}
Gr: 1,7,37,187,921,4511

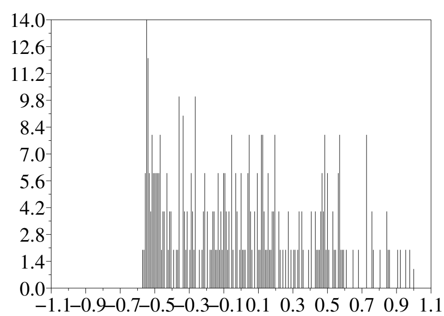




Automaton number 2239

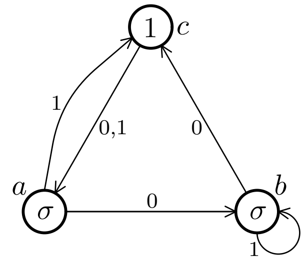
$a = \sigma(a, c)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: ca^2, $[ca^{-2}cba^{-1}, a^{-1}ca^{-2}cb]$
SF: 2^0,2^1,2^2,2^3,2^5,2^8,2^{14},2^{25},2^{47}
Gr: 1,7,29,115,441,1695

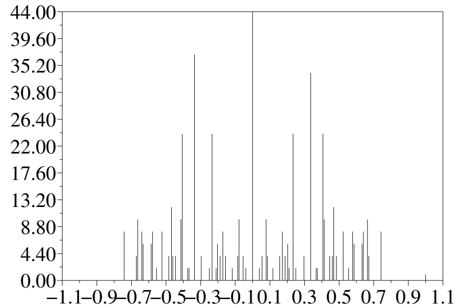




Automaton number 2240

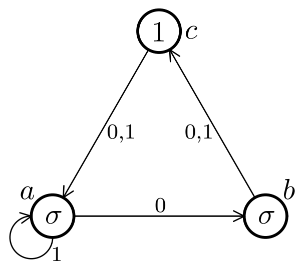
$a = \sigma(b, c)$ Group: F_3
$b = \sigma(c, b)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *no*
Rels:
SF: 2^0,2^1,2^2,2^4,2^7,2^{10},2^{14},2^{21},2^{34}
Gr: 1,7,37,187,937,4687

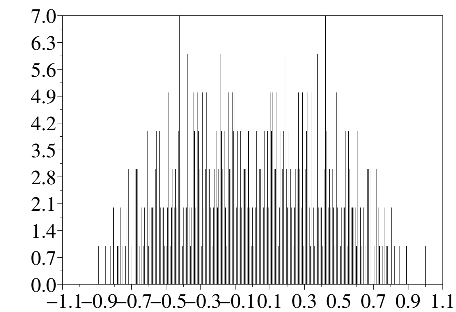




Automaton number 2261

$a = \sigma(b, a)$ Group:
$b = \sigma(c, c)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*
Rels: $acac^{-1}a^{-2}cac^{-1}aca^{-2}c^{-1}$,
$acac^{-1}a^{-2}cba^{-1}c^{-1}aca^{-1}cb^{-1}aca^{-1}c^{-1}$,
$bc^{-1}ac^{-1}a^{-1}cab^{-1}c^{-1}$,
$bcac^{-1}a^{-1}b^{-1}cac^{-1}a^{-1}baca^{-1}c^{-1}b^{-1}aca^{-1}c^{-1}$
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,7,37,187,937,4687





Automaton number 2265

$a = \sigma(c, b)$ Group:

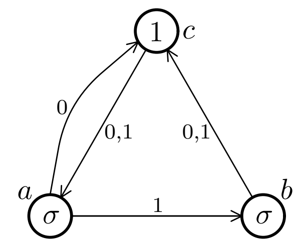
$b = \sigma(c, c)$ Contracting: *no*

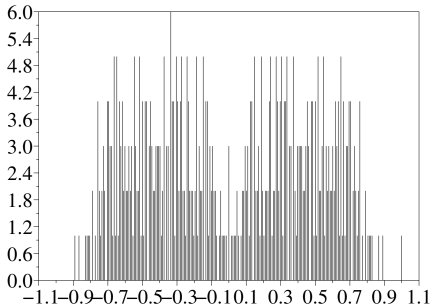
$c = (a, a)$ Self-replicating: *no*

Rels: $[b^{-1}a, ba^{-1}]$, $a^{-1}ca^{-1}cb^{-1}ac^{-1}ac^{-1}b$, $a^{-1}cb^{-1}cb^{-1}ac^{-1}bc^{-1}b$

SF: 2^0,2^1,2^3,2^6,2^9,2^{14},2^{22},2^{36},2^{63}

Gr: 1,7,37,187,929,4579,22521





Automaton number 2271

$a = \sigma(c, a)$ Group:

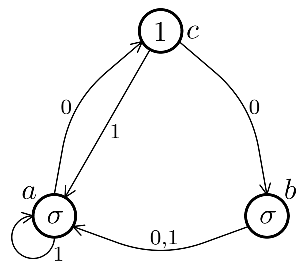
$b = \sigma(a, a)$ Contracting: *no*

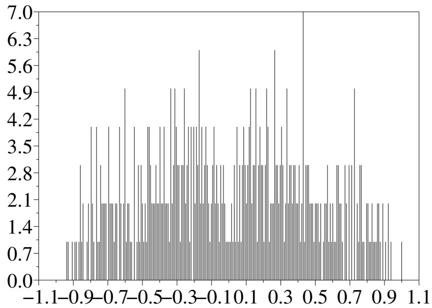
$c = (b, a)$ Self-replicating: *yes*

Rels: $[b^{-1}a, ba^{-1}]$, $a^{-1}c^2a^{-1}b^{-1}a^2c^{-2}b$, $a^{-1}c^2b^{-2}abc^{-2}b$

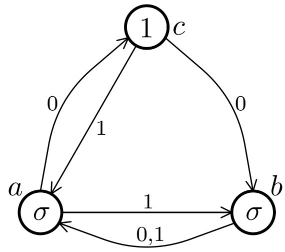
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}

Gr: 1,7,37,187,929,4583

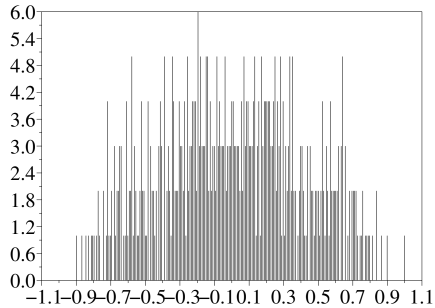




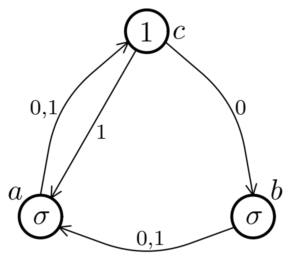
Automaton number 2274



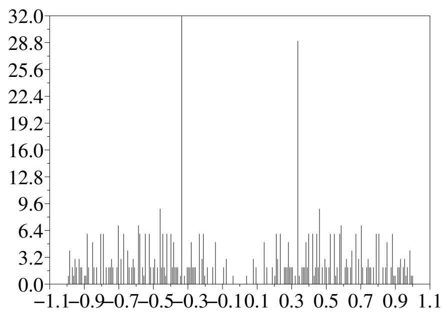
$a = \sigma(c, b)$ Group:
$b = \sigma(a, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $ac^3b^{-1}c^{-2}b^3c^{-3}a^{-1}c^3b^{-1}c^{-2}b^3c^{-3}ac^3b^{-3}\cdot$
$c^2bc^{-3}a^{-1}c^3b^{-3}c^2bc^{-3}$,
$ac^3b^{-1}c^{-2}b^3c^{-3}a^{-1}c^2ab^{-2}c^{-1}b^3c^{-3}ac^3b^{-3}\cdot$
$c^2bc^{-3}a^{-1}c^3b^{-3}cb^2a^{-1}c^{-2}$,
$bc^3b^{-1}c^{-2}b^3c^{-3}b^{-1}c^3b^{-1}c^{-2}b^3c^{-3}\cdot$
$bc^3b^{-3}c^2bc^{-3}b^{-1}c^3b^{-3}c^2bc^{-3}$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,937,4687



Automaton number 2277

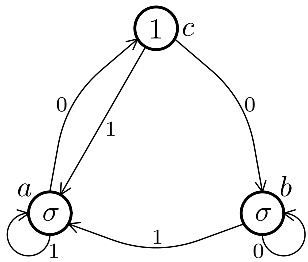


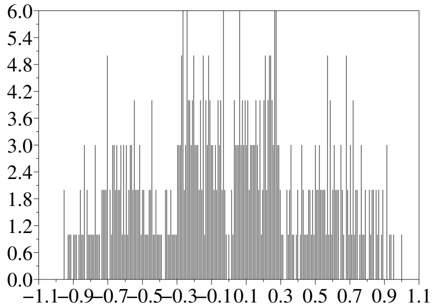
$a = \sigma(c, c)$ Group: $C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$
$b = \sigma(a, a)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, b^2, c^2, $(acb)^2$
SF: 2^0,2^1,2^2,2^4,2^5,2^6,2^7,2^8,2^9
Gr: 1,4,10,19,31,46,64,85,109,136,166
Limit space: 2-dimensional sphere S_2



Automaton number 2280

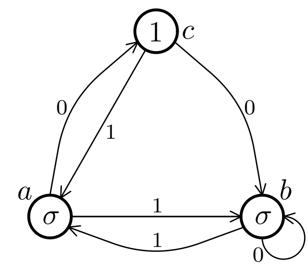
$a = \sigma(c, a)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[a, b]^2$, $[b, c]^2$, $(a^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,597,2465

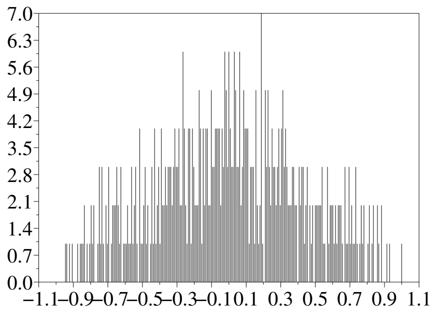




Automaton number 2283

$a = \sigma(c, b)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[b, c]^2$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,604,2534





Automaton number 2284

$a = \sigma(a, c)$ Group:

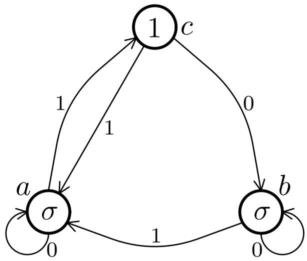
$b = \sigma(b, a)$ Contracting: *no*

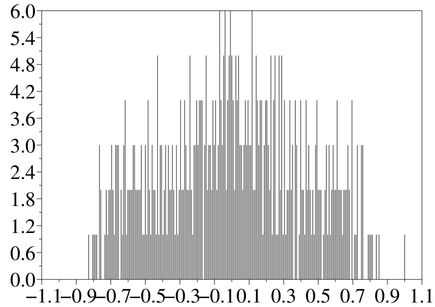
$c = (b, a)$ Self-replicating: *yes*

Rels: $(b^{-1}c)^2$, $(a^{-1}b)^4$, $(bc^{-2}a)^2$, $(a^{-1}c)^4$

SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}

Gr: 1,7,35,165,758,3460





Automaton number 2285

$a = \sigma(b, c)$ Group:

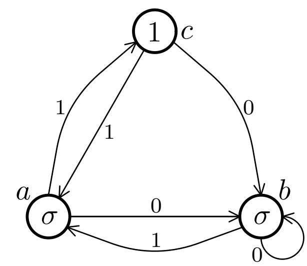
$b = \sigma(b, a)$ Contracting: *no*

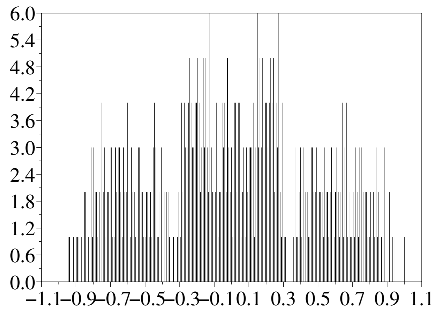
$c = (b, a)$ Self-replicating: *yes*

Rels: $(b^{-1}c)^2$, $[b^{-1}a, ba^{-1}]$, $[(c^{-1}a)^2, c^{-1}b]$, $[(ca^{-1})^2, cb^{-1}]$

SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}

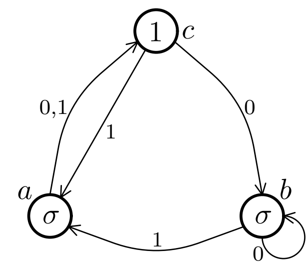
Gr: 1,7,35,165,761,3479

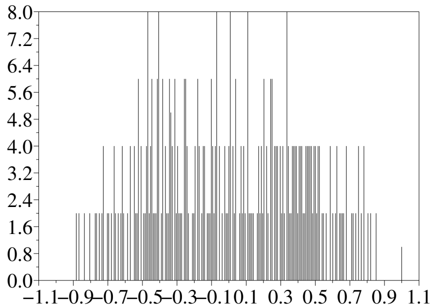




Automaton number 2286

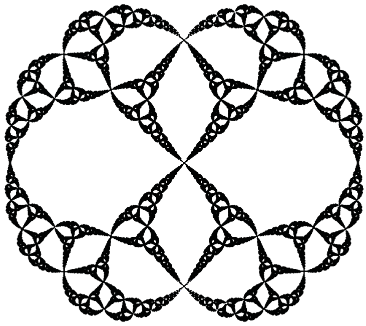
$a = \sigma(c, c)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $(b^{-1}c)^2$, $[a, bc^{-1}]$
SF: 2^0,2^1,2^2,2^3,2^5,2^9,2^{15},2^{27},2^{49}
Gr: 1,7,35,159,705,3107

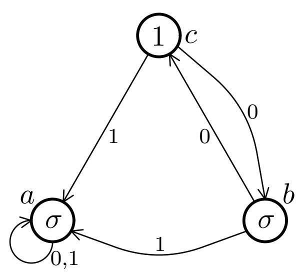


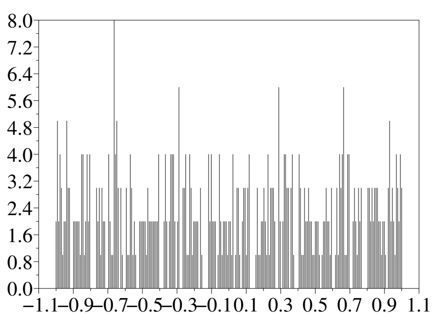


Automaton number 2287

$a = \sigma(a, a)$ Group: $IMG\left(\frac{z^2+2}{1-z^2}\right)$
$b = \sigma(c, a)$ Contracting: *yes*
$c = (b, a)$ Self-replicating: *yes*
Rels: a^2, $[a, b^2]$, $(b^{-1}ac)^2$, $[ba, c^2]$, $[c^2, aca]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,6,26,100,362,1246
Limit space:







Automaton number 2293

$a = \sigma(a, c)$ Group:

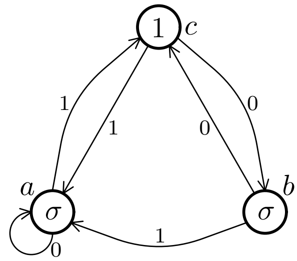
$b = \sigma(c, a)$ Contracting: *no*

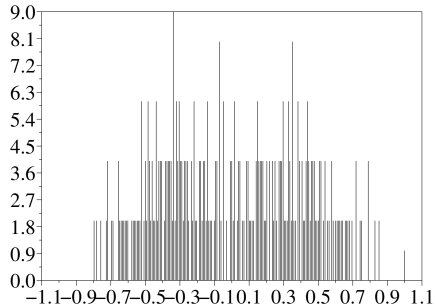
$c = (b, a)$ Self-replicating: *yes*

Rels:
$cb^{-1}a^{-1}ca^{-1}cb^{-1}a^{-1}cac^{-1}abc^{-1}a^{-1}c^{-1}abc^{-1}a$,
$cb^{-1}a^{-1}c^2a^{-1}c^2b^{-1}a^{-1}c^2b^{-1}a^{-1}ca^{-2}c^{-1}a\cdot$
$b^2c^{-2}ab^{-1}a^{-1}ca^2c^{-1}abc^{-2}abc^{-2}ac^{-1}$,
$ba^{-1}cb^{-1}a^{-1}cab^{-1}a^{-1}cb^{-1}a^{-1}c\cdot$
$aba^{-1}c^{-1}abc^{-1}ab^{-1}a^{-1}c^{-1}abc^{-1}a$

SF: 2^0,2^1,2^2,2^4,2^8,2^{13},2^{23},2^{41},2^{76}

Gr: 1,7,37,187,937,4687





Automaton number 2294

$a = \sigma(b, c)$ Group: $BS(1, -3)$

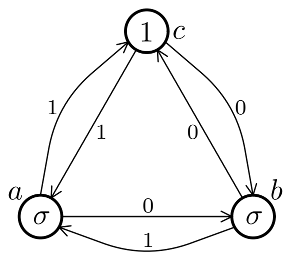
$b = \sigma(c, a)$ Contracting: *no*

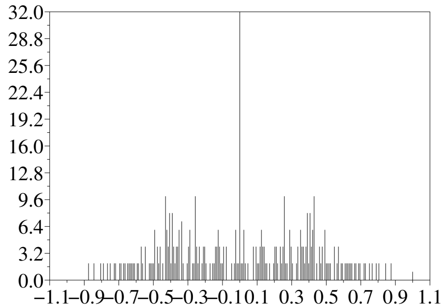
$c = (b, a)$ Self-replicating: *yes*

Rels: $b^{-1}ca^{-1}c$, $(ca^{-1})^a(ca^{-1})^3$

SF: 2^0,2^1,2^2,2^4,2^6,2^8,2^{10},2^{12},2^{14}

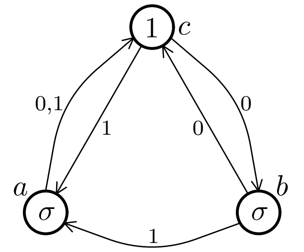
Gr: 1,7,33,127,433,1415

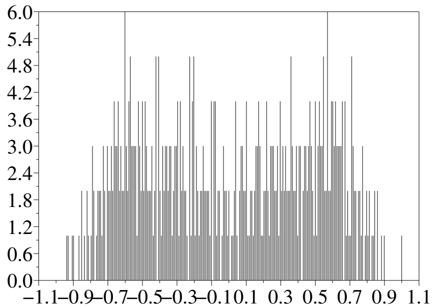




Automaton number 2295

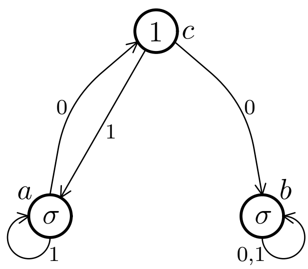
$a = \sigma(c, c)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $[b^{-1}a, ba^{-1}]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,929,4599

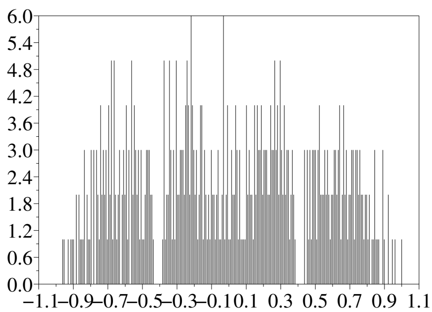




Automaton number 2307

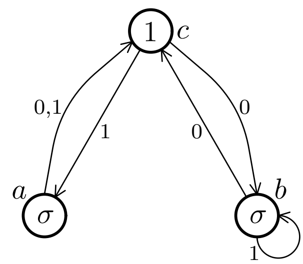
$a = \sigma(c, a)$ Group:
$b = \sigma(b, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: b^2, $a^{-2}c^{-1}bca^2c^{-1}bc$, $a^{-1}c^{-1}bc^{-2}bcac^2$,
$a^{-1}cbc^{-2}bc^{-1}ac^2$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,6,26,106,426,1681

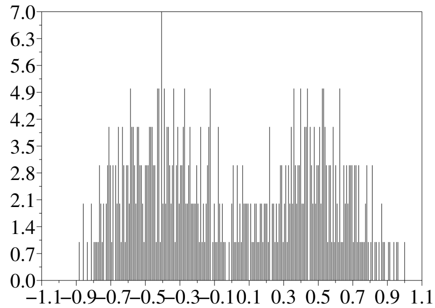




Automaton number 2322

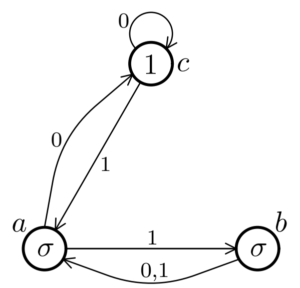
$a = \sigma(c, c)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $[b^{-1}a, ba^{-1}]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,929,4599

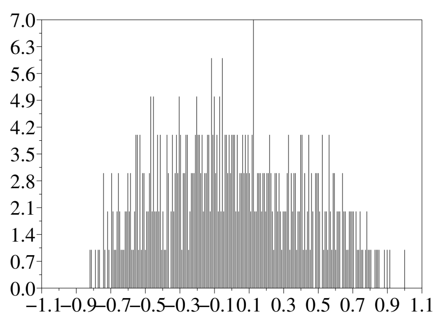




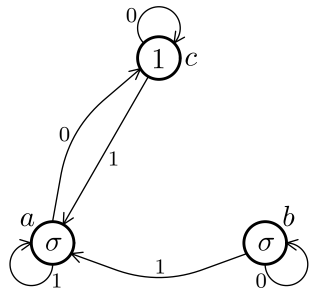
Automaton number 2355

$a = \sigma(c, b)$ Group:
$b = \sigma(a, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels:
$bca^{-2}c^{-1}bcac^{-1}b^{-2}cac^{-1}$,
$aca^{-1}c^{-1}ba^{-1}ca^{-1}c^{-1}bab^{-1}cac^{-1}a^{-1}b^{-1}cac^{-1}$,
$abac^{-1}bc^{-1}b^{-1}a^{-1}ca^{-1}c^{-1}bab\cdot$
$cb^{-1}ca^{-1}b^{-1}a^{-1}b^{-1}cac^{-1}$,
$aca^{-1}c^{-1}ba^{-1}bac^{-1}bc^{-1}b^{-1}a\cdot$
$b^{-1}cac^{-1}a^{-1}bcb^{-1}ca^{-1}b^{-1}$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,937,4687

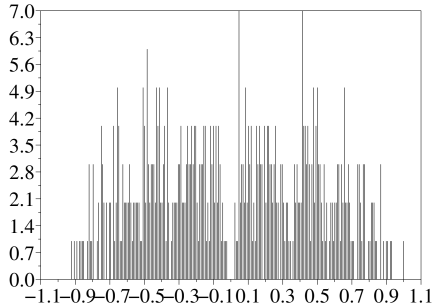




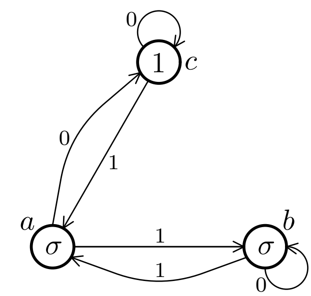
Automaton number 2361



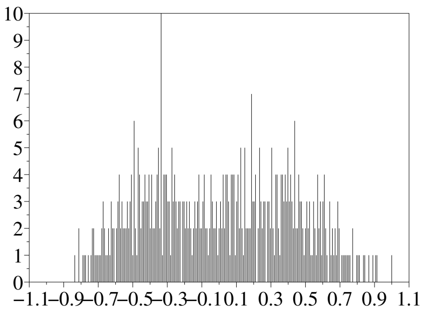
$a = \sigma(c, a)$ Group:
$b = \sigma(b, a)$ Contracting: *n/a*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}c)^2$, $[b^{-1}a, ba^{-1}]$, $[a, c]^2$,
$(b^{-1}a^{-1}c^2)^2$, $[ac^{-1}, bc^{-1}ba^{-1}]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,749,3343



Automaton number 2364

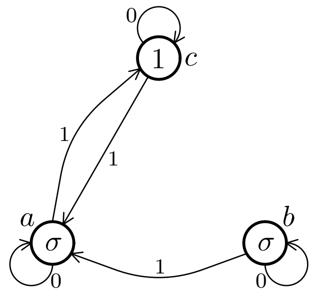


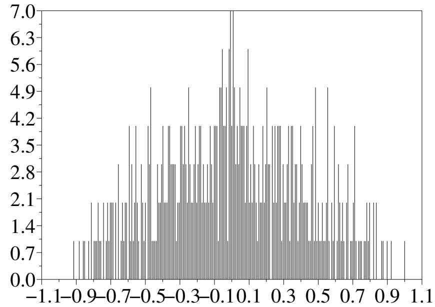
$a = \sigma(c, b)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels:
$aca^{-1}cb^{-1}a^{-1}ca^{-1}cb^{-1}abc^{-1}ac^{-1}a^{-1}bc^{-1}ac^{-1}$,
$bca^{-1}cb^{-2}ca^{-2}ca^{-1}b^3c^{-1}ac^{-1}b^{-2}ac^{-1}a^2c^{-1}$,
$bca^{-2}ca^{-1}ca^{-2}ca^{-1}bac^{-1}a^2c^{-1}b^{-2}ac^{-1}a^2c^{-1}$,
$bca^{-2}ca^{-1}ca^{-1}cb^{-1}ac^{-1}a^2c^{-2}ac^{-1}$,
$bca^{-1}cb^{-2}ca^{-1}cbc^{-1}ac^{-2}ac^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{24},2^{46},2^{90},2^{176}
Gr: 1,7,37,187,937,4687



Automaton number 2365

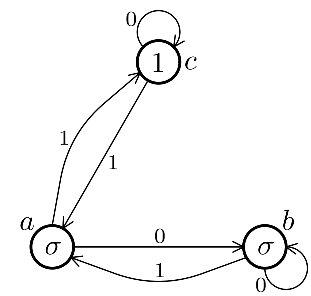
$a = \sigma(a, c)$ Group:
$b = \sigma(b, a)$ Contracting: *n/a*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(a^{-1}c)^2$, $[a, c]^2$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,604,2534

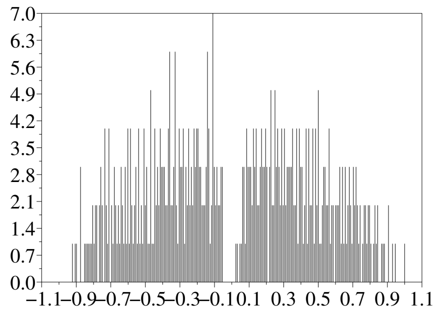




Automaton number 2366

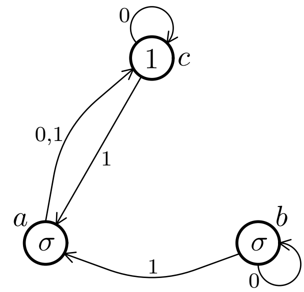
$a = \sigma(b, c)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $[b^{-1}a, ba^{-1}]$, $a^{-1}c^{-1}acb^{-1}ac^{-1}a^{-1}cb$,
$a^{-1}cbc^{-1}b^{-1}acb^{-1}c^{-1}b$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,37,187,929,4579

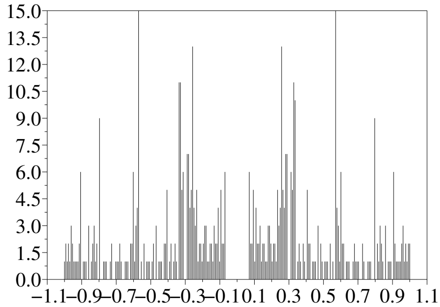




Automaton number 2367

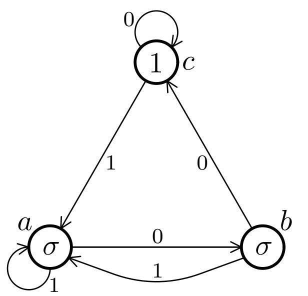
$a = \sigma(c, c)$ Group:
$b = \sigma(b, a)$ Contracting: *yes*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, c^2, $b^{-2}cacb^2cac$
SF: 2^0,2^1,2^3,2^5,2^8,2^{14},2^{25},2^{47},2^{90}
Gr: 1,5,17,53,161,480,1422

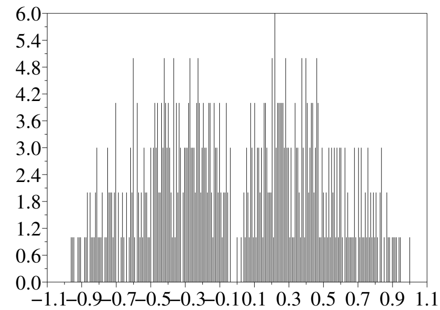




Automaton number 2369

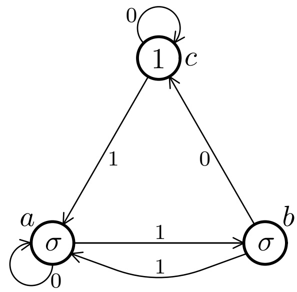
$a = \sigma(b, a)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[a, b]^2$, $(a^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,602,2514

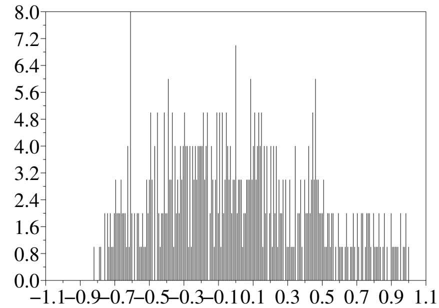




Automaton number 2371

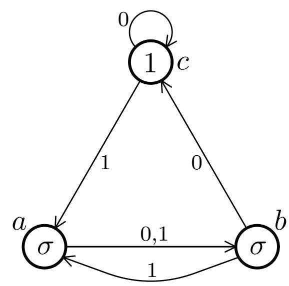
$a = \sigma(a, b)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(b^{-1}c)^2$, $(a^{-1}b)^4$, $(b^{-1}c^{-1}ac)^2$, $(a^{-1}c)^4$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,35,165,758,3460

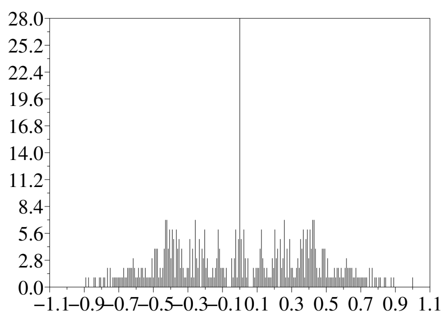




Automaton number 2372

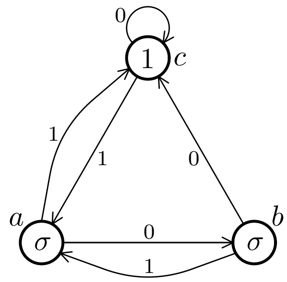
$a = \sigma(b, b)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(a^{-1}b)^2$, $(b^{-1}c)^2$, $[c, ab^{-1}]$, $[cb^{-1}, a]$, $[c^{-1}, b^{-1}] \cdot [a^{-1}, b^{-1}]$, $[a, c^{-1}] \cdot [b, a^{-1}]$, $[b^{-1}, a^{-1}] \cdot [c^{-1}, a^{-1}]$
SF: 2^0,2^1,2^3,2^5,2^7,2^9,2^{11},2^{13},2^{15}
Gr: 1,7,33,127,433,1415

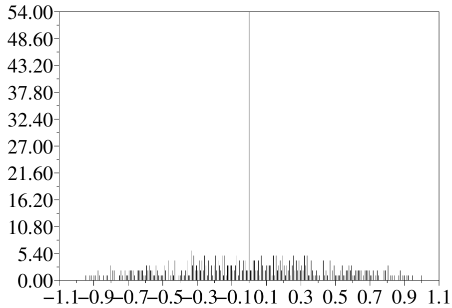




Automaton number 2375

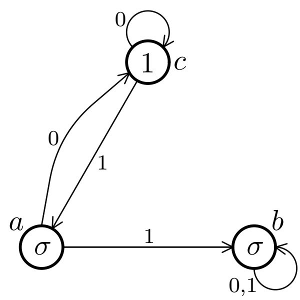
$a = \sigma(b, c)$ Group:
$b = \sigma(c, a)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $(b^{-1}c)^2$
SF: 2^0,2^1,2^3,2^5,2^9,2^{15},2^{26},2^{48},2^{92}
Gr: 1,7,35,165,769,3575

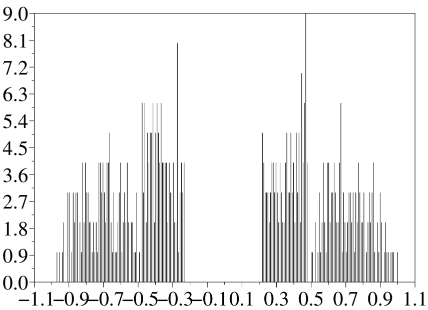




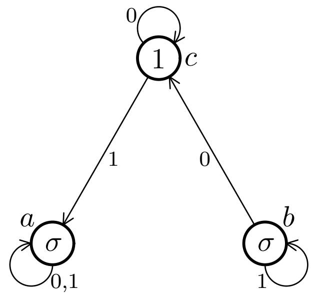
Automaton number 2391

$a = \sigma(c, b)$ Group:
$b = \sigma(b, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: b^2, $[a^2, b]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,6,26,103,399,1538

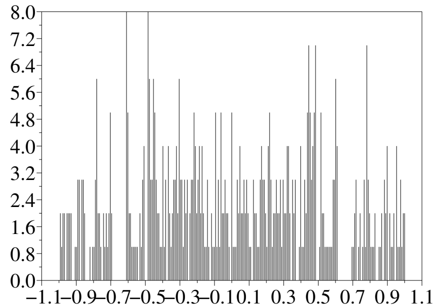




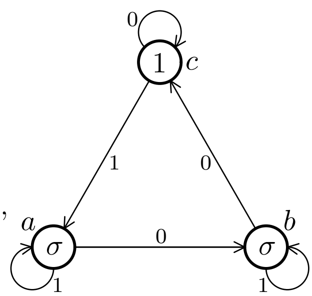
Automaton number 2395



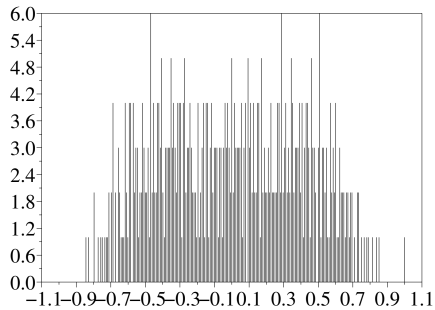
$a = \sigma(a, a)$ Group:
$b = \sigma(c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: a^2, c^2, $(acb)^2$, $[b^2, cac]$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,5,17,50,140,377,995,2605



Automaton number 2396



$a = \sigma(b, a)$ Group: *A. Boltenkov group*
$b = \sigma(c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $acb^{-1}ca^{-2}cb^{-1}cac^{-1}bc^{-2}bc^{-1}$,
$acb^{-1}ca^{-2}cb^{-1}a^2c^{-1}b^{-1}a^2c^{-1}bc^{-1}a^{-1}bca^{-2}bc^{-1}$,
$acb^{-1}a^2c^{-1}b^{-1}a^{-1}cb^{-1}cbca^{-2}bc^{-2}bc^{-1}$,
$bcb^{-1}ca^{-1}b^{-1}cb^{-1}a^2c^{-1}ac^{-1}ba^{-2}bc^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{24},2^{46},2^{90},2^{176}
Gr: 1,7,37,187,937,4687



Automaton number 2398

$a = \sigma(a, b)$ Group: *F.Dahmani Group*

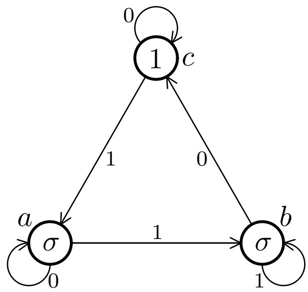
$b = \sigma(c, b)$ Contracting: *no*

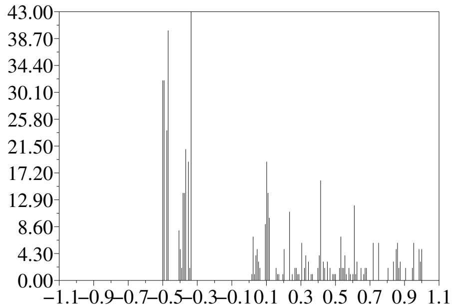
$c = (c, a)$ Self-replicating: *yes*

Rels: cba, $b^{-1}a^{-1}b^2a^{-1}b^{-1}a^2$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,31,127,483,1823





Automaton number 2399

$a = \sigma(b, b)$ Group:

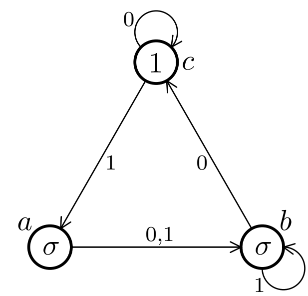
$b = \sigma(c, b)$ Contracting: *no*

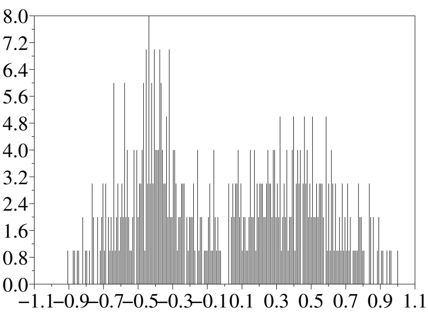
$c = (c, a)$ Self-replicating: *yes*

Rels: $[b^{-1}a, ba^{-1}]$

SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}

Gr: 1,7,37,187,929,4599





Automaton number 2401

$a = \sigma(a, c)$ Group:

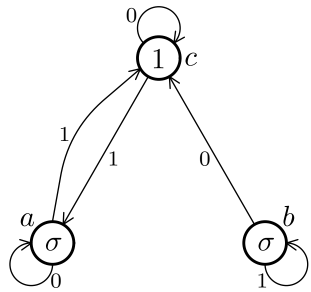
$b = \sigma(c, b)$ Contracting: *no*

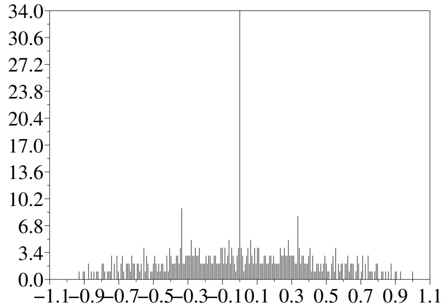
$c = (c, a)$ Self-replicating: *yes*

Rels: $(a^{-1}c)^2$, $[a, c]^2$, $(c^{-2}ba)^2$

SF: 2^0,2^1,2^3,2^5,2^9,2^{15},2^{26},2^{48},2^{92}

Gr: 1,7,35,165,757,3447





Automaton number 2402

$a = \sigma(b, c)$ Group:

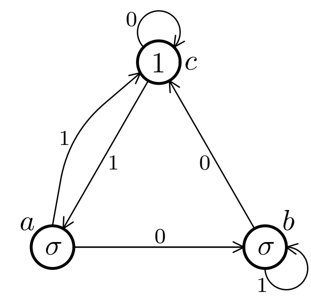
$b = \sigma(c, b)$ Contracting: *n/a*

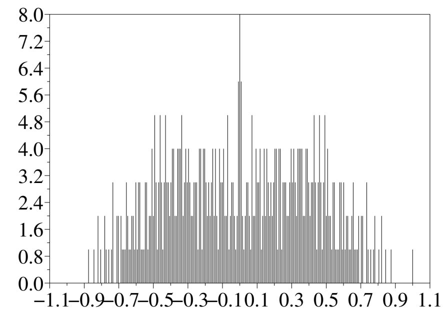
$c = (c, a)$ Self-replicating: *yes*

Rels: $ac^2b^{-1}a^{-2}c^2b^{-1}abc^{-2}bc^{-2}$,
$ac^2b^{-1}a^{-2}cb^{-2}c^{-1}a^4bc^{-2}a^{-3}cb^2c^{-1}$,
$acb^{-2}c^{-1}ac^2b^{-1}a^{-2}cb^2c^{-1}bc^{-2}$,
$acb^{-2}c^{-1}acb^{-2}c^{-1}acb^2c^{-1}a^{-3}cb^2c^{-1}$

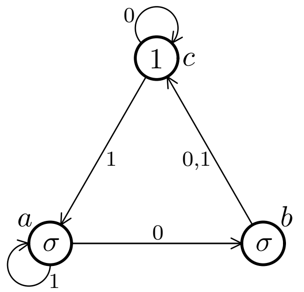
SF: 2^0,2^1,2^3,2^5,2^7,2^{10},2^{15},2^{25},2^{41}

Gr: 1,7,37,187,937,4687

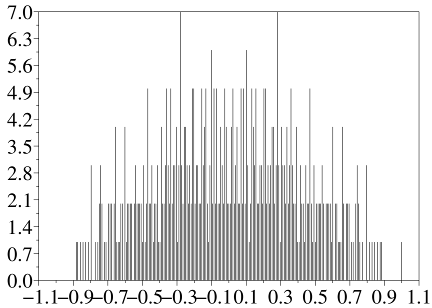




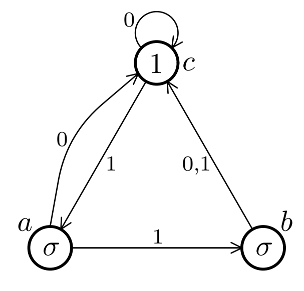
Automaton number 2423



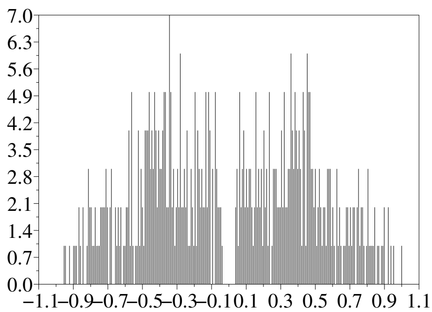
$a = \sigma(b, a)$ Group:
$b = \sigma(c, c)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $ac^{-1}bca^{-2}c^{-1}bcac^{-1}b^{-2}c$,
$ac^{-1}bca^{-1}c^{-1}bac^{-1}b^{-1}a^{2}c^{-1}b^{-1}ca^{-1}b\cdot$
$ca^{-1}b^{-1}ca^{-1}$,
$bc^{-1}bca^{-1}b^{-1}ac^{-1}bac^{-1}ac^{-1}b^{-1}c^{2}a^{-1}\cdot$
$b^{-1}ca^{-1}$,
$bac^{-1}bac^{-1}b^{-2}c^{-1}bca^{-1}b^{2}ca^{-1}\cdot$
$b^{-1}ca^{-1}b^{-1}ac^{-1}b^{-1}c$,
$bac^{-1}bac^{-1}b^{-2}ac^{-1}bac^{-1}bca^{-1}\cdot$
$b^{-1}ca^{-1}ca^{-1}b^{-1}ca^{-1}$
SF: 2^0,2^1,2^3,2^5,2^8,2^{14},2^{25},2^{47},2^{90}
Gr: 1,7,37,187,937,4687



Automaton number 2427

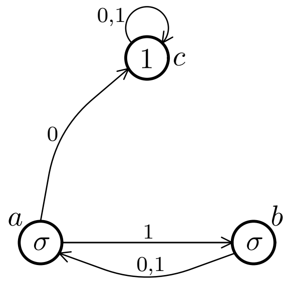


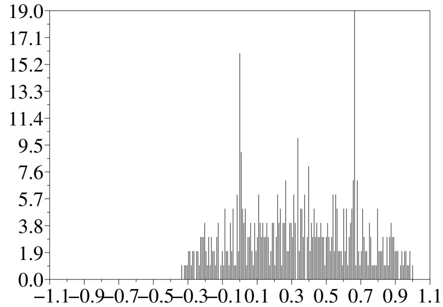
$a = \sigma(c, b)$ Group:
$b = \sigma(c, c)$ Contracting: *n/a*
$c = (c, a)$ Self-replicating: *yes*
Rels: $[b^{-1}a, ba^{-1}]$, $a^{-1}c^{2}a^{-1}b^{-1}a^{2}c^{-2}b$
SF: 2^0,2^1,2^3,2^7,2^{13},2^{24},2^{46},2^{89},2^{175}
Gr: 1,7,37,187,929,4583



Automaton number 2841

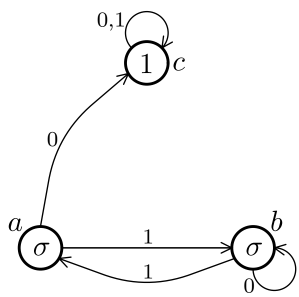
$a = \sigma(c, b)$ Group:
$b = \sigma(a, a)$ Contracting: *no*
$c = (c, c)$ Self-replicating: *yes*
Rels: c, $a^{-1}b^{-1}a^{-2}ba^{-1}b^{-1}aba^2b^{-1}ab$,
$a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}babab^{-2}abab$,
$a^{-1}ba^{-1}b^{-2}a^{-1}ba^{-1}bab^{-1}a^2b^{-1}ab$
SF: 2^0,2^1,2^3,2^5,2^8,2^{13},2^{23},2^{42},2^{79}
Gr: 1,5,17,53,161,485,
1457,4359,12991

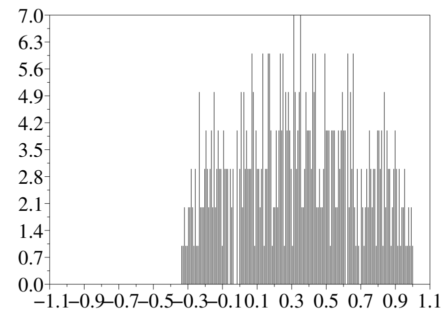




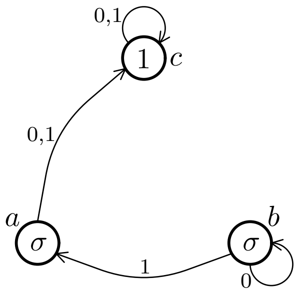
Automaton number 2850

$a = \sigma(c, b)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (c, c)$ Self-replicating: *yes*
Rels: c, $a^{-4}bab^{-1}a^2b^{-1}ab$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,5,17,53,161,485,1445

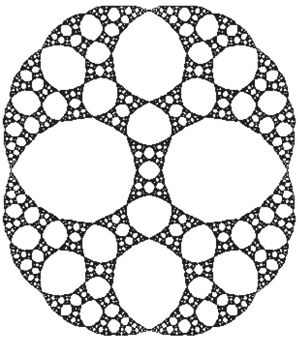


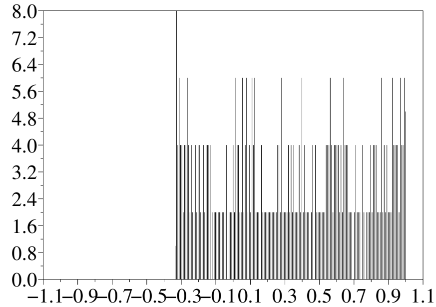


Automaton number 2853



$a = \sigma(c, c)$ Group: $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = \sigma(b, a)$ Contracting: *yes*
$c = (c, c)$ Self-replicating: *yes*
Rels: c, a^2, $ab^{-1}ab^{-2}ab^{-1}abab^2ab$
SF: 2^0,2^1,2^2,2^3,2^5,2^8,2^{14},2^{25},2^{47}
Gr: 1,4,10,22,46,94,190,375,731,
1422,2752,5246,9908
Limit space:





9. Proofs

This section contains proofs of many of the claims contained in the tables in Section 7 and Section 8 and some additional information.

We sometimes encounter one of the following four binary tree automorphisms

$$a = \sigma(1, a), \qquad b = \sigma(b, 1), \qquad c = \sigma(c^{-1}, 1), \qquad d = \sigma(1, d^{-1}).$$

The first one is the binary adding machine, the second is its inverse, and all are conjugate to the adding machine and therefore act level transitively on the binary tree and have infinite order.

We freely use the known classification of groups generated by 2-state automata over a 2-letter alphabet.

Theorem 7 ([GNS00]). *Up to isomorphism, there are six $(2,2)$-automaton groups: the trivial group, the cyclic group of order 2 (we denote it by C_2), Klein group $C_2 \times C_2$ of order 4, the infinite cyclic group $\mathbb{Z}$, the infinite dihedral group D_∞ and the Lamplighter group $\mathbb{Z} \wr C_2$.*

In particular the sixteen 2-state automata for which both states are inactive generate the trivial group, and the sixteen 2-state automata in which both states are active generate C_2 (since both states in that case describe the mirror automorphism $\mu = \sigma(\mu, \mu)$ of order 2.

The automata given by either of the wreath recursions

$$\begin{aligned} a &= \sigma(a, a), \qquad b = (a, a), \\ a &= \sigma(b, b), \qquad b = (a, a), \end{aligned}$$

generate the Klein group $C_2 \times C_2$.

The automata given by the wreath recursions

$$\begin{aligned} a &= \sigma(a, a), \qquad b = (a, b), \\ a &= \sigma(a, a), \qquad b = (b, a), \\ a &= \sigma(b, b), \qquad b = (a, b), \\ a &= \sigma(b, b), \qquad b = (b, a), \end{aligned}$$

generate the infinite dihedral group D_∞.

The automata given by the wreath recursions

$$\begin{aligned} a &= \sigma(a, a), \qquad b = (b, b), \\ a &= \sigma(b, b), \qquad b = (b, b), \end{aligned}$$

generate the cyclic group C_2.

The automata given by the wreath recursions

$$\begin{aligned} a &= \sigma(a,b), & b &= (a,a),\\ a &= \sigma(b,a), & b &= (a,a),\\ a &= \sigma(a,b), & b &= (b,b),\\ a &= \sigma(b,a), & b &= (b,a), \end{aligned}$$

generate the infinite cyclic group $\mathbb{Z}$. Moreover, in the first two cases we have $b = a^{-2}$, in the fourth case $b = 1$ and a is the adding machine, and in the third case $b = 1$ and a is the inverse of the adding machine.

The automata given by the wreath recursions

$$\begin{aligned} a &= \sigma(a,b), & b &= (a,b),\\ a &= \sigma(a,b), & b &= (b,a),\\ a &= \sigma(b,a), & b &= (a,b),\\ a &= \sigma(b,a), & b &= (b,a), \end{aligned}$$

generate the Lamplighter group $\mathbb{Z} \wr C_2 = \mathbb{Z} \ltimes (\oplus_{\mathbb{Z}} C_2)$.

The results on the next few pages concern the existence of elements of infinite order and the level transitivity of the action. They are used in some of the proofs that follow.

Lemma 1 ([BGK$^+$a])**.** *Let G be a group generated by an automaton $\mathcal{A}$ over a 2-letter alphabet. Assume that the set of states S of $\mathcal{A}$ splits into two nonempty parts P and Q such that*

(i) *one of the parts consists of the active states (those with nontrivial vertex permutation) and the other consists of the inactive states;*

(ii) *for each state from P, both arrows go to states in the same part (either both to P or both to Q);*

(iii) *for each state from Q, one arrow goes to a state in P and the other to a state in Q.*

Then any element of the group that can be written as a product of odd number of active generators or their inverses and odd number of inactive generators and their inverses, in any order, has infinite order. In particular, the group G is not a torsion group.

Proof. Denote by D the set of elements in G that can be represented as a product of odd number of active generators or their inverses and odd number of inactive generators and their inverses, in any order.

We note that if $g \in D$ then both sections of g^2 are in D. Indeed, for such an element, $g = \sigma(g_0, g_1)$ and $g^2 = (g_1 g_0, g_0 g_1)$. Both sections of g^2 are products (in some order) of the first level sections of the generators (and/or their inverses) used to express g as an element in D. By assumption, among these generators, there are odd number of active and odd number of inactive ones. The generators from P, by condition (ii), produce even number of active and even number of inactive sections on level 1, while the generators from Q, by condition (iii), produce odd number of active sections and odd number of inactive sections. Thus both sections of g are in D.

By way of contradiction, assume that h is an element of D of finite order 2^n, for some $n \geq 0$. If $n > 0$ the sections of h^2 are elements in D of order 2^{n-1}. Thus, continuing in this fashion, we reach an element in D that is trivial. This is contradiction since all elements in D act nontrivially on level 1. $\square$

There is a simple criterion that determines whether a given element of a self-similar group generated by a finite automaton over the 2-letter alphabet $X = \{0, 1\}$ acts level transitively on the tree. The criterion is based on the image of the given element in the abelianization of $\mathrm{Aut}(X^*)$, which is isomorphic to the infinite Cartesian product $\prod_{i=0}^{\infty} C_2$. The canonical isomorphism sends $g \in G$ to $(a_i \mod 2)_{i=0}^{\infty}$, where a_i is the number of active sections of g at level i. We also make use of the ring structure on $\prod_{i=0}^{\infty} C_2$ obtained by identifying $(b_i)_{i=0}^{\infty}$ with $\sum_{i=0}^{\infty} b_i t^i$ in the ring of formal power series $C_2[[t]]$. It is known that a binary tree automorphism g acts level transitively on X^* if and only if $\bar{g} = (1, 1, 1, \ldots)$, where $\bar{g}$ be the image of g in the abelianization $\prod_{i=0}^{\infty} C_2$ of $\mathrm{Aut}(X^*)$.

Lemma 2 (Element transitivity, [BGK+a]). *Let G be a group generated by an automaton $\mathcal{A}$ over a 2-letter alphabet. There exists an algorithm that decides if g acts level transitively on X^*.*

Proof. Let $g = \sigma^i(g_0, g_1)$, where $i \in \{0, 1\}$. Then

$$\overline{g} = i + t \cdot (\overline{g_0} + \overline{g_1}).$$

Similar equations hold for all sections of g. Since G is generated by a finite automaton, g has only finitely many different sections, say k. Therefore we obtain a linear system of k equations over the k variables $\{g_v, v \in X^*\}$. The solution of this system expresses $\bar{g}$ as a rational function $P(t)/Q(t)$, where P an Q are polynomials of degree not higher than k. The element g acts level transitively if and only if $\bar{g} = \frac{1}{1-t}$. $\square$

We often need to show that a given group of tree automorphisms is level transitive. Here is a very convenient necessary and sufficient condition for this in the case of a binary tree.

Lemma 3 (Group transitivity, [BGK+a]). *A self-similar group of binary tree automorphisms is level transitive if and only if it is infinite.*

Proof. Let G be a self-similar group acting on a binary tree.

If G acts level transitively then G must be infinite (since the size of the levels is not bounded).

Assume now that the group G is infinite.

We first prove that all level stabilizers $\mathrm{Stab}_G(n)$ are different. Note that, since all level stabilizers have finite index in G and G is infinite, all level stabilizers are infinite. In particular, each contains a nontrivial element.

Let $n > 0$ and $g \in \mathrm{Stab}_G(n-1)$ be an arbitrary nontrivial element. Let $v = x_1 \dots x_k$ be a word of shortest length such that $g(v) \neq v$. Since $g \in \mathrm{Stab}_G(n-1)$, we must have $k \geq n$. The section $h = g_{x_1 x_2 \dots x_{k-n}}$ is an element of G by the self-similarity of G. The minimality of the word v implies that $g \in \mathrm{Stab}_G(k-1)$, and therefore $h \in \mathrm{Stab}_G(n-1)$. On the other hand h acts nontrivially on $x_{k-n+1} \dots x_k$ and we conclude that $h \in \mathrm{Stab}_G(n-1) \setminus \mathrm{Stab}_G(n)$. Thus all level stabilizers are different.

We now prove level transitivity by induction on the level.

The existence of elements in $\mathrm{Stab}_G(0) \setminus \mathrm{Stab}_G(1)$ shows that G acts transitively on level 1.

Assume that G acts transitively on level n. Select an arbitrary element $h \in \mathrm{Stab}_G(n) \setminus \mathrm{Stab}_G(n+1)$ and let $w =\in X^n$ be a word of length n such that $h(w1) = w0$.

Let u be an arbitrary word of length n and let x be a letter in $X = \{0, 1\}$. We will prove that ux is mapped to $w0$ by some element of G, proving the transitivity of the action at level $n+1$. By the inductive assumption there exists $f \in G$ such that $f(u) = w$. If $f(ux) = w0$ we are done. Otherwise, $hf(ux) = h(w1) = w0$ and we are done again. □

Consider the infinitely iterated permutational wreath product $\wr_{i \geq 1} C_d$, consisting of the automorphisms of the d-ary tree for which the activity at every vertex is a power of some fixed cycle of length d. The last proof works, mutatis mutandis, for the self-similar subgroups of $\wr_{i \geq 1} C_d$ and may be easily adapted in other situations.

The following lemma is used often when we want to prove that some automaton group is not free.

Lemma 4. *If a self-similar group contains two nontrivial elements of the form $(1, u), (v, 1)$, then the group is not free.*

Proof. Suppose $a = (1, u), b = (v, 1)$ are two nontrivial elements of a self-similar group G and G is free. Obviously $[a, b] = 1$, hence a and b are powers of some element $x \in G$: $a = x^m$, $b = x^n$. Then $a^n = b^m$, so $a^n = (1, u^n) = b^m = (v^m, 1)$. This implies that $u^n = v^m = 1$, which is a contradiction, since u and v are nontrivial elements of a free group. $\square$

In most case when the corresponding group is finite we do not offer a full proof. In all such cases the proof can be easily done by direct calculations. As an example, a detailed proof is given in the case of the automaton [748].

We now proceeds to individual analysis of the properties of the automaton groups in our classification.

1. Trivial group.

730. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (a, a)$, $c = (a, a)$.

The claim follows from the relations $b = c$, $a^2 = b^2 = abab = 1$.

731 $\cong \mathbb{Z}$. Wreath recursion: $a = \sigma(b, a)$, $b = (a, a)$, $c = (a, a)$.

We have $c = b$ and $b = a^{-2}$. The states a and b form a 2-state automaton generating $\mathbb{Z}$ (see Theorem 7).

734 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(b, b)$, $b = (a, a)$, $c = (a, a)$.

The claim follows from the relations $b = c$, $a^2 = b^2 = abab = 1$.

739 $\cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, a)$, $c = (a, a)$.

All generators have order 2. The elements $u = acba = (1, ba)$ and $v = bc = (ba, 1)$ generate $\mathbb{Z}^2$. This is clear since $ba = \sigma(1, ba)$ is the adding machine and therefore has infinite order. Further, we have $ac = \sigma$ and $\langle u, v\rangle$ is normal in $H = \langle u, v, \sigma\rangle$, since $u^\sigma = v$ and $v^\sigma = u$. Thus $H \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z}) = C_2 \wr \mathbb{Z}$.

We have $G_{739} = \langle H, a\rangle$ and H is normal in G_{739}, since it has index 2. Moreover, $u^a = v^{-1}$, $v^a = u^{-1}$ and $\sigma^a = \sigma$. Thus $G_{739} = C_2 \ltimes (C_2 \wr \mathbb{Z})$, where the action of C_2 on H is specified above.

740. Wreath recursion: $a = \sigma(b, a)$, $b = (b, a)$, $c = (a, a)$.

The states a, b form a 2-state automaton generating the Lamplighter group (see Theorem 7). Thus G_{740} has exponential growth and is neither torsion nor contracting.

Since $c = (a, a)$ we obtain that G_{740} can be embedded into the wreath product $C_2 \wr (\mathbb{Z} \wr \mathbb{C}_2)$. Thus G_{740} is solvable.

741. Wreath recursion: $a = \sigma(c, a)$, $b = (b, a)$, $c = (a, a)$.

The states a and c form a 2-state automaton generating the infinite cyclic group $\mathbb{Z}$ in which $c = a^{-2}$ (see Theorem 7).

Since $b = (b, a)$, we see that b has infinite order and that G_{741} is not contracting).

We have $c = a^-2$ and $b^{-1}a^{-3}b^{-1}ababa = 1$. Since a and b do not commute the group is not free.

743 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(b, b)$, $b = (b, a)$, $c = (a, a)$.

All generators have order 2. The elements $u = acba = (1, ba)$ and $v = bc = (ba, 1)$ generate $\mathbb{Z}^2$ because $ba = \sigma(ab, 1)$ is conjugate to the adding machine and has infinite order. Further, we have $babc = \sigma$ and $\langle u, v\rangle$ is normal in $H = \langle u, v, \sigma\rangle$ because $u^\sigma = v$ and $v^\sigma = u$. In other words, $H \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z}) = C_2 \wr \mathbb{Z}$.

Furthermore, $G_{743} = \langle H, a\rangle$ and H is normal in G_{743} because $u^a = v^{-1}$, $v^a = u^{-1}$ and $\sigma^a = \sigma$. Thus $G_{743} = C_2 \ltimes (C_2 \wr \mathbb{Z})$, where the action of C_2 on H is specified above and coincides with the one in G_{739}. Therefore $G_{743} \cong G_{739}$.

744. Wreath recursion: $a = \sigma(c, b)$, $b = (b, a)$, $c = (a, a)$.

Since $(a^{-1}c)^2 = (c^{-1}ab^{-1}a, b^{-1}ac^{-1}a)$ and $c^{-1}ab^{-1}a = ((c^{-1}ab^{-1}a)^{-1}, a^{-1}c)$, the element $(a^{-1}c)^2$ fixes the vertex 01 and its section at this vertex is equal to $a^{-1}c$. Hence, $a^{-1}c$ has infinite order.

The element $c^{-1}ab^{-1}a$ also has infinite order, fixes the vertex 00 and its section at this vertex is equal to $c^{-1}ab^{-1}a$. Therefore G_{744} is not contracting.

We have $b^{-1}c^{-1}ba^{-1}ca = (1, a^{-1}c^{-1}ac)$, $ab^{-1}c^{-1}ba^{-1}c = (ca^{-1}c^{-1}a, 1)$, hence by Lemma 4 the group is not free.

747 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(c, c)$, $b = (b, a)$, $c = (a, a)$.

All generators have order 2 and a commutes with c. Conjugating this group by the automorphism $\gamma = (\gamma, c\gamma)$ yields an isomorphic group generated by automaton $a' = \sigma$, $b' = (b', a')$ and $c' = (a', a')$. On the other hand we obtain the same automaton after conjugating G_{739} by $\mu = (\mu, a\mu)$ (here a denotes the generator of G_{739}).

748 $\cong D_4 \times C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, a)$, $c = (a, a)$.

Since a is nontrivial and b and c have a as a section, none of the generators is trivial. All generators have order 2. Indeed, we have $a^2 = (a^2, a^2)$, $b^2 = (c^2, a^2)$, $c^2 = (a^2, a^2)$, showing that a^2, b^2 and c^2 generate a self-similar group in which no element is active. Therefore $a^2 = b^2 = c^2 = 1$. Since $ac = \sigma$ we have that $(ac)^2 = 1$. Therefore a and c commute. Since $(bc)^2 = ((ca)^2, 1) = 1$, we see that b and c also commute. Further, the relations $(ab)^2 = (ac, 1) = (\sigma, 1) \neq 1$ and $(ab)^4 = 1$ show that a and b generate the dihedral group D_4. It remains to be shown that $c \notin \langle a, b\rangle$. Clearly c could only be equal to one of the four elements 1, b, aba, and $abab$ in D_4 that stabilize level 1. However, c is nontrivial, differs from b at 0 (the section $b|_0 = c$ is not active, while $c|_0 = a$ is active), differs from aba at 1 (the section $(aba)|_1 = aca$ is not active, while $c|_1 = a$ is active), and differs from $abab$ at 1 (the section of $abab$ at 1 is trivial). This completes the proof.

749. Wreath recursion: $a = \sigma(b, a)$, $b = (c, a)$, $c = (a, a)$.

The element $(a^{-1}c)^4$ stabilizes the vertex 000 and its section at this vertex is equal to $a^{-1}c$. Hence, $a^{-1}c$ has infinite order.

We have $ac^{-1} = \sigma(ba^{-1}, 1)$, $ba^{-1} = \sigma(1, cb^{-1})$, $cb^{-1} = (ac^{-1}, 1)$, Thus the subgroup generated by these elements is isomorphic to $IMG(1 - \frac{1}{z^2})$ (see [BN06]).

We have $c^{-1}b = (a^{-1}c, 1)$, $ac^{-1}ba^{-1} = (1, ca^{-1})$. Thus, by Lemma 4 the group is not free.

748 $\cong G_{848} \cong C_2 \wr \mathbb{Z}$. Wreath recursion:$a = \sigma(c, a)$, $b = (c, a)$, $c = (a, a)$.

It is proven below that $G_{848} \cong G_{2190}$ and for G_{2190} we have $a = \sigma(c, a)$, $b = \sigma(a, a)$, $c = (a, a)$. Therefore $G_{2190} = \langle a, b, c\rangle = \langle a, c, c^{-1}b = \sigma\rangle = \langle a = (c, a)\sigma, c = (a, a), a\sigma = (c, a)\rangle = G_{750}$.

752. Wreath recursion: $a = \sigma(b, b)$, $b = (c, a)$, $c = (a, a)$.

The group G_{752} is a contracting group with nucleus consisting of 41 elements. It is a virtually abelian group, containing $\mathbb{Z}^3$ as a subgroup of index 4.

All generators have order 2.

Let $x = ca$, $y = babc$, and $K = \langle x, y\rangle$. Since $xy = ((cbab)^{ca}, abcb) = ((y^{-1})^x, abcb)$ and $yx = (cbab, abcb) = (y^{-1}, abcb)$ the elements x and y commute. Conjugating by $\gamma = (\gamma, bc\gamma)$ yields the self-similar copy K' of K generated by $x' = \sigma((y')^{-1}, (x')^{-1})$ and $y' = \sigma((y')^{-1}x', 1)$, where $x' = x^\gamma$ and $y' = y^\gamma$. Since $(x')^2 = ((x')^{-1}(y')^{-1}, (y')^{-1}(x')^{-1})$ and $(y')^2 = ((y')^{-1}x', (y')^{-1}x')$, the virtual endomorphism of K' is given by

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

The eigenvalues $\lambda = -\frac{1}{2} \pm \frac{1}{2}i$ of this matrix are not algebraic integers, and therefore, by the results in [NS04], the group $K' \cong K$ is free abelian of rank 2.

Let $H = \langle ba, cb\rangle$. The index of $\mathrm{Stab}_H(1)$ in G is 4, since the index of $\mathrm{Stab}_H(1)$ in H is 2 and the index of H in G is 2 (the generators have order 2). We have $\mathrm{Stab}_H(1) = \langle cb, cb^{ba}, (ba)^2\rangle$. If we conjugate the generators of $\mathrm{Stab}_H)(1)$ by $g = (1, b)$, we obtain

$$\begin{aligned} g_1 &= (cb)^g &&= (x^{-1}, 1), \\ g_2 &= \left((cb)^{ba}\right)^g &&= (1, \quad x), \\ g_3 &= \left((ba)^2\right)^g &&= (y^{-1}, y). \end{aligned}$$

Therefore, g_1, g_2, and g_3 commute. If $g_1^{n_1}g_2^{n_2}g_3^{n_3} = 1$, then we must have $x^{-n_1}y^{-n_3} = x^{n_2}y^{n_3} = 1$. Since K is free abelian, this implies $n_1 = n_2 = n_3 = 0$. Thus, $\mathrm{Stab}_H(1)$ is a free abelian group of rank 3.

753. Wreath recursion: $a = \sigma(c, b)$, $b = (c, a)$, $c = (a, a)$.

Since $ab^{-1} = \sigma(1, ba^{-1})$, this element is conjugate to the adding machine.

For a word w in $w \in \{a^{\pm1}, b^{\pm1}, c^{\pm1}\}^*$, let $|w|_a$, $|w|_b$ and $|w|_c$ denote the sum of the exponents of a, b and c in w. Let w represents the element $g \in G$. If $|w|_a$ and $|w|_b$ are odd, then g acts transitively on the first level, and $g^2|_0$ is represented by a word w_0, which is the product (in some order) of all first level sections of all generators appearing in w. Hence, $|w_0|_a = |w|_b + 2|w|_c$ and $|w_0|_b = |w|_a$ are odd again. Therefore, similarly to Lemma 1, any such element has infinite order.

In particular c^2ba has infinite order. Since $a^4 = (caca, a^4, acac, a^4)$ and $caca = (baca, c^2ba, bac^2, caba)$, the element a^4 has infinite order (and so does a). Since a^4 fixes the vertex 01 and its section at that vertex is equal to a^4, the group G_{753} is not contracting.

We have $cb^{-1} = (ac^{-1}, 1), acb^{-1}a^{-1} = (1, bac^{-1}b^{-1})$, hence by Lemma 4 the group is not free.

756 $\cong G_{748} \cong D_4 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = (c, a)$, $c = (a, a)$.

All generators have order 2. The generator c commutes with both a and b. Since $(ab)^2 = (ca, ca)$ the order of ca is 4 and the group is isomorphic to $D_4 \times C_2$

766 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, b)$, $c = (a, a)$.

The state b is trivial. The states a and c form a 2-state automaton generating $C_2 \times C_2$ (see Theorem 7).

767 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(1, a)$, $b = (b, b)$, $c = (a, a) = a^2$.

The state b is trivial. The automorphism a is the binary adding machine.

768 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(c, a)$, $b = (b, b)$, $c = (a, a)$.

The states a and c form a 2-state automaton generating $\mathbb{Z}$ (see Theorem 7) in which $c = a^{-2}$.

770 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(b, b)$, $b = (b, b)$, $c = (a, a)$.

The state b is trivial. The states a and c form a 2-state automaton generating $C_2 \times C_2$ (see Theorem 7).

771 $\cong \mathbb{Z}^2$. Wreath recursion: $a = \sigma(c, b)$, $b = (b, b)$, $c = (a, a)$.

The group G_{771} is finitely generated, abelian, and self-replicating. Therefore, it is free [NS04]. Since $b = 1$ the rank is 1 or 2. We prove that the rank is 2, by showing that $c^n \neq a^m$, unless $n = m = 0$. By way of contradiction, let $c^n = a^m$ for some integer n and m and choose such integers with minimal $|n| + |m|$. Since c^n stabilizes level 1, m must be even and we have $(a^n, a^n) = c^n = a^m = (c^{m/2}, c^{m/2})$, implying $a^n = c^{m/2}$. By the minimality assumption, m must be 0, which then implies that n must be 0 as well.

774 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = (b, b)$, $c = (a, a)$.

The state b is trivial. The states a and c form a 2-state automaton generating $C_2 \times C_2$ (see Theorem 7).

775 $\cong C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, b)$, $c = (a, a)$.

All generators have order 2. Further, $ac = ca = \sigma(1, 1)$ and $ba = \sigma(ba, ca)$. Hence, for the subgroup $H = \langle ba, ca\rangle \cong G_{2853} \cong IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$.

Since the generators have order 2, H is normal subgroup of index 2 in G_{775}. Moreover $(ba)^a = (ba)^{-1}$ and $(ca)^a = ca$. Therefore $G \cong C_2 \ltimes H$, where C_2 is generated by a and the action of a on H is given above.

Conjugating the generators by $g = \sigma(g, g)$ we obtain the wreath recursion

$$a' = \sigma(a', a'), \qquad b' = (b', c'), \qquad c' = (a', a'),$$

where $a' = a^g$, $b' = b^g$ and $c' = c^g$. This is the wreath recursion defining G_{793}. Denote G_{793} by G and its generators by a, b, and c (we continue working only with G_{793}). Thus

$$a = \sigma(a, a), \qquad b = (b, c), \qquad c = (a, a).$$

The generators have order 2. Moreover $ac = ca$ and $\langle a, c\rangle = C_2 \times C_2$ is the Klein group. Denote $A = \langle a, c\rangle$.

The element $x = ba$ has infinite order, since x^2 fixes 00, and has itself as a section at 00. Note that

$$x = ba = (b, c)\sigma(a, a) = \sigma(ca, ba) = \sigma(\sigma, x).$$

and, therefore, $x^2 = (x\sigma, \sigma x) = (x, \sigma, \sigma, x)$.

Proposition 1. *The subgroup $H = \langle x, y\rangle$ of G, where $x = ba$ and $y = cabc$ is torsion free.*

Proof. The first level decompositions of $x^{\pm1}$ and $y^{\pm1}$ and the second level decompositions of x and y are given by

$$\begin{aligned}
x &= \sigma(\sigma, x)\\
y &= cabc = \sigma aaba\sigma = \sigma ba\sigma = x^\sigma = \sigma(x,\sigma)\\
x^{-1} &= \sigma(x^{-1},\sigma)\\
y^{-1} &= \sigma(\sigma, x^{-1})\\
x &= \sigma(\sigma(1,1),\sigma(\sigma,x)) = \mu(1,1,\sigma,x)\\
y &= x^\sigma = \mu(\sigma,x,1,1),
\end{aligned}$$

where $\mu = \sigma(\sigma,\sigma)$ permutes the first two levels of the tree as $00 \leftrightarrow 11$, $10 \leftrightarrow 01$. We encode this as the permutation $\mu = (03)(12)$.

For a word w over $\{x^{\pm 1},\sigma\}$, denote by $\#_x(w)$ and $\#_\sigma(w)$ the total number of appearances of x and x^{-1} and the number of appearances of σ in w, respectively.

Note that x and x^{-1} act as the permutation $(03)(12)$ on the second level, and σ acts as the permutation $(02)(13)$. These permutations have order 2, commute, and their product is $(01)(23)$, which is not trivial. Thus, a tree automorphisms represented by a word w over $\{x^{\pm 1},\sigma\}$ cannot be trivial unless both $\#_x(w)$ and $\#_\sigma(w)$ are even.

Let g be an element of H that can be written as $g = z_1z_2\dots z_n$, for some $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$, $i = 1,\dots,n$.

If n is odd, the element g cannot have order 2. By way of contradiction assume otherwise. For z in $\{x^{\pm 1}, y^{\pm 1}\}$ denote $z' = \sigma z$. Thus, for instance $x' = (\sigma, x)$ and $y' = (x,\sigma)$. Note that

$$g^2 = (z_1z_2\dots z_n)^2 = (z_1')^\sigma z_2'(z_3')^\sigma z_4' \dots (z_n')^\sigma z_1'(z_2')^\sigma \dots z_n' = (w_0,w_1),$$

where the words w_i over $\{x^{\pm 1},\sigma\}$ are such that

$$\#_x(w_i) = \#_\sigma(w_i) = n, \tag{8}$$

for $i = 1, 2$. The last claim holds because exactly one of z_i' and $(z_i')^\sigma$ contributes $x^{\pm 1}$ to w_0 and σ to w_1, respectively, while the other contributes the same letters to w_1 and w_0, respectively. Since n is odd, (8) shows that neither w_0 nor w_1 can be 1 and therefore g^2 cannot be 1.

Assume that H contains an element of finite order. In particular, this implies that H must contain an element of order 2. Let $g = z_1z_2\dots z_n$ be such an element of the shortest possible length, where $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$, $i = 1,\dots,n$.

Note that n must be even. Therefore,

$$g = z_1z_2\dots z_n = (z_1')^\sigma z_2' \dots (z_{n-1}')^\sigma z_n' = (w_0,w_1),$$

where w_0 and w_1 are words over $\{x^{\pm 1}, \sigma\}$. Moreover, as elements in H, the orders of w_0 and w_1 divide 2 and the order of at least one of them is 2. We claim that

$$\#_x(w_0) \equiv \#_\sigma(w_0) \equiv \#_x(w_1) \equiv \#_\sigma(w_0) \mod 2. \tag{9}$$

The congruence $\#_x(w_i) \equiv \#_\sigma(w_i) \bmod 2$ holds because $\#_x(w_i) + \#_\sigma(w_i) = n$ is even. For the other congruences, observe that whenever z_i' or $(z_i')^\sigma$ contributes $x^{\pm 1}$ or σ to w_0, respectively, it contributes σ or $x^{\pm 1}$ to w_1, respectively. Therefore $\#_x(w_0) = \#_\sigma(w_1)$ and $\#_\sigma(w_0) = \#_x(w_1)$.

If the numbers in (9) are even, then w_0 and w_1 represent elements in H and can be rewritten as words over $\{x^{\pm 1}, y^{\pm 1}\}$ of lengths at most $\#_x(w_0) = n - \#_\sigma(w_0)$ and $\#_x(w_1) = n - \#_\sigma(w_1)$, respectively. If both of these lengths are shorter than n then none of them can represent an element of order 2 in H. Otherwise, one of the words w_i is a power of x and the other is trivial. Sice x has infinite order this shows that g cannot have order 2.

If the numbers in (9) are odd, then, for $i = 1, 2$, w_i can be rewritten as σu_i, where u_i are words of odd length over $\{x^{\pm 1}, y^{\pm 1}\}$. Let $w_0 = \sigma t_1 \dots t_m$, where m is odd, and t_j are letters in $\{x^{\pm 1}, y^{\pm 1}\}$, $j = 1, \dots, m$. We have

$$w_0 = t_1'(t_2')^\sigma \dots (t_{m-1}')^\sigma t_m' = (w_{00}, w_{01}),$$

where w_{00} and w_{01} are words of odd length m over $\{x^{\pm 1}, \sigma\}$. Moreover, exactly one of the words w_{00} and w_{01} has even number of σ's and this word can be rewritten as a word over $\{x^{\pm 1}, y^{\pm 1}\}$ of odd length. However, an element in H represented by such a word cannot have order dividing 2. This completes the proof. □

Since

$$\begin{aligned}
x^a &= abaa = ab = x^{-1}, & y^a &= acabca = cbac = y^{-1}, \\
x^b &= bbab = ab = x^{-1}, & y^b &= bcabcb = bacbacab = xy^{-1}x^{-1}, \\
x^c &= cbac = y^{-1}, & y^c &= ccabcc = ab = x^{-1},
\end{aligned}$$

we see that H is the normal closure of x in G. Further, $G = \{x, y, a, c\}$ and $G = AH$. It follows from Proposition 1 that $A \cap H = 1$ (since A is finite) and therefore $G = A \ltimes H$.

Proposition 2. *The group G is a regular, weakly branch group, branching over H''.*

Proof. The group G is infinite self-similar group acting on a binary three. Therefore it is level transitive by Lemma 3.

Since

$$x^2 = (x, \sigma, \sigma, x)$$
$$y^{-1}x^2y = (y, x^{-1}\sigma x, \sigma, x)$$

we have that

$$H'' \times \langle \sigma, x^{-1}\sigma x\rangle'' \times \langle \sigma\rangle'' \times \langle x\rangle'' \preceq H''.$$

On the other hand, $\langle \sigma, x^{-1}\sigma x\rangle$ is metabelian (in fact dihedral, since the generators have order 2) and $\langle \sigma\rangle$ and $\langle x\rangle$ are abelian (cyclic). Therefore

$$H'' \times 1 \times 1 \times 1 \preceq H''.$$

The group H'' is normal in G, since it is characteristic in the normal subgroup H. Finally, H'' is not trivial. For instance it is easy to show that $[[x, y], [x, y^{-1}]] \neq 1$ (see [BGK+b]). □

776. Wreath recursion: $a = \sigma(b, a)$, $b = (c, b)$, $c = (a, a)$.

The element $(b^{-1}a)^4$ stabilizes the vertex 00 and its section at this vertex is equal to $(b^{-1}a)^{-1}$. Hence, $b^{-1}a$ has infinite order. Furthermore, by Lemma 1 ab has infinite order, which yields that a,c and b also have infinite order, because $a^2 = (ab, ba)$. Since $b^n = (c^n, b^n)$ we obtain that b^n belong to the nucleus for all $n \geq 1$. Thus G_{776} is not contracting.

We have $a^{-1}ba^{-1}c = (1, b^{-1}c), ba^{-1}ca^{-1} = (cb^{-1}, 1)$, hence by Lemma 4 the group is not free.

777. Wreath recursion: $a = \sigma(c, a)$, $b = (c, b)$, $c = (a, a)$.

The states a, c form the 2-state automaton generating $\mathbb{Z}$ (see Theorem 7). So the group is not torsion and $G_{777} = \langle a, b\rangle$. Since c has infinite order, so has b. Therefore the relation $b^n = (c^n, b^n)$ implies that b^n belong to the nucleus for all $n \geq 1$. Thus G_{777} is not contracting.

Also we have $ab^{-1} = \sigma(1, ab^{-1})$ is the adding machine. Since $a^{-3} = \sigma(1, a^3)$ elements ab^{-1} and a^{-3} generate the Brunner-Sidki-Vierra group (see [BSV99]).

779. Wreath recursion: $a = \sigma(b, b)$, $b = (c, b)$, $c = (a, a)$.

The element $(ab^{-1})^2$ stabilizes the vertex 01 and its section at this vertex is equal to $(ab^{-1})^{-1}$. Hence, ab^{-1} has infinite order.

780. Wreath recursion: $a = \sigma(c, b)$, $b = (c, b)$, $c = (a, a)$.

The element $(c^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $c^{-1}a$. Hence, $c^{-1}a$ has infinite order. Since $[c, a]\big|_{100} = (c^{-1}a)^a$ and 100 is fixed under the action of $[c, a]$ we obtain that $[c, a]$ also has infinite order. Finally, $[c, a]$ stabilizes the vertex 00 and its section at this vertex is $[c, a]$. Therefore G_{780} is not contracting.

783 $\cong G_{775} \cong C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. Wreath recursion: $a = \sigma(c, c)$, $b = (c, b)$, $c = (a, a)$.

All generators have order 2 and $ac = ca$. If we conjugate the generators of this group by the automorphism $\gamma = (c\gamma, \gamma)$ we obtain the wreath recursion

$$a' = \sigma(1, 1), \qquad b' = (c', b'), \qquad c' = (a', a'),$$

where $a' = a^\gamma$, $b' = b^\gamma$, and $c' = c^\gamma$. The same wreath recursion is obtained after conjugating G_{775} by $\mu = (a\mu, \mu)$ (where a denotes the generator of G_{775}).

Since $bca = \sigma(bca, a)$, $G_{783} = \langle acb, a, c\rangle \cong G_{2205}$.

802 $\cong C_2 \times C_2 \times C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, c)$, $c = (a, a)$.

Direct calculation.

803 $\cong G_{771} \cong \mathbb{Z}^2$. Wreath recursion: $a = \sigma(b, a)$, $b = (c, c)$, $c = (a, a)$.

The group G_{771} is finitely generated, abelian, and self-replicating. Therefore, it is free abelian [NS04]. Let $\phi : \mathrm{Stab}_{G_{803}}(1) \to G_{803}$ be the $\frac{1}{2}$-endomorphism associated to the vertex 0, given by $\phi(g) = h$, provided $g = (h, *)$. The matrix of the linear map $\mathbb{C}^3 \to \mathbb{C}^3$ induced by ϕ with to the basis corresponding to the triple $\{a, b, c\}$ is given by

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{4} - \frac{1}{4}i\sqrt{7}$ and $\lambda_3 = -\frac{1}{4} + \frac{1}{4}i\sqrt{7}$. Let v_i, $i = 1, 2, 3$, be eigenvectors corresponding to the eigenvalues λ_i, $i = 1, 2, 3$. Note that v_1 may be selected to be equal to $v_1 = (2, 1, 1)$. This shows that $a^2bc = 1$ in G_{803} and the rank of $G_{803} = \langle a, c\rangle$ is at most 2. We will prove that $a^{2m}c^n \neq 1$ (except when $m = n = 0$) by proving that iterations of the action of A eventually push the vector $v = (2m, 0, n)$ out of the set $D = \{(2i, j, k), i, j, k \in \mathbb{Z}\}$ corresponding to the first level stabilizer.

Let $v = a_1v_1 + a_2v_2 + a_3v_3$. The vector v is not a scalar multiple of v_1. Therefore either $a_2 \neq 0$ or $a_3 \neq 0$. Since $|\lambda_2| = |\lambda_3| < 1$, we have $A^t(v) = a_1v_1 + \lambda_2^t a_2v_2 + \lambda_3^t a_3v_3 \to a_1v_1$, as $t \to \infty$. Note that, since $a_2 \neq 0$ or $a_3 \neq 0$, $A^t(v)$is never equal to a_1v_1. Choose a neighborhood U of a_1v_1 that does not contain vectors from D, except possibly the vector a_1v_1. For t large enough t, the vector $A^t(v)$ is in U and is therefore outside of D.

Thus the rank of G_{803} is 2.

804 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(c, a)$, $b = (c, c)$, $c = (a, a)$.

Indeed, the states a and c form a 2-state automaton generating the cyclic group $\mathbb{Z}$ (see Theorem 7). Since $b = a^4$ we are done.

806 $\cong G_{802} \cong C_2 \times C_2 \times C_2$. Wreath recursion: $a = \sigma(b, b)$, $b = (c, c)$, $c = (a, a)$.

Direct calculation.

807 $\cong G_{771} \cong \mathbb{Z}^2$. Wreath recursion: $a = \sigma(c, b)$, $b = (c, c)$, $c = (a, a)$.

The same arguments as in the case of G_{771} show that G_{807} is free abelian. It has a relation $c^2ba^2 = 1$ and, hence, it has either rank 1 or rank 2. Analogically to G_{803} we consider a $\frac{1}{2}$-endomorphism $\phi : \mathrm{Stab}_{G_{807}}(1) \to G_{807}$, and a linear map $A : \mathbb{C}^3 \to \mathbb{C}^3$ induced by ϕ. It has the following matrix representation with respect to the basis corresponding to the triple $\{a, b, c\}$:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial $\chi_A(\lambda) = -\lambda^3 + \frac{1}{2}\lambda + \frac{1}{2}$ has three distinct complex roots $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2} - \frac{1}{2}i$ and $\lambda_3 = -\frac{1}{2} + \frac{1}{2}i$. Analogically for $v = (2m, 0, n)$ we get that $A^t(v)$ will be pushed out from the domain corresponding to $\mathrm{Stab}_{G_{807}}(1)$. Thus $c^n a^{2m} \neq 1$ in G_{807} and $G_{807} \cong \mathbb{Z}^2$.

810 $\cong G_{802} \cong C_2 \times C_2 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = (c, c)$, $c = (a, a)$.

Direct calculation.

820 $\cong D_\infty$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, a)$, $c = (b, a)$.

The states a and b form a 2-state automaton generating D_∞ (see Theorem 7) and $c = b$.

821. Lamplighter group $\mathbb{Z} \wr C_2$. Wreath recursion: $a = \sigma(b, a)$, $b = (b, a)$, $c = (b, a)$.

The states a and b form a 2-state automaton generating the Lamplighter group (see Theorem 7) and $c = b$.

824 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, a)$, $c = (b, a)$.

The states a and b form a 2-state automaton generating D_∞ (see Theorem 7) and $c = b$.

838 $\cong C_2 \ltimes \langle s, t \mid s^2 = t^2 \rangle$. Wreath recursion: $a = \sigma(a, a)$, $b = \sigma(a, b)$, $c = (b, a)$.

All generators have order 2. Consider the subgroup $H = \langle ba = \sigma(ba, 1), ca = \sigma(1, ab) \rangle \cong G_{2860} = \langle s, t \mid s^2 = t^2 \rangle$. This subgroup is normal in G_{838} because the generators have order 2. Since $G_{838} = \langle H, a \rangle$, it has a structure of a semidirect product $\langle a \rangle \ltimes H = C_2 \ltimes \langle s, t \mid s^2 = t^2 \rangle$ with the action of a on H as $(ba)^b = (ba)^{-1}$ and $(ca)^b = (ca)^{-1}$.

839 $\cong G_{821}$. Lamplighter group $\mathbb{Z} \wr C_2$. Wreath recursion: $a = \sigma(b, a)$, $b = (a, b)$, $c = (b, a)$.

The states a and b form a 2-state automaton generating the Lamplighter group (see Theorem 7). Since $b^{-1}a = \sigma = ac^{-1}$, we see that

$c = a^{-1}ba$ and $G = \langle a, b\rangle$.

840. Wreath recursion: $a = \sigma(c, a)$, $b = (a, b)$, $c = (b, a)$.

The element $(b^{-1}a)^2$ stabilizes the vertex 01 and its section at this vertex is equal to $b^{-1}a$. Hence, $b^{-1}a$ has infinite order.

The element $(c^{-1}b)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(c^{-1}b)^{-1}$. Hence, $c^{-1}b$ has infinite order. Since $(b^{-1}a^{-1}b^{-1}cba)^2|_{00000000} = c^{-1}b$ and the vertex 00000000 is fixed under the action of $(b^{-1}a^{-1}b^{-1}cba)^2$ we obtain that $b^{-1}a^{-1}b^{-1}cba$ also has infinite order. Finally, $b^{-1}a^{-1}b^{-1}cba$ stabilizes the vertex 0001 and has itself as a section at this vertex. Therefore G_{840} is not contracting.

We have $b^{-1}a^{-1}ca = (1, b^{-1}c^{-1}bc), ab^{-1}a^{-1}c = (cb^{-1}c^{-1}b, 1)$, hence by Lemma 4 the group is not free.

842 $\cong G_{838} \cong C_2 \ltimes \langle s, t \mid s^2 = t^2\rangle$. Wreath recursion: $a = \sigma(b, b)$, $b = \sigma(a, b)$, $c = (b, a)$.

All generators have order 2. Consider the subgroup $H = \langle u = ba = \sigma(1, ba) = \sigma(1, u^{-1}), v = ca = \sigma(ab, 1) = \sigma(u^{-1}, 1)\rangle$. Let us prove that $H \cong W = \langle s, t \mid s^2 = t^2\rangle$. Indeed, the relation $u^2 = v^2$ is satisfied, so H is a homomorphic image of W with respect to the homomorphism induced by $s \mapsto u$ and $t \mapsto v$. Each element of W can be written in its normal form $t^r(st)^l s^n$, $n \in \mathbb{Z}, l \geq 0, r \in \{0, 1\}$. It suffices to prove that images of these words (except for the identity word, of course) represent nonidentity elements in H.

We have $u^{2n} = (u^{-n}, u^{-n})$, $u^{2n+1} = \sigma(a^{-n}, a^{-n-1})$ for any integer n; $(uv)^l = (u^{2l}, 1)$ for any integer l. Thus

$$(uv)^l u^{2n} = (u^{-2l-n}, u^{-n}) \neq 1$$

in G if $n \neq 0$ or $l \neq 0$ since u has infinite order, as it is conjugate to the adding machine.

Furthermore,

$$v(uv)^l u^{2n} = \sigma(u^{-2l-n-1}, u^{-n}) \neq 1,$$

$$(uv)^l u^{2n+1} = \sigma(u^{-n}, u^{-2l-n-1}) \neq 1$$

since they act nontrivially on the first level of the tree.

Finally, $v(uv)^l u^{2n+1} = (u^{-2l-n-2}, u^{-n}) = 1$ if and only if $n = 0$ and $l = -1$, which is not the case, because l must be nonnegative. Thus $H \cong W$.

The subgroup H is normal in G_{842} because generators are of order 2. Since $G_{842} = \langle H, a\rangle$, it has a structure of a semidirect product $\langle a\rangle \ltimes H = C_2 \ltimes \langle s, t \mid s^2 = t^2\rangle$ with the action of a on H as $(ba)^b = (ba)^{-1}$ and $(ca)^b = (ca)^{-1}$. Therefore it has the same structure as G_{838}.

843. Wreath recursion: $a = \sigma(c, b)$, $b = (a, b)$, $c = (b, a)$.

The element $c^{-1}a = \sigma(a^{-1}c, 1)$ is a conjugate of the adding machine. Therefore, it acts transitively on the level of the tree and has infinite order.

Since $(c^{-1}ab^{-1}a)^2$ fixes the vertex 000 and its section at this vertex is equal to $c^{-1}a$, we obtain that $c^{-1}ab^{-1}a$ has infinite order. Since the element $c^{-1}ab^{-1}a$ fixes the vertex 10 and has itself as a section at this vertex, G_{843} is not contracting.

We have $c^{-1}a^{-1}ba = (1, a^{-1}c^{-1}ac)$, $ac^{-1}a^{-1}b = (ca^{-1}c^{-1}a, 1)$, hence by Lemma 4 the group is not free.

846 $\cong C_2 * C_2 * C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = (a, b)$, $c = (b, a)$.

The automaton [846] was studied during the Advanced Course on Automata Groups in Bellaterra, Spain, in the summer of 2004 and is since called the Bellaterra automaton. We present here a proof that $G_{846} = C_2 * C_2 * C_2$, based on the concept of dual automata. A different proof, still based on dual automata, is given in [Nek05].

Let $\mathcal{A} = (Q, X, \pi, \tau)$ be a finite automaton. Its *dual* automaton, by definition, is $\mathcal{A}' = (X, Q, \pi', \tau')$, where $\pi'(x, q) = \tau(q, x)$, and $\tau'(x, q) = \pi(q, x)$. Thus the dual automaton is obtained by exchanging the roles of the states and the alphabet (and the roles of the transition and output function) in a given automaton. The notion od dual automata is not new, but there is a recent renewed interest based on the new results and applications in [MNS00, GM05, BŠ06, VV05].

If in addition to $\mathcal{A}$, both $\mathcal{A}'$ and $(\mathcal{A}^{-1})'$ are invertible, the automaton $\mathcal{A}$ is called *fully invertible* (or *bi-reversible*). Examples of such automata are the automaton 2240 generating a free group with three generators [VV05], Bellaterra automaton [846], and various automata constructed in [GM05], generating free groups of various ranks.

We now consider the automaton [846] and its dual more closely. Since the generators a, b, and c have order 2, in order to prove that $G_{846} \cong C_2 * C_2 * C_2$ we need to show that no word in $w \in R_n$, $n \geq 1$, is trivial in G_{846}, where R_n is the set of reduced words over $\{a, b, c\}$ of length n (here a word is reduced if it does not contain aa, bb, or cc). For every $n > 0$, the set of words in R_n that are nontrivial in G_{846} is nonempty, since the word $r_n = acbcbcb\cdots$ of length n acts nontrivially on level 1. If we prove that the dual automaton acts transitively on the sets R_n, $n \geq 1$, this would mean that r_n is a section of every element of G_{846} that can be represented as a reduced word of length n. Therefore, every word in R_n would represent a nontrivial element in G_{846} and our proof would be complete.

The automaton dual to 846 is the invertible automaton defined by the wreath recursion

$$\begin{array}{rcl} A & = & (acb)(B, A, A), \\ B & = & (ac)(A, B, B), \end{array} \tag{10}$$

where the three coordinates in the recursion represent the sections at a, b, and c, respectively. Denote $D = \langle A, B\rangle$. The set $R = \bigcup_{n\geq 0} R_n$ of all reduced words over $\{a, b, c\}$ is a subtree of the ternary tree $\{a, b, c\}^*$ and this subtree R is invariant under the action of D (this is because the set $\{aa, bb, cc\}$ is invariant under the action of D). The structure of R is as follows. The root of R has three children a, b and c, each of which is a root of a binary tree. We want to understand the actio of D on the subtree R. It is given by

$$\begin{array}{rcl} A & = & (acb)(B_a, A_b, A_c) \\ B & = & (ac)(A_a, B_b, B_c) \end{array} \tag{11}$$

where $A_a, A_b, A_c, B_a, B_b, B_c$ are automorphisms of the binary trees hanging down from the vertices a, b and c. After identification of these three trees with the binary tree $\{0, 1\}^*$, the action of $A_a, A_b, \dots, B_c$ is defined by

$$\begin{array}{rcl} A_a & = & (A_b, A_c), \\ A_b & = & \sigma(B_a, A_c), \\ A_c & = & \sigma(B_a, A_b), \\ B_a & = & \sigma(B_b, B_c), \\ B_b & = & \sigma(A_a, B_c), \\ B_c & = & \sigma(A_a, B_b). \end{array} \tag{12}$$

Using Lemma 2 one can verify that B_b acts level transitively on the binary tree. This is sufficient to show that D acts transitively on R, since it acts transitively on the first level, B stabilizes the vertex b, and its section at b is B_b.

The fact that G_{846} is not contracting follows now from the result of Nekrashevych [Nek07a], that a contracting group can not have free subgroups. Alternatively, it is sufficient to observe that aba has infinite order, stabilizes the vertex 01 and has itself as a section at this vertex.

847 $\cong D_4$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, b)$, $c = (b, a)$.

The state b is trivial. The states a and c form a 2-state automaton generating D_4 (see Theorem 7).

848 $\cong C_2 \wr \mathbb{Z}$. Wreath recursion: $a = \sigma(b, a)$, $b = (b, b)$, $c = (b, a)$.

The state b is trivial and a is the adding machine. Every element $g \in G_{848}$ has the form $g = \sigma^i(a^n, a^m)$. On the other hand, $c = (1, a)$, $c^{ac^{-1}} = (a, 1)$, so $\mathrm{Stab}_G(1) = \{(a^n, a^m)\} \cong \mathbb{Z}^2$. Since $ac^{-1} = \sigma$ we see that $G \cong C_2 \wr \mathbb{Z}$.

849. Wreath recursion: $a = \sigma(c, a)$, b=(b,b), $c = (b, a)$.

The state b is trivial. The element $a^2c = (ac, ca^2)$ is nontrivial because its section at 0 is ac, and ac acts nontrivially on level 1. The automorphism $(a^2c)^2$ fixes the vertex 00 and its section at this vertex is equal to a^2c. Therefore a^2c has infinite order. Further, the section of a^2c at 100 coincides with a^2c, implying that G_{849} is not contracting.

The group G_{849} is regular weakly branch group over its commutator G'_{849}. This is clear since the group is self-replicating and $[a^{-1}, c] \cdot [c, a] = ([a, c], 1)$.

Conjugation of the generators of G_{849} by $\mu = \sigma(\mu, c^{-1}\mu)$ yields the wreath recursion

$$x = \sigma(yx, 1), \qquad y = (x, 1),$$

where $x = a^{\mu}$ and $y = c^{\mu}$. Further, we have

$$x = \sigma(yx, 1), \qquad yx = \sigma(yx, x),$$

and the last wreath recursion coincides with the one defining the automaton 2852. Therefore $G_{849} \cong G_{2852}$ (see G_{2852} for more information on this group).

851 $\cong G_{847} \cong D_4$. Wreath recursion: $a = \sigma(b, b)$, b=(b,b), $c = (b, a)$.

Direct calculation.

852. Basilica group $\mathcal{B} = IMG(z^2 - 1)$. Wreath recursion: $a = \sigma(c, b)$, $b = (b, b)$, $c = (b, a)$.

This group was studied in [GŻ02a], where it is shown that $\mathcal{B}$ is not a sub-exponentially amenable group, it does not contain free subgroups of rank 2, and that the monoid generated by a and b is free. Some spectral considerations are provided in [GŻ02b]. Bartholdi and Virág showed in [BV05] that $\mathcal{B}$ is amenable, distinguishing the Basilica group as the first example of an amenable group that is not sub-exponentially amenable.

855 $\cong G_{847} \cong D_4$. Wreath recursion: $a = \sigma(c, c)$, b=(b,b), $c = (b, a)$.

Direct calculation.

856 $\cong C_2 \ltimes G_{2850}$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, b)$, $c = (b, a)$.

All generators have order 2, hence $H = \langle ba, ca \rangle$ is normal in G_{856}. Furthermore, $ba = \sigma(ba, ca)$, $ca = \sigma(1, ba)$, and therefore $H = G_{2850}$. Thus $G_{856} = \langle a \rangle \ltimes H \cong C_2 \ltimes G_{2850}$, where $(ba)^a = (ba)^{-1}$ and $(ca)^a = (ca)^{-1}$. The group is not contracting since G_{2850} is not contracting.

857. Wreath recursion: $a = \sigma(b, a)$, $b = (c, b)$, $c = (b, a)$.

By using the approach used for G_{875}, we can show that the forward orbit of 10^{∞} under the action of a is infinite, and therefore a has infinite order.

Since $c = (b, a)$ and $b = (c, b)$, both b and c have infinite order and G_{857} is not a contracting group.

858. Wreath recursion: $a = \sigma(c, a)$, $b = (c, b)$, $c = (b, a)$.

The element $ab^{-1} = \sigma(1, ab^{-1})$ is the adding machine.

By using the approach used for G_{875}, we can show that the forward orbit of 10^∞ under the action of a is infinite, and therefore a has infinite order.

Since $c = (b, a)$ and $b = (c, b)$, both b and c have infinite order and G_{857} is not a contracting group.

We have $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}b^{-1}aca^{-1}b)$, $a^{-1}c^{-1}b^{-1}aba^{-1}ba = (a^{-2}b^{-1}aca^{-1}ba, 1)$, hence by Lemma 4 the group is not free.

860. Wreath recursion: $a = \sigma(b, b)$, $b = (c, b)$, $c = (b, a)$.

The element $(ba^{-1})^2$ stabilizes the vertex 11 and its section at this vertex is equal to $(ba^{-1})^{-1}$. Hence, ba^{-1} has infinite order.

Furthermore, $bc^{-1} = (cb^{-1}, ba^{-1})$ implies that the order of bc^{-1} is infinite. Since this element stabilizes vertex 00 and its section at this vertex is equal to bc^{-1}, all its powers belong to the nucleus. Thus, G_{860} is not contracting.

861. Wreath recursion: $a = \sigma(b, b)$, $b = (a, a)$, $c = (b, a)$.

The element $a^{-1}c = \sigma(1, c^{-1}a)$ is conjugate to the adding machine and has infinite order.

864. Wreath recursion: $a = \sigma(c, c)$, $b = (c, b)$, $c = (b, a)$.

The element $(ab^{-1})^2$ stabilizes the vertex 11 and its section at this vertex is equal to ab^{-1}. Hence, ab^{-1} has infinite order.

Furthermore, $cb^{-1} = (bc^{-1}, ab^{-1})$ implies that the order of cb^{-1} is infinite. Since this element stabilizes vertex 00 and its section at this vertex is equal to cb^{-1}, G_{864} is not contracting.

865 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, a), b = (a, c), c = (b, a)$.

All generators have order 2. Since $abac = (acab, 1)$ and $acab = (1, abac)$, we see that $c = aba$ and $G_{865} = \langle a, b\rangle$. The section of $(ba)^2$ at the vertex 0 is $(ba)^{-1}$, so ba has infinite order and $G_{865} \cong D_\infty$.

Note that the group is conjugate to G_{932} by the automorphism $\delta = (a\delta, \delta)$.

866. Wreath recursion: $a = \sigma(b, a)$, $b = (a, c)$, $c = (b, a)$.

The element $(c^{-1}b)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $c^{-1}b$, which is nontrivial. Hence, $c^{-1}b$ has infinite order.

The element $(b^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $b^{-1}a$. Hence, $b^{-1}a$ has infinite order. Since $b^{-1}c^{-1}ba^{-1}ba\big|_{10} = (b^{-1}a)^b$ and vertex 10 is fixed under the action of $b^{-1}c^{-1}ba^{-1}ba$ we obtain that $b^{-1}c^{-1}ba^{-1}ba$ also has infinite order. Finally, $b^{-1}c^{-1}ba^{-1}ba$ stabilizes the vertex 00 and has itself as a section at this vertex. Therefore G_{866} is not contracting.

869. Wreath recursion: $a = \sigma(b, b)$, $b = (a, c)$, $c = (b, a)$.

All generators have order 2. By Lemma 1 ab has infinite order, which implies that $babcba$ also has infinite order, because it fixes the vertex 000 and its section at this vertex is equal to ab. But $babcba$ fixed 10 and has itself as a section at this vertex. Thus, G_{869} is not contracting.
870: Baumslag-Solitar group $BS(1,3)$. Wreath recursion: $a = \sigma(c,b)$, $b = (a,c)$, $c = (b,a)$.

The automaton satisfies the conditions of Lemma 1. In particular ab has infinite order. Since $bc = (ab, ca)$, $a^2 = (bc, cb)$, we obtain that bc and a have infinite order. Since $b = (a,c)$, b also has infinite order. Since b has infinite order, fixes the vertex 10 and has itself as a section at this vertex, G_{870} is not contracting.

The element $\mu = b^{-1}a = \sigma(1, a^{-1}b) = \sigma(1, \mu^{-1})$ is conjugate to the adding machine and therefore has infinite order. Since $a^{-1}c = \sigma(1, c^{-1}a)$ we see that $a^{-1}c = \mu$. Therefore $c = ab^{-1}a$ and $G_{870} = \langle a, b\rangle = \langle \mu, b\rangle$.

We claim that $b^{-1}\mu b = \mu^3$. Since $c = ab^{-1}a$, we have

$$ab^{-1}ab^{-1}ab^{-1}a^{-1}b = (ba^{-1}bc^{-1}b^{-1}a, ca^{-1}ba^{-1}) = (ba^{-1}ba^{-1}ba^{-1}b^{-1}a, 1).$$

But $ba^{-1}ba^{-1}ba^{-1}b^{-1}a$ is a conjugate of the inverse of $ab^{-1}ab^{-1}ab^{-1}a^{-1}b$, which shows that $ab^{-1}ab^{-1}ab^{-1}a^{-1}b = 1$, and the last relation is equivalent to $b^{-1}\mu b = \mu^3$.

Since b and μ have infinite order, $G_{870} \cong BS(1,3)$.

See [BŠ06] for realizations of $BS(1,m)$ for any value of m, $m \neq \pm 1$.
874 $\cong C_2 \ltimes G_{2852}$. Wreath recursion: $a = \sigma(a,a)$, $b = (b,c)$, $c = (b,a)$.

All the generators have order 2, hence $H = \langle ba, ca\rangle$ is normal in G_{874}. Furthermore, $ba = \sigma(ca, ba)$, $ca = \sigma(1, ba)$, therefore $H = G_{2852}$. Thus $G_{874} = \langle a\rangle \ltimes H \cong C_2 \ltimes G_{2852}$, where $(ba)^a = (ba)^{-1}$ and $(ca)^a = (ca)^{-1}$. In particular, G_{874} is not contracting and has exponential growth.
875. Wreath recursion: $a = \sigma(b,a)$, $b = (b,c)$, $c = (b,a)$.

The equalities

$$a(10^\infty) = 010^\infty, \qquad b(10^\infty) = 10^\infty, \qquad c(10^\infty) = 110^\infty,$$

show that all members of the forward orbit of 10^∞ under the action of a have only finitely many 1's and that the position of the rightmost 1 cannot decrease under the action of a. Since $a(10^\infty) = 010^\infty$, the forward orbit of 10^∞ under the action of a can never return to 10^∞ and a has infinite order.

Note that the above equalities also show that no nonempty words w over $\{a, b, c\}$ satisfies a relation of the form $w = 1$ in G_{875}. First note that $c = (b,a)$ and $b = (b,c)$, implying that b and c have infinite order. Thus $b^n \neq 1$, for $n > 0$. On the other hand, for any word w that contains

a or c, $w(10^\infty) \neq 10^\infty$ (again, since the position of the rightmost 1 moves to the right and never decreases).

Since b has infinite order and $b = (b, c)$, G_{875} is not contracting.

876. Wreath recursion: $a = \sigma(c, a)$, $b = (b, c)$, $c = (b, a)$.

By Lemma 2 the elements ba and acb^2a^2cb act transitively on the levels of the tree and, hence, have infinite order. Since $(b^8)\big|_{1100001100} = acb^2a^2cb$ and vertex 1100001100 is fixed under the action of b^8 we obtain that b also has infinite order. Finally, b stabilizes the vertex 0 and has itself as a section at this vertex. Therefore G_{876} is not contracting.

We have $c^{-1}b = (1, a^{-1}c), ac^{-1}ba^{-1} = (ca^{-1}, 1)$, hence by Lemma 4 the group is not free.

878 $\cong C_2 \ltimes IMG(1 - \frac{1}{z^2})$. Wreath recursion: $a = \sigma(b, b)$, $b = (b, c)$, $c = (b, a)$.

Let $x = bc$ and $y = ca$. Since all generators have order 2, the index of the subgroup $H = \langle x, y\rangle$ in G_{878} is 2, H is normal and $G_{878} \cong C_2 \ltimes H$, where C_2 is generated by c. The action of C_2 on H is given by $x^c = x^{-1}$ and $y^c = y^{-1}$. We have $x = bc = (1, ca) = (1, y)$ and $y = ca = \sigma(ab, 1) = \sigma(y^{-1}x^{-1}, 1)$. An isomorphic copy of H is obtained by exchanging the letters 0 and 1, yielding the wreath recursion $x = (y, 1)$ and $y = \sigma(1, y^{-1}x^{-1})$. The last recursion defines $IMG(1 - \frac{1}{z^2})$ [BN06]. Thus, $G_{878} \cong C_2 \ltimes IMG(1 - \frac{1}{z^2})$.

879. Wreath recursion: $a = \sigma(c, b)$, $b = (b, c)$, $c = (b, a)$.

The element $c^{-1}a = \sigma(a^{-1}c, 1)$ is conjugate to the adding machine and has infinite order.

By Lemma 2 the element ca acts transitively on the levels of the tree and, hence, has infinite order. Since $(b^2)\big|_{1101} = ca$ and vertex 1101 is fixed under the action of b^2 we obtain that b also has infinite order. Finally, b stabilizes the vertex 0 and has itself as a section at this vertex. Therefore G_{879} is not contracting.

882. Wreath recursion: $a = \sigma(c, c)$, $b = (b, c)$, $c = (b, a)$.

The element $(ca^{-1}cb^{-1})^2$ stabilizes the vertex 00 and its section at this vertex is equal to $ca^{-1}cb^{-1}$. Hence, $ca^{-1}cb^{-1}$ has infinite order.

883 $\cong C_2 \ltimes G_{2841}$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, c)$, $c = (b, a)$.

All generators have order 2, hence $H = \langle ba, ca\rangle$ is normal in G_{883}. Furthermore, $ba = \sigma(ca, ca)$, $ca = \sigma(1, ba)$, therefore $H = G_{2841}$. Thus $G_{883} = \langle a\rangle \ltimes H \cong C_2 \ltimes G_{2841}$, where $(ba)^a = (ba)^{-1}$ and $(ca)^a = (ca)^{-1}$. In particular, G_{883} is not contracting and has exponential growth.

884. Wreath recursion: $a = \sigma(b, a)$, $b = (c, c)$, $c = (b, a)$.

The element $(b^{-1}ca^{-1}c)^2$ stabilizes the vertex 0 and its section at this vertex is equal to $(b^{-1}ca^{-1}c)^{-1}$. Hence, $b^{-1}ca^{-1}c$ has infinite order. Since $[b, a]^2\big|_{0100} = (b^{-1}ca^{-1}c)^c$ and 0100 is fixed under the action of $[b, a]^2$ we obtain that $[b, a]$ also has infinite order. Finally, $[b, a]$ stabilizes the vertex

00 and its section at this vertex is $[b,c]=[b,a]$. Therefore G_{884} is not contracting.

885. Wreath recursion: $a=\sigma(c,a)$, $b=(c,c)$, $c=(b,a)$.

The element $(c^{-1}b)^2$ stabilizes the vertex 10 and its section at this vertex is equal to $c^{-1}b$. Hence, $c^{-1}b$ has infinite order. Furthermore, $c^{-1}b$ stabilizes the vertex 00 and has itself as a section at this vertex. Therefore G_{885} is not contracting.

We have $b^{-1}aba^{-1}=(1,c^{-1}aca^{-1}), a^{-1}b^{-1}ab=(a^{-1}c^{-1}ac,1)$, hence by Lemma 4 the group is not free.

887. Wreath recursion: $a=\sigma(b,b)$, $b=(c,c)$, $c=(b,a)$.

The element $(ac^{-1})^4$ stabilizes the vertex 001 and its section at this vertex is equal to $(ac^{-1})^2$, which is nontrivial. Hence, ac^{-1} has infinite order.

888. Wreath recursion: $a=\sigma(c,b)$, $b=(c,c)$, $c=(b,a)$.

The element $a^{-1}c=\sigma(1,c^{-1}a)$ is conjugate to the adding machine and has infinite order. Since $c^{-1}b\big|_1=a^{-1}c$ and vertex 1 is fixed under the action of $c^{-1}b$ we obtain that $c^{-1}b$ also has infinite order. Finally, $c^{-1}b$ stabilizes the vertex 00 and has itself as a section at this vertex. Therefore G_{888} is not contracting.

We have $c^{-1}ab^{-1}a=(1,a^{-1}b), ac^{-1}ab^{-1}=(ca^{-1}bc^{-1},1)$, hence by Lemma 4 the group is not free.

891 $\cong C_2\ltimes(\mathbb{Z}\wr C_2)$. Wreath recursion: $a=\sigma(c,c)$, $b=(c,c)$, $c=(b,a)$.

Let $x=ac$ and $y=cb$. Since all generators have order 2, the index of the subgroup $H=\langle x,y\rangle$ in G_{891} is 2, H is normal and $G_{891}\cong C_2\ltimes H$, where C_2 is generated by c. The action of C_2 on H is given by $x^c=x^{-1}$ and $y^c=y^{-1}$.

In fact, to support the claim that H has index 2 in G_{891} we need to prove that $c\notin H$. We will prove a little bit more than that. Let $w=1$ be a relation in G_{891} where w is a word over $\{a,b,c\}$. The number of occurrences of a in w must be even (otherwise w would act nontrivially on level 1). Similarly, the number of occurrences of c in w is even. Indeed, if it were odd, then exactly one of the words w_0 and w_1 in the decomposition $w=(w_0,w_1)$ would have odd number of occurrences of the letter a, and the action of w would be nontrivial on level 2. Finally, we claim that the number of occurrences of b in w is also even. Otherwise the number of c's in both w_0 and w_1 would be odd and the action of w would be nontrivial on level 3. Thus every word over $\{a,b,c\}$ representing 1 must have even number of occurrences of each of the three letters. Note that this implies that the abelianization of G_{891} is $C_2\times C_2\times C_2$.

We now prove that H is isomorphic to the Lamplighter group $\mathbb{Z}\wr C_2$.

The group H is self-similar, which can be seen from

$$x = ac = \sigma(cb, ca) = \sigma(y, x^{-1}), \qquad y = cb = (bc, ac) = (y^{-1}, x).$$

Consider the elements $s_n = \sigma^{y^n} = y^{-n}xy^{n+1}$, $n \in \mathbb{Z}$ (note that $xy = \sigma$). For $n > 0$, we have $s_0 s_1 \cdots s_{n-1} = x^n y^n$ and $s_{-n}s_{-n+1}\cdots s_{-1} = y^n x^n$. On the other hand, $s_n = y^{-n}\sigma y^n = \sigma(x^{-n}y^{-n}, y^n x^n)$ and $s_{-n} = y^n \sigma y^{-n} = \sigma(x^n y^n, y^{-n}x^{-n})$, implying

$$s_n = \sigma(s_{-1}s_{-2}\cdots s_{-n}, s_{-n}\cdots s_{-2}s_{-1})$$

and

$$s_{-n} = \sigma(s_0 s_1 \cdots s_{n-1}, s_{n-1}\cdots s_1 s_0).$$

By induction on n we obtain that the depth of s_n is $2n+1$ for $n \geq 0$ and the depth of s_{-n} is $2n$ for $n > 0$ (*depth* of a finitary element is the lowest level at which all sections of the element are trivial). This implies that all s_i, $i \in \mathbb{Z}$ are different, have order 2 (they are conjugates of σ), and commute (for each i and each level m all sections of s_i at level m are equal). Therefore y has infinite order and $H = \langle x, y\rangle = \langle y, \sigma\rangle \cong \mathbb{Z} \wr C_2$.

Since y has infinite order, stabilizes the vertex 00 and has itself as a section at this vertex, G_{891} is not contracting.

919 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, a), b = (a, b), c = (c, a)$.

The states a, b form a 2-state automaton generating D_∞ (see Theorem 7) and $c = aba$.

920. Wreath recursion: $a = \sigma(b, a)$, $b = (a, b)$, $c = (c, a)$.

The element $(ac^{-1})^2$ stabilizes the vertex 00 and its section at this vertex is equal to ac^{-1}. Hence, ba^{-1} has infinite order.

923. Wreath recursion: $a = \sigma(b, b)$, $b = (a, b)$, $c = (c, a)$.

The states a and b form a 2-state automaton generating D_∞ (see Theorem 7).

924 $\cong G_{870}$. Baumslag-Solitar group $BS(1, 3)$. Wreath recursion: $a = \sigma(c, b)$, $b = (a, b)$, $c = (c, a)$.

This fact is proved in [BŠ06].

928 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, b)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7) and b is trivial.

929 $\cong G_{2851}$. Wreath recursion: $a = \sigma(b, a)$, $b = (b, b)$, $c = (c, a)$.

See G_{2851} for an isomorphism (in fact the groups coincide as subgroups of $\mathrm{Aut}(X^*)$).

930 $\cong G_{821}$. Lamplighter group $\mathbb{Z} \wr C_2$. Wreath recursion: $a = \sigma(c, a)$, $b = (b, b)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating the Lamplighter group (see Theorem 7) and b is trivial.

932 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(b,b)$, $b = (b,b)$, $c = (c,a)$.

We have $b = 1$ and $a^2 = c^2 = 1$. The element $ac = \sigma(c,a)$ is clearly nontrivial. Since $(ac)^2 = (ac, ca)$, this element has infinite order. Thus $G \cong D_\infty$.

933 $\cong G_{849}$. Wreath recursion: $a = \sigma(c,b)$, $b = (b,b)$, $c = (c,a)$.

See G_{2852} for an isomorphism between G_{933} and G_{2852} and G_{849} for an isomorphism between G_{2852} and G_{849}.

936 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(c,c)$, $b = (b,b)$, $c = (c,a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7) and b is trivial.

937 $\cong C_2 \ltimes G_{929}$. Wreath recursion: $a = \sigma(a,a)$, $b = (c,b)$, $c = (c,a)$.

All generators have order 2, hence $H = \langle ca, ba \rangle = \langle ca, caba \rangle$ is normal in G_{937}. Furthermore, $ca = \sigma(1, ca)$, $caba = \sigma(caba, ca)$, therefore $H = G_{929}$. Thus $G_{937} = \langle a \rangle \ltimes H \cong C_2 \ltimes G_{929}$, where $(ba)^a = (ba)^{-1}$ and $(ca)^a = (ca)^{-1}$. In particular, G_{937} is regular weakly branch over H', has exponential growth and is not contracting.

938. Wreath recursion: $a = \sigma(b,a)$, $b = (c,b)$, $c = (c,a)$.

The element $(b^{-1}a^{-1}ca)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $\left((b^{-1}a^{-1}ca)^{-1}\right)^{a^{-1}c}$. Hence, $b^{-1}a^{-1}ca$ has infinite order. Furthermore, $b^{-1}a^{-1}ca$ stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{938} is not contracting.

We have $c^{-1}b = (1, a^{-1}b), a^{-1}c^{-1}ba = (a^{-2}ba, 1)$, hence by Lemma 4 the group is not free.

939. Wreath recursion: $a = \sigma(c,a)$, $b = (c,b)$, $c = (c,a)$.

The states a and c form a 2-state automaton generating the Lamplighter group (see Theorem 7). Hence, G_{939} is neither torsion, nor contracting, and has exponential growth.

941. Wreath recursion: $a = \sigma(b,b)$, $b = (c,b)$, $c = (c,a)$.

The second iteration of the wreath recursion is

$$a = (02)(13)(c,b,c,b), \qquad b = (c,a,c,b), \qquad c = (23)(c,a,b,b).$$

Conjugation by $g = (cg, g, g, bg)$ gives the wreath recursion

$$a' = (02)(13), \qquad b = (c', a', c', b'), \qquad c = (23)(c', a', 1, 1),$$

where $a' = a^g$, $b' = b^g$, and $c' = c^g$. The last recursion coincides with the second iteration of the recursion

$$\alpha = \sigma, \qquad \beta = (\gamma, \beta), \qquad \gamma = (\gamma, \alpha).$$

Conjugating the last recursion by $h = (\gamma h, h)$ yields the recursion defining G_{945}. Thus, $G_{941} \cong G_{945} \cong C_2 \ltimes IMG(z^2 - 1)$ (see G_{945}). The limit space is half of the Basilica.

942. Wreath recursion: $a = \sigma(c, b)$, $b = (c, b)$, $c = (c, a)$.

The Lamplighter group $L = \mathbb{Z} \wr C_2$ can be defined as the group generated by a' and b' given by the wreath recursion (see Theorem 7)

$$\begin{aligned} a' &= \sigma(a', b'), \\ b' &= \quad (a', b'). \end{aligned}$$

Let $H = \langle a, b\rangle \le G_{942}$. We will show that H and L are isomorphic. Let Y^* be the subtree of X^* consisting of all words over the alphabet $Y = \{01, 11\}$. The element b fixes the letter in Y, while a swaps them. Since $a_{01} = b_{01} = a$, $a_{11} = b_{11} = b$, the tree Y^* is invariant under the action of H. Moreover, the action of H on Y^* coincides with the action of the Lamplighter group $L = \langle a', b'\rangle$ on X^* (after the identification $01 \leftrightarrow 0$, $11 \leftrightarrow 1$). This implies that the map $\phi : H \to L$ given by $a \mapsto a'$, $b \mapsto b'$ can be extended to a homomorphism. We claim that this homomorphism is in fact an isomorphism. Let $w = w(a, b)$ be a group word representing an element of the kernel of ϕ. Since $w(a', b')$ represents the identity in the lamplighter group L, the total exponent of a in w must be even and the total exponent ε of both a and b in w must be 0. Therefore the element $g = w(a, b)$ stabilizes the top two levels of the tree X^* and can be decomposed as

$$g = (c^{\varepsilon}, *, c^{\varepsilon}, *),$$

where the $*$'s are words over a and b representing the identity in H (these words correspond precisely to the first level sections of $w(a', b')$ in L). Since $\varepsilon = 0$, we see that $g = 1$ and the kernel of ϕ is trivial.

Thus, the Lamplighter group is a subgroup of G_{942}, which shows that G_{942} is not a torsion group, it is not free, and has exponential growth. Since $b = (c, b)$ and b has infinite order, G_{942} is not a contracting group.

945 $\cong G_{941} \cong C_2 \ltimes IMG(z^2 - 1)$. Wreath recursion: $a = \sigma(c, c)$, $b = (c, b)$, $c = (c, a)$.

All generators have order 2. Since $ab = \sigma(1, cb)$ and $cb = (1, ab)$ we see that $H = \langle ab, cb\rangle \cong G_{852} = IMG(z^2 - 1)$. This subgroup is normal in G_{945} because the generators have order 2. Since $G_{945} = \langle H, b\rangle$, it has a structure of a semidirect product $\langle b\rangle \ltimes H = C_2 \ltimes IMG(z^2 - 1)$ with the action of b on H given by $(ab)^b = (ab)^{-1}$ and $(cb)^b = (cb)^{-1}$. It follows that G_{945} is regular weakly branch over H' and has exponential growth. See G_{941} for an isomorphism.

955 $\cong G_{937} \cong C_2 \ltimes G_{929}$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, c)$, $c = (c, a)$.

All generators have order 2. Consider the subgroup $H = \langle ba = \sigma(ca, ba), ca = \sigma(1, ca)\rangle \cong G_{929}$. This subgroup is normal in G_{955} because all generators have order 2. Since $G_{955} = \langle H, a\rangle$, it has a structure of a semidirect product $\langle a\rangle \ltimes H = C_2 \ltimes G_{929}$ with the action of a on H given by $(ba)^b = (ba)^{-1}$ and $(ca)^b = (ca)^{-1}$. It is proved above that G_{937} has the same structure. It follows that G_{955} is regular weakly branch over H' and has exponential growth.

956. Wreath recursion: $a = \sigma(b,a)$, $b = (b,c)$, $c = (c,a)$.

The element $(c^{-1}b)^2$ stabilizes the vertex 10 and its section at this vertex is equal to $(c^{-1}b)^{-1}$. Hence, $c^{-1}b$ has infinite order. Furthermore, $c^{-1}b$ stabilizes the vertex 0 and has itself as a section at this vertex. Therefore G_{956} is not contracting.

We have $c^{-1}b^{-1}aba^{-1}b = (1, a^{-1}c^{-1}aba^{-1}c), a^{-1}c^{-1}b^{-1}aba^{-1}ba = (a^{-2}c^{-1}aba^{-1}ca, 1)$, hence by Lemma 4 the group is not free.

957. Wreath recursion: $a = \sigma(c,a)$, $b = (b,c)$, $c = (c,a)$.

The states a, c form a 2-state automaton generating the Lamplighter group (see Theorem 7). Hence, G_{957} is neither torsion, nor contracting and has exponential growth.

959. Wreath recursion: $a = \sigma(b,b)$, $b = (b,c)$, $c = (c,a)$.

The element $(a^{-1}c)^4$ stabilizes the vertex 00 and its section at this vertex is equal to $(a^{-1}c)^{-1}$. Hence, $a^{-1}c$ has infinite order.

Furthermore, since $c^{-1}b = (c^{-1}b, a^{-1}c)$, this element also has infinite order. Thus, G_{959} is not contracting.

960. Wreath recursion: $a = \sigma(c,b)$, $b = (b,c)$, $c = (c,a)$.

Define $x = ac^{-1}$, $y = ba^{-1}$ and $z = cb^{-1}$. Then $x = \sigma(1,y)$, $y = \sigma(z,z^{-1})$ and $z = (z,x)$.

The element $(zxy)^8$ stabilizes the vertex 001010 and its section at this vertex is equal to $xy^{-1}z = xyz = (zxy)^{z^{-1}}$ (since $y^2 = 1$). Hence, zxy has infinite order.

Denote $t = (b^{-1}c)^4(b^{-1}a)(c^{-1}a)^5(b^{-1}c)$. Then t^2 stabilizes the vertex 00 and $t^2\big|_{00} = t^{b^{-1}c}$. Hence, t has infinite order. Let $s = c^{-2}b^2$. Since $s^{32}\big|_{111000000100} = t^c$ and s^{32} fixes 111000000100, we obtain that s also has infinite order. Finally, s stabilizes the vertex 00 and has itself as a section at this vertex. Therefore G_{960} is not contracting.

963. Wreath recursion: $a = \sigma(c,c)$, $b = (b,c)$, $c = (c,a)$.

All generators have order 2. The element $ac = \sigma(1,ca)$ is conjugate to the adding machine and has infinite order.

Furthermore, since $cb = (cb, ac)$, this element also has infinite order. Thus, G_{963} is not contracting.

964 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(a,a)$, $b = (c,c)$, $c = (c,a)$.

All generators have order 2. The elements $u = acba = (ca, 1)$ and $v = bc = (1, ca)$ generate $\mathbb{Z}^2$ because $ca = \sigma(1, ca)$ is the adding machine and has infinite order. We have $cacb = \sigma$ and $\langle u, v\rangle$ is normal in $H = \langle u, v, \sigma\rangle$

because $u^\sigma = v$ and $v^\sigma = u$. In other words, $H \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z}) = C_2 \wr \mathbb{Z}$.

Furthermore, $G_{964} = \langle H, a\rangle$ and H is normal in G_{972} because $u^a = v^{-1}$, $v^a = u^{-1}$ and $\sigma^a = \sigma$. Thus $G_{964} = C_2 \ltimes (C_2 \wr \mathbb{Z})$, where the action of C_2 on H is specified above and coincides with the one in G_{739}. Therefore $G_{964} \cong G_{739}$.

965. Wreath recursion: $a = \sigma(b, a)$, $b = (c, c)$, $c = (c, a)$.

The element $(ac^{-1})^2$ stabilizes the vertex 01 and its section at this vertex is equal to $(ac^{-1})^{-1}$. Hence, ac^{-1} has infinite order.

By Lemma 2 the element a acts transitively on the levels of the tree and, hence, has infinite order. Since $c = (c, a)$ we obtain that c also has infinite order. Therefore G_{965} is not contracting.

We have $bc^{-1} = (1, ca^{-1}), a^{-1}bc^{-1}a = (a^{-1}c, 1)$, hence by Lemma 4 the group is not free.

966. Wreath recursion: $a = \sigma(c, a)$, $b = (c, c)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating the Lamplighter group (see Theorem 7). Hence, G_{966} is neither torsion, nor contracting, and has exponential growth.

Since $b = (c, c)$ we obtain that G_{966} can be embedded into the wreath product $C_2 \wr (\mathbb{Z} \wr \mathbb{C}_2)$. This shows that G_{966} is solvable.

968. Wreath recursion: $a = \sigma(b, b)$, $b = (c, c)$, $c = (c, a)$.

We will show that this group contains $\mathbb{Z}^5$ as a subgroup of index 16. It is a contracting group, with nucleus consisting of 73 elements (the self-similar closure of the nucleus consists of 77 elements).

All generators have order 2. Let $x = (ac)^2$, $y = bcba$, and $K = \langle x, y\rangle$. Conjugating x and y by $\gamma = (b\gamma, a\gamma)$ yields the self-similar copy K' of K generated by $x' = ((y')^{-1}, (y')^{-1})$ and $y = \sigma(x', y')$, where $x' = x^\gamma$ and $y' = y^\gamma$. Since $[x', y'] = ([x', y']^{(y')^{-1}}, 1)$ K' is abelian. The matrix of the corresponding virtual endomorphism is given by

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues $\lambda = \frac{1}{4} \pm \frac{1}{4}\sqrt{7}i$ of this matrix are not algebraic integers. Therefore K' (ad therefore K as well) is free abelian of rank 2, by the results in [NS04].

The subgroup $H = \langle ab, bc\rangle$ has index 2 in G_{968} (the generators of G_{968} have order 2). The second level stabilizer $\mathrm{Stab}_H(2)$ has index 8 in H (the quotient group is isomorphic to the dihedral group D_4). The stabilizer $\mathrm{Stab}_H(2)$, is generated by $(bc)^2$, $\left((bc)^2\right)^{ba}$, $(ab)^2$, $\left((ab)^2\right)^{bc}$, $\left((ab)^2\right)^{(bc)^{ba}}$, and $\big((ab)^2\big)^{bc(bc)^{ba}}$. Conjugating these elements by $g = (b, c, b, 1)$ gives

$$
\begin{array}{lll}
g_1 = \big((bc)^2\big)^g & = (bcbc)^g = & (1,\ 1,\ y,\ y^{-1}\ ),\\
g_2 = \big((bc)^2\big)^{bag} & = (acbcba)^g = & (y,\ y,\ 1,\ 1\ ),\\
g_3 = \big((ab)^2\big)^{bcg} & = (cbabac)^g = & (1,\ x,\ x,\ 1\ ),\\
g_4 = \big((ab)^2\big)^g & = (abab)^g = & (1,\ x,\ 1,\ x^{-1}\ ),\\
g_5 = \big((ab)^2\big)^{(bc)^{ba}g} & = (abcbabacba)^g = & (x,\ 1,\ 1,\ x^{-1}\ ),\\
g_6 = \big((ab)^2\big)^{bc(bc)^{ba}g} & = (abcacbabacacba)^g = & (x,\ 1,\ x,\ 1\ ).
\end{array}
$$

Therefore, $\mathrm{Stab}_H(2)$ is abelian and $g_6 = g_5 g_3 g_4^{-1}$. If $\prod_{i=1}^5 g_i^{n_i} = 1$, then $x^{n_5}y^{n_2} = x^{n_3+n_4}y^{n_2} = x^{n_3}y^{n_1} = x^{n_4+n_5}y^{n_1} = 1$. Since K is free abelian, we obtain $n_i = 0$, $i = 1, \dots, 5$. Therefore $\mathrm{Stab}_H(2)$ is a free abelian group of rank 5.

969. Wreath recursion: $a = \sigma(c, b)$, $b = (c, c)$, $c = (c, a)$.

The element $(cb^{-1})^4$ stabilizes the vertex 100 and its section at this vertex is equal to cb^{-1}. Hence, cb^{-1} has infinite order.

We have $bc^{-1} = (1, ca^{-1})$, $ca^{-1} = \sigma(ab^{-1}, 1)$, $ab^{-1} = \sigma(1, bc^{-1})$, hence the subgroup generated by these elements is isomorphic to $IMG(1 - \frac{1}{z^2})$ (see [BN06]).

We also have $c^{-1}b = (1, a^{-1}c), a^{-1}c^{-1}ba = (b^{-1}a^{-1}cb, 1)$, hence by Lemma 4 the group is not free.

972 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion : $a = \sigma(c, c)$, $b = (c, c)$, $c = (c, a)$.

All generators have order 2. The elements $u = acba = (ca, 1)$ and $v = bc = (1, ac)$ generate $\mathbb{Z}^2$ because $ca = \sigma(ac, 1)$ is conjugate to the adding machine and has infinite order. Also we have $ba = \sigma$ and $\langle u, v\rangle$ is normal in $H = \langle u, v, \sigma\rangle$ because $u^\sigma = v$ and $v^\sigma = u$. In other words, $H \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z}) = C_2 \wr \mathbb{Z}$.

Furthermore, $G_{972} = \langle H, a\rangle$ and H is normal in G_{972} because $u^a = v^{-1}$, $v^a = u^{-1}$ and $\sigma^a = \sigma$. Thus $G_{972} = C_2 \ltimes (C_2 \wr \mathbb{Z})$, where the action of C_2 on H is specified above and coincides with the one in G_{739}. Therefore $G_{972} \cong G_{739}$.

1090 $\cong C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (b, b)$, $c = (b, b)$.

Both b and c are trivial and $a^2 = 1$.

1091 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(b, a)$, $b = (b, b)$, $c = (b, b)$.

Both b and c are trivial and a is the adding machine.

1094 $\cong G_{1090} \cong C_2$. Wreath recursion: $a = \sigma(b, b)$, $b = (b, b)$, $c = (b, b)$.

Both b and c are trivial and $a^2 = 1$.

2190 $\cong G_{848} \cong C_2 \wr \mathbb{Z}$. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(a, a)$, $c = (a, a)$.

First note that $c = a^{-2}$. Therefore $G = \langle a, b\rangle$, where $a = \sigma(a^{-2}, a)$, and $b = \sigma(a, a)$. Also, a has infinite order.

Consider the subgroup $H = \langle ba, ab\rangle < G$. The generators of H commute since $ba = (a^{-1}, a^2)$ and $ab = (a^2, a^{-1})$. Furthermore, $(ba)^n(ab)^m = (a^{-n+2m}, a^{2n-m}) = 1$ if and only if $m = n = 0$. Therefore $H \cong \mathbb{Z}^2$.

Consider the element $ba^2 = bc^{-1} = \sigma$. This element does not belong to H, since H stabilizes the first level of the tree. On the other hand $a = (ba)^{-1}ba^2 = (ba)^{-1}\sigma$ and $b = a^{-1}(ab)$ so $G = \langle\sigma, H\rangle$. Finally, $(ba)^\sigma = ab$ and $(ab)^\sigma = ba$ implies that H is normal in G and $G = C_2 \wr H \cong C_2 \wr \mathbb{Z} \cong G_{848}$.

Also note that $\langle a, a^b\rangle = G_{2212} \cong \mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$.

2193. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(a, a)$, $c = (a, a)$.

Let $x = ca^{-1}$ and $y = ab^{-1}$. Then $x = \sigma(ab^{-1}, ac^{-1}) = \sigma(y, x^{-1})$ and $y = (ba^{-1}, ca^{-1}) = (y^{-1}, x)$. It is already shown (see G_{891}), that $\langle x, y\rangle$ is not contracting and is isomorphic to the Lamplighter group. Therefore G_{2193} is not a torsion group, it is not contracting, and has exponential growth.

2196 $\cong G_{802} \cong C_2 \times C_2 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(a, a)$, $c = (a, a)$.

Direct calculation.

2199. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, a)$, $c = (a, a)$.

By Lemma 2 the element ac acts transitively on the levels of the tree and, hence, has infinite order. Since $ba = (ac, ba)$ we obtain that ba also has infinite order. Therefore G_{2199} is not contracting.

We have $b^{-2}abcba = b^{-2}aba^{-2}ba = 1$, and a and b do not commute, hence the group is not free.

2202. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(b, a)$, $c = (a, a)$.

The element $(b^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $b^{-1}a$. Hence, $b^{-1}a$ has infinite order. Furthermore, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2202} is not contracting.

We have $cb^{-1}c^{-1}b = (1, ab^{-1}a^{-1}b), bcb^{-1}c^{-1} = (bab^{-1}a^{-1}, 1)$, hence by Lemma 4 the group is not free.

2203. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(b, a)$, $c = (a, a)$.

The states a and c form a 2-state automaton generating the infinite cyclic group $\mathbb{Z}$ in which $c = a^{-2}$ (see Theorem 7).

Since $b^{-1}a\big|_1 = a^{-1}c$ and vertex 1 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 0 and has itself as a section at this vertex. Therefore G_{2203} is not contracting.

We have $c^{-2}ab = (1, a^{-2}cb), bc^{-2}a = (ba^{-2}c, 1)$, hence by Lemma 4 the group is not free.

2204. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(b, a)$, $c = (a, a)$.

The element $(b^{-1}ac^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $b^{-1}ac^{-1}a$. Hence, $b^{-1}ac^{-1}a$ has infinite order. Since $[c,a]^2\big|_{000}= (b^{-1}ac^{-1}a)^{a^{-1}cb}$ and 000 is fixed under the action of $[c,a]^2$ we obtain that $[c,a]$ also has infinite order. Finally, $[c,a]$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2204} is not contracting.

We have $ab^{-1} = (1, ca^{-1}), b^{-1}a = (a^{-1}c, 1)$, hence by Lemma 4 the group is not free.

2205 $\cong G_{775} \cong C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. Wreath recursion: $a = \sigma(c,c)$, $b = \sigma(b,a)$, $c = (a,a)$.

See G_{783} for an isomorphism between G_{783} and G_{2205}.

2206 $\cong G_{748} \cong D_4 \times C_2$. Wreath recursion: $a = \sigma(a,a)$, $b = \sigma(c,a)$, $c = (a,a)$.

Direct calculation.

2207. Wreath recursion: $a = \sigma(b,a)$, $b = \sigma(c,a)$, $c = (a,a)$.

The element $(c^{-1}a)^4$ stabilizes the vertex 000 and its section at this vertex is equal to $c^{-1}a$. Hence, $c^{-1}a$ has infinite order.

Since $b^{-1}a^{-1}b^{-1}aba\big|_{001}= (c^{-1}a)^a$ and the vertex 001 is fixed under the action of $b^{-1}a^{-1}b^{-1}aba$ we obtain that $b^{-1}a^{-1}b^{-1}aba$ also has infinite order. Finally, $b^{-1}a^{-1}b^{-1}aba$ stabilizes the vertex 000 and has itself as a section at this vertex. Therefore G_{2207} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2209. Wreath recursion: $a = \sigma(a,b)$, $b = \sigma(c,a)$, $c = (a,a)$.

The element $(b^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(b^{-1}a)^{-1}$. Hence, $b^{-1}a$ has infinite order. Furthermore, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2209} is not contracting.

We have $aca^{-2}c^{-1}acac^{-1}a^{-2}cac^{-1} = 1$, and a and c do not commute, hence the group is not free.

2210. Wreath recursion: $a = \sigma(b,b)$, $b = \sigma(c,a)$, $c = (a,a)$.

The element $(a^{-1}c)^2$ stabilizes the vertex 000 and its section at this vertex is equal to $a^{-1}c$. Hence, $a^{-1}c$ has infinite order. Since $(b^{-1}a)^2\big|_{00}= a^{-1}c$ and 00 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2210} is not contracting.

We have $c^{-1}b^{-1}cb = (1, a^{-1}c^{-1}ac), bc^{-1}b^{-1}c = (ca^{-1}c^{-1}a, 1)$, hence by Lemma 4 the group is not free.

2212. Klein bottle group, $\langle a, b \mid a^2 = b^2 \rangle$. Wreath recursion: $a = \sigma(a,c)$, $b = \sigma(c,a)$, $c = (a,a)$.

The states a and c form a 2-state automaton generating the infinite cyclic group $\mathbb{Z}$ in which $c = a^{-2}$ (see Theorem 7).

We have $a = \sigma(a, a^{-2})$, $b = \sigma(a^{-2}, a)$, and $x = ab^{-1} = (a^{-3}, a^3)$. Finally, since $x^a = b^{-1}a = (a^3, a^{-3}) = x^{-1}$, we have $G_{2212} = \langle x, a \mid x^a = x^{-1}\rangle$ and G_{2212} is the Klein bottle group. Tietze transformations yield the presentation $G_{2212} = \langle a, b \mid a^2 = b^2\rangle$ in terms of the generators a and b.

2213. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, a)$, $c = (a, a)$.

By Lemma 2 the element cb acts transitively on the levels of the tree and, hence, has infinite order. Since $(ba)\big|_{100} = cb$ and the vertex 100 is fixed under the action of ba we obtain that ba also has infinite order. Finally, ba stabilizes the vertex 01 and has itself as a section at this vertex. Therefore G_{2213} is not contracting.

We have $c^{-1}b^{-1}cb = (1, a^{-1}c^{-1}ac), bc^{-1}b^{-1}c = (ca^{-1}c^{-1}a, 1)$, hence by Lemma 4 the group is not free.

2214 $\cong G_{748} \cong D_4 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, a)$, $c = (a, a)$.

Direct calculation.

2226 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, b)$, and $c = (a, a)$.

We have $ba = (bc, ba)$, $bc = \sigma(ba, ba)$, and $b = \sigma(b, b)$. Therefore x, y and b satisfy the wreath recursion defining the automaton $\mathcal{A}_{2394}$. Thus $G_{2226} = G_{2394} \cong G_{820}$.

2229. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(b, b)$, $c = (a, a)$.

Note that b is of order 2. Post-conjugating the recursion by $(1, b)$ (which is equivalent to conjugating by the tree automorphism $g = (g, bg)$ in $\mathrm{Aut}(X^*)$ gives a copy of G_{2229} defined by

$$a = \sigma(bc, 1), \qquad b = \sigma, \qquad c = (a, bab)$$

The stabilizer of the first level is generated by

$$a^2 = (bc, bc), \qquad c = (a, bab), \qquad ba = (bc, 1), \qquad bcb = (bab, a).$$

Its projection on the first level is generated by

$$bc = \sigma(a, bab), \qquad a = \sigma(bc, 1), \qquad bab = \sigma(1, bc).$$

Furthermore,

$$bcbc = (baba, abab), \qquad abab = (1, bcbc), \qquad baba = (bcbc, 1),$$

which implies that bc is of order 2 and $a^{-1} = bab$. Hence, the projection of the stabilizer on the first level is generated by the recursion

$$a = \sigma(bc, 1), \qquad bc = \sigma(a, a^{-1}).$$

Post-conjugating by $(1,a)$, we obtain the recursion

$$a = \sigma(a^{-1} \cdot bc, a), \qquad bc = \sigma,$$

which is the group $C_4 \ltimes \mathbb{Z}^2$ of all orientation preserving automorphisms of the integer lattice (see [BN06]). Note that the nucleus of G_{2229} consists of 52 elements.

2232 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(c,c)$, $b = \sigma(b,b)$, $c = (a,a)$.

Direct calculation.

2233. Wreath recursion: $a = \sigma(a,a)$, $b = \sigma(c,b)$, $c = (a,a)$.

Therefore, $\langle ba = (ba, ca), ca = \sigma\rangle = G_{932} \cong D_\infty$.

Conjugating by $g = (ag, g)$, we obtain the recursion

$$\alpha = \sigma, \qquad \beta = \sigma(\gamma\beta, \alpha\beta), \qquad \gamma = (\alpha, \alpha),$$

where $\alpha = a^g$, $\beta = b^g$, and $\gamma = c^g$. Therefore

$$\alpha = \sigma, \qquad \alpha\beta = (\gamma\alpha, \alpha\beta), \qquad \gamma\alpha = \sigma(\alpha, \alpha),$$

and the last wreath recursion defines a bounded automaton (see Section 3 for a definition). It follows from [BKN] that G_{2233} is amenable.

2234. Wreath recursion: $a = \sigma(b,a)$, $b = \sigma(c,b)$, $c = (a,a)$.

The element $(c^{-1}b)^4$ stabilizes the vertex 00 and its section at this vertex is equal to $(c^{-1}b)^{-1}$. Hence, $c^{-1}b$ has infinite order. Since $(b^{-1}a)\big|_0 = c^{-1}b$ and 0 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{2234} is not contracting.

We have $c^{-1}b^{-1}ac^{-1}a^2 = (1, a^{-1}c^{-1}b^2), ac^{-1}b^{-1}ac^{-1}a = (ba^{-1}c^{-1}b, 1)$, hence by Lemma 4 the group is not free.

2236. Wreath recursion: $a = \sigma(a,b)$, $b = \sigma(c,b)$, $c = (a,a)$.

By Lemma 2 the element b acts transitively on the levels of the tree and, hence, has infinite order.

By Lemma 2 the element cb acts transitively on the levels of the tree and, hence, has infinite order. Since $ba = (ba, cb)$ we obtain that ba also has infinite order. Since ba has itself as a section at 0 the group is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2237. Wreath recursion: $a = \sigma(b,b)$, $b = \sigma(c,b)$, $c = (a,a)$.

By Lemma 2 the elements b and $(bc)^3$ acts transitively on the levels of the tree and, hence, have infinite order.

Since $(cba)^2\big|_{00000} = (bc)^3$ and 00000 is fixed under the action of $(cba)^2$ we obtain that cba also has infinite order. Finally, cba stabilizes the vertex 101 and has itself as a section at this vertex. Therefore G_{2237} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2239. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, b)$, $c = (a, a)$.

The group contains elements of infinite order by Lemma 1. In particular, ca has infinite order. Since $(ba)\big|_{100} = ca$ and the vertex 100 is fixed under the action of ba we obtain that ba also has infinite order. Finally, ba stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{2239} is not contracting.

We have $ca^{-2}cba^{-1} = (1, c^{-1}abc^{-1}), a^{-1}ca^{-2}cb = (c^{-2}ab, 1)$, hence by Lemma 4 the group is not free.

We can also simplify the wreath recursion in the following way. Since $c = a^{-2}$ we have

$$a = \sigma(a, a^{-2}), \qquad b = \sigma(a^{-2}, b).$$

Therefore

$$ab = (a^{-4}, ab), \qquad a = \sigma(a, a^{-2}),$$

which can be written as

$$ab = (a^{-4}, ab), \qquad a = \sigma(1, a^{-1}),$$

which is a subgroup of

$$\beta = (a, \beta), \qquad a = \sigma(1, a^{-1}).$$

2240. Free group of rank 3. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, b)$, $c = (a, a)$.

The automaton appeared for the first time in [Ale83]. The fact that G_{2240} is free group of rank 3 with basis $\{a, b, c\}$ is proved in [VV05]. This is the smallest automaton among all automata over a 2-letter alphabet generating a free nonabelian group.

The fact that G_{2240} is not contracting follows now from the result of Nekrashevych [Nek07a], that a contracting group cannot have free subgroups. Alternatively, $b^{-1}ca$ has infinite order, stabilizes the vertex 11 and has itself as a section at this vertex. Hence, the group is not contracting.

2241 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, b)$, $c = (a, a)$.

Consider G_{747}. Its wreath recursion is given by $a = \sigma(c, c)$, $b = (b, a)$, $c = (a, a)$. All generators have order 2 and a commutes with c. Therefore

$acb = \sigma(cab, c) = \sigma(acb, c)$ and wa have $G_{747} = \langle a, acb, c\rangle = G_{2241}$. Thus $G_{2241} = G_{747} \cong G_{739}$.

2260 $\cong G_{802} \cong C_2 \times C_2 \times C_2$. Wreath recursion: $a = \sigma(a, a)$, $b = (c, c)$, $c = (a, a)$.

Direct calculation.

2261. Wreath recursion: $a = \sigma(b, a)$, $b = \sigma(c, c)$, $c = (a, a)$.

The element $(ac^{-1})^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(ac^{-1})^{-1}$. Hence, ac^{-1} and $c^{-1}a$ have infinite order.

Since $b^{-1}c^{-1}ac^{-1}ba\big|_{001} = \left((c^{-1}a)^2\right)^a$ and the vertex 001 is fixed under the action of $b^{-1}c^{-1}ac^{-1}ba$ we obtain that $b^{-1}c^{-1}ac^{-1}ba$ also has infinite order. Finally, $b^{-1}c^{-1}ac^{-1}ba$ stabilizes the vertex 000 and has itself as a section at this vertex. Therefore G_{2261} is not contracting.

We have $acac^{-1}a^{-2}cac^{-1}aca^{-2}c^{-1} = 1$, and a and c do not commute, hence the group is not free.

2262 $\cong G_{848} \cong C_2 \wr \mathbb{Z}$. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(c, c)$, $c = (a, a)$.

The states a and c form a 2-state automaton (see Theorem 7). Moreover, $c = a^{-2}$ and a has infinite order.

Thus $a = \sigma(a^{-2}, a)$, $b = \sigma(a^{-2}, a^{-2})$ and $G_{2262} = \langle a, b\rangle$. Further, $b^{-1}a = (1, a^3)$ and $a^{-3} = \sigma(1, a^3)$, yielding $a^{-4}b = \sigma$. Therefore $G = \langle a, \sigma\rangle$. Since $\langle a, a^\sigma\rangle = \mathbb{Z}^2$, we obtain that $G_{2262} \cong C_2 \wr Z^2 \cong G_{848}$.

2264 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(b, b)$, $b = \sigma(c, c)$, $c = (a, a)$.

Direct calculation.

2265. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(c, c)$, $c = (a, a)$.

The element $(c^{-1}b)^4$ stabilizes the vertex 0000 and its section at this vertex is equal to $\left((c^{-1}b)^{-1}\right)^{c^{-1}a}$. Hence, $c^{-1}b$ has infinite order. Since $[c, a]\big|_{10} = (c^{-1}b)^c$ and 10 is fixed under the action of $[c, a]$ we obtain that $[c, a]$ also has infinite order. Finally, $[c, a]$ stabilizes the vertex 00 and has itself as a section at this vertex. Therefore G_{2265} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2271. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(a, a)$, $c = (b, a)$.

The element $(ac^{-1})^4$ stabilizes the vertex 001 and its section at this vertex is equal to ac^{-1}. Hence, ac^{-1} has infinite order.

The element $(a^{-1}b)^4$ stabilizes the vertex 000 and its section at this vertex is equal to $a^{-1}b$. Hence, $a^{-1}b$ has infinite order. Since $b^{-1}c^{-1}ac^{-1}a^2\big|_{001} = (a^{-1}b)^a$ and the vertex 001 is fixed under the action of $b^{-1}c^{-1}ac^{-1}a^2$ we obtain that $b^{-1}c^{-1}ac^{-1}a^2$ also has infinite order. Finally, $b^{-1}c^{-1}ac^{-1}a^2$ stabilizes the vertex 000 and has itself as a section at this vertex. Therefore G_{2271} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2274. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(a, a)$, $c = (b, a)$.

The element $a^{-1}c = \sigma(1, c^{-1}a)$ is conjugate to the adding machine and has infinite order. Since $(b^{-1}a)\big|_0 = a^{-1}c$ and 0 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2274} is not contracting.

We have $bc^{-2}b = (1, ab^{-2}a), b^2c^{-2} = (a^2b^{-2}, 1)$, hence by Lemma 4 the group is not free.

2277 $\cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(a, a)$, $c = (b, a)$.

All generators have order 2. Let $x = cb$, $y = ab$ and $H = \langle x, y\rangle$. We have $x = \sigma(1, y^{-1})$ and $y = (xy^{-1}, xy^{-1})$. The elements x and y commute and the matrix of the associated virtual endomorphism is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1/2 & -1 \end{pmatrix}.$$

The eigenvalues $-\frac{1}{2} \pm \frac{1}{2}i$ are not algebraic integers, and therefore, according to [NS04], H is free abelian of rank 2.

The subgroup H is normal of index 2 in G_{2277}. Therefore $G_{2277} = \langle H, b\rangle = C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$, where C_2 is generated by b, which acts on H is inversion of the generators.

2280. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, a)$, $c = (b, a)$.

We prove that a has infinite order by considering the forward orbit of 10^∞ under the action of a^2. We have

$$\begin{aligned} a^2 &= (ac, ca), & ac &= \sigma(cb, a^2), & ca &= \sigma(ac, ba) \\ cb &= \sigma(ab, ba), & ba &= (ac, ba), & ab &= (ab, ca). \end{aligned}$$

The equalities

$$\begin{aligned} &a^2(10^\infty) = ab(10^\infty) = 1110^\infty, \\ &ac(10^\infty) = ca(10^\infty) = cb(10^\infty) = 0010^\infty, \text{ and} \\ &ba(10^\infty) = 10110^\infty \end{aligned}$$

show that all members of the forward orbit of 10^∞ under the action of a^2 have only finitely many 1's and that the position of the rightmost 1 cannot decrease under the action of a^2. Since $a^2(10^\infty) = 1110^\infty$, the forward orbit of 10^∞ under the action of a^2 can never return to 10^∞ and a^2 has infinite order.

Since $a^2 = (ac, ca)$, the elements ca and $ab = (ab, ca)$ have infinite order, showing that G_{2280} is not contracting.

2283. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(b, a)$, $c = (b, a)$.

By Lemma 2 the element ac acts transitively on the levels of the tree and, hence, has infinite order. Since $ba = (ac, b^2)$ we obtain that ba also has infinite order. Finally, ba stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2283} is not contracting.

2284. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(b, a)$, $c = (b, a)$.

Define $u = b^{-1}a$, $v = a^{-1}c$ and $w = c^{-1}b$. Then $u = (u, v)$, $v = \sigma(w, 1)$ and $w = \sigma(u^{-1}, u)$. The group $\langle u, v, w\rangle$ is generated by the automaton symmetric to the one generating the subgroup $\langle x, y, z\rangle$ of G_{960} (see G_{960} for the definition). It is shown above that zxy has infinite order. Therefore wvu also has infinite order.

The element $(b^{-1}ac^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(b^{-1}ac^{-1}a)^{a^{-1}b}$. Hence, $b^{-1}ac^{-1}a$ has infinite order. Let $t = b^{-1}ab^{-2}a^2$. Since $t|_{110} = b^{-1}ac^{-1}a$ and the vertex 110 is fixed under the action of t we see that t also has infinite order. Finally, t stabilizes the vertex 11101000 and has itself as a section at this vertex. Therefore G_{2284} is not contracting.

2285. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(b, a)$, $c = (b, a)$.

The element $ac^{-1} = \sigma(1, ca^{-1})$ is conjugate to the adding machine and has infinite order.

By Lemma 2 the element $abcb$ acts transitively on the levels of the tree and, hence, has infinite order. Since $(ba)^2|_{000} = (ac, b^2)$ and the vertex 000 is fixed under the action of $(ba)^2$ we obtain that ba also has infinite order. Finally, ba stabilizes the vertex 01 and has itself as a section at this vertex. Therefore G_{2285} is not contracting.

2286. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(b, a)$, $c = (b, a)$.

The element $(c^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(c^{-1}a)^{a^{-1}b}$. Hence, $c^{-1}a$ has infinite order. Since $(c^{-2}a^2)|_{000} = (c^{-1}a)^{b^{-1}}$ and 000 is fixed under the action of $c^{-2}a^2$ we obtain that $c^{-2}a^2$ also has infinite order. Finally, $c^{-2}a^2$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2286} is not contracting.

2287. Wreath recursion: $a = \sigma(a, a)$, $b = \sigma(c, a)$, $c = (b, a)$.

The element $bc^{-1} = \sigma(cb^{-1}, 1)$ is conjugate to the adding machine and has infinite order.

Conjugating the generators by $g = (g, ag)$, we obtain the wreath recursion

$$a' = \sigma, \qquad b' = \sigma(a'c', 1), \qquad c' = (b', a'),$$

where $a' = a^g$, $b' = b^g$, and $c' = c^g$. Therefore

$$a' = \sigma, \qquad b' = \sigma(a'c', 1), \qquad a'c' = \sigma(b', a')$$

A direct computation shows that the iterated monodromy group of $\frac{z^2+2}{1-z^2}$ is generated by

$$\alpha = \sigma, \qquad \beta = \sigma(\gamma^{-1}\beta^{-1}, \alpha), \qquad \gamma = (\beta\gamma\beta^{-1}, \alpha),$$

where α, β, and γ are loops around the post-critical points 2, -1 and -2, respectively (recall the definition of iterated monodromy group in Section 5). We see that

$$\alpha = \sigma, \qquad \beta\gamma = \sigma(\beta^{-1}, 1), \qquad \beta = \sigma(\gamma^{-1}\beta^{-1}, \alpha)$$

satisfy the same recursions as a, b and ac, only composed with taking inverses. If we take second iteration of the wreath recursions, we obtain isomorphic self-similar groups.

It follows that the group G_{2287} is isomorphic to $IMG\left(\frac{z^2+2}{1-z^2}\right)$ and the limit space is homeomorphic to the Julia set of this rational function.

2293. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, a)$, $c = (b, a)$.

The element $(b^{-1}c)^2$ stabilizes the vertex 0 and its section at this vertex is equal to $(b^{-1}c)^{-1}$. Hence, $b^{-1}c$ has infinite order. Since $(c^{-1}bc^{-1}a)^2\big|_{000} = b^{-1}c$ and 000 is fixed under the action of $(c^{-1}bc^{-1}a)^2$ we obtain that $c^{-1}bc^{-1}a$ also has infinite order. Finally, $c^{-1}bc^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2293} is not contracting.

We have $b^{-1}c^2a^{-1} = (1, c^{-1}b^2c^{-1}), c^2a^{-1}b^{-1} = (b^2c^{-2}, 1)$, hence by Lemma 4 the group is not free.

2294. Baumslag-Solitar group $BS(1, -3)$. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, a)$, $c = (b, a)$.

The automaton satisfies the conditions of Lemma 1. Therefore cb has infinite order. Since $a^2 = (cb, bc)$, $c = (b, a)$ and $ba = (ab, c^2)$, the elements a, c and ba have infinite order. Finally, ba fixes the vertex 01 and has itself as a section at this vertex, showing that G_{2294} is not contracting.

Let $\mu = ca^{-1}$. We have $\mu = ca^{-1} = \sigma(ac^{-1}, 1) = \sigma(\mu^{-1}, 1)$, and therefore μ is conjugate of the adding machine and has infinite order. Further, we have $bc^{-1} = \sigma(cb^{-1}, 1) = \sigma((bc^{-1})^{-1}, 1)$, showing that $bc^{-1} = \mu = ca^{-1}$. Therefore $G_{2294} = \langle \mu, a\rangle$.

It can be shown that $a\mu a^{-1} = \mu^{-3}$ in G_{2294} (compare to G_{870}. Since both a and μ have infinite order $G_{2294} \cong BS(1, -3)$.

2295. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, a)$, $c = (b, a)$.

The element $cb^{-1} = \sigma(1, bc^{-1})$ is conjugate to the adding machine and has infinite order. Hence, its conjugate $a^{-1}cb^{-1}a$ also has infinite order. Since $c^{-1}ac^{-1}b = \left(c^{-1}ac^{-1}b, a^{-1}cb^{-1}a\right)$, the element $c^{-1}ac^{-1}b$ has infinite order and G_{2295} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2307. Contains G_{933}. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, b)$, $c = (b, a)$.

We have $ba = (bc, ba)$, and $bc = \sigma(1, ba)$. Therefore G_{933} is a subgroup of G_{2307} (the wreath recursion for ba and bc defines an automaton that is symmetric to the one defining the automaton [993]).

The element $(a^{-1}b)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $a^{-1}b$. Hence, $a^{-1}b$ has infinite order. Furthermore, $a^{-1}b$ stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{2307} is not contracting.

2313 $\cong G_{2277} \cong C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(b, b)$, $c = (b, a)$.

Since all generators have order 2 the subgroup $H = \langle ba, bc\rangle$ is normal in G_{2313}. Furthermore, $ba = \sigma(bc, bc)$ and $bc = \sigma(1, ba)$. Hence, $H = G_{771} \cong \mathbb{Z}^2$.

Finally, $G_{2313} = \langle H, b\rangle = \langle b\rangle \ltimes H = C_2 \ltimes (\mathbb{Z} \times \mathbb{Z})$, where b inverts the generators of H. This action coincides with the one for G_{2277}, which proves that these groups are isomorphic.

2320 $\cong G_{2294}$. Baumslag-Solitar group $BS(1, -3)$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, b)$, $c = (b, a)$.

It is proved in [BŠ06] that the automaton [2320] generates $BS(1, -3)$.

2322. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, b)$, $c = (b, a)$.

The element $(a^{-1}c)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(a^{-1}c)^{b^{-1}}$. Hence, $a^{-1}c$ has infinite order. Since $(c^{-2}a^2)^2\big|_{000} = a^{-1}c$ and 000 is fixed under the action of $c^{-2}a^2$ we obtain that $c^{-2}a^2$ also has infinite order. Finally, $c^{-2}a^2$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2322} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2352 $\cong G_{740}$. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(a, a)$, $c = (c, a)$.

We have $ac^{-1}b = (a, a)$. Therefore $G_{2352} = \langle a, ac^{-1}b, c\rangle = G_{740}$.

2355. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(a, a)$, $c = (c, a)$.

The element $(b^{-1}a)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $(b^{-1}a)^{a^{-1}c}$. Hence, $b^{-1}a$ has infinite order. Furthermore, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2355} is not contracting.

We have $a^{-1}cb^{-1}c = (b^{-1}c, 1), cb^{-1}ca^{-1} = (1, cb^{-1})$, hence by Lemma 4 the group is not free.

2358 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(a, a)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7) and $b = aca$.

2361. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, a)$, $c = (c, a)$.

The element $bc^{-1} = \sigma(bc^{-1}, 1)$ is conjugate to the adding machine and has infinite order.

2364. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(b, a)$, $c = (c, a)$.

The element $cb^{-1} = \sigma(1, cb^{-1})$ is the adding machine and has infinite order. Therefore its conjugate $b^{-1}c$ also has infinite order. Since $(b^{-1}a)\big|_0 = b^{-1}c$ and 0 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2364} is not contracting.

We have $c^{-1}ac^{-1}b = (1, a^{-1}bc^{-1}b), bc^{-1}ac^{-1} = (ba^{-1}bc^{-1}, 1)$, hence by Lemma 4 the group is not free.

2365. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(b, a)$, $c = (c, a)$.

By Lemma 2 the element cb acts transitively on the levels of the tree and, hence, has infinite order.

2366. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(b, a)$, $c = (c, a)$.

By Lemma 2 the element a acts transitively on the levels of the tree and, hence, has infinite order. Since $c = (c, a)$ we obtain that c also has infinite order and G_{2366} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2367. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(b, a)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7).

Also we have $bc = \sigma(bc, 1)$ and $ca = \sigma(ac, 1)$. Therefore the elements bc and ca generate the Brunner-Sidki-Vierra group (see [BSV99]).

2368 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(a, a)$, $b = \sigma(c, a)$, $c = (c, a)$.

We have $bc^{-1}a = (a, a)$. Therefore $G_{2368} = \langle a, c, bc^{-1}a \rangle = G_{739}$.

2369. Wreath recursion: $a = \sigma(b, a)$, $b = \sigma(c, a)$, $c = (c, a)$.

By using the approach already used for G_{875}, we can show that the forward orbit of 10^∞ under the action of a is infinite, and therefore a has infinite order.

Since $a^2 = (ab, ba)$, the element ab also has infinite order. Furthermore, ab fixes 00 and has itself as a section at this vertex. Therefore G_{2369} is not contracting.

2371. Wreath recursion: $a = \sigma(a, b)$, $b = \sigma(c, a)$, $c = (c, a)$.

The element $(c^{-1}ab^{-1}a)^2$ stabilizes the vertex 01 and its section at this vertex is equal to $c^{-1}ab^{-1}a$, which is nontrivial. Hence, $c^{-1}ab^{-1}a$ has infinite order.

Let $t = b^{-1}c^{-1}a^2c^{-1}ba^{-1}ca^{-1}ca^{-2}cbc^{-1}ab^{-1}a$. Then t^2 stabilizes the vertex 00 and $t^2|_{00} = t^{a^{-1}ba^{-1}c}$. Hence, t has infinite order. Let $s = b^{-1}c^{-2}a^3$ Since $s^8|_{00100001} = t$ and s fixes the vertex 00100001 we see that s also has infinite order. Finally, s stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2371} is not contracting.

2372. Wreath recursion: $a = \sigma(b, b)$, $b = \sigma(c, a)$, $c = (c, a)$.

By Lemma 2 the elements b and ac act transitively on the levels of the tree and, hence, have infinite order. Since $(c^2)|_{100} = ac$ and the vertex 100 is fixed under the action of c^2 we obtain that c also has infinite order. Finally, c stabilizes the vertex 0 and has itself as a section at this vertex. Therefore G_{2372} is not contracting.

2374 $\cong G_{821}$. Lamplighter group $\mathbb{Z} \wr C_2$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, a)$, $c = (c, a)$.

The states a and c form a 2-state automaton that generates the Lamplighter group (see Theorem 7). Since $bc^{-1} = \sigma = c^{-1}a$, we have $b = a^c$ and $G = \langle a, c\rangle$.

2375. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, a)$, $c = (c, a)$.

The element $(a^{-1}c)^2$ stabilizes the vertex 01 and its section at this vertex is equal to $a^{-1}c$. Hence, $a^{-1}c$ and $c^{-1}a$ have infinite order. Since $c^{-1}b^{-1}ac^{-1}a^2|_{00} = c^{-1}a$ and the vertex 00 is fixed under the action of $c^{-1}b^{-1}ac^{-1}a^2$ we obtain that $c^{-1}b^{-1}ac^{-1}a^2$ also has infinite order. Finally, $c^{-1}b^{-1}ac^{-1}a^2$ stabilizes the vertex 11 and has itself as a section at this vertex. Therefore G_{2375} is not contracting.

2376 $\cong G_{739} \cong C_2 \ltimes (C_2 \wr \mathbb{Z})$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, a)$, $c = (c, a)$.

Since $\sigma = bc^{-1}$, we have $G_{2376} = \langle a, c, \sigma\rangle$. We already proved that $G_{972} = \langle a, c, \sigma\rangle$. Therefore $G_{2376} = G_{972} \cong G_{739}$.

2388 $\cong G_{821}$. Lamplighter group $\mathbb{Z} \wr C_2$. Wreath recursion: $a = \sigma(c, a)$, $b = \sigma(b, b)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating the Lamplighter group (see Theorem 7) and $b = \sigma = ac^{-1}$.

2391. Wreath recursion: $a = \sigma(c, b)$, $b = \sigma(b, b)$, $c = (c, a)$.

The element $(c^{-1}ba^{-1}b)^2$ stabilizes the vertex 00 and its section at this vertex is equal to $c^{-1}ba^{-1}b$. Hence, $c^{-1}ba^{-1}b$ has infinite order. Since $(bc^{-2}b)^2|_{000} = c^{-1}ba^{-1}b$ and 000 is fixed under the action of $bc^{-2}b$ we obtain that $bc^{-2}b$ also has infinite order. Finally, $bc^{-2}b$ stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{2391} is not contracting.

2394 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(b, b)$, $c = (c, a)$.

All generators have order 2, hence $H = \langle ba, bc\rangle$ is normal in G_{2394}. Furthermore, $ba = (bc, bc)$, $bc = \sigma(bc, ba)$, and therefore $H = G_{731} \cong \mathbb{Z}$.

Thus $G_{2394} = \langle b \rangle \ltimes H \cong C_2 \ltimes \mathbb{Z} \cong D_\infty$ since $(bc)^b = (bc)^{-1}$.

2395. Wreath recursion: $a = \sigma(a,a)$, $b = \sigma(c,b)$, $c = (c,a)$.

By Lemma 2 the element ca acts transitively on the levels of the tree.

The element $(c^{-1}a)^2$ stabilizes the vertex 0 and its section at this vertex is equal to $c^{-1}a$. Hence, $c^{-1}a$ has infinite order. Since $(b^{-1}a)\big|_0 = c^{-1}a$ and 0 is fixed under the action of $b^{-1}a$ we obtain that $b^{-1}a$ also has infinite order. Finally, $b^{-1}a$ stabilizes the vertex 1 and has itself as a section at this vertex. Therefore G_{2395} is not contracting.

Note that $ab = (ac, ab), ac = \sigma(ac, 1)$ and $ba = (ba, ca), ca = \sigma(1, ca)$, i.e., G_{2395} contains copies of G_{929}.

2396. Boltenkov group. Wreath recursion: $a = \sigma(b,a)$, $b = \sigma(c,b)$, $c = (c,a)$.

This group was studied by A. Boltenkov (under direction of R. Grigorchuk), who showed that the monoid generated by $\{a,b,c\}$ is free, and the group G_{2396} is torsion free.

Proposition 2. *The monoid generated by a, b, and c is free.*

Proof. By way of contradiction, assume that there are some relations and let $w = u$ be a relation for which $\max(|w|,|u|)$ minimal.

We first consider the case when neither w nor u is empty. Because of cancelation laws, the words w and u must end in different letters. We have $w = \sigma_w(w_0, w_1) = \sigma_u(u_0, u_1) = u$, where σ_w, and σ_u are permutations in $\{1, \sigma\}$. Clearly, $w_0 = u_0$ and $w_1 = u_1$ must also be relations.

Assume that w ends in b and u ends in c. Then w_0 and u_0 both end in c. Therefore, by minimality, $w_0 = u_0$ as words and $|u| = |w|$. Since $b \neq c$ in G_{2396} the length of w and u is at least 2. We can recover the second to last letter in w and u. Indeed, the second to last letter in u_0 can be only b or c (these are the possible sections at 0 of the three generators), while the second to last letter of w_0 can be only a or b (these are the possible sections at 1 of the three generators). Therefore $w_0 = u_0 = \ldots bc$, $w = \ldots bb$, and $u = \ldots ac$. Since $bb \neq ac$ in G_{2396} (look at the action at level 1), the length of w and u must be at least 3. Continuing in the same fashion we obtain that $w_0 = u_0 = b \ldots bbc$, $w = \ldots ababb$, and $u = \ldots babac$. Since the lengths of w and u are equal, they have different action on level 1, which is a contradiction.

Assume that w ends in a and u ends in b or c. Then u_0 and w_0 end in b and c, respectively, and we may proceed as before.

It remains to show that, say, u cannot be empty word. If this is the case then $w_0 = 1 = w_1$, implying that $w_0 = w_1$ is also a minimal relation. But this is impossible since both w_0 and w_1 are nonempty. $\square$

For a group word w over $\{a,b,c\}$, define the exponent $\exp_a(w)$ of

a in w as the sum of the exponents in all occurrences of a and a^{-1} in w. Define $\exp_b(w)$ and $\exp_c(w)$ in analogous way and let $\exp(w) = \exp_a(w) + \exp_b(w) + \exp_c(w)$.

Lemma 5. *If* $w = 1$ *in* G_{2396} *then* $\exp(w) = 0$.

Proof. By way of contradiction, assume otherwise and choose a freely reduced group word w over $\{a, b, c\}$ such that $w = 1$ in G_{2396}, $\exp(w) \neq 0$, and w has minimal length among such words. If $w = (w_0, w_1)$, w_0 and w_1 also represent 1 in G_{2396} and $\exp(w_0) = \exp(w_1) = \exp(w) \neq 0$. Since the exponents is nonzero, the words w_0 and w_1 are nonempty and, by minimality, their length must be equal to $|w|$. Note that $ac^{-1} = \sigma(bc^{-1}, 1)$ and $bc^{-1} = \sigma(1, ba^{-1})$. This implies that w cannot ac^{-1}, bc^{-1}, ca^{-1}, or cb^{-1} as a subword (otherwise the length of w_0 or w_1 would be shorter than the length of w). By the same reason, w_0 and w_1 cannot have the above 4 words as subwords, which implies that w does not have $ab^{-1} = (ab^{-1}, bc^{-1})$ or its inverse ba^{-1} as a subword. Therefore w has the form $w = W_1(a^{-1}, b^{-1}, c^{-1})W_2(a, b, c)$, and since $w = 1$ in G_{2396}, we obtain a relation between positive words over $\{a, b, c\}$, which contradicts Proposition 2. □

Lemma 6. *If* $w = 1$ *in* G_{2396} *then* $\exp_a(w)$, $\exp_b(w)$ *and* $\exp_c(w)$ *are even.*

Proof. Indeed, $\exp_a(w) + \exp_b(w)$ must be even (since both a and b are active at the root). By Lemma 5, $\exp_c(w)$ must be even. If $w = (w_0, w_1)$, then $\exp_a(w_0)+\exp_b(w_0)$ and $\exp_a(w_1)+\exp_b(w_1)$ must be even. Since $\exp_a(w) + \exp_b(w) = \exp_b(w_0) + \exp_b(w_1)$, $\exp_a(w) + \exp_c(w) = \exp_a(w_0)+\exp_a(w_1)$ we obtain that $2\exp_a(w)+\exp_b(w)+\exp_c(w)$ is even, which then implies that $\exp_b(w)$ is even. Finally, since both $\exp_b(w)$ and $\exp_c(w)$ are even, $\exp_a(w)$ must be even as well (by Lemma 5). □

Proposition 3. *The group* G_{2396} *is torsion free.*

Proof. By way of contradiction, assume otherwise. Let w be an element of order 2. We may assume that w does not belong to the stabilizer of the first level (otherwise we may pass to a section of w). Let $w = \sigma(w_0, w_1)$. Since $w^2 = (w_1w_0, w_0w_1) = 1$, we have the modulo 2 equalities $\exp_b(w_0w_1) = \exp_b(w_0) + \exp_b(w_1) = \exp_a(w) + \exp_b(w)$. Since $\exp_b(w_0w_1)$ is even, $\exp_a(w) + \exp_b(w)$ must be even, implying that w stabilizes level 1, a contradiction. □

Since $b^{-1}a = (c^{-1}b, b^{-1}a)$, the group G_{2396} is not contracting (our considerations above show that $b^{-1}a$ is not trivial and therefore has infinite order).

We have $c^{-1}bc^{-1}a = (1, a^{-1}bc^{-1}b)$, $ac^{-1}bc^{-1} = (ba^{-1}bc^{-1}, 1)$, hence by Lemma 4 the group is not free.

2398. Dahmani group. Wreath recursion: $a = \sigma(a, b)$, $b = \sigma(c, b)$, $c = (c, a)$.

This group is self-replicating, not contracting, weakly regular branch group over its commutator subgroup. It was studied by Dahmani in [Dah05].

2399. Wreath recursion: $a = \sigma(b, b)$, $b = \sigma(c, b)$, $c = (c, a)$.

By Lemma 2 the elements ca and $c^4bc^2bc^2b^2cb^2cb^3acba^2$ act transitively on the levels of the tree and, hence, have infinite order. Since $(cba)^8|_{000010001} = c^4bc^2bc^2b^2cb^2cb^3acba^2$ and vertex 000010001 is fixed under the action of $(cba)^8$ we obtain that cba also has infinite order. Finally, cba stabilizes the vertex 01001 and has itself as a section at this vertex. Therefore G_{2399} is not contracting.

We have $a^{-2}bab^{-2}ab = 1$, and a and b do not commute, hence the group is not free.

2401. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, b)$ and $c = (c, a)$.

The states a and c form a 2-state automaton generating the Lamplighter group (see Theorem 7). Hence, G_{2401} is neither torsion, nor contracting and has exponential growth.

2402. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, b)$, $c = (c, a)$.

The element $(bc^{-1})^2$ stabilizes the vertex 00 and its section at this vertex is equal to bc^{-1}. Hence, bc^{-1} has infinite order.

We have $c^{-2}ba = (1, a^{-2}b^2)$, $ac^{-2}b = (ba^{-2}b, 1)$, hence by Lemma 4 the group is not free.

2403 $\cong G_{2287}$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, b)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7).

Also we have $bc = \sigma(1, ba)$ and $ba = (bc, 1)$. Therefore the elements bc and ba generate the Basilica group G_{852}.

By conjugating by $g = (cg, g)$, we obtain

$$a' = \sigma, \qquad b' = \sigma(1, c'b'), \qquad c' = (c', a'),$$

where $a' = a^g$, $b' = b^g$, and $c' = c^g$. Therefore

$$a' = \sigma, \qquad b' = \sigma(1, c'b'), \qquad c'b' = \sigma(a', b'),$$

and G_{2402} is isomorphic to G_{2287}, i.e., to $IMG(\frac{z^2+2}{1-z^2})$.

2422 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, a)$, $b = \sigma(c, c)$, $c = (c, a)$.

The states a and c form a 2-state automaton generating D_∞ (see Theorem 7) and $b = aca$.

2423. Wreath recursion: $a=\sigma(b,a)$, $b=\sigma(c,c)$, $c=(c,a)$.

Contains elements of infinite order by Lemma 1. In particular, ac has infinite order. Since $c^2\big|_{100}=ac$ and the vertex 100 is fixed under the action of c^2 we obtain that c also has infinite order. Since $c=(c,a)$ the group is not contracting.

We have $c^{-1}bc^{-1}a = (1,a^{-1}b)$, $ac^{-1}bc^{-1} = (ba^{-1},1)$, hence by Lemma 4 the group is not free.

2424 $\cong G_{966}$. Wreath recursion $a=\sigma(c,a)$, $b=\sigma(c,c)$, $c=(c,a)$.

We have $ac^{-1}b=(c,c)$. Therefore $G_{2424}=\langle a, ac^{-1}b, c\rangle = G_{966}$.

2426 $\cong G_{2277}\cong C_2\ltimes(\mathbb{Z}\times\mathbb{Z})$. Wreath recursion: $a=\sigma(b,b)$, $b=\sigma(c,c)$, $c=(c,a)$.

Since all generators have order 2 the subgroup $H=\langle ab, cb\rangle$ is normal in G_{2426}. Furthermore, $ab=(bc,bc)$, $cb=\sigma(ac,1)=\sigma(ab(cb)^{-1},1)$, so H is self-similar. Since $acb=bca$ in G_{2426} we obtain $ab\cdot cb=abcaab=aacbab=cb\cdot ab$, hence, H is an abelian self-similar 2-generated group.

Consider the $\frac{1}{2}$-endomorphism $\phi:\mathrm{Stab}_H(1)\to H$, given by $\phi(g)=h$, provided $g=(h,*)$ and consider the linear map $A:\mathbb{C}^2\to\mathbb{C}^2$ induced by ϕ. It has the following matrix representation with respect to the basis corresponding to the generating set $\{ab, cb\}$:

$$A=\begin{pmatrix} 0 & \frac{1}{2} \\ -1 & -\frac{1}{2}\end{pmatrix}.$$

Its eigenvalues are not algebraic integers and, therefore, by [NS04], H is a free abelian group of rank 2.

Finally, $G_{2426}=\langle H,b\rangle=\langle b\rangle\ltimes H=C_2\ltimes(\mathbb{Z}\times\mathbb{Z})$, where b inverts the generators of H. This action coincides with the one for G_{2277}, which proves that these groups are isomorphic.

2427. The element $(bc^{-1})^4$ stabilizes the vertex 000 and its section at this vertex is equal to bc^{-1}. Hence, bc^{-1} has infinite order.

We have $a^{-2}bab^{-2}ab=1$, and a and b do not commute, hence the group is not free.

2838 $\cong G_{848}\cong C_2\wr\mathbb{Z}$. Wreath recursion: $a=\sigma(c,a)$, $b=\sigma(a,a)$, $c=(c,c)$.

Since c is trivial, we have $G=\langle a, ba^{-1}\rangle$, where $a=\sigma(1,a)$ is the adding machine and $ba^{-1}=(1,a)$. Therefore $G_{2838}=G_{848}$.

2841. Wreath recursion: $a=\sigma(c,b)$, $b=\sigma(a,a)$, $c=(c,c)$.

The element c is trivial. Since $a^2=(b,b)$, $b^2=(a^2,a^2)$ and a^2 is nontrivial, the elements a and b have infinite order. Also we have $ba=(a,ab)$ and $ab=(ba,a)$, hence ba has infinite order and G_{2841} is not contracting.

We claim that the monoid generated by a and b is free. Hence, G_{2841} has exponential growth.

Proof. We can first prove (analogous to G_{2851}) that $w \neq 1$ for any nonempty word $w \in \{a,b\}^*$.

By way of contradiction, let w and v be two nonempty words in $\{a,b\}^*$ with minimal $|w|+|v|$ such that $w = v$ in G_{2841}. Assume that w ends with a and v ends with b. Consider the following cases.

1. If $w = a$ then $v|_0 = 1$ in G_{2841} and $v|_0$ is nontrivial word.

2. If w ends with a^2 then $w|_1 = v|_1$ in G_{2841}, $\big|w|_1\big| + \big|v|_1\big| < |w|+|v|$ and $w|_1$ ends with b, $v|_1$ with a.

3. If w ends with ba and v ends with ab, then $w|_1 = v|_1$ in G_{2841}, $\big|w|_1\big| + \big|v|_1\big| < |w|+|v|$ (because $\big|v|_1\big| < |v|$) and $w|_1$ ends with b, $v|_1$ with a.

4. If w ends with ba and v ends with b, then $w|_1 = v|_1$ in G_{2841}, $\big|w|_1\big|+\big|v|_1\big| \le |w|+|v|$ and $w|_1$ ends with ab, $|v_1|$ with a. Therefore, words $v|_1$ and $w|_1$ satisfy one of the first three cases.

In all cases we obtain either a shorter relation, which contradicts to our assumption, or a relation of the form $v = 1$, which is also impossible. □

There are non-trivial group relations, e.g. $a^{-1}b^{-1}a^{-2}ba^{-1}b^{-1}aba^2b^{-1}ab = 1$, while a and b do not commute, hence the group is not free.

2284 $\cong G_{730}$. Klein Group $C_2 \times C_2$.

Direct calculation.

2847 $\cong G_{929}$. Wreath recursion: $a = \sigma(c,a)$, $b = \sigma(b,a)$, $c = (c,c)$.

Since c is trivial, the generator $a = \sigma(1,a)$ is the adding machine and $b = \sigma(b,a)$. We have $ab = (ab,a)$. Therefore $G_{2847} = \langle a, ab\rangle = G_{929}$.

2850. Wreath recursion: $a = \sigma(c,b)$, $b = \sigma(b,a)$, $c = (c,c)$.

Since c is trivial, we have $a^2 = (b,b)$, $b^2 = (ab,ba)$, $ab = (b^2,a)$ and $ba = (a,b^2)$. Therefore the elements a, b, ab and ba have infinite order. Since ba fixes the vertex 11 and has itself as a section at that vertex, G_{2850} is not contracting.

The group is regular weakly branch over G'_{2850}, since it is self-replicating and $[b,a^2] = (1,[a,b])$.

Semigroup $\langle a,b\rangle$ is free. Hence, G_{2850} has exponential growth.

Proof. We can first prove (analogous G_{2851}) that $w \neq 1$ for any nonempty word $w \in \{a,b\}^*$.

By way of contradiction, let w and v be two nonempty words in $\{a,b\}^*$ with minimal $|w|+|v|$ such that $w = v$ in G_{2850}. Assume that w ends with a and v ends with b. Consider the following cases.

1. If $w = a$ then $v|_0 = 1$ in G and $v|_0$ is nontrivial word.

2. If w ends with a^2 then $w|_1 = v|_1$ in G, $\big|w|_1\big| + \big|v|_1\big| < |w| + |v|$ and $w|_1$ ends with b, $v|_1$ with a.

3. If w ends with ba then $w|_0 = v|_0$ in G, $\big|w|_0\big| + \big|v|_0\big| < |w| + |v|$ and $w|_0$ ends with a, $v|_0$ with b.

In all cases we obtain either a shorter relation, which contradicts to our assumption, or a relation of the form $v = 1$, which is also impossible. □

Since $a^{-4}bab^{-1}a^2b^{-1}ab = 1$ and a and b do not commute, the group is not free.

2851 $\cong G_{929}$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(b, a)$, $c = (c, c)$.

The automorphism c is trivial. Therefore $a = \sigma(a, 1)$ is the inverse of the adding machine. Since $ba^{-1} = (a, ba^{-1})$, the order of ba^{-1} is infinite and G_{2851} is not contracting.

Since G_{2851} is self-replicating and $[a^2, b] = ([a, b], 1)$, the group is a regular weakly branch group over its commutator.

The monoid $\langle a, b\rangle$ is free.

Proof. By way of contradiction, assume that w be a nonempty word over $\{a, b\}$ such that $w = 1$ in G_{2851} and w has the smallest length among all such words. The word w must contain both a and b (since they have infinite order). Therefore, one of the projections of w is be shorter than w, nonempty, and represents the identity in G_{2851}, a contradiction.

Assume now that w and v are two nonempty words over $\{a, b\}$ such that $w = v$ in G_{2851} and they are chosen so that the sum $|w| + |v|$ is minimal. Assume that w ends in a and v ends in b. Then

- if w ends in a^2, then w_0 is a nonempty word that is shorter than w ending in a, while v_0 is a nonempty word of length no greater than $|v|$ ending in b. Since $w_0 = v_0$ in G_{2851}, this contradicts the minimality assumption.

- if w ends in ba, then w_1 is a word that is shorter than w ending in b, while v_1 is a nonempty word of length no greater than $|v|$ ending in a. Since $w_1 = v_1$ in G_{2851}, this contradicts the minimality assumption.

- if $w = a$ then $v_1 = 1$ in G and v_1 is a nonempty word. Thus we obtain a relation $v_1 = 1$ in G_{2851}, a contradiction.

□

This shows that G has exponential growth, while the orbital Schreier graph $\Gamma(G, 000\ldots)$ has intermediate growth (see [BH05, BCSN]).

The groups G_{2851} and G_{929} coincide as subgroups of $\mathrm{Aut}(X^*)$. Indeed, $a^{-1} = \sigma(1, a^{-1})$ is the adding machine and $b^{-1}a = (b^{-1}a, a^{-1})$, showing that $G_{929} = \langle a^{-1}, b^{-1}a\rangle = G_{2851}$.

2852 $\cong G_{849}$. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(b, a)$, $c = (c, c)$.

The automorphism c is trivial. Therefore $a = \sigma(b, 1)$, $a^2 = (b, b)$ and $ab = (b, ba)$, which implies that G_{2852} is self-replicating and level transitive.

The group G_{2852} is a regular weakly branch group over its commutator. This follows from $[a^{-1}, b] \cdot [b, a] = ([a, b], 1)$, together with the self-replicating property and the level transitivity. Moreover, the commutator is not trivial, since G_{2852} is not abelian (note that $[b, a] = (b^{-1}ab, a^{-1}) \neq 1$).

We have $b^2 = (ab, ba)$, $ba = (ab, b)$, and $ab = (b, ba)$. Therefore b^2 fixes the vertex 00 and has b as a section at this vertex. Therefore b has infinite order (since it is nontrivial), and so do ab and a (since $a^2 = (b, b)$). Since ab fixes the vertex 10 and has itself as a section at that vertex, G_{2852} is not contracting.

The monoid generated by a and b is free (and therefore the group has exponential growth).

Proof. By way of contradiction assume that w is a word of minimal length over all nonempty words over $\{a, b\}$ such that $w = 1$ in G_{2851}. Then w must have occurrences of both a and b (since both have infinite order). This implies that one of the sections of w is shorter than w (since $a|_1$ is trivial), nonempty (since both $b|_0$ and $b|_1$ are nontrivial), and represents the identity in G_{2851}, a contradiction.

Assume now that there are two nonempty words $w, v \in \{a, b\}^*$ such that $w = u$ in G_{2852} and choose such words with minimal sum $|w| + |v|$. Let $w = \sigma_w(w_0, w_1)$ and $u = \sigma_u(u_0, u_1)$, where $\sigma_w, \sigma_w \in \{1, \sigma\}$. Assume that w ends in a and v ends in b (they must end in different letters because of the cancelation property and the minimality of the choice). Then $w_1 = v_1$ in G_{2851}, the word v_1 is nonempty, $|v_1| \leq |v|$, and $|w_1| < |w|$. Thus we either obtain a contradiction with the minimality of the choice of w and v or we obtain a relation of the type $v_1 = 1$, also a contradiction. □

See G_{849} for an isomorphism between G_{2852} and G_{849}.

If we conjugate the generators of G_{2852} by the automorphism $\mu = \sigma(b\mu, \mu)$, we obtain the wrath recursion

$$x = \sigma(y, 1), \qquad y = \sigma(xy, 1),$$

where $x = a^\mu$ and $y = b^\mu$. Further,

$$y = \sigma(xy, 1), \qquad xy = (xy, y),$$

and the last recursion defines the automaton 933. Therefore $G_{2852} \cong G_{933}$.

2853 $\cong IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$. Wreath recursion $a = \sigma(c, c)$, $b = \sigma(b, a)$ and $c = (c, c)$.

The automorphism c is trivial and $a = \sigma$.

It is shown in [BN06] that $IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$ is generated by $\alpha = \sigma(1, \beta)$ and $\beta = (\alpha^{-1}\beta^{-1}, \alpha)$.

We have then $\beta\alpha = \sigma(\alpha, \alpha^{-1})$. If we conjugate by $\gamma = (\gamma, \alpha\gamma)$, we obtain the wreath recursion

$$A = \sigma, \qquad B = \sigma(B^{-1}, A)$$

where $A = (\beta\alpha)^\gamma$ and $B = \alpha^\gamma$. The group $\langle A, B\rangle$ is conjugate to G_{2853} by the element $\delta = (\delta_1, \delta_1)$, where $\delta_1 = \sigma(\delta, \delta)$ (this is the automorphism of the tree changing the letters on even positions).

Therefore $G_{2852} \cong IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$ and the limit space of G_{2852} is the Julia set of the rational map $z \mapsto \left(\frac{z-1}{z+1}\right)^2$.

Note that G_{2853} is contained in G_{775} as a subgroup of index 2. Therefore it is virtually torsion free (it contains the torsion free subgroup H mentioned in the discussion of G_{775} as a subgroup of index 2) and is a weakly branch group over H''.

The diameters of the Schreier graphs on the levels grow as $\sqrt{2}^n$ and have polynomial growth of degree 2 (see [BN, Bon07]).

2854 $\cong G_{847} \cong D_4$. Wreath recursion: $a = \sigma(a, a)$, $b = \sigma(c, a)$, $c = (c, c)$.

Direct calculation.

2860 $\cong G_{2212}$. Klein bottle group $\langle s, t \mid s^2 = t^2\rangle$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, a)$, $c = (c, c)\rangle$.

Note that c is trivial and therefore $a = \sigma(a, 1)$ and $b = \sigma(1, a)$. The element a has infinite order since a is inverse of the adding machine.

Let us prove that $G_{2860} \cong H = \langle s, t \mid s^2 = t^2\rangle$. Indeed, the relation $a^2 = b^2$ is satisfied, so G_{2860} is a homomorphic image of H with respect to the homomorphism induced by $s \mapsto a$ and $t \mapsto b$. Each element of H can be written in the form $t^r(st)^l s^n$, $n \in \mathbb{Z}, l \geq 0, r \in \{0, 1\}$. It suffices to prove that images of these words (except for the identity word, of course) represent nonidentity elements in G_{2860}.

We have $a^{2n} = (a^n, a^n)$, $a^{2n+1} = \sigma(a^{n+1}, a^n)$, $(ab)^l = (1, a^{2l})$. We only need to check words of even length (those of odd length act nontrivially on level 1). We have $(ab)^\ell a^{2n} = (a^n, a^{n+2\ell}) \neq 1$ in G if $n \neq 0$ or $\ell \neq 0$, since a has infinite order. On the other hand, $b(ab)^l a^{2n+1} = (a^{n+1+2l+1}, a^n) = 1$ if and only if $n = 0$ and $l = -1$, which is not the case, because l must be nonnegative. This finishes the proof.

2861 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(b, c)$, $b = \sigma(c, a)$, $c = (c, c)\rangle$.

Since c is trivial, $ba = (ab, 1)$, $ab = (1, ba)$, which yields $a = b^{-1}$. Also $a^{2n} = (b^n, b^n), b^{2n} = (a^n, a^n)$ and $a^{2n+1} \neq 1, b^{2n+1} \neq 1$. Thus a has infinite order and $G_{2861} \cong \mathbb{Z}$.

2862 $\cong G_{847} \cong D_4$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, a)$, $c = (c, c)\rangle$.

Direct calculation.

2874 $\cong G_{820} \cong D_\infty$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(b, b)$, $c = (c, c)$.

Since c is trivial, $G_{2874} = \langle b, ba\rangle$. Since $ba = (ba, b)$, the elements b and ba form a 2-state automaton generating D_∞ (see Theorem 7).

2880 $\cong G_{730}$. Klein Group $C_2 \times C_2$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(b, b)$, $c = (c, c)$.

Direct calculation.

2887 $\cong G_{731} \cong \mathbb{Z}$. Wreath recursion: $a = \sigma(a, c)$, $b = \sigma(c, b)$, $c = (c, c)$.

Note that c is trivial, b is the adding machine and $a = b^{-1}$.

2889 $\cong G_{848} \cong C_2 \wr \mathbb{Z}$. Wreath recursion: $a = \sigma(c, c)$, $b = \sigma(c, b)$, $c = (c, c)$.

Note that c is trivial. Since b is the adding machine and $ab = (1, b)$, we have $G_{2889} = \langle b, ab\rangle = G_{848}$.

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