Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 16 **(2013)**. Number 1. pp. 96 – 102

On Markov graphs

Sergiy Kozerenko

Communicated by V. V. Kirichenko

Abstract. We investigate graph-theoretical properties of Markov graphs from dynamical systems.

1. Introduction

This paper is concerned with the so-called Markov graphs which are very useful in one-dimension dynamics. Using them we can prove in combinatorial way the famous Sharkovsky's theorem (see [1]). More, considering the interval as homeomorphic image of a path-graph one can define Markov graphs in a similar way for topological trees and then prove an analogue of Sharkovsky's theorem for them (see [3]). Finally, Markov graphs can be naturally defined for special class of continuous maps on arbitrary topological graphs (see [4]).

In this paper we investigate the properties of Markov graphs from graph-theoretical point of view.

2. Basic definitions

Recall some basic definitions from graph theory.

An undirected graph G is a pair (V, E), where E is a collection of two-element subsets from V. The sets $V = V(G)$ and $E = E(G)$ are

The author would like to thank M. Plakhotnyk for his outstanding support during the work on this paper, V. V. Kirichenko for many helpful suggestions and V. V. Fedorenko for a general discussions on this topic.

2010 MSC: 37E25.

Key words and phrases: dynamics on graphs, Markov graph.

called the vertex set and the edge set of a graph G. Two vertices u and v in G are adjacent if they are joined by an edge, i.e. if $uv \in E(G)$. The edge $e = uv \in E(G)$ is incident to vertices u and v. A vertex is called a leaf if it incident exactly to one edge.

Two graphs G_1 and G_2 are isomorphic if there exists adjacency preserving bijection between their vertex sets. Every such bijection is called an isomorphism. We write $G_1 \simeq G_2$ if G_1 and G_2 are isomorphic. Automorphism is a graph isomorphism to itself. Automorphism group of graph G is denoted by $Aut\, G$.

A graph is called connected if for every pair of vertices there exists a path joining them. A connected component of a graph is its maximal connected subgraph. A bridge is an edge whose deletion increases the number of connected components. Tree is a connected graph without cycles. Forest is a disjoint union of trees.

A directed graph D is a pair $D = (V, A)$, where $A \subset V \times V$. The sets $V = V(D)$ and $A = A(D)$ are called the vertex set and the arc set of D. If $(u, v) \in A(D)$ then u is a predecessor of v and v is a successor of u. The number of predecessors of a given vertex is called its indegree and the number of successors is its outdegree. If for a vertex $v \in V(D)$ we have $(v, v) \in A(D)$ then we say that v has a loop.

Now let us precisely define what we will call by topological graph. The closed interval $[a, b]$ is called an edge. Its end-points are called vertices. A topological graph is a collection of edges with the property that intersection of two of them is empty or consists of one vertex. Topological tree is a topological graph which is contractible to a point. A (unique) shortest path between two vertices u and v in topological tree X will be denoted as $[u, v]_X$. A graph $X \setminus v$ is obtained from X by deletion the vertex v with all interiors of edges incident to v. Similarly, a graph $X \setminus e$ is obtained from X by deletion only the interior of the edge e.

In this paper we consider only finite topological graphs and their maps. More, every topological graph will be called simply as graph and obviously it shares all the definitions above. Now we proceed to the maps.

Definition 1. Let X be a graph and $f : X \to X$ be a continuous map.

We will call f a

- *vertex map* if $f(V(X)) \subset V(X)$;

- *permutation map* if $f(V(X)) = V(X)$ (i.e. if restriction $f|_{V(X)}$ is a permutation of $V(X)$);

- *cyclic permutation map* if $f|_{V(X)}$ is a cyclic permutation.

Definition 2. Let X be a graph and f be a vertex map on X. The *Markov graph* $\Gamma(X, f)$ is a directed graph with the vertex set $E(X)$, and there exists an arc $e_1 \to e_2$, if e_1 "covers" e_2 under f, i.e. if $e_2 \subset f(e_1)$.

Note that since f is continuous, then for every edge $e = uv \in E(X)$ we have $P = P[f(u), f(v)] \subset f(e)$ for some $f(u) - f(v)$ path P. In case of X is a tree it means that $[f(u), f(v)]_X \subseteq f(e)$.

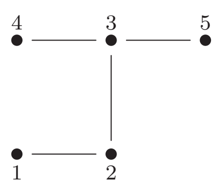
Definition 3. Let X be a tree and f be a vertex map on X. The map f is called *minimal* if $[f(u), f(v)]_X = f(e)$ for every edge $e = uv \in E(X)$.

It is easy to see that if f_1 and f_2 are two minimal maps on a tree X with $f_1|_{V(X)} = f_2|_{V(X)}$ then $\Gamma(X, f_1) = \Gamma(X, f_2)$. So one can define a discrete Markov graph for every map on $V(X)$.

Definition 4. Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. The *discrete Markov graph* $\Gamma(X, \sigma)$ is a graph $\Gamma(X, f)$, where f is some minimal map on X with $f|_{V(X)} = \sigma$.

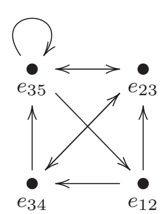
Thus $\Gamma(X, \sigma)$ is a purely combinatorial object and obviously it can be defined for graph-theoretical trees and their maps.

Example 1. Consider the following tree X:



and the map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 2 & 1 \end{pmatrix}$ on $V(X)$.

Then the correspond discrete Markov graph $\Gamma(X, \sigma)$ is following:



3. Results

First of all, let us formulate an analogue of Sharkovsky's theorem for trees. Note, that this result was proven using signed discrete Markov graphs.

Theorem 1 ([3]). *Let X be a tree with n vertices. Let f be a cyclic permutation map on X. Then:*

1) If n is not a divisor of 2^k then f has a periodic point with period 2^k.

2) If $n = 2^k m$, where $m > 1$ is odd and $k \geqslant 0$, then f has a periodic point with period $2^k l$ for every $l \geqslant m$.

3) If n is odd then f has a periodic point with period $n - 1$.

Also, using discrete Markov graphs one can easily construct an irreducible representation of S_n. In what follows we denote by M_G an adjacency matrix of a graph G.

Theorem 2 ([3, 6]). *Let X be a tree with n vertices. Then $M : S_n \to Gl(n-1, 2)$, where $M(\sigma) := M_{\Gamma(X,\sigma)}$ is an irreducible representation of S_n.*

Remark 1. Since X is arbitrary tree, we have a family of monomorphisms from S_n into $GL(n-1, 2)$.

Corollary 1. *Let X be a forest with n vertices and $X_1, \ldots, X_m$ be its trees. Consider the set Π of all permutations σ of $V(X)$ with $\sigma(V(X_i)) \subset V(X_i)$ for every $i \in \{1, \ldots, m\}$. Then $M : S_n \to Gl(n-m, 2)$, where $M(\sigma) := M_{\Gamma(X,\sigma)}$ is a representation of a group Π.*

Proof. We can apply Theorem 2 for each of the X_i. □

Corollary 2. *Let X be a tree and σ_1, σ_2 be two permutations of $V(X)$.*

1) $M^{-1}_{\Gamma(X,\sigma_1)} = M_{\Gamma(X,\sigma_1^{-1})}$.

2) If $\Gamma(X, \sigma_1) \simeq \Gamma(X, \sigma_2)$ then $\Gamma(X, \sigma_1^k) \simeq \Gamma(X, \sigma_2^k)$ for every $k \in \mathbb{N}$.

3) $Aut\, \Gamma(X, \sigma_1) \simeq Aut\, \Gamma(X, \sigma_1^{-1})$.

Proof. 1) Denote that as I is an identity matrix and as id is identity permutation. So, we have $I = M_{\Gamma(X,id)} = M_{\Gamma(X,\sigma_1)} M_{\Gamma(X,\sigma_1^{-1})}$. Whence, $M^{-1}_{\Gamma(X,\sigma_1)} = M_{\Gamma(X,\sigma_1^{-1})}$.

2) Let $\Gamma(X, \sigma_1) \simeq \Gamma(X, \sigma_2)$. Then there exists a permutation matrix T such that

$$T M_{\Gamma(X,\sigma_1)} = M_{\Gamma(X,\sigma_2)} T. \tag{1}$$

Now we prove by induction that equality $TM_{\Gamma(X,\sigma_1^k)} = M_{\Gamma(X,\sigma_2^k)}T$ holds for every $k \in \mathbb{N}$.

Equality (1) gives us a basis. Induction step:

$$\begin{aligned} TM_{\Gamma(X,\sigma_1^{k+1})} &= TM_{\Gamma(X,\sigma_1)}M_{\Gamma(X,\sigma_1^k)} = M_{\Gamma(X,\sigma_2)}TM_{\Gamma(X,\sigma_1^k)} \\ &= M_{\Gamma(X,\sigma_2)}M_{\Gamma(X,\sigma_2^k)}T = M_{\Gamma(X,\sigma_2^{k+1})}T. \end{aligned}$$

3) Immediately follows from 1). □

Proposition 1. *Let X be a connected graph and f be a permutation map on $V(X)$. Then*

1) Every vertex from $\Gamma(X, f)$ has non-zero outdegree.

2) If edge $e \in E(X)$ is a bridge, then the correspond vertex in $\Gamma(X, f)$ has a non-zero indegree.

Proof. 1) Let $e = uv \in E(X)$. Since f is a bijection on $V(X)$, we have $\sigma(u) \neq \sigma(v)$ for every $u, v \in V(X)$. But X is connected. Thus the collection of all $f(u) - f(v)$ paths in X is nonempty, which means that e has a non-zero outdegree in $\Gamma(X, f)$.

2) Now let $e \in E(X)$ be a bridge. Deleting the edge e from X we obtain a graph $X \setminus e$ with two connected components:

$$X \setminus e = X_1 \cup X_2, X_1 \cap X_2 = \varnothing.$$

Therefore, since f is a bijection on $V(X)$, we have a partition $V(X) = f^{-1}(V(X_1)) \cup f^{-1}(V(X_2))$. Again, X is connected, so there exists an edge $e' = u'v' \in E(X)$ such that $u' \in f^{-1}(V(X_1)), v' \in f^{-1}(V(X_2))$. Obviously, e' "covers" e under f. □

Corollary 3. *Let X be a tree and σ be a permutation of $V(X)$. Then every vertex from $\Gamma(X, \sigma)$ has a non-zero both indegree and outdegree.*

Proof. Tree is a connected graph in which every edge is a bridge. □

Denote the set of all leaf vertices of X as $L(X)$.

Lemma 1. *Let X be a graph with $L(X) \neq \varnothing$ and f be a vertex map on X. Let us fix a vertex $v \in L(X)$ and the (unique) edge $e_v = vv_0 \in E(X)$. Define a new map $f_v : X \setminus v \to X \setminus v$ as follows*

$$f_v(x) = \begin{cases} f(x), & \text{if } x \notin f^{-1}(e_v) \\ v_0, & \text{if } x \in f^{-1}(e_v). \end{cases}$$

Then f_v is a vertex map on $X \setminus v$ and

$$\Gamma(X, f) \setminus e_v = \Gamma(X \setminus v, f_v).$$

Proof. It is obvious that f_v is a vertex map on $X \setminus v$.

Now put $\Gamma_1 = \Gamma(X, f) \setminus e_v$ and $\Gamma_2 = \Gamma(X \setminus v, f_v)$. Since v is a leaf vertex, then $V(\Gamma_1) = E(G) \setminus \{e_v\} = E(X \setminus v) = V(\Gamma_2) =: Y$. Also, directly from the definition of f_v it follows that for every $e \in Y$ we have $f(e) \cap (X \setminus v) = f_v(e)$. Thus $A(\Gamma_1) = A(\Gamma_2)$. □

Further, the symbol $fix\, \sigma$ denotes the set of all fixed points of σ.

Theorem 3. *Let X be a tree and σ be a permutation of $V(X)$ with $fix\, \sigma = \varnothing$. Then $\Gamma(X, \sigma)$ has a vertex with a loop.*

Proof. Since X is a tree, it is sufficient to find two adjacent vertices u and v, such that the edge uv lies on $\sigma(u) - \sigma(v)$ path. If so, then the vertex in $\Gamma(X, \sigma)$ which corresponds to uv has a loop.

Fix a root $v_0 \in V(X)$ and define a partial order on $V(X)$ as follows:

$$v_1 \leqslant v_2, \text{ if } v_1 \in [v_0, v_2]_X.$$

Consider the set

$$M = \{v \in V(X) | \sigma(v) > v\}.$$

Since $fix\, \sigma = \varnothing$, then $v_0 \in M$. Therefore M is nonempty, which means that M has maximal elements. Let v' be some maximal element in M. Consider the upper cone of v'

$$N = \{v \in V(X) | v > v'\}.$$

Analogously, $\sigma(v') \in N$. Therefore N is nonempty, which means that N has minimal elements. Let u' be some minimal element in N. Then v' and u' are adjacent in X and $\sigma(v') > v', \sigma(u') \leqslant u'$. Again, from $fix\, \sigma = \varnothing$ it follows that $\sigma(u') < u'$. So, we obtain the next path:

$$\sigma(u') \leqslant v' < u' \leqslant \sigma(v').$$

Now, from the definition of partial order $\leqslant$ it follows that $u'v'$ lies on $\sigma(u') - \sigma(v')$ path. □

Proposition 2. *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map with $fix\, \sigma = \varnothing$. Also, let $u \in L(X)$. Then the vertex in $\Gamma(X, \sigma)$, correspond to the edge $e = uv$ has a loop if and only if $\sigma(v) = u$.*

Proof. If $\sigma(v) \neq u$, then, since $fix\,\sigma = \varnothing$, we have that $\sigma(u) \neq u$. But the vertex u is a leaf. Therefore $u \notin [\sigma(v), \sigma(u)]_X$. It means that the vertex in $\Gamma(X, \sigma)$, correspond to $e = uv$ doesn't have a loop. Conversely, let $\sigma(v) = u$. But $\sigma(u) \neq u$. Thus, $u, v \in [u, \sigma(u)]_X = [\sigma(v), \sigma(u)]_X$. □

Corollary 4. *Let X be a tree and σ be a permutation of $V(X)$ with $fix\,\sigma = \varnothing$. Also, let there exists two leaves in X with the distance 2 between them. Then $\Gamma(X, \sigma)$ has a vertex without loop.*

Proof. Let $u_1, u_2 \in L(X)$ and $u_1v, u_2v \in E(X)$ for some $v \in V(X)$. If the vertex in $\Gamma(X, \sigma)$, corresponds to u_1v has a loop, then $\sigma(v) = u_1$. So, we have $\sigma(v) \neq u_2$ and thus the vertex in $\Gamma(X, \sigma)$ which corresponds to u_2v doesn't have a loop. □

References

[1] *A. N. Sharkovsky, A. G. Sivak, S. F. Kolyada, V. V. Fedorenko*, Dynamics of one-dimensional maps // - Kluwer Academic Publishers, 1997. - 272 p.

[2] *A. N. Sharkovsky*, Co-existence of the cycles of a continuous mapping of the line onto itself // Ukrain. Math. Zh. 16(1), 61–71 (1964)

[3] *C. Bernhardt*, Vertex maps for trees: algebra and periods of periodic orbits // Discrete Contin. Dyn. Syst, Vol. 14, No.3, pp. 399-408, March 2006.

[4] *C. Bernhardt*, Zero-entropy vertex maps on graphs // Journal of Difference Equations and Applications, 15 (1), 17-22 (2009).

[5] *C. Bernhardt*, A Sharkovsky theorem for vertex maps on trees // Journal of Difference Equations and Applications, 17 (1), 103-113 (2011).

[6] *C. Bernhardt*, A simple construction of a faithful, irreducible representation of S_n // J. Algebra Appl., 05, 125 (2006).

[7] *Ll. Alseda, D. Juher, P. Mumbru*, Sets of periods for piecewise monotone tree maps // International Journal of Bifurcation and Chaos, Vol 13, No. 2 (2003), 311-341.

[8] *C-W. Ho, C. Morris*, A graph theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions // Pacific J. of Math. 96 (1981), 361-370.

Contact information

S. Kozerenko — Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko Univ., Volodymyrska str., 64, 01033 Kyiv, Ukraine
E-Mail: `kozerenkosergiy@ukr.net`

Received by the editors: 11.04.2013
and in final form 11.04.2013.