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RESEARCH ARTICLE

Algorithms computing $\mathrm{O}(n, \mathbb{Z})$-orbits of P-critical edge-bipartite graphs and P-critical unit forms using Maple and C#

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Abstract. We present combinatorial algorithms constructing loop-free P-critical edge-bipartite (signed) graphs Δ', with $n \geq 3$ vertices, from pairs (Δ, w), with Δ a positive edge-bipartite graph having n-1 vertices and w a sincere root of Δ, up to an action $* : \mathcal{UB}igr_n \times \mathrm{O}(n, \mathbb{Z}) \to \mathcal{UB}igr_n$ of the orthogonal group $\mathrm{O}(n, \mathbb{Z})$ on the set $\mathcal{UB}igr_n$ of loop-free edge-bipartite graphs, with $n \geq 3$ vertices. Here $\mathbb{Z}$ is the ring of integers. We also present a package of algorithms for a Coxeter spectral analysis of graphs in $\mathcal{UB}igr_n$ and for computing the $\mathrm{O}(n, \mathbb{Z})$-orbits of P-critical graphs Δ in $\mathcal{UB}igr_n$ as well as the positive ones. By applying the package, symbolic computations in Maple and numerical computations in C#, we compute P-critical graphs in $\mathcal{UB}igr_n$ and connected positive graphs in $\mathcal{UB}igr_n$, together with their Coxeter polynomials, reduced Coxeter numbers, and the $\mathrm{O}(n, \mathbb{Z})$-orbits, for $n \leq 10$. The computational results are presented in tables of Section 5.

1. Introduction

Throughout, we denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ the field of the rational, real, and the complex numbers, respectively. We view $\mathbb{Z}^n$, with $n \geq 1$, as a free

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abelian group. We denote by $e_1, \ldots, e_n$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Given $n \geq 1$, we denote by $\mathbb{M}_n(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square n by n matrices, and by $E \in \mathbb{M}_n(\mathbb{Z})$ the identity matrix.

Following some ideas of the spectral graph theory [7], the graph coloring techniques [2] and [14], and of the unit quadratic forms investigation given in [3], [5]-[6], [9], [12], [21]-[22], [28], [29], [36], [40], we study in [38] the category of loop-free edge-bipartite (signed [42]) graphs Δ by means of the Coxeter spectrum of Δ, that is, the finite set $\mathbf{specc}_\Delta \subset \mathbb{C}$ of all complex roots of the Coxeter polynomial $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$ of Δ, see Section 2. In particular, we are interested in an orthogonal and Coxeter spectral classification of positive and non-negative loop-free edge-bipartite graphs Δ (in the sense of Definition 1.1). In the present paper we study in details P-critical loop-free edge-bipartite graphs Δ, by applying a technique developed in [5]-[6], [21], [22], [35]–[38]. In this case, the Coxeter spectrum $\mathbf{specc}_\Delta$ of Δ is a subset of the unit circle and consists of roots of unity.

Our main inspiration for the study comes from the representation theory of posets (see [5], [8], [28], [31]-[32], [40]), groups and algebras (see [1], [16]-[19], [29], [39]), Lie theory, and Diophantine geometry problems, see [35]-[37].

Our motivation comes also from the fact that positive (and non-negative) edge-bipartite graphs, P-critical graphs, positive unit forms, and P-critical forms (and their positive roots) have important applications in the study of indecomposable representations of posets, tame algebras, tame vector space categories and tame bimodule matrix problems, see [1], [4], [8], [10], [20], [22], [28], [30]–[34], [39]. [43].

Here we use the terminology and notation introduced in [38]. In particular, by an **edge-bipartite graph** (bigraph, in short) we mean a pair $\Delta = (\Delta_0, \Delta_1)$, where Δ_0 is a finite non-empty set of vertices and Δ_1 is a finite set of edges equipped with a disjoint union bipartition $\Delta_1 = \Delta_1^- \cup \Delta_1^+$ such that the set

$$\Delta_1(i,j) = \Delta_1^-(i,j) \cup \Delta_1^+(i,j)$$

of edges connecting the vertices i and j does not contain edges lying in $\Delta_1^-(i,j) \cap \Delta_1^+(i,j)$, for each pair of vertices $i, j \in \Delta_0$, and either $\Delta_1(i,j) = \Delta_1^-(i,j)$ or $\Delta_1(i,j) = \Delta_1^+(i,j)$. Note that the edge-bipartite graph Δ can be viewed as signed multi-graph satisfying a separation property, see [38] and [42].

We define an edge-bipartite graph $\Delta = (\Delta_0, \Delta_1)$ to be **simply-laced** if

(b1) the set $\Delta_1(i,j)$ contains at most one edge, for each pair of vertices $i, j \in \Delta_0$, and

(b2) $\Delta_1(j,j)$ is empty, for any $j \in \Delta_0$, that is, Δ is loop-free.
We say that Δ has no isolated loop if $|\Delta_1^-(j,j)| \neq 1$, for $j = 1,\dots,n$. If Δ has no loop, we call it **loop-free**.

We visualize Δ as a (multi-)graph in a Euclidean space $\mathbb{R}^m$, $m \geq 2$, with the vertices numbered by the integers $1,\dots,n$; usually, we simply write $\Delta_0 = \{1,\dots,n\}$. Any edge in $\Delta_1^-(i,j)$ is visualised as a continuous one $\bullet_i \!\!-\!\!-\!\! \bullet_j$, and any edge in $\Delta_1^+(i,j)$ is visualised as a dotted one $\bullet_i\text{- - -}\bullet_j$.

We view any finite graph $\Delta = (\Delta_0, \Delta_1)$ as an edge-bipartite one by setting $\Delta_1^-(i,j) = \Delta_1(i,j)$ and $\Delta_1^+(i,j) = \emptyset$, for each pair of vertices $i, j \in \Delta_0$.

We denote by $\mathcal{B}igr_n$ the category of finite edge-bipartite graphs, with $n \geq 2$ vertices, and usual edge-bipartite graph maps as morphisms, see [38] for details. We denote by $\mathcal{UB}igr_n$ the full subcategory of $\mathcal{B}igr_n$ whose objects are the loop-free edge-bipartite graphs.

Following the representation theory of finite-dimensional K-algebras over an algebraically closed field K (see [1], [8],[31], [39]) and the mesh-geometry study of roots of integral unit quadratic forms described in [35]–[37], the edge-bipartite graphs $\Delta \in \mathcal{UB}igr_n$ are studied in [38] by means of the Coxeter(-Gram) transformation $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$ (and its spectral properties, compare with [7]) associated with the non-symmetric **adjacency matrix** $\check{D}_\Delta$ and the non-symmetric **Gram matrix** $\check{G}_\Delta$ of Δ defined as follows.

Definition 1.1 (see [38]). *Let $\Delta = (\Delta_0, \Delta_1)$ be an edge-bipartite graph in $\mathcal{B}igr_n$, with $\Delta_0 = \{a_1,\dots,a_n\}$, $n \geq 1$, and the bipartition $\Delta_1 = \Delta_1^- \cup \Delta_1^+$.*

(a) *The non-symmetric* **adjacency matrix** $\check{D}_\Delta$ *and the non-symmetric* **Gram matrix** $\check{G}_\Delta$ *of Δ are defined to be the square matrices*

$$\check{D}_\Delta = \begin{bmatrix} d_{11}^\Delta & d_{12}^\Delta & \dots & d_{1n}^\Delta \\ 0 & d_{22}^\Delta & \dots & d_{2n}^\Delta \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn}^\Delta \end{bmatrix}, \quad \check{G}_\Delta = E + \check{D}_\Delta = \begin{bmatrix} 1+d_{11}^\Delta & d_{12}^\Delta & \dots & d_{1n}^\Delta \\ 0 & 1+d_{22}^\Delta & \dots & d_{2n}^\Delta \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1+d_{nn}^\Delta \end{bmatrix}$$

where $d_{ij}^\Delta = -|\Delta_1^-(a_i,a_j)|$, if there is an edge $a_i \!-\!\!-\!\!-\! a_j$ and $i \leq j$, $d_{ij}^\Delta = |\Delta_1^+(a_i,a_j)|$, if there is an edge $a_i\text{- - -}a_j$ and $i \leq j$. We set $d_{ij}^\Delta = 0$, if $\Delta_1(a_i,a_j)$ is empty or $j < i$.

(b) *The matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ is called the* **symmetric Gram matrix** *of Δ. By the* **symmetric adjacency matrix** (*or, signed adjacency matrix, see* [42]) *of $\Delta \in \mathcal{UB}igr_n$ we mean the symmetric matrix*

$$Ad_\Delta := \check{D}_\Delta + \check{D}_\Delta^{tr} \in \mathbb{M}_n(\mathbb{Z}).$$

The **spectrum** *of* $\Delta \in \mathcal{UB}igr_n$ *is defined to be the set* $\mathbf{spec}_\Delta \subset \mathbb{R}$ *of* n *real eigenvalues of the symmetric matrix* $Ad_\Delta \in \mathbb{M}_n(\mathbb{Z})$ (*see* [42]), *i.e., the set of all* n *roots of the polynomial*

$$P_\Delta(t) = \det(t \cdot E - Ad_\Delta) \in \mathbb{Z}[t],$$

called the **characteristic polynomial** *of the edge-bipartite graph* Δ.

(c) $\Delta = (\Delta_0, \Delta_1)$ *is defined to be* **positive** (*resp.* **non-negative**), *if the symmetric Gram matrix* $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ *of* Δ *is positive definite* (*resp. positive semi-definite*).

(d) *We define* Δ *to be* P-**critical**, *if* Δ *has no loop, is not positive, and any proper full* (*induced*) *edge-bipartite subgraph of* Δ *is positive.*

Any edge-bipartite graph $\Delta = (\Delta_0, \Delta_1) \in \mathcal{UB}igr_n$, with $\Delta_1 = \Delta_1^- \cup \Delta_1^+$, is uniquely determined by its non-symmetric adjacency matrix $\check{D}_\Delta$ and its non-symmetric Gram matrix $\check{G}_\Delta$.

If Δ is simply-laced, we have $d_{11}^\Delta = \ldots = d_{nn}^\Delta = 0$, $d_{ij}^\Delta = -1$, if $i < j$ and there is an edge a_i —— a_j, and $d_{ij}^\Delta = 1$, if $i < j$ and there is an edge a_i- - -a_j.

Throughout this paper, for simplicity of the presentation, we assume that $\Delta_0 = \{1, \ldots, n\}$, that is, $a_1 = 1, \ldots, a_n = n$.

In Sections 2 and 3, we study in details P-critical edge-bipartite graphs Δ, by applying unit quadratic form results obtained in [21] and [22]. In particular, in Theorem 2.8 we give a useful characterization of P-critical edge-bipartite graphs, and we present an algorithm in the form $(\Delta, w) \mapsto \Delta[[w]]$ that associates to any pair (Δ, w), with $\Delta \in \mathcal{UB}igr_n$ positive, $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ its sincere root (i.e., $w \cdot G_\Delta \cdot w^{tr} = 1$ and $w_1 \neq 0, \ldots, w_n \neq 0$), a P-critical edge-bipartite graph $\Delta[[w]]$ in $\mathcal{UB}igr_{n+1}$. We show that the map $(\Delta, w) \mapsto \Delta[[w]]$ is invariant under the orthogonal group action $\mathrm{O}(n, \mathbb{Z}) \times \mathcal{UB}igr_n \to \mathcal{UB}igr_n$ (2.1), and any P-critical edge-bipartite graph Δ' in $\mathcal{UB}igr_{n+1}$ has the form $\Delta' = \Delta[[w]]$.

We also present a package of algorithms for a Coxeter spectral analysis of graphs in $\mathcal{UB}igr_n$ and for computing the $\mathrm{O}(n, \mathbb{Z})$-orbits of P-critical graphs Δ in $\mathcal{UB}igr_n$, as well as the positive ones. By applying the package, symbolic computations in Maple and numerical computations in C#, we compute P-critical graphs in $\mathcal{UB}igr_n$, the connected positive ones, their unit quadratic forms, their Coxeter polynomials, and the $\mathrm{O}(n, \mathbb{Z})$-orbits, for $n \leq 10$. The computing results are presented in tables of Section 5.

Main results of the paper are announced in [24]-[25] and were presented at the Sixth European Conference on Combinatorics, Graphs Theory and Applications, EuroComb′11, Budapest, August 2011, and at International

Conference Combinatorics 2012, Perugia, September 2012, see also [26]. Some applications of the results presented here are given in [27].

2. Preliminaries

Given $n \geq 1$, we denote by

$$\mathrm{Gl}(n, \mathbb{Z}) = \{A \in \mathbb{M}_n(\mathbb{Z});\ \det A \in \{-1, 1\}\} \subset \mathbb{M}_n(\mathbb{Z})$$

the group of $\mathbb{Z}$-invertible matrices with integer coefficients, and by

$$\mathrm{O}(n, \mathbb{Z}) = \{B \in \mathbb{M}_n(\mathbb{Z});\ B \cdot B^{tr} = E\}$$

its subgroup formed by the orthogonal matrices. Recall from [36] that $\mathrm{O}(n, \mathbb{Z})$ is generated by:

• the diagonal matrices $\widehat{\varepsilon}^{(j)} = \mathrm{diag}(1, \ldots, 1, -1_j, 1, \ldots, 1) \in \mathbb{M}_n(\mathbb{Z})$, where $1 \leq j \leq n$, $-1_j = -1$ is the jth coordinate of the vector $\varepsilon^{(j)} = \mathrm{diag}(1, \ldots, 1, -1_j, 1, \ldots, 1)$, and

• the permutation matrices $\widehat{\sigma} = M_\sigma$, where $\sigma \in \mathbf{S}_n$ is a permutation of the set $\{1, \ldots, n\}$.

We define the right action

$$* : \mathcal{B}igr_n \times \mathrm{O}(n, \mathbb{Z}) \longrightarrow \mathcal{B}igr_n \tag{2.1}$$

of the orthogonal group $\mathrm{O}(n, \mathbb{Z})$ on $\mathcal{B}igr_n$ as follows. Let $\Delta = (\Delta_0, \Delta_1)$ be an edge-bipartite graph in $\mathcal{B}igr_n$, with $n \geq 2$.

(i) If $\widehat{\varepsilon}^{(j)} = \mathrm{diag}(1, \ldots, 1, -1_j, 1, \ldots, 1)$, where $1 \leq j \leq n$, we define $\Delta * \widehat{\varepsilon}^{(j)}$ to be the graph in $\mathcal{B}igr_n$ obtained from Δ by replacing every dotted edge $\bullet_i$- - -$\bullet_j$, with $i \neq j$, by a continuous one, and every continuous edge $\bullet_s$—— $\bullet_j$, with $s \neq j$, by a dotted one. The remaining edges (in particular the loops at j) remain unchanged.

(ii) If $\widehat{\sigma} = M_\sigma$ is a permutation matrix defined by $\sigma \in \mathbf{S}_n$, we define $\Delta * \widehat{\sigma}$ to be the the graph in $\mathcal{B}igr_n$ obtained from Δ by the permutation σ^{-1} of the vertices of Δ as well as the corresponding edges between them.

It is shown in [38] that (i) and (ii) uniquely define the action (2.1), because every matrix $B \in \mathrm{O}(n, \mathbb{Z})$ has a unique decomposition $B = \widehat{\sigma} \cdot \widehat{\varepsilon}$, where $\widehat{\sigma} \in \widehat{\mathbf{S}}_n := \{\widehat{\sigma} \in \mathrm{O}(n, \mathbb{Z});\ \sigma \in \mathbf{S}_n\}$ and $\widehat{\varepsilon}$ is a finite product of matrices of the type $\widehat{\varepsilon}^{(j)}$, see [36, Lemma 2.3].

It is easy to see that the full subcategories $\mathcal{SLB}igr_n$ and $\mathcal{UB}igr_n$ of $\mathcal{B}igr_n$, whose objects are the simply-laced edge-bipartite graphs and the loop-free edge-bipartite graphs, are $\mathrm{O}(n, \mathbb{Z})$-invariant.

Assume that Δ has no isolated loop. The **Coxeter polynomial** of Δ is the characteristic polynomial

$$\mathrm{cox}_\Delta(t) = \det(t \cdot E - \mathrm{Cox}_\Delta) \tag{2.2}$$

of the **Coxeter matrix**

$$\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr} \in \mathrm{Gl}(n, \mathbb{Z})$$

of Δ. The group automorphism $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by the formula $\Phi_\Delta(v) = v \cdot \mathrm{Cox}_\Delta$ is called the **Coxeter transformation** of Δ, compare with Drozd [8]. The set $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ of n complex eigenvalues of Φ_Δ is called the **Coxeter spectrum** of Δ. The order $\mathbf{c}_\Delta$ of Cox_Δ in $\mathrm{Gl}(n, \mathbb{Z})$ is called the **Coxeter number** of Δ. In other words, $\mathbf{c}_\Delta$ is a minimal integer $c \geq 1$ such that $\mathrm{Cox}_\Delta^c = E$. If there is no such integer, we set $\mathbf{c}_\Delta = \infty$.

It is shown in [38] that the matrices $\check{G}_\Delta$, Cox_Δ, the Coxeter polynomial $\mathrm{cox}_\Delta(t)$, and the Coxeter number $\mathbf{c}_\Delta$ depend on the numbering of the vertices of the bigraph Δ. Moreover, the Coxeter spectrum $\mathbf{specc}_\Delta$ of Δ is a subset of the unit circle, if Δ is non-negative and loop-free.

We associate with $\Delta \in \mathcal{UB}igr_n$ the **Gram forms** $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$ and $b_\Delta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula

$$\begin{aligned} q_\Delta(x) &= x \cdot \check{G}_\Delta \cdot x^{tr} = x_1^2 + \cdots + x_n^2 + \textstyle\sum_{i<j} d_{ij}^\Delta x_i x_j, \\ b_\Delta(x, y) &= x \cdot \check{G}_\Delta \cdot y^{tr}. \end{aligned} \tag{2.3}$$

The set of roots of Δ (and of q_Δ) is defined to be the set $\mathcal{R}_\Delta = \{v \in \mathbb{Z}^n;\ q_\Delta(v) = 1\}$. The set $\mathrm{Ker}\, q_\Delta = \{v \in \mathbb{Z}^n;\ q_\Delta(v) = 0\}$ is called the **kernel** of q_Δ.

In the study of the subcategory $\mathcal{UB}igr_n$ of $\mathcal{B}igr_n$ consisting of the loop-free edge-bipartite graphs Δ, we essentially use the results on unit integral quadratic forms obtained in [3], [9], [12], [21]-[22], [28]-[29], [33]-[36], [39]. For this purpose we recall some definitions.

A unit integral quadratic form is a map $q : \mathbb{Z}^n \longrightarrow \mathbb{Z},\ n \geq 1$, defined by the formula

$$q(x) = q(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2 + \textstyle\sum_{i<j} q_{ij} x_i x_j, \tag{2.4}$$

where $q_{ij} \in \mathbb{Z}$, for $i < j$. Obviously, q is uniquely determined by its **non-symmetric Gram matrix**

$$\check{G}_q = \begin{bmatrix} 1 & q_{12} & \cdots & q_{1n} \\ 0 & 1 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{M}_n(\mathbb{Z}) \tag{2.5}$$

and by the **symmetric Gram matrix** $G_q = \frac{1}{2}[\check{G}_q + \check{G}_q^{tr}]$ of q, because $q(x) = x \cdot \check{G}_q \cdot x^{tr} = x \cdot G_q \cdot x^{tr}$, where $\check{G}_q^{tr}$ means the transpose of $\check{G}_q$ and $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. The **polar form** of q is the symmetric $\mathbb{Z}$-bilinear form $b_q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \frac{1}{2} \cdot \mathbb{Z}$ defined by the formula

$$b_q(x, y) = x \cdot G_q \cdot y^{tr} = \frac{1}{2}[q(x + y) - q(x) - q(y)],$$

where the vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ are viewed as one-row matrices, and we set $y^{tr} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

We call q **positive** (resp. non-negative) if $q(v) > 0$ (resp. $q(v) \geq 0$), for all non-zero vectors $v \in \mathbb{Z}^n$. A vector $v \in \mathbb{Z}^n$ is said to be a q-**root** (of unity), if $q(v) = v \cdot G_q \cdot v^{tr} = 1$. A vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ is said to be **sincere**, if $v_1 \neq 0, \ldots, v_n \neq 0$. We say that v is **positive**, if $v \neq 0$ and $v_1 \geq 0, \ldots, v_n \geq 0$. Given $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and $s \in \{1, \ldots, n\}$, we set

$$v^{(s)} := (v_1, \ldots, v_{s-1}, v_{s+1}, \ldots, v_n) \in \mathbb{Z}^{n-1}.$$

Given $n \geq 1$, we denote by $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ the set of all unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$.

We recall from [36, Section 2], that the **Coxeter(-Gram) polynomial** of $q : \mathbb{Z}^n \to \mathbb{Z}$ is the characteristic polynomial

$$\mathrm{cox}_q(t) := \det(t \cdot E - \mathrm{Cox}_q) \in \mathbb{Z}[t]$$

of the **Coxeter(-Gram) matrix**

$$\mathrm{Cox}_q := -\check{G}_q \cdot \check{G}_q^{-tr}.$$

The **Coxeter number** of q is the order $\mathbf{c}_q$ of the matrix Cox_q in the group $\mathrm{Gl}(n, \mathbb{Z})$, i.e., a minimal integer $c \geq 1$ such that $\mathrm{Cox}_q^c = E$. If there is no such integer, we set $\mathbf{c}_q = \infty$.

To any unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, we associate in [36, Section 2] an edge-bipartite graph $\mathbf{bigr}(q)$ of q with the vertices $1, \ldots, n$ as follows. Two vertices $s \neq j$ are joined by $|q_{sj}|$ continuous edges $\bullet_s$—— $\bullet_j$ if $q_{sj} < 0$, and by q_{sj} dotted edges $\bullet_s$- - -$\bullet_j$ if $q_{sj} > 0$. There is no edge between s and j, if $q_{sj} = 0$, or $s = j$. We say that q is **connected** if the graph $\mathbf{bigr}(q)$ is connected.

We recall from [36] that a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, with $n \geq 2$, is defined to be P-**critical** (critical with respect to the positivity) if q is not positive, and each of the restrictions

$$q^{(1)}, \ldots, q^{(n)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$$

of q is positive, where $q^{(j)} = q|_{x_j=0}$. The form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is defined to be **principal** if q is non-negative and $\operatorname{Ker} q = \{v \in \mathbb{Z}^n;\ q(v) = 0\}$ is an infinite cyclic subgroup of $\mathbb{Z}^n$.

Positive unit forms, P-critical unit forms and their roots are essentially applied in the representation theory of finite-dimensional algebras R, coalgebras C, and their derived categories $\mathcal{D}^b(\text{mod } R)$, see [1], [11], [17], [31]–[35], [39], [43]. They are mainly used as a computational tool for determining the representation type (finite, tame, wild) of an algebra R and in a combinatorial description of the Auslander-Reiten quiver $\Gamma(\text{mod } R)$ of its module category mod R, see [1], [10], [31], [34]-[35], [39]-[40], and [43].

Given $a \in \Delta_0$, we denote by $\Delta^{(a)}$ the edge-bipartite graph obtained from Δ by removing the vertex a together with all edges $\bullet_i$- - - -$\bullet_a$ and $\bullet_a$—— $\bullet_j$ connected with a.

In the study of the category $\mathcal{UB}igr_n$, the following result is of importance, see also [38].

Proposition 2.6. *The correspondence $\Delta \mapsto q_\Delta$ defines a bijection*

$$q_\bullet : \mathcal{UB}igr_n \xrightarrow{1-1} \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \tag{2.7}$$

with the following properties:

(a) $\check{G}_{q_\Delta} = \check{G}_\Delta$, $G_{q_\Delta} = G_\Delta$, $\mathcal{R}_{q_\Delta} = \mathcal{R}_\Delta$, $\mathrm{Cox}_{q_\Delta} = \mathrm{Cox}_\Delta$, $\mathbf{c}_{q_\Delta} = \mathbf{c}_\Delta$, *and* $\mathrm{cox}_{q_\Delta}(t) = \mathrm{cox}_\Delta(t)$,

(b) *an edge bipartite graph $\Delta \in \mathcal{UB}igr_n$ is positive* (*resp. non-negative, principal, P-critical*) *if and only if the unit form $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$ is positive* (*resp. non-negative, principal, P-critical*).

Proof. It is easy to see that $q_\Delta^{(a)} = q_{\Delta^{(a)}}$, for $a = 1, \ldots, n$. In view of [38, Lemma 2.1], the unit form q_Δ is P-critical if and only if Δ is not positive and the edge bipartite graphs $\Delta^{(1)}, \ldots, \Delta^{(n)}$ are positive, or equivalently, every full edge-bipartite subgraph Δ' of Δ is positive, apply [38, Lemma 2.1(c)] and its proof. The remaining properties of $q_\bullet$ follow by applying the foregoing definitions. □

Throughout the paper, we often use the bijection (2.7) as an identification $\Delta \equiv q_\Delta$. In particular, we often describe the bigraph Δ by presenting its unit form q_Δ, or the matrix $\check{G}_\Delta$.

Now we are able to present a useful characterization of P-critical edge-bipartite graphs.

Theorem 2.8. *Assume that Δ is a loop-free edge-bipartite graph in $\mathcal{UB}igr_n$, with $n \geq 3$. The following four conditions are equivalent.*

(a) *Δ is P-critical.*

(b) *The bigraph Δ is non-negative and the free abelian group* $\operatorname{Ker} q_\Delta = \{v \in \mathbb{Z}^n, q(v) = 0\}$ *is infinite cyclic generated by a sincere vector $\mathbf{h}_\Delta = (h_1, \dots, h_n)$, such that $-6 \leq h_j \leq 6$, for all $j \in \{1, \dots, n\}$, and $h_s \in \{-1, 1\}$, for some $s \in \{1, \dots, n\}$.*

(c) *The set $\mathcal{R}_\Delta := \{v \in \mathbb{Z}^n;\ q_\Delta(v) = 1\}$ of roots of Δ is infinite, and each of the subbigraphs $\Delta^{(1)}, \dots, \Delta^{(n)}$ of Δ has only finitely many roots.*

(d) *There exist a Euclidean diagram $D\Delta \in \{\widetilde{\mathbb{A}}_{n-1}, \widetilde{\mathbb{D}}_{n-1}, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$, see* (5.1)*, and a group isomorphism $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $q_\Delta \circ T$ is the quadratic form $q_{D\Delta} : \mathbb{Z}^n \to \mathbb{Z}$, $n = |D\Delta_0|$, of the diagram $D\Delta$ and T carries a sincere vector $\mathbf{h}' \in \operatorname{Ker} q_{D\Delta}$ to a sincere vector in* $\operatorname{Ker} q_\Delta$.

Proof. Since we have $q_\Delta^{(a)} = q_{\Delta^{(a)}}$, for $a = 1, \dots, n$, then $\mathcal{R}_{q_\Delta^{(a)}} = \mathcal{R}_{\Delta^{(a)}}$. On the other hand, by Proposition 2.6, Δ is P-critical if and only if the unit form q_Δ is P-critical. Then the equivalence of (a)–(d) follows by applying [21, Theorem 2.3] and [38]. □

In the classification of P-critical unit forms $q \in \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$, we use the right action

$$* : \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \times \mathrm{O}(n, \mathbb{Z}) \longrightarrow \mathcal{U}(\mathbb{Z}^n, \mathbb{Z}) \qquad (2.9)$$

(defined in [36, Section 2]), that associates to $q \in \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ and $B \in \mathrm{O}(n, \mathbb{Z})$ the unit form $q * B : \mathbb{Z}^n \to \mathbb{Z}$ by setting $(q * B)(x) = q(x \cdot B^{tr})$, for $x \in \mathbb{Z}^n$ (see also [21]). It is easy to see that the action (2.9) coincides with the action (2.1) under the identification $q_\bullet : \mathcal{UB}igr_n \xrightarrow{1-1} \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ (2.7).

By [36, Lemma 2.3], the group $\mathrm{O}(n, \mathbb{Z})$ is generated by the following two subgroups:

- the group $\widehat{\mathbf{C}}_\mathbf{2}^\mathbf{n}$ of all matrices $\widehat{\varepsilon} = \varepsilon \cdot E$, where $E \in \mathbb{M}_n(\mathbb{Z})$ is the identity matrix and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbf{C}_\mathbf{2}^\mathbf{n}$ runs through all vectors with coefficients $\varepsilon_1, \dots, \varepsilon_n \in \mathbf{C_2} = \{-1, 1\}$, the cyclic group of order two, and
- the group $\widehat{\mathbf{S}}_\mathbf{n}$ of all matrices $\hat{\sigma} = M_\sigma$ of the group homomorphisms $\sigma : \mathbb{Z}^n \to \mathbb{Z}^n$ given by the permutation $\sigma \in \mathbf{S}_n$ and defined by $\sigma(x) = x \cdot M_\sigma^{tr} = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $\mathbf{S}_n$ is the symmetric group of order $n!$.

Moreover, every matrix $B \in \mathrm{O}(n,\mathbb{Z})$ has a unique form $B = \widehat{\sigma} \cdot \widehat{\varepsilon}$, with $\widehat{\sigma} \in \widehat{\mathbf{S}}_n$ and $\widehat{\varepsilon} \in \widehat{\mathbf{C}}_2^n$, the group $\mathrm{O}(n,\mathbb{Z})$ is finite of order $|\mathrm{O}(n,\mathbb{Z})| = n! \cdot 2^n$, and $\mathrm{O}(n,\mathbb{Z})$ admits the semi-direct product decomposition $\mathrm{O}(n,\mathbb{Z}) = \widehat{\mathbf{S}}_n \rtimes \widehat{\mathbf{C}}_2^n$, with respect to the right group action $\bullet : \widehat{\mathbf{C}}_2^n \times \widehat{\mathbf{S}}_n \to \widehat{\mathbf{C}}_2^n$ of $\widehat{\mathbf{S}}_n$ on the group $\widehat{\mathbf{C}}_2^n$ defined by the formula $\widehat{\varepsilon}\bullet\widehat{\sigma} = \mathrm{diag}(\varepsilon_{\sigma^{-1}(1)}, \ldots, \varepsilon_{\sigma^{-1}(n)})$. Note that $\widehat{\varepsilon} \cdot \widehat{\sigma} = \widehat{\sigma} \cdot (\widehat{\varepsilon}\bullet\widehat{\sigma})$, for all $\widehat{\varepsilon} \in \widehat{\mathbf{C}}_2^n$ and $\widehat{\sigma} \in \widehat{\mathbf{S}}_n$.

It follows from the description of $\mathrm{O}(n,\mathbb{Z})$ given above that $q * B \in \mathcal{U}(\mathbb{Z}^n,\mathbb{Z})$, if $q \in \mathcal{U}(\mathbb{Z}^n,\mathbb{Z})$ and $B \in \mathrm{O}(n,\mathbb{Z})$. It is easy to see that the set P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$ of all P-critical unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$, and the set $\mathbf{posit}(\mathbb{Z}^n,\mathbb{Z})$ of all positive unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$ are $\mathrm{O}(n,\mathbb{Z})$-invariant subsets of $\mathcal{U}(\mathbb{Z}^n,\mathbb{Z})$.

Following [21] and [38], we investigate recursive algorithmic procedures that construct all $\mathrm{O}(n,\mathbb{Z})$-orbits of the P-critical bigraphs and $\mathrm{O}(n,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n \geq 3$. We do it by applying the following correspondence

$$\mathrm{ind}_{s,\varepsilon_s} : \mathcal{Z}_{n-1} \longrightarrow P\text{-}\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z}), \quad (p,w) \mapsto q := q_{p,s,w,\varepsilon_s} \tag{2.10}$$

defined by the formula (3.2), from the set

$$\mathcal{Z}_{n-1} = \{(p,w);\ p \in \mathbf{posit}(\mathbb{Z}^{n-1},\mathbb{Z}), w \in \mathbb{Z}^{n-1} \text{ a sincere root of } p\}, \tag{2.11}$$

to the set P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$ of all P-critical unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$. We show that $\mathrm{ind}_{s,\varepsilon_s}$ (2.10) defines a surjective map

$$\mathrm{ind} : \mathrm{O}(n\text{-}1,\mathbb{Z})\text{-}\mathcal{O}rb(\mathcal{Z}_{n-1}) \longrightarrow \mathrm{O}(n,\mathbb{Z})\text{-}\mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})),$$

with $n \geq 3$, between the set of $\mathrm{O}(n\text{-}1,\mathbb{Z})$-orbits in $\mathcal{Z}_{n-1}$ and the set of $\mathrm{O}(n,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$. By applying a package of algorithms presented in Section 4, symbolic computations in Maple, and numerical computations in C#, we compute in Section 5 all P-critical unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$, together with their Coxeter-Gram polynomials and the $\mathrm{O}(n,\mathbb{Z})$-orbits, for $n \leq 10$.

Under the identification $q_\bullet : \mathcal{UB}igr_{n-1} \xrightarrow{1-1} \mathcal{U}(\mathbb{Z}^{n-1},\mathbb{Z})$ (2.7), we can identify the set $\mathcal{Z}_{n-1}$ with the set

$$\mathcal{Z}'_{n-1} = \{(\Delta,w);\ \Delta \in \mathcal{UB}igr_{n-1} \text{ positive}, w \in \mathbb{Z}^{n-1} \text{ a sincere root of } \Delta\}, \tag{2.12}$$

$n \geq 3$, and the correspondence $\mathrm{ind}_{s,\varepsilon_s}$ (2.10) can be viewed as a map

$$\mathrm{ind}_{s,\varepsilon_s} : \mathcal{Z}'_{n-1} \longrightarrow \mathcal{UB}igr_n, \quad (\Delta,w) \mapsto \Delta[[w,s,\varepsilon_s]], \tag{2.13}$$

see (3.6). It follows from our next result that $\Delta[[w,s,\varepsilon_s]]$ is a P-critical bigraph.

A connection between P-critical bigraphs and positive ones, and between P-critical unit forms and positive ones is given by the following result proved in [21, Section 3], see also [12].

Proposition 2.14. *Assume that $q : \mathbb{Z}^n \to \mathbb{Z}$ is a P-critical unit form* (2.4)*, with $n \geq 3$, $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$ and $s \in \{1, \dots, n\}$ is such that $h_s \in \{-1, 1\}$.*

(a) *The vector $\mathbf{h}^{(s)} := (h_1, \dots, h_{s-1}, h_{s+1}, \dots h_n) \in \mathbb{Z}^{n-1}$ is a sincere root of the positive unit form $q^{(s)} : \mathbb{Z}^{n-1} \to \mathbb{Z}$.*

(b) *The form q can be reconstructed from the triple $(q^{(s)}, s, \mathbf{h}^{(s)})$ by the formula $q(x) = q^{(s)}(x^{(s)}) + x_s^2 - 2 \cdot b_{q^{(s)}}(x^{(s)}, \mathbf{h}^{(s)}) \cdot h_s \cdot x_s$, where $x^{(s)} = (x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_n) \in \mathbb{Z}^{n-1}$ and $b_{q^{(s)}}$ is the symmetric bilinear polar form of $q^{(s)}(x^{(s)}) = q^{(s)}(x_1, \dots, x_{s-1}, x_{s+1}, \dots x_n)$.*

3. Main theorem and main algorithm

For $n \geq 3$, the correspondence $\operatorname{ind}_{s,\varepsilon_s} : \mathcal{Z}_{n-1} \longrightarrow P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ (2.10) between positive unit forms $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$, with a sincere root, and P-critical forms $q : \mathbb{Z}^n \to \mathbb{Z}$, and the correspondence (2.13) between positive loop-free bigraphs Δ, with a sincere root, and P-critical bigraphs are described in the following theorem (its proof is outlined in [21]).

Theorem 3.1. (a) *Given $n \geq 3$, $s \in \{0, 1, \dots, n-1\}$, $\varepsilon_s \in \{-1, 1\}$, a positive connected loop-free bigraph $\Delta \in \mathcal{UB}igr_{n-1}$, with the unit form $p := q_\Delta : \mathbb{Z}^{n-1} \to \mathbb{Z}$ and a sincere root $w = (w_1, \dots, w_{n-1}) \in \mathbb{Z}^{n-1}$ of Δ and of $p := q_\Delta$, the bigraph Δ' corresponding via* (2.7) *to the unit form $q_{\Delta'} := q_{p,s,w,\varepsilon_s} : \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula*

$$q(x) := q_{\Delta'}(x_1, \dots, x_n) = p(x^{(s)}) + x_s^2 - 2 \cdot b_p(x^{(s)}, w) \cdot \varepsilon_s \cdot x_s, \qquad (3.2)$$

is P-critical and $\operatorname{Ker} q_{\Delta'} = \mathbb{Z} \cdot \widehat{w}^{\varepsilon_s}$, where $b_p : \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \to \mathbb{Z}$ is the symmetric polar form of p and $\widehat{w}^{\varepsilon_s} = (w_1, \dots, w_{s-1}, \varepsilon_s, w_s, \dots w_{n-1}) \in \mathbb{Z}^n$.

(b) *The set $\mathcal{Z}_{n-1}$* (2.11) *is an $\mathrm{O}(n, \mathbb{Z})$-invariant subset of $\mathbf{posit}(\mathbb{Z}^{n-1}, \mathbb{Z}) \times \mathbb{Z}^{n-1}$ under the action $(p, w) * B := (p * B, w \cdot B)$, with $B \in \mathrm{O}(n, \mathbb{Z})$. The map $(p, w) \mapsto \operatorname{ind}_{s,\varepsilon_s}(p, w) := q_{p,s,w,\varepsilon_s}$ described in* (3.2) *defines a surjection*

$$\operatorname{ind} : \mathrm{O}(n{-}1, \mathbb{Z})\text{-}\mathcal{O}rb(\mathcal{Z}_{n-1}) \longrightarrow \mathrm{O}(n, \mathbb{Z})\text{-}\mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})) \quad (3.3)$$

between the set of $\mathrm{O}(n{-}1, \mathbb{Z})$-orbits of $\mathcal{Z}_{n-1}$ and the set of $\mathrm{O}(n, \mathbb{Z})$-orbits in the set $P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ of P-critical unit forms.

(c) *A right inverse of* ind *is given by* $q \mapsto \mathrm{res}_s(q) := (q^{(s)}, \mathbf{h}^{(s)})$ *defined in* (a)*, that associates to any P-critical form* $q = q_\Delta$*, with* $\mathrm{Ker}\, q = \mathbb{Z} \cdot \mathbf{h}$ *and* $h_s \in \{-1, 1\}$*, the pair* $(q^{(s)}, \mathbf{h}^{(s)}) \in \mathcal{Z}_{n-1}$.

Proof. Assume that $n \geq 3$, Δ is a connected positive loop-free bigraph in $\mathcal{UB}igr_{n-1}$, $p := q_\Delta : \mathbb{Z}^{n-1} \to \mathbb{Z}$ is its unit form and $w = (w_1, \dots, w_{n-1}) \in \mathbb{Z}^{n-1}$ a sincere root of p. Let $\Delta' \in \mathcal{UB}igr_n$ be the bigraph corresponding via (2.7) to the unit form $q := q_{p,s,w,\varepsilon_s} : \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula (3.2), that is, $q_{\Delta'} = q_{p,s,w,\varepsilon_s}$. Throughout the proof, we set $p := q_\Delta$ and $q = q_{\Delta'} = q_{p,s,w,\varepsilon_s}$.

(a) Assume that $p \in \mathbf{posit}(\mathbb{Z}^{n-1}, \mathbb{Z})$, $w \in \mathbb{Z}^{n-1}$ is a sincere root of p, $s \in \{0, 1, \dots, n\}$, $\varepsilon_s \in \{-1, 1\}$, and $q := q_{p,s,w,\varepsilon_s} : \mathbb{Z}^n \to \mathbb{Z}$ is defined by the formula (3.2).

We prove that q is P-critical by showing that q is non-negative and Ker $q = \mathbb{Z} \cdot \mathbf{h}$, where $\mathbf{h}$ is sincere vector, see Theorem 2.8. For simplicity of the presentation, we assume that $s = n$, that is, q is defined by the formula

$$q(x_1, \dots, x_n) = p(x^{(n)}) + x_n^2 - 2 \cdot b_p(x^{(n)}, w) \cdot \varepsilon_n \cdot x_n.$$

It follows that $q^{(n)} = p$, $\check{G}_{q^{(n)}} = \check{G}_p$, $G_{q^{(n)}} = G_p$, and

$$\check{G}_q = \left[\begin{array}{c|c} \check{G}_p & -2\check{G}_p \cdot w^{tr} \cdot \varepsilon_n \\ \hline 0 & 1 \end{array}\right] \text{ and } G_q = \left[\begin{array}{c|c} G_p & -G_p \cdot w^{tr} \cdot \varepsilon_n \\ \hline -w \cdot G_p \cdot \varepsilon_n & 1 \end{array}\right]$$

are the non-symmetric and the symmetric Gram matrix of q, respectively. We split the proof in three steps.

Step 1.1° First, we show that det $G_q = 0$ and q is not positive. Since $q^{(n)} = p$ is positive and $v \cdot G_p \cdot v^{tr} = p(v)$, for all $v \in \mathbb{Z}^{n-1}$, then $\det G_p > 0$, and, in view of the Cauchy theorem, the obvious equality

$$\left[\begin{array}{c|c} G_p & -G_p \cdot w^{tr} \cdot \varepsilon_n \\ \hline -w \cdot G_p \cdot \varepsilon_n & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} G_p & 0 \\ \hline -w \cdot G_p \cdot \varepsilon_n & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} E & -G_p^{-1} G_p \cdot w^{tr} \cdot \varepsilon_n \\ \hline 0 & 1 - (-wG_p\varepsilon_n) \cdot G_p^{-1} \cdot (-G_p w^{tr} \varepsilon_n) \end{array}\right]$$

yields

$$\det G_q = (\det G_p) \cdot (1 - w \cdot G_p \cdot w^{tr}) = (\det G_p) \cdot (1 - p(w)) = 0,$$

because $p(w) = 1$. Hence, by Sylvester′s criterion, the unit form q is not positive.

Step 1.2° We prove that the unit form q is non-negative. Consider the rational subspace $W = \mathbb{Q}^{n-1} \times \{0\}$ of $\mathbb{Q}^n$. Since $q^{(n)} = p$ then $b_{q^{(n)}} = b_p$ and the rational bilinear form $b_q|_W = b_p : \mathbb{Q}^{n-1} \times \mathbb{Q}^{n-1} \to \mathbb{Q}$

is positive. Hence, there is a direct sum decomposition $\mathbb{Q}^n = W \oplus W^\perp$, where $W^\perp = \{v \in \mathbb{Q}^n;\ b_q(v,w) = 0, \text{ for all } w \in W\}$ is the orthogonal complement of W. It follows that $W^\perp = \mathbb{Q}\cdot\eta$, for some non-zero vector $\eta \in W^\perp$.

(i) First we show that $b_q(\eta,\eta) = 0$. For, let $\mathbf{e} = \{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{Q}^n$. Note that the Gram matrix of q in the $\mathbb{Q}$-basic $\mathbf{e}' = \{e_1,\dots,e_{n-1},\eta\}$ of $\mathbb{Q}^n$ has the diagonal form

$$G_q^{\mathbf{e}'} = \left[\begin{array}{c|c} & b_q(e_1,\eta) \\ G_p & \vdots \\ & b_q(e_{n-1},\eta) \\ \hline b_q(\eta,e_1)\dots b_q(\eta,e_{n-1}) & b_q(\eta,\eta) \end{array}\right] = \left[\begin{array}{c|c} G_p & 0 \\ \hline 0 & b_q(\eta,\eta) \end{array}\right].$$

If $C \in \mathbb{M}_n(\mathbb{Q})$ is the translation matrix from $\mathbf{e}$ to $\mathbf{e}'$, then $\det C \neq 0$, $G_b^{\mathbf{e}'} = C^{tr}\cdot G_b \cdot C$ and Cauchy's theorem yields

$(\det G_p)\cdot b_q(\eta,\eta) = \det G_b^{\mathbf{e}'} = \det(C^{tr}\cdot G_b^{\mathbf{e}}\cdot C) = (\det C)^2\cdot(\det G_b) = 0$, because $\det G_b = 0$, by 1.1°. It follows that $b_q(\eta,\eta) = 0$, because p is positive and $\det G_p > 0$.

(ii) Next, by applying (i), we prove that $q : \mathbb{Q}^n \to \mathbb{Q}$ is non-negative, i.e., $q(v) \geq 0$, for each $v \in \mathbb{Q}^n$. Since $\mathbb{Q}^n = W \oplus \mathbb{Q}\cdot\eta$, the vector v has the form $v = w + \lambda\cdot\eta$, where $w \in W$, $\lambda \in \mathbb{Z}$, and we get

$$\begin{aligned} q(v) = q(w+\lambda\cdot\eta) &= b_q(w+\lambda\cdot\eta, w+\lambda\cdot\eta) \\ &= b_q(w,w) + \lambda^2\cdot b_q(\eta,\eta) + 2\lambda\cdot b_q(w,\eta) \\ &= p(w) \geq 0, \end{aligned}$$

because $b_q(\eta,\eta) = 0$, $b_q(w,\eta)$, $b_q(w,w) = q(w) = p(w) \geq 0$ and the positivity of $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ implies that the rational form $p : \mathbb{Q}^{n-1} \to \mathbb{Q}$ is positive. The proof of Step 1.2° is complete.

Step 1.3° We show that $\operatorname{Ker} q = \mathbb{Z}\cdot\widehat{w}^{\varepsilon_s}$, where

$$\widehat{w}^{\varepsilon_s} = (w_1,\dots,w_{s-1},\varepsilon_s,w_s,\dots w_{n-1}) \in \mathbb{Z}^n.$$

Since $q : \mathbb{Z}^n \to \mathbb{Z}$ is non-negative, then

$$\operatorname{Ker} q = \operatorname{rad} q = \bigcap_{i=1}^n \operatorname{Ker}\, b_q(e_i,-)$$

(see [36, Proposition 2.8]) and therefore $\operatorname{Ker} q$ coincides with the abelian subgroup $\mathcal{U}_\mathbb{Z} \subseteq \mathbb{Z}^n$ of all solutions $v \in \mathbb{Z}^n$ of the system of $\mathbb{Z}$-linear equations

$$\begin{cases} b_q(e_1,x) = 0 \\ \vdots \\ b_q(e_n,x) = 0 \end{cases} \tag{3.4}$$

Denote by $\mathcal{U}_\mathbb{Q}$ the rational subspace of $\mathbb{Q}^n$ of all solutions $v \in \mathbb{Q}^n$ of the system (3.4).

Since $\det G_q = 0$ and $\det G_{q^{(n)}} = \det G_p > 0$ then $\operatorname{rank}(G_q)=n-1$ and the system (3.4) has a non-zero solution $\xi \in \mathbb{Q}^n$ such that $\mathcal{U}_\mathbb{Q} = \mathbb{Q}\cdot\xi$, because $\dim_{\mathbb{Q}} \mathcal{U}_{\mathbb{Q}} = n - \operatorname{rank}(G_q) = 1$. Let $\xi = (\frac{k_1}{r_1}, \dots, \frac{k_n}{r_n})$, where $r_1 \dots, r_n \in \mathbb{N}\backslash\{0\}$ and $k_1, \dots, k_n \in \mathbb{Z}$. If $r = \operatorname{lcm}(r_1, \dots, r_n)$, then $\xi' = r \cdot \xi \in \mathbb{Z}^n$ is non-zero and $0 = q(\xi) = q(\frac{1}{r} \cdot \xi') = \frac{1}{r^2} \cdot q(\xi')$, that is, $\xi' \in \mathcal{U}_{\mathbb{Z}}$. Then the group $\mathcal{U}_{\mathbb{Z}} \subseteq \mathbb{Z}^n$ is non-zero and, hence, it is free of rank $\leq n$. It follows that $\mathcal{U}_{\mathbb{Z}}$ is of rank one, because $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{Q}} = \mathbb{Q} \cdot \xi$.

We show that $\operatorname{Ker} q = \mathcal{U}_{\mathbb{Z}} = \mathbb{Z} \cdot \widehat{w}^{\varepsilon_s}$. Since the formula (3.2) yields $q(\widehat{w}^{\varepsilon_s}) = p(w) + \varepsilon_s^2 - 2b_p(w, w) \cdot \varepsilon_s \cdot \varepsilon_s = 1 + 1 - 2 = 0$ then $\widehat{w}^{\varepsilon_s} \in \operatorname{Ker} q$ and $\operatorname{Ker} q \supseteq \mathbb{Z} \cdot \widehat{w}^{\varepsilon_s}$. To prove the inverse inclusion, assume that $\mathbf{h}$ is a $\mathbb{Z}$-generator of the rank one group $\operatorname{Ker} q = \mathcal{U}_{\mathbb{Z}}$, i.e., $\operatorname{Ker} q = \mathbb{Z} \cdot \mathbf{h}$. Then there exists $\lambda \in \mathbb{Z}$ such that $\widehat{w}^{\varepsilon_s} = \lambda \cdot \mathbf{h}$. Since $\mathbf{h} \in \mathbb{Z}^n$ and the sth coordinate of $\widehat{w}^{\varepsilon_s}$ equals $\varepsilon_s \in \{-1, 1\}$ then the equality $\varepsilon_s = \lambda \cdot h_s$ yields $\varepsilon_s, \lambda, h_s \in \{-1, 1\}$. It follows that $\widehat{w}^{\varepsilon_s}$ generates the group $\operatorname{Ker} q = \mathbb{Z} \cdot v$.

Since the vector $\widehat{w}^{\varepsilon_s}$ is sincere, q is non-negative, and $\operatorname{Ker} q = \mathbb{Z} \cdot \widehat{w}^{\varepsilon_s}$ then, by Theorem 2.8, q is P-critical and the proof of (a) is complete.

(b) We recall from [36] that given $q : \mathbb{Z}^n \to \mathbb{Z}$, $\widehat{\sigma} \in \widehat{\mathbf{S}}_n$, and $\widehat{\varepsilon} \in \widehat{\mathbf{C}}_2^n$, with $\varepsilon_1, \dots, \varepsilon_n \in \mathbf{C}_2 = \{-1, 1\}$, we have

(A) the form $(q * \widehat{\sigma})(x) = q(x \cdot \widehat{\sigma}^{tr}) = q(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is obtained from $q(x)$ by permuting the variables of x under $\sigma \in \mathbf{S}_n$, and

(B) $(q * \widehat{\varepsilon})(x) = q(x \cdot \widehat{\varepsilon}) = q(\varepsilon_1 \cdot x_1, \dots, \varepsilon_n \cdot x_n)$.

(C) two integral unit forms $q_1, q_2 : \mathbb{Z}^n \to \mathbb{Z}$ lie in the same $\mathrm{O}(n, \mathbb{Z})$-orbit if and only if $G_{q_1} = B^{tr} \cdot G_{q_2} \cdot B$, for some matrix $B \in \mathrm{O}(n, \mathbb{Z})$.

Assume that $p : \mathbb{Z}^{n-1} \to \mathbb{Z}$ is positive, $w \in \mathbb{Z}^{n-1}$ is a sincere root of p, and

$$q_1 := q_{p,s_1,w,\varepsilon_{s_1}}, \quad q_2 := q_{p,s_2,w,\varepsilon_{s_2}}$$

are P-critical forms associated with (p, w) by applying the construction (3.2), with some $s_1, s_2 \leq n$ and $\varepsilon_{s_1}, \varepsilon_{s_2} \in \mathbf{C}_2 = \{-1, 1\}$. We split the proof of (b) in several steps.

<u>Step 2.1°</u> We show that P-critical forms $q_1, q_2 : \mathbb{Z}^n \to \mathbb{Z}$ lie in the same $\mathrm{O}(n, \mathbb{Z})$-orbit.

2.1.1° First, we assume that $s := s_1 = s_2$ and $\varepsilon_{s_1} \neq \varepsilon_{s_2}$. It follows that $q_1(x)$ is obtained from $q_2(x)$ by $x_s \mapsto -x_s$, that is, $q_1 = q_2 * \widehat{\varepsilon}'$, where $\varepsilon'_s = -1$, and $\varepsilon'_j = 1$, for all $j \neq s$, see **(B)**. Consequently, the forms $q_1, q_2 : \mathbb{Z}^n \to \mathbb{Z}$ lie in the same $\mathrm{O}(n, \mathbb{Z})$-orbit.

2.1.2° Next, we assume that $s_1 \neq s_2$ and $\varepsilon_{s_1} = \varepsilon_{s_2}$. We find a matrix $B \in \mathrm{O}(n, \mathbb{Z})$ such that $G_{q_1} = B^{tr} \cdot G_{q_2} \cdot B$.

In case $s_1 = 1$ and $s_2 \in \{2, \dots, n\}$, we set

$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_i \\ e_1 \\ e_{i+1} \\ \vdots \\ e_n \end{bmatrix}, \text{ where } i = s_2.$$

For simplicity of the presentation, we assume that $s_2 = n$. Then $G_{q_1} = \left[\begin{array}{c|c} 1 & -w \cdot G_p \\ \hline -G_p \cdot w^{tr} & G_p \end{array}\right]$, $G_{q_2} = \left[\begin{array}{c|c} G_p & -G_p \cdot w^{tr} \\ \hline -w \cdot G_p & 1 \end{array}\right]$ are the symmetric Gram matrices of the forms q_1, q_2, respectively, and simple calculation yields

$B^{tr} \cdot G_{q_2} \cdot B =$

$$= \begin{bmatrix} 0 & 0 & 0\cdots0 & 1 \\ 1 & 0 & 0\cdots0 & 0 \\ 0 & 1 & 0\cdots0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\dots1 & 0 \end{bmatrix} \cdot \left[\begin{array}{c|c} G_p & -G_p \cdot w^{tr} \\ \hline -w \cdot G_p & 1 \end{array}\right] \cdot \begin{bmatrix} 0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 \\ \vdots & \vdots & \vdots & \ddots\ \vdots \\ 0 & 0 & 0 & \dots\ 1 \\ 1 & 0 & 0 & \dots\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} -w \cdot G_p & 1 \\ \hline G_p & -G_p \cdot w^{tr} \end{array}\right] \cdot \begin{bmatrix} 0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 \\ \vdots & \vdots & \vdots & \ddots\ \vdots \\ 0 & 0 & 0 & \dots\ 1 \\ 1 & 0 & 0 & \dots\ 0 \end{bmatrix} = G_{q_1}.$$

In case $s_1 \neq s_2$ and $s_1, s_2 \in \{2, \dots, n\}$, we set

$$B = \begin{bmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{i-1} \\ e_j \\ e_{i+1} \\ \vdots \\ e_{j-1} \\ e_i \\ e_{j+1} \\ \vdots \\ e_n \end{bmatrix} = \widehat{\tau},$$

where $i = s_1$, $j = s_2$, and τ is the transpose permutation (i, j). For simplicity of the presentation, we assume that $s_1 = n-1$, $s_2 = n$ and, given $a \leq k \leq n-1$ and $b \leq s \leq n-1$, we denote by $G_p^{[(a,k)(b,s)]}$ the $(k-a+1) \times (s-b+1)$ matrix obtained from G_p by removing the rows enumerated by $1, \dots, a-1, k+1, \dots, n-1$ and the columns enumerated by $1, \dots, b-1, s+1, \dots, n-1$.

Given $w \in \mathbb{Z}^n$ and $1 \le k \le s \le n$, we set $\widetilde{w} = -G_p \cdot w^{tr}$ and $\widetilde{w}^{(k,s)} = (w_k, \dots, w_s)$. Then

$$G_{q_1} = \left[\begin{array}{c|c|c} G_p^{[(1,n-2)(1,n-2)]} & \widetilde{w}^{(1,n-2)} & G_p^{[(1,n-2)(n-1,n-1)]} \\ \hline [\widetilde{w}^{(1,n-2)}]^{tr} & 1 & \widetilde{w}_{n-1}^{(n-1,n-1)} \\ \hline G_p^{[(n-1,n-1)(1,n-2)]} & \widetilde{w}_{n-1}^{(n-1,n-1)} & G_p^{[(n-1,n-1)(n-1,n-1)]} \end{array}\right], \text{and}$$

$$G_{q_2} = \left[\begin{array}{c|c} G_p & \widetilde{w} \\ \hline \widetilde{w}^{tr} & 1 \end{array}\right].$$

Note that $B^{tr} = B = \widehat{\tau}$ and, given a matrix $A \in \mathbb{M}_n(\mathbb{Z})$, the matrix $A \cdot B = A \cdot \widehat{\tau}$ (resp. $B \cdot A = \widehat{\tau} \cdot A$) is obtained from A by the transpose of its ith column with the jth column (resp. of its ith row with the jth row). Then a straightforward computation yields

$B^{tr} \cdot G_{q_2} \cdot B =$

$$= \begin{bmatrix} 1 & 0 & 0 & 0\dots0 & 0 \\ 0 & 1 & 0 & 0\dots0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1\dots0 & 0 \\ 0 & 0 & 0 & 0\dots0 & 1 \\ 0 & 0 & 0 & 0\dots1 & 0 \end{bmatrix} \cdot \left[\begin{array}{c|c} G_p & \widetilde{w} \\ \hline \widetilde{w}^{tr} & 1 \end{array}\right] \cdot \begin{bmatrix} 1 & 0 & 0 & 0\dots0 & 0 \\ 0 & 1 & 0 & 0\dots0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1\dots0 & 0 \\ 0 & 0 & 0 & 0\dots0 & 1 \\ 0 & 0 & 0 & 0\dots1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} G_p^{[(1,n-2)(1,n-1)]} & \widetilde{w}^{(1,n-2)} \\ \hline \widetilde{w}^{tr} & 1 \\ \hline G_p^{[(n-1,n-1)(1,n-1)]} & \widetilde{w}^{(n-1,n-1)} \end{array}\right] \cdot \begin{bmatrix} 1 & 0 & 0 & 0\dots0 & 0 \\ 0 & 1 & 0 & 0\dots0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1\dots0 & 0 \\ 0 & 0 & 0 & 0\dots0 & 1 \\ 0 & 0 & 0 & 0\dots1 & 0 \end{bmatrix} = G_{q_1}.$$

Step 2.2° We show that the P-critical forms $q_1 := q_{p_1,s_1,w_1,\varepsilon_{s_1}}$, $q_2 := q_{p_2,s_2,w_2,\varepsilon_{s_2}}$ lie in the same $\mathrm{O}(n,\mathbb{Z})$-orbit, if $(p_1,w_1),(p_2,w_2) \in \mathcal{Z}_{n-1}$ lie in the same $\mathrm{O}(n{-}1,\mathbb{Z})$-orbit.

In view of Step 2.1°, without loss of generality, we can assume that $s_1 = s_2 = 1$ and $\varepsilon_{s_1} = \varepsilon_{s_2} = 1$. Then

$$G_{q_1} = \left[\begin{array}{c|c} 1 & -w_1 \cdot G_{p_1} \\ \hline -(w_1 \cdot G_{p_1})^{tr} & G_{p_1} \end{array}\right], \quad G_{q_2} = \left[\begin{array}{c|c} 1 & -w_2 \cdot G_{p_2} \\ \hline -(w_2 \cdot G_{p_2})^{tr} & G_{p_2} \end{array}\right]$$

is the symmetric Gram matrix of the form q_1 and q_2, respectively.

Assume that $B' \in \mathrm{O}(n{-}1,\mathbb{Z})$ is a matrix such that $(p_1,w_1) * B' = (p_1 * B', w_1 \cdot B') = (p_2,w_2)$, and we set $B = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & B' \end{array}\right] \in \mathrm{O}(n,\mathbb{Z})$. We show that $q_2(x) = q_1(x \cdot B^{tr})$, for all $x \in \mathbb{Z}^n$, by proving that the equality $B^{tr} \cdot G_{q_1} \cdot B = G_{q_2}$ holds, see **(A3)**.

Since $(B')^{tr} = (B')^{-1}$, $(B')^{tr} \cdot G_{p_1} \cdot B' = G_{p_2}$ and $w_2 = w_1 \cdot B'$, we obtain

$$\begin{aligned}
G_{q_2} = & \left[\begin{array}{c|c} 1 & -w_2 \cdot G_{p_2} \\ \hline -(w_2 \cdot G_{p_2})^{tr} & G_{p_2} \end{array}\right] \\
= & \left[\begin{array}{c|c} 1 & -w_2 \cdot (B')^{tr} \cdot G_{p_1} \cdot B' \\ \hline -(w_2 \cdot (B')^{tr} \cdot G_{p_1} \cdot B')^{tr} & (B')^{tr} \cdot G_{p_1} \cdot B' \end{array}\right] \\
= & \left[\begin{array}{c|c} 1 & -w_1 \cdot B' \cdot (B')^{-1} \cdot G_{p_1} \cdot B' \\ \hline -(w_1 B' \cdot (B')^{-1} \cdot G_{p_1} \cdot B')^{tr} & (B')^{tr} \cdot G_{p_1} \cdot B' \end{array}\right] \\
= & \left[\begin{array}{c|c} 1 & -w_1 \cdot G_{p_1} \cdot B' \\ \hline -(w_1 \cdot G_{p_1} \cdot B')^{tr} & (B')^{tr} \cdot G_{p_1} \cdot B' \end{array}\right] = B^{tr} \cdot G_{q_1} \cdot B
\end{aligned}$$

and the proof of Step 2.2° is complete. Consequently, we have a well defined map (3.3)

$$\text{ind} : \mathrm{O}(n-1, \mathbb{Z})\text{-}\mathcal{O}rb(\mathcal{Z}_{n-1}) \longrightarrow \mathrm{O}(n, \mathbb{Z})\text{-}\mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z}))$$

induced by the map $(p, w) \mapsto \text{ind}(p, w) := q_{p,s,w,\varepsilon_s}$ on the orbit representatives, see (3.2).

To finish the proof of (b), it remains to show that the map ind (3.3) is surjective. Hence (b) follows, because the surjectivity of ind is a consequence of Proposition 2.14. Since the statement (c) follows from Proposition 2.14, the proof is complete. □

Construction 3.5. Assume that $n \geq 3$. It follows from the proof of Theorem 3.1 that the correspondence $\text{ind}_{s,\varepsilon_s} : \mathcal{Z}'_{n-1} \longrightarrow \mathcal{UB}igr_n$ (2.13), with $s = n$, associates to any pair $(\Delta, w) \in \mathcal{Z}'_{n-1}$ the P-critical bigraph

$$\Delta' := \Delta[[w, n, \varepsilon_n]] \in \mathcal{UB}igr_n \tag{3.6}$$

defined as follows. The set Δ'_0 of vertices of Δ' is obtained from $\Delta_0 = \{1, 2, \ldots, n-1\}$ by adding a new vertex n, that is, $\Delta'_0 = \{1, 2, \ldots, n-1, n\}$.

We set $w' := -2\check{G}_\Delta \cdot w$. By Theorem 3.1, each of the coordinates $w'_1, \ldots, w'_{n-1}$ of the vector $w' = (w'_1, \ldots, w'_{n-1}) \in \mathbb{Z}^{n-1}$ lies in $\{-1, 0, 1\}$, because the bigraph Δ' corresponds to the unit form q defined by the formula (3.2). Hence, $q_{1n} = w'_1, \ldots, q_{n-1n} = w'_{n-1}$, Δ' is P-critical,

$$\check{G}_{\Delta'} = \left[\begin{array}{c|c} \check{G}_\Delta & -2\check{G}_\Delta \cdot w^{tr} \cdot \varepsilon_n \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \check{G}_\Delta & w'^{tr} \\ \hline 0 & 1 \end{array}\right] \tag{3.7}$$

and $q_{1n}, \dots, q_{n-1n} \in \{-1, 0, 1\}$, because the restriction $q^{[j,n]}(x_j, x_n) := x_j^2 + x_n^2 + q_{jn}x_jx_n$ of $q(x)$ is positive, for $j = 1, \dots, n-1$, see also [38, Theorem 4.1].

It follows that the set of edges of $\Delta' := \Delta[[w, n, \varepsilon_n]]$ is the enlargement of Δ_1 by adding a new continuous edge $\bullet_s$—— $\bullet_n$, for any s such that $w'_s < 0$, and a new dotted edge $\bullet_r$- - -$\bullet_n$, for any r such that $w'_r > 0$. □

It follows from [21, Corollary 4.10] and our calculation in Section 5 that the surjective map (3.3) is far from being injective, see also Remark 4.10. However, by applying Theorem 3.1, its proof, and the correspondence $(p, w) \mapsto q_{p,w,s,\varepsilon_s}$ defined by (3.2), we get Algorithm 3.8 constructing the set P-**crit**$(\mathbb{Z}^{n+1}, \mathbb{Z})$ of all P-critical unit forms $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ from the unit forms $p \in$ **posit**$(\mathbb{Z}^n, \mathbb{Z})$, with a sincere root w, for $n \geq 3$, as well as the set of P-critical bigraphs $\Delta' \in \mathcal{UB}igr_n$ from the positive ones $\Delta \in \mathcal{UB}igr_{n-1}$, with a sincere root w, for $n \geq 2$. Its implementation is presented in the following section as Algorithms 4.8 and 4.9.

By applying Algorithm 3.8 and the package of algorithms presented in the following section, we compute in Section 5 all P-critical unit forms $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$, for $n \leq 9$, up to the action (2.9) of the orthogonal group $\mathrm{O}(n+1, \mathbb{Z})$ on $\mathcal{U}(\mathbb{Z}^{n+1}, \mathbb{Z})$, and the set of P-critical bigraphs $\Delta' \in \mathcal{UB}igr_n$, for $n \geq 2$, up to the action (2.1) of the orthogonal group $\mathrm{O}(n, \mathbb{Z})$ on $\mathcal{B}igr_n$.

Algorithm 3.8. Input: An integer $n \geq 3$ and the finite sets of matrices $\mathrm{O}(n, \mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$ and $\mathrm{O}(n+1, \mathbb{Z}) \subseteq \mathbb{M}_{n+1}(\mathbb{Z})$ (see [21, 3.1]).

Output: A finite set P-**crit**$^\bullet_{n+1} \subseteq P$-**crit**$(\mathbb{Z}^{n+1}, \mathbb{Z})$ of pairwise different representatives of all $\mathrm{O}(n+1, \mathbb{Z})$-orbits in P-**crit**$(\mathbb{Z}^{n+1}, \mathbb{Z})$.

Step 1° Construct a finite set **posit**$^\bullet_n \subseteq$ **posit**$(\mathbb{Z}^n, \mathbb{Z})$ of pairwise different representatives $p : \mathbb{Z}^n \to \mathbb{Z}$ of $\mathrm{O}(n, \mathbb{Z})$-orbits in **posit**$(\mathbb{Z}^n, \mathbb{Z})$.

Step 2° Given $p \in$ **posit**$^\bullet_n$, construct the finite set

$$\mathcal{R}_p = \{w \in \mathbb{Z}^n; p(w) = 1\}$$

of roots of p, and then construct the list $\mathcal{SR}_p$ of all sincere vectors in $\mathcal{R}_p$.

Step 3° Construct a finite set $\mathcal{Z}_n^-$ of pairwise different representatives of all $\mathrm{O}(n, \mathbb{Z})$-orbits in the finite set

$$\mathcal{Z}_n^\bullet = \{(p, w);\ p \in \mathbf{posit}^\bullet_n,\ w \in \mathcal{SR}_p\},$$

see [21, Proposition 3.7(c)].

Step 4° Given $(p, w) \in \mathcal{Z}_n^-$, construct the Gram matrices $\check{G}_p$ and G_p of p, then construct the matrix $G_{p,w} = \left[\begin{array}{c|c} 1 & -w \cdot 2 \cdot G_p \\ \hline 0 & \check{G}_p \end{array}\right]$, Step 5° Given $(p, w) \in \mathcal{Z}_n^-$, construct the unit form $q_{p,w} : \mathbb{Z}^{n+1} \to \mathbb{Z}$, by applying the formula $q_{p,w}(x) = x \cdot G_{p,w} \cdot x^{tr}$.

Step 6° Take for P-$\mathbf{crit}_{n+1}^{\bullet}$ the finite set $\{q_{p,w}\}_{(p,w)\in\mathcal{Z}_n^-}$.

Remark 3.9. (i) We implement the groups $\mathrm{O}(n, \mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$ and $\mathrm{O}(n{+}1, \mathbb{Z}) \subseteq \mathbb{M}_{n+1}(\mathbb{Z})$ in the C# programming language, by applying Algorithm 4.5 of the following section.

(ii) In Step 1° we can apply the correspondence

$$(p, \mu) \mapsto \widehat{p}^{\mu} \in \mathbf{posit}(\mathbb{Z}^{n+1}, \mathbb{Z})$$

that associates to any $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ and any vector $\mu = (\mu_1, \ldots, \mu_n)$, such that $\mu_1, \ldots, \mu_n \in \{-1, 0, 1\}$ and $\det \begin{bmatrix} 2G_p & \mu^{tr} \\ \mu & 2 \end{bmatrix} > 0$, the one-point extension positive form

$$\widehat{p}^{\mu}(x) = x \cdot \begin{bmatrix} \check{G}_p & \mu^{tr} \\ 0 & 1 \end{bmatrix} \cdot x^{tr},$$

see [21, Theorem 4.1]. In fact, we should apply an implementation of [21, Algorithm 4.5]. The positivity of the unit form $\widehat{p}^{\mu}$ is checked by applying Algorithm 4.1 based on Sylvester′s citerion. To get the subset $\mathbf{posit}_n^{\bullet}$ of $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$, we apply Algorithm 4.7 of the following section.

(iii) In Step 2° we can apply the restrictively counting algorithm [35, Algorithm 4.2] and [36, Algorithm 3.7, Remark 3.8].

(iv) In Step 3°, note that two P-critical edge-bipartite graphs constructed from pairs (p, w) and $(p, -w)$ lie in the same $\mathrm{O}(n + 1, \mathbb{Z})$-orbit.

4. A package of algorithms

We present a package of algorithms that we apply in our construction of all $\mathrm{O}(n, \mathbb{Z})$-orbits of a P-critical unit form for $n \geq 3$. We use the standard convention: when a unit quadratic form q is operated upon in a computer program, it is implemented as a two-dimensional array in C# representing the corresponding symmetric Gram matrix G_q (or non-symmetric Gram matrix $\check{G}_q$) of q.

The following algorithm verifies in a polynomial time $\mathcal{O}(n^4)$ wheather or not a quadratic form $p : \mathbb{Z}^n \to \mathbb{Z}$ is positive or a bigraph $\Delta \in \mathcal{UB}igr_n$ is positive.

Algorithm 4.1 (SylvesterCriterion). Input: An integer $n \geq 2$ and the non-symmetric Gram matrix $\check{G}_p \in \mathbb{M}_n(\mathbb{Z})$ of a unit form $p : \mathbb{Z}^n \to \mathbb{Z}$.

Output: True if p is positive definite, false otherwise.

1° $\widehat{G}_p = \check{G}_p + \check{G}_p^{tr}$

2° $k = 0$; $N = n$;

$\underline{3^\circ}$ **if** ($\det \widehat{G}_p > 0$) $k++$;
$\underline{4^\circ}$ $i = n$;
$\underline{5^\circ}$ **while**($i > 1$){
$\underline{6^\circ}$ $W =$ the matrix (of size $N \geq 2$) obtained from $\widehat{G}_p$ by choosing the first N rows and the first N columns of $\widehat{G}_p$;
$\underline{7^\circ}$ $W =$ the matrix (of size N-1) obtained from W by removing the i-th row and the i-th column;
$\underline{8^\circ}$ $N--$;
$\underline{10^\circ}$ **if** ($\det W > 0$) $k++$;
$\underline{11^\circ}$ $i--$;}
$\underline{12^\circ}$ **if** ($n == k$) **return** true;
$\underline{13^\circ}$ **return** false;

By applying Algorithm 4.1, we check in a polynomial time $\mathcal{O}(n^5)$ wheather or not $q = q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$ is P-critical or a bigraph $\Delta \in \mathcal{UB}igr_n$ is P-critical.

Algorithm 4.2 (IsPcritical)**. Input:** An integer $n \geq 2$ and the non-symmetric Gram matrix $\check{G}_q \in \mathbb{M}_n(\mathbb{Z})$ of a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$.

Output: True if q is P-critical, false otherwise.
$\underline{1^\circ}$ $\widehat{G}_q = \check{G}_q + \check{G}_q^{tr}$
$\underline{2^\circ}$ **if** (SylvesterCriterion($\widehat{G}_q$,n)==false){
$\underline{3^\circ}$ **for**($i = 1$; $i <= n$; $++i$){
$\underline{4^\circ}$ $L =$ the matrix obtained from $\widehat{G}_q$ by removing i-th row and i-th column;
$\underline{5^\circ}$ **if**(SylvesterCriterion($L, n-1$)==false)
$\underline{6^\circ}$ **return** false;}
$\underline{7^\circ}$ **return** true; }
$\underline{8^\circ}$ **else return** false;

Now, we present three algorithms that we need in our implementation of the orthogonal group

$$\mathrm{O}(n, \mathbb{Z}) = \widehat{\mathbf{S}}_n \rtimes \widehat{\mathbf{C}}_2^n,$$

for a given $n \geq 2$, see (2.9) and [36, Lemma 2.3]. First (see Algorithm 4.3), we present an implementation of the symmetric group $\mathbf{S}_n$ by applying an algorithm of Johnson [13] and Trotter [41] based on the idea of adjacent transpositions; originally it generates the symmetric group $\mathbf{S}_n$. A simple modification of the algorithm constructs the subgroup $\widehat{\mathbf{S}}_n \subseteq \mathrm{O}(n, \mathbb{Z})$ of $\mathrm{O}(n, \mathbb{Z})$ as follows.

Algorithm 4.3 (GeneretePermutation)**. Input:** An integer $n \geq 2$.

Output: The group $\widehat{\mathbf{S}}_n \subseteq \mathrm{O}(n, \mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$, see (2.9).
$\underline{1^\circ}$ **for**($i = 1$;$i <= n$;$i++$){
$\underline{2^\circ}$ $P[i] = i$; //P is a vector of integers

```
    C[i] = 1; //C is a vector of integers
    PR[i] = true;} //PR is a vector of booleans
C[n] = 0;
for(t = 1; t <= n; ++t)
    M[t, P[t]] = 1; //M is a matrix of integers
PMatrix.Add(M); //PMatrix is a list of matrices
i = 1;
while(i < n){
    i = 1; x = 0;
    while(C[i] == n - i + 1){
        PR[i]=(PR[i]==true)?false:true;
        C[i] = 1;
        if(PR[i]) x++;
        i++; }
    if (i < n){
        if (PR[i]) k = C[i] + x;
        else k = n - i + 1 - C[i] + x;
        P[k] <-> P[k + 1]
        for(t = 1; t <= n; ++t)
            M[t, P[t]] = 1;
        PMatrix.Add(M);
        C[i] = C[i] + 1;}
}
return PMatrix;
```

Now we present an implementation of the finite group $\widehat{\mathbf{C}}_2^n \subseteq \mathrm{O}(n,\mathbb{Z})$ defined in Section 2 (see (2.9)), which uses the Gray binary generation (see [15]). We modify the Gray algorithm by interchanging all zeros with -1. Obviously, the modified algorithm constructs the group $\widehat{\mathbf{C}}_2^n \subseteq \mathrm{O}(n,\mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$.

Algorithm 4.4 (GenereteSignMatrices)**. Input:** An integer $n \geq 2$.
Output: The group $\widehat{\mathbf{C}}_2^n \subseteq \mathrm{O}(n,\mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$, see (2.9).

```
for(t = 1; t <= n; ++t)
    B[t] = -1;
i = 0; p = 1;
do{
    for(t = 1; t <= n; ++t){
        C[t - 1] = B[t]; //C is a vector of integers
        M[t, t] = C[t - 1];} //M is a matrix of integers
    SMatrix.Add(M); //SMatrix is a list of matrices
    i++;
    j = i;
    p = 1;
    while(j%2 == 0){
        j = j/2;
        p++;}
```

```
    if (p ≤ n){
        if (B[p] == −1) B[p] = 1;
        else B[p] = −1;}
}while(p ≤ n);
return SMatrix;
```

Finally, by applying Algorithm 4.3 and Algorithm 4.4, we define an algorithm constructing the orthogonal group $\mathrm{O}(n,\mathbb{Z}) = \widehat{\mathbf{S}}_n \rtimes \widehat{\mathbf{C}}_2^n$, for a given integer $n \geq 2$.

Algorithm 4.5 (OrthogonalGroup)**. Input:** An integer $n \geq 2$ and a list of permutation matrices PMatrix.

Output: The matrices $B \in \mathrm{O}(n,\mathbb{Z})$.

```
SMatrix=GenereteSignMatrices(n);
foreach(int[,] P in PMatrix){
    foreach(int[,] S in SMatrix){
        W = P * S ∈ M_n(Z);
        All.Add(W);} } //All is a list of matrices
    return All;
```

Hint. We use the fact that every matrix $B \in \mathrm{O}(n,\mathbb{Z}) = \widehat{\mathbf{S}}_n \rtimes \widehat{\mathbf{C}}_2^n$ can be uniquely represented as the product $B = P \cdot S$, where $P = \widehat{\sigma} \in \widehat{\mathbf{S}}_n$ and $S = \widehat{\varepsilon} \in \widehat{\mathbf{C}}_2^n$, see [36, Lemma 2.3] and Sections 2-3. A list of permutation matrices PMatrix is a subset of the set of all permutation matrices $\widehat{\mathbf{S}}_n$.

With bigraph $\Delta \in \mathcal{UB}igr_n$ we associate the graph $\overline{\Delta}$ which is constructed from Δ by replacing all broken edges •- - -• with full edges •—— •.

We denote by $\check{G}^{\bullet}_{\Delta}$=NonSymmetricGramMatrixOfGraph($\check{G}_{\Delta}$) the non-symmetric Gram matrix of the graph $\overline{\Delta}$ constructed from the bigraph Δ with non-symmetric Gram matrix $\check{G}_{\Delta}$.

We denote by $\widehat{G}^{\bullet}_{\Delta}$=SymmetricGramMatrixOfGraph($\widehat{G}_{\Delta}$) the duplicate symmetric Gram matrix $2G_{\overline{\Delta}}$ of the graph $\overline{\Delta}$ constructed from the bigraph Δ with the duplicate symmetric Gram matrix $2G_{\Delta}$.

The following algorithm checks, if two unit forms $q_1,\ q_2 : \mathbb{Z}^n \to \mathbb{Z}$ lie in the same $\mathrm{O}(n,\mathbb{Z})$-orbit under the action (2.9), or two bigraphs $\Delta, \Delta' \in \mathcal{UB}igr_n$ lie in the same $\mathrm{O}(n,\mathbb{Z})$-orbit under the action (2.1). Here (Algorithm 4.6 and Algorithm 4.7) we use the fact that if two bigraphs $\Delta, \Delta' \in \mathcal{UB}igr_n$ lie in the same $\mathrm{O}(n,\mathbb{Z})$-orbit then the associated graphs $\overline{\Delta}$ and $\overline{\Delta}'$ are isomorphic.

Algorithm 4.6 (CompareOrbits). **Input:** An integer $n \geq 2$, the non-symmetric Gram matrices $\check{G}_{q_1}, \check{G}_{q_2} \in \mathbb{M}_n(\mathbb{Z})$ of unit forms $q_1, q_2 : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, and the orthogonal group $\mathrm{O}(n,\mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$ (4.5).

Output: True if q_1, q_2 lie in the same $\mathrm{O}(n,\mathbb{Z})$-orbit, false otherwise.

1° $\widehat{G}_{q_1} = \check{G}_{q_1} + \check{G}^{tr}_{q_1}$; $\widehat{G}_{q_2} = \check{G}_{q_2} + \check{G}^{tr}_{q_2}$;
2° **if**(the numbers of zero coefficients in the matrices $\widehat{G}_{q_1}$ and $\widehat{G}_{q_1}$ are not equal)
3° **return** false;
4° **else**{
5° $\widehat{G}_{q_1} = \check{G}_{q_1} + \check{G}^{tr}_{q_1}$, $\widehat{G}_{q_2} = \check{G}_{q_2} + \check{G}^{tr}_{q_2}$; let PMatrix be an empty list;
6° $\widehat{G}^{\bullet}_{q_1}$=SymmetricGramMatrixOfGraph($\widehat{G}_{q_1}$); $\widehat{G}^{\bullet}_{q_2}$=SymmetricGramMatrixOfGraph($\widehat{G}_{q_2}$);
7° Find all permutation matrices $P \in \mathbb{M}_n(\mathbb{Z})$ such that $P^{tr} \cdot \widehat{G}^{\bullet}_{q_1} \cdot P = \widehat{G}^{\bullet}_{q_2}$ and add P to PMatrix;
8° **if** (PMatrix is empty) **return** false;
9° Orthogonal=OrthogonalGroup(n, PMatrix);
6° **foreach**([,] B in Orthogonal)
7° **if**($B^{tr} \cdot \widehat{G}_{q_1} \cdot B == \widehat{G}_{q_2}$)
8° **return** true;}
9° **return** false;

Algorithm 4.7 (SetOfOrbitRepresentatives). **Input:** An integer $n \geq 2$, a finite non-empty subset $\mathcal{H} \subset \mathcal{U}(\mathbb{Z}^n,\mathbb{Z})$ of P-critical (or positive) quadratic forms, and the orthogonal group $\mathrm{O}(n,\mathbb{Z}) \subseteq \mathbb{M}_n(\mathbb{Z})$ (4.5).

Output: A list AMatrix of non-symmetric Gram matrices $\check{G}_q$ of pairwise different representatives $q : \mathbb{Z}^n \to \mathbb{Z}$ of all $\mathrm{O}(n,\mathbb{Z})$-orbits $\mathrm{O}(n,\mathbb{Z}) * q$, with $\check{G}_q \in \mathcal{H}$.

1° Let AMatrix, AMatrix$^{\bullet}$, SMatrix be the empty lists;
2° Add first a matrix $H[0]$ from $\mathcal{H}$ to AMatrix; $H^{\bullet}[0]$=NonSymmetricGramMatrixOfGraph($H[0]$); add $H^{\bullet}[0]$ to AMatrix$^{\bullet}$;
3° SMatrix:=GeneretePermutation(n);
4° **for**($i = 1$; $i < |H|$; $i++$){
5° Let WMatrix$^{\bullet}$ be the list of matrices such that $P^{tr} \cdot \widehat{G}^{\bullet}_{\Delta} \cdot P$, where $P \in$SMatrix, $\widehat{G}^{\bullet}_{\Delta}$=SymmetricGramMatrixOfGraph($\widehat{H}[i]$) and $\widehat{H}[i] = H[i] + H[i]^{tr}$;
6° For each matrix $A^{\bullet} \in$AMatrix$^{\bullet}$ do steps $7° - 11°$;
7° For each matrix $W^{\bullet} \in$WMatrix$^{\bullet}$ do {
8° Let PMatrix be an empty list of matrices;
9° **if** ($W^{\bullet} = \widehat{A}^{\bullet}$, where $\widehat{A}^{\bullet} = A^{\bullet} + A^{\bullet tr}$) add permutation matrix P (corresponding to $W^{\bullet}$) to PMatrix; }
10° **if** (PMatrix is not empty){

- *Ort*=OrthogonalGroup(n, PMatrix);
- Generate a list W_1 of matrices such that $B^{tr} \cdot \widehat{H}[i] \cdot B$, where $B \in Ort$ and $\widehat{H}[i] = H[i] + H[i]^{tr}$;
- Check if there exists a matrix in W_1 which is equal to matrix $\widehat{A} = A + A^{tr}$ (we verify this equality for (duplicate) symmetric Gram matrices coding bigraphs) and if

so, go to Step 4°; if such matrix does not exist go to Step 6° (i.e., repeat Steps 7° − 11° for another matrix from $\text{AMatrix}^\bullet$); }

11° **else** (i.e, PMatrix is empty) go to Step 6°;

12° If in Steps 7° − 11° no orthogonal matrix has been found to witness that $H[i]$ is in the same orbit as one of the matrices in AMatrix, then add $H[i]$ to AMatrix and add $H^\bullet[i]$ to $\text{AMatrix}^\bullet$, where

$H^\bullet[i]$=NonSymmetricGramMatrixOfGraph($H[i]$);

13° }

13° **return** AMatrix;

Algorithm 4.8 (PcriticalFormsFromPositiveForm)**. Input:** An integer $n \geq 2$, the non-symmetric Gram matrix $\check{G}_p$ of a positive form $p : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, and the list W_p of all sincere roots of p.

Output: A list of P-critical forms $q : \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}$ constructed from p by applying (3.3).

1° $\widehat{G}_p = \check{G}_p + \check{G}_p^{tr}$;
2° **for**($i = 1$;$i <= n + 1$;$+ + i$)
3° **for**($j = 1$;$j <= n + 1$;$+ + j$){
4° **if**($i > 1$ && $j > 1$)
5° $Q[i, j] = \check{G}_p[i - 1, j - 1]$;
6° **else** {
7° **if**(($i > 1$ && $j == 1$)||($i == 1$ && $j > 1$))
8° $Q[i, j] = 0$;
9° **else** $Q[i, j] = 1$;}
10° }
11° **foreach**(int[] w in W_p){
12° $\hat{w} = -w \cdot \widehat{G}_p$;
13° **for**($i = 1$;$i <= n + 1$;$+ + i$)
14° **if**($i > 1$)
15° $Q[1, i] = \hat{w}[i - 1]$; //Q is a matrix of integers
16° **if**(IsPcritical($Q, n + 1$)==1)
17° All.Add(Q); } //All is a list of matrices
18° **return** All;

Algorithm 4.9 (AllPcriticalForms)**. Input:** An integer $n \geq 3$ and a list GP of non-symmetric Gram matrices $\check{G}_p$ of pairwise different representatives $p : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ of all $\mathrm{O}(n, \mathbb{Z})$-orbits in $\mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$.

Output: A list of non-symmetric Gram matrices $\check{G}_q$ of pairwise different representatives $q : \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}$ of all $\mathrm{O}(n{+}1, \mathbb{Z})$-orbits in the set P-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$.

1° **foreach**(int[,] P in GP){
2° M=PcriticalFormsFromPositiveForm(P,n)
3° All.Add(M); }//All is a list of matrices

4° $\widehat{\text{All}}$=list of pairwise different representatives of all $\mathrm{O}(n{+}1,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ selected from All by apllying Algorithm 4.6;

5° **return** $\widehat{\text{All}}$;

Remark 4.10. (a) It follows from Proposition 2.6, (2.13), and Construction 3.7 that, in view of the identification

$$q_\bullet : \mathcal{UB}igr_{n-1} \xrightarrow{1-1} \mathcal{U}(\mathbb{Z}^{n-1}, \mathbb{Z}),$$

see (2.7), Algorithms 4.6-4.9 apply to edge-bipartite graphs in $\mathcal{UB}igr_n$ and construct all P-critical bigraphs $\Delta \in \mathcal{UB}igr_n$.

(b) The complexity of Algorithms 4.8 is dominated by the cardinality $|W_p|$ of the set W_p of all sincere roots of a given $p \in \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$; more precisely, it is bounded by $\mathcal{O}(|W_p| * n^5)$, where the consecutive values of $|W_p|$ can be computed by the restrictively counting algorithm [36, Algorithm 4.2].

(c) The complexity of Algorithm 4.6 is $\mathcal{O}(|\mathrm{O}(n, \mathbb{Z})|)$.

(d) The complexity of Algorithms 4.7 and 4.9 is also dominated by $|\mathrm{O}(n, \mathbb{Z})|$.

5. A classification and tables of computing results

In this section we present a classification of P-critical unit forms in $\mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ (and hence P-critical bigraphs $\Delta \in \mathcal{UB}igr_n$), for $n \leq 10$. We get it by applying the package of algortithms of the previous section. By symbolic computations in Maple and numerical computations in C#, we compute P-critical unit forms and connected positive unit forms, together with their Coxeter polynomials and the $\mathrm{O}(n, \mathbb{Z})$-orbits, for $n \leq 10$. The results are presented in tables 5.6-5.9.

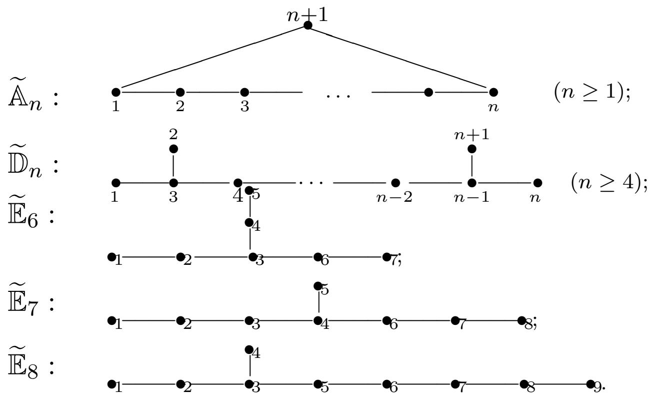
We recall from [36, Section 2], that the **Coxeter polynomial** of $q : \mathbb{Z}^n \to \mathbb{Z}$, is the characteristic polynomial

$$\mathrm{cox}_q(t) = \det(t \cdot E - \mathrm{Cox}_q) \in \mathbb{Z}[t]$$

of the **Coxeter(-Gram) matrix**

$$\mathrm{Cox}_q := -\check{G}_q \cdot \check{G}_q^{-tr},$$

where $\check{G}_q$ is the Gram matrix (2.5), see also [35]. In general, Cox_q depends on the numbering of the indeterminates in the form $q(x_1, \dots, x_n)$. In our classification of P-critical forms we use the Coxeter polynomials of the Euler quadratic form $q_\Delta : \mathbb{Z}^{n+1} \to \mathbb{Z}$, with $n+1 = |\Delta_0|$, of the following simply-laced Euclidean graphs (extended Dynkin diagrams)



(5.1)

Note that $\widetilde{\mathbb{A}}_1$ is the Kronecker graph $\bullet\!=\!=\!=\!\bullet$.

We recall that the Euler quadratic form $q_\Delta : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ of a graph $\Delta = (\Delta_0, \Delta_1)$, with the set of vertices Δ_0 and the set of edges Δ_1, is defined by the formula

$$q_\Delta(x) = \sum_{i\in\Delta_0} x_i^2 + \sum_{i<j} d_{ij}^\Delta x_i x_j,$$

where $x = (x_j)_{j\in\Delta_0} \in \mathbb{Z}^{\Delta_0} \equiv \mathbb{Z}^{n+1}$, $n+1 = |\Delta_0|$, and $d_{ij}^\Delta = -|\Delta_1(\bullet_i, \bullet_j)|$, and $|\Delta_1(\bullet_i, \bullet_j)|$ is the number of edges between the vertices $\bullet_i$, $\bullet_j$ in Δ.

If Δ is any of the diagrams $\widetilde{\mathbb{D}}_n$, $n \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$ then the Coxeter polynomial $F_\Delta(t) := \mathrm{cox}_{q_\Delta}(t)$ of Δ has the form

$$F_\Delta(t) = \begin{cases} t^{n+1} + t^n - t^{n-1} - t^{n-2} - t^3 - t^2 + t + 1, & \text{for } \Delta = \widetilde{\mathbb{D}}_n, \\ t^7 + t^6 - 2t^4 - 2t^3 + t + 1, & \text{for } \Delta = \widetilde{\mathbb{E}}_6, \\ t^8 + t^7 - t^5 - 2t^4 - t^3 + t + 1, & \text{for } \Delta = \widetilde{\mathbb{E}}_7, \\ t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1, & \text{for } \Delta = \widetilde{\mathbb{E}}_8. \end{cases} \tag{5.2}$$

Note that, if $\Delta = \widetilde{\mathbb{D}}_4$ or $\Delta = \widetilde{\mathbb{D}}_5$, we have

$$F_\Delta(t) := \begin{cases} t^5 + t^4 - 2t^3 - 2t^2 + t + 1, & \text{for } n = 4, \\ t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1, & \text{for } n = 5. \end{cases} \tag{5.3}$$

If $n \geq 1$ and $\Delta = \widetilde{\mathbb{A}}_n$, then the Coxeter polynomial $\mathrm{cox}_{q_\Delta}(t)$ of Δ is of one of the forms $F_\Delta^{(1)}(t), F_\Delta^{(2)}(t), \ldots, F_\Delta^{(m_n)}(t)$, where

$$m_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n+1}{2}, & \text{if } n+1 \text{ is even}, \end{cases}$$

$F_{\Delta}^{(j)}(t)=t^{n+1}-t^{n-j+1}-t^j+1=(t-1)^2\cdot\mathfrak{v}_j(t)\cdot\mathfrak{v}_{n-j+1}(t)$, for $j=1,\dots,m_n$, and $\mathfrak{v}_m(t)=t^{m-1}+t^{m-2}+t^{m-3}+\dots+t^2+t+1$, see [20], [34], and [38].

In particular, if $n+1$ is even and $j=m_n=\frac{n+1}{2}$, then $t^{n-j+1}=t^j$ and $F_{\Delta}^{(j)}(t)$ has the form

$$F_{\Delta}^{(m_n)}(t)=F_{\Delta}^{(\frac{n+1}{2})}(t)=t^{n+1}-2t^{\frac{n+1}{2}}+1.$$

If $\widetilde{\mathbb{A}}_1 : \bullet \!=\!=\!=\! \bullet$ is the Kronecker graph, we have $n=1$, $m_n=2$ and $F_{\Delta}(t)=F_{\Delta}^{(2)}(t)=t^2-2t+1$.

Now we are able to present a complete classification of P-critical unit forms (and P-critical bigraphs $\Delta\in\mathcal{UB}igr_n$), their Coxeter polynomials, and the $\mathrm{O}(n,\mathbb{Z})$-orbits in the set $P\text{-}\mathbf{crit}_n := P\text{-}\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n\le 10$. We get it as a result of computer computation.

Theorem 5.4. (a) *If $n\le 9$ then, up to the action of the groups $\mathrm{O}(n,\mathbb{Z})$ and $\mathrm{O}(n{+}1,\mathbb{Z})$, the number of elements in the set $\mathcal{Z}_n$ (Algorithm 3.8, Step 3°) and the number of $\mathrm{O}(n{+}1,\mathbb{Z})$-orbits in $P\text{-}\mathbf{crit}(\mathbb{Z}^{n+1},\mathbb{Z})$ are as shown in the following table*

n	2	3	4	5	6	7	8	9
$\lvert\mathcal{Z}_n^-\rvert$	1	2	4	10	72	639	7980	95
$\lvert P\text{-}\mathbf{crit}_{n+1}^{\bullet}\rvert$	1	1	3	5	24	152	1730	17

where $P\text{-}\mathbf{crit}_{n+1}^{\bullet}$ is the finite set of pairwise different representatives of all $\mathrm{O}(n+1,\mathbb{Z})$-orbits in $P\text{-}\mathbf{crit}(\mathbb{Z}^{n+1},\mathbb{Z})$.

(b) *If $n=3,4,5$ then, up to the action of $\mathrm{O}(n,\mathbb{Z})$, the number of P-critical unit forms $q:\mathbb{Z}^n\longrightarrow\mathbb{Z}$ and the number of P-critical bigraphs $\Delta\in\mathcal{B}igr_n$ equals 1, 1, 3, respectively, and the forms $q=q_\Delta$ are listed in* [21, Corollary 4.10], *together with their Coxeter polynomials and a generator* $\mathbf{h}$ *of* $\mathrm{Ker}\,q=\mathrm{Ker}\,q_\Delta$.

(c) *If $n=6$ or $n=7$, the list of P-critical unit forms $q:\mathbb{Z}^n\longrightarrow\mathbb{Z}$ and P-critical bigraphs $\Delta\in\mathcal{UB}igr_n$, up to the action of $\mathrm{O}(n,\mathbb{Z})$, is presented in Table 5.7 and Table 5.9, respectively, together with their Coxeter polynomials.*

(d) *For $n=8,9,10$, a list of P-critical unit forms $q:\mathbb{Z}^n\longrightarrow\mathbb{Z}$, up to the action of $\mathrm{O}(n,\mathbb{Z})$, is available on request from the authors (see [23]).*

Outline of proof. (a) The table is obtained by computer computation using implementations of Algorithm 4.8 and Algorithm 4.9 in Maple and C#.

(b) The result for $n = 3$ and $n = 4$ can be obtained by simple hand calculation using Proposition 2.14 and Theorem 3.1. For this purpose we need to compute the sets $\mathcal{Z}_2$ and $\mathcal{Z}_3$. We do it directly by using Maple and Sylvester′s criterion (Algorithm 4.1), see also [21, Algorithm 4.5].

The statements (c) and (d) are obtained by computer calculations using our package of algorithms presented in Section 4. □

The following tables illustrate the algorithmic construction $\mathrm{ind} : \mathcal{Z}_n \longrightarrow P\text{-}\mathbf{crit}_{n+1}$ of Theorem 3.1, in cases $n = 6$ and $n = 7$. By applying our package of algorithms of Section 4, we also construct a complete set of representatives of the $\mathrm{O}(n{+}1, \mathbb{Z})$-orbits in $P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ and of P-critical bigraphs in $\mathcal{UB}igr_n$, for $n = 6$ and $n = 7$. In view of the identification $q_\bullet : \mathcal{UB}igr_n \xrightarrow{1-1} \mathcal{U}(\mathbb{Z}^n, \mathbb{Z})$ (2.7), we present the classification results for the unit forms $q : \mathbb{Z}^n \to \mathbb{Z}$ only.

Throughout, we set $P\text{-}\mathbf{crit}_n := P\text{-}\mathbf{crit}(\mathbb{Z}^n, \mathbb{Z})$ and we denote by

$$\mathcal{SR}_p = \{w = (w_1, \ldots, w_n) \in \mathbb{Z}^n;\ p(w) = 1 \,\mathrm{and}\, w_1 \neq 0, \ldots, w_n \neq 0\}$$

the set of sincere roots w of a positive form $p : \mathbb{Z}^n \to \mathbb{Z}$, by $\mathbf{h} \in \mathbb{Z}^{n+1}$ a sincere vector such that $\mathrm{Ker}\, q = \mathbb{Z} \cdot \mathbf{h}$, and by $\mathrm{cox}_q(t) = \det(t \cdot E - \mathrm{Cox}_q) \in \mathbb{Z}[t]$ the Coxeter polynomial of $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$.

We recall from Sections 3 and 4 the following notations:

- $\mathbf{posit}_n^\bullet \subseteq \mathbf{posit}(\mathbb{Z}^n, \mathbb{Z})$ is a complete finite set of pairwise different representatives $p : \mathbb{Z}^n \to \mathbb{Z}$ of all $\mathrm{O}(n, \mathbb{Z})$-orbits in $\mathbf{posit}(\mathbb{Z}^{\mathbf{n}}, \mathbb{Z})$, and
- $P\text{-}\mathbf{crit}_{n+1}^\bullet \subseteq P\text{-}\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$ is a complete finite set of pairwise different representatives of all $\mathrm{O}(n{+}1, \mathbb{Z})$-orbits in $P\text{-}\mathbf{crit}(\mathbb{Z}^{n+1}, \mathbb{Z})$.
- In the last column of each of the tables, we use the notation introduced in (5.2) for the Coxeter polynomials $F_\Delta(t)$ (5.2), with $\Delta \in \{\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$, and $F_\Delta^{(1)}(t), \ldots, F^{(m_n)}\Delta(t)$, with $\Delta = \widetilde{\mathbb{A}}_n$, of the Euler form q_Δ, where the vertices of the extended Dynkin diagram Δ are enumerated as in (5.1). The simply-laced Euclidean diagram $D\Delta$ associated with any P-critical bigraph Δ (see Definition 5.5) and with its unit quadratic form q_Δ can be determined by applying the inflation algorithm $\Delta \mapsto \mathbf{t}_{ab}^{-}\Delta$ described in [16, Algorithm 5.9] and [38, Theorem 3.7]. Following [38], we introduce the following definition.

Definition 5.5. *The* **Euclidean type** (*or the E-type, in short*) *of a P-critical bigraph* $\Delta \in \mathcal{UB}igr_{n+1}$ *is a unique simply-laced Euclidean diagram*

$$D\Delta \in \{\widetilde{\mathbb{A}}_n, n \geq 1,\ \widetilde{\mathbb{D}}_m, m \geq 4,\ \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$$

such that the symmetric Gram matrix G_Δ *of* Δ *is* $\mathbb{Z}$*-congruent with the symmetric Gram matrix* $G_{D\Delta}$ *of the diagram* $D\Delta$.

We recall from Theorem 2.8 that any loop-free P-critical bigraph Δ in $\mathcal{UB}igr_{n+1}$ is principal and connected. Then, by [38, Section 3], the existence and the uniqueness of a Euclidean diagram $D\Delta$ is a consequence of [16, Section 5] and [38, Section 3].

A complete list of the Coxeter polynomials of P-critical bigraphs Δ in $\mathcal{UB}igr_{n+1}$, with at most 10 vertices, is presented in Table 5.12 and 5.13 at the end of this section.

TABLE 5.6. A list of P-critical unit forms constructed by ind: $\mathcal{Z}_5 \longrightarrow P$-$\mathbf{crit}_6$, for $n = 6$

$p \in \mathbf{posit}_5^\bullet$	$w \in \mathcal{SR}_p$	$\mathrm{ind}(p, w)$	$\mathrm{cox}_q(t)$ and E-type
$p_1(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_3x_4 - x_4x_5$	$w = (1, 1, 1, \hat{1}, \hat{1})$	$q_1(x) = \sum_{i=1}^{6} x_i^2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5 - x_5x_6$	$t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
$p_2(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_5 + x_2x_4 + x_3x_4$	$w = (1, 1, 1, \hat{1}, \hat{1})$	$q_2(x) = \sum_{i=1}^{6} x_i^2 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_3x_5 + x_4x_5$	$t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
$p_3(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_2x_4 + x_2x_5 + x_3x_4$	$w_1 = (1, 1, 1, \hat{1}, \hat{1})$	$q_3(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 + x_1x_3 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_5 + x_3x_6 + x_4x_5$	$t^6 - t^5 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
	$w_2 = (1, 2, 1, \hat{1}, \hat{1})$	$q_4(x) = \sum_{i=1}^{6} x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_5 + x_3x_6 + x_4x_5$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
$p_4(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_2x_5 + x_3x_4 - x_4x_5$	$w = (1, 1, 1, \hat{1}, \hat{1})$	$q_5(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 + x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_6 + x_4x_5 - x_5x_6$	$t^6 - t^5 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
$p_5(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_2x_4 + x_3x_5$	$w = (1, 1, 1, 1, \hat{1})$	$q_6(x) = \sum_{i=1}^{6} x_i^2 - x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_3x_5 + x_4x_6$	$t^6 - 2t^3 + 1 = F^{(3)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{A}}_5$
$p_6(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_5 + x_3x_4$	$w = (2, 1, 1, 1, \hat{1})$	$q_7(x) = \sum_{i=1}^{6} x_i^2 - x_1x_4 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_4x_5$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
$p_7(x) = \sum_{i=1}^{5} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_2x_5$	$w_1 = (1, 1, 1, 1, \hat{1})$	$q_8(x) = \sum_{i=1}^{6} x_i^2 + x_1x_2 - x_1x_4 - x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_6$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
	$w_2 = (2, 1, 1, 1, \hat{1})$	$q_9(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 + x_1x_3 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_6$	$t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
	$w_3 = (2, 2, 1, 1, \hat{1})$	$q_{10}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_6$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$

Table 5.7. A set of representatives of $\mathrm{O}(n,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n=6$

(p,w)	$\mathrm{ind}(p,w)\in \mathrm{O}(6,\mathbb{Z})$-$\mathcal{O}rb(q)\subseteq P$-$\mathbf{crit}_6^\bullet$	$\mathrm{cox}_q(t)$, E-type
(p_5,w)	$q_6(x)=\sum_{i=1}^6 x_i^2 - x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_3x_5 + x_4x_6$	$t^6 - 2t^3 + 1 = F^{(3)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{A}}_5$
(p_1,w), (p_3,w_1)	$q_1(x)=\sum_{i=1}^6 x_i^2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5 - x_5x_6$	$t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
(p_2,w), (p_4,w), (p_7,w_1)	$q_2(x)=\sum_{i=1}^6 x_i^2 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_3x_5 + x_4x_5$	$t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
(p_7,w_3)	$q_{10}(x)=\sum_{i=1}^6 x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_6$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$, E-type $=\widetilde{\mathbb{D}}_5$
(p_3,w_2), (p_6,w), (p_7,w_2)	$q_4(x)=\sum_{i=1}^6 x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_5 + x_3x_6 + x_4x_5$	$t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_5}(t)$,E-type $=\widetilde{\mathbb{D}}_5$

Hint. The unit forms p_j and the vectors w and w_j used in 5.7 are the same as presented in 5.6.

It follows that the surjective correspondence

$$\mathrm{ind}_{s,\varepsilon_s} : \mathrm{O}(5,\mathbb{Z})\text{-}\mathcal{O}rb(\mathcal{Z}_5) \longrightarrow \mathrm{O}(6,\mathbb{Z})\text{-}\mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^6,\mathbb{Z}))$$

(3.3) induced by the map $(p,w)\mapsto \mathrm{ind}_{s,\varepsilon_s}(p,w) := q_{p,s,w,\varepsilon_s}$, is not injective, see also Remark 5.10 at the end of this section.

Table 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P$-$\mathbf{crit}_7$, for $n=7$

$p\in \mathbf{posit}_6^\bullet$	$w\in\mathcal{SR}_p$	$q=\mathrm{ind}(p,w)\in P$-$\mathbf{crit}_7$
$p_1(x)=\sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5 - x_4x_6 - x_5x_6$	$w_1=(1,1,1,\hat{1},\hat{1},\hat{2})$	$q_1(x)=\sum_{i=1}^7 x_i^2 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 - x_5x_7 - x_6x_7$
	$w_2=(1,1,1,\hat{1},\hat{1},\hat{1})$	$q_2(x)=\sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_5 + x_1x_6 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 - x_5x_7 - x_6x_7$

TABLE 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P\text{-}\mathbf{crit}_7$, for $n = 7$

$p \in \mathbf{posit}_6^\bullet$	$w \in \mathcal{SR}_p$	$q = \mathrm{ind}(p, w) \in P\text{-}\mathbf{crit}_7$
$p_3(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_5 - x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5 - x_5x_6$	$w_1 = (1, 1, 1, 1, \hat{2}, \hat{1})$	$q_4(x) = \sum_{i=1}^7 x_i^2 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6 - x_6x_7$
	$w_2 = (1, 1, 1, 1, \hat{1}, \hat{1})$	$q_5(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6 - x_6x_7$
	$w_3 = (2, 1, 1, 1, \hat{1}, \hat{1})$	$q_6(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6 - x_6x_7$
$p_4(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 - x_3x_6 + x_4x_5 - x_4x_6$	$w = (1, 1, 1, 1, \hat{1}, 1)$	$q_7(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 - x_4x_7 + x_5x_6 - x_5x_7$
$p_5(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5$	$w_1 = (1, 2, 1, \hat{1}, \hat{1}, \hat{1})$	$q_8(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_4 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_5 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_2 = (1, 2, 2, \hat{1}, \hat{1}, \hat{1})$	$q_9(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_5 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
$p_6(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$	$w_1 = (1, 1, 1, 1, \hat{1}, \hat{1})$	$q_{10}(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_4 - x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_2 = (2, 1, 1, 1, \hat{1}, \hat{1})$	$q_{11}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_3 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_3 = (2, 2, 1, 1, \hat{1}, \hat{1})$	$q_{12}(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$
$p_7(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5$	$w_1 = (1, 1, 1, 1, \hat{1}, \hat{1})$	$q_{13}(x) = \sum_{i=1}^7 x_i^2 + x_1x_3 - x_1x_4 - x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_2 = (1, 2, 1, 1, \hat{2}, \hat{1})$	$q_{14}(x) = \sum_{i=1}^7 x_i^2 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_3 = (1, 2, 1, 1, \hat{1}, \hat{1})$	$q_{15}(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_4 = (2, 2, 1, 1, \hat{1}, \hat{1})$	$q_{16}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
$p_8(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5$	$w = (1, 1, 1, 1, \hat{1}, \hat{1})$	$q_{17}(x) = \sum_{i=1}^7 x_i^2 - x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_4x_5 + x_4x_6 + x_4x_7 + x_5x_6$

Table 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P\text{-}\mathbf{crit}_7$, for $n = 7$

$p \in \mathbf{posit}_6^{\bullet}$	$w \in \mathcal{SR}_p$	$q = \mathrm{ind}(p, w) \in P\text{-}\mathbf{crit}_7$
$p_9(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5 - x_5x_6$	$w = (1,1,1,1,\hat{1},\hat{1})$	$q_{18}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_4 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6 - x_6x_7$
$p_{10}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_5 + x_3x_4 - x_3x_6 + x_4x_5$	$w_1 = (1,1,\hat{1},1,\hat{1},\hat{1})$	$q_{19}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_4 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 - x_4x_7 + x_5x_6$
	$w_2 = (1,1,\hat{1},2,\hat{1},\hat{1})$	$q_{20}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 - x_4x_7 + x_5x_6$
$p_{11}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_2x_6 + x_3x_4 + x_4x_5 + x_5x_6$	$w = (1,1,1,\hat{1},1,\hat{1})$	$q_{21}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 + x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_5x_6 + x_6x_7$
$p_{12}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_2x_6 + x_3x_4 + x_4x_5 - x_4x_6$	$w = (1,1,1,\hat{1},1,\hat{1})$	$q_{22}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_5x_6 - x_5x_7$
$p_{13}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_6 - x_2x_5 - x_2x_6 + x_3x_4 + x_3x_5 + x_4x_5$	$w_1 = (1,\hat{1},1,1,\hat{2},\hat{1})$	$q_{23}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 - x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
	$w_2 = (1,\hat{1},1,1,\hat{1},\hat{1})$	$q_{24}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_3 - x_1x_4 - x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 - x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$
$p_{14}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_3x_4 + x_4x_5 + x_4x_6$	$w = (1,1,\hat{1},2,\hat{1},\hat{1})$	$q_{25}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_4x_5 + x_5x_6 + x_5x_7$
$p_{15}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_6 - x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$	$w = (2,1,1,1,\hat{1},\hat{1})$	$q_{26}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 - x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$
$p_{17}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_3x_5 + x_3x_6 + x_4x_5$	$w_1 \in (1,1,1,1,\hat{1},\hat{1})$	$q_{28}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 + x_1x_4 + x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_4x_7 + x_5x_6$
	$w_2 = (1,1,2,1,\hat{1},\hat{1})$	$q_{29}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_4x_7 + x_5x_6$
	$w_3 = (2,1,2,1,\hat{1},\hat{1})$	$q_{30}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_4x_7 + x_5x_6$
	$w_4 = (2,1,2,2,\hat{1},\hat{1})$	$q_{31}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_4x_7 + x_5x_6$
$p_{20}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_1x_6 - x_2x_5 + x_3x_4 + x_4x_5$	$w = (1,1,1,\hat{1},1,\hat{1})$	$q_{36}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 - x_3x_6 + x_4x_5 + x_5x_6$

TABLE 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P\text{-}\mathbf{crit}_7$, for $n = 7$

$p \in \mathbf{posit}_6^\bullet$	$w \in \mathcal{SR}_p$	$q = \mathrm{ind}(p, w) \in P\text{-}\mathbf{crit}_7$
$p_{21}(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_2x_6 + x_3x_4 + x_4x_5$	$w_1 = (1,1,1,\hat{1},1,\hat{1})$	$q_{37}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_3 + x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 + x_4x_5 + x_5x_6$
	$w_2 = (1,2,1,\hat{1},1,\hat{1})$	$q_{38}(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 + x_4x_5 + x_5x_6$
	$w_3 = (1,2,1,\hat{1},2,\hat{1})$	$q_{39}(x) = \sum_{i=1}^7 x_i^2 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 + x_4x_5 + x_5x_6$
$p_{22}(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_3x_4 + x_3x_6 + x_4x_5$	$w_1 = (1,1,1,\hat{1},1,\hat{1})$	$q_{40}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_4 + x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_5 + x_4x_7 + x_5x_6$
	$w_2 = (1,1,2,\hat{1},1,\hat{1})$	$q_{41}(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_5 + x_4x_7 + x_5x_6$
$p_{23}(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_2x_6 + x_3x_4 - x_3x_5 - x_4x_5$	$w_1 = (1,1,1,1,1,\hat{1})$	$q_{42}(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_4 - x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_7 + x_4x_5 - x_4x_6 - x_5x_6$
	$w_2 = (2,1,1,1,1,\hat{1})$	$q_{43}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_3 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_7 + x_4x_5 - x_4x_6 - x_5x_6$
	$w_3 = (2,2,1,1,1,\hat{1})$	$q_{44}(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_7 + x_4x_5 - x_4x_6 - x_5x_6$
$p_{24}(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 + x_3x_4 - x_3x_5 - x_4x_5 + x_5x_6$	$w_1 = (1,1,1,1,1,\hat{1})$	$q_{45}(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
	$w_2 = (1,1,1,1,2,\hat{1})$	$q_{46}(x) = \sum_{i=1}^7 x_i^2 + x_1x_2 - x_1x_3 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
	$w_3 = (2,1,1,1,1,\hat{1})$	$q_{47}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
	$w_4 = (2,1,1,1,2,\hat{1})$	$q_{48}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_4 + x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
	$w_5 = (2,1,1,2,2,\hat{1})$	$q_{49}(x) = \sum_{i=1}^7 x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
	$w_6 = (2,1,2,1,2,\hat{1})$	$q_{50}(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 + x_4x_5 - x_4x_6 - x_5x_6 + x_6x_7$
$p_{25}(x) = \sum_{i=1}^6 x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_2x_6 - x_3x_6 + x_4x_5$	$w = (1,\hat{1},1,1,\hat{1},1)$	$q_{51}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 - x_4x_7 + x_5x_6$

TABLE 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P\text{-}\mathbf{crit}_7$, for $n = 7$

$p \in \mathbf{posit}_6^\bullet$	$w \in \mathcal{SR}_p$	$q = \mathrm{ind}(p, w) \in P\text{-}\mathbf{crit}_7$
$p_{27}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_6 + x_4x_5$	$w_1 = (2, 1, 1, 1, 1, \hat{1})$	$q_{53}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_3 - x_1x_5 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_5x_6$
	$w_2 = (2, 2, 1, 1, 1, \hat{1})$	$q_{54}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_3 - x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_5x_6$
	$w_3 = (3, 2, 1, 1, 1, \hat{1})$	$q_{55}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_5x_6$
	$w_4 = (3, 2, 2, 1, 1, \hat{1})$	$q_{56}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_5x_6$
$p_{28}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_4x_5 + x_4x_6$	$w = (2, 1, 1, 1, 1, \hat{1})$	$q_{57}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_5x_6 + x_5x_7$
$p_{29}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_2x_6 + x_4x_5$	$w = (2, 2, 1, 1, 1, \hat{1})$	$q_{58}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 + x_5x_6$
$p_{31}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_3x_6$	$w_1 = (1, 1, 1, 1, 1, \hat{1})$	$q_{60}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_5 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_2 = (2, 1, 1, 1, 1, \hat{1})$	$q_{61}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_3 + x_1x_4 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_3 = (2, 1, 2, 1, 1, \hat{1})$	$q_{62}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_3 - x_1x_4 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_4 = (2, 2, 1, 1, 1, \hat{1})$	$q_{63}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 + x_1x_4 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_5 = (2, 2, 2, 1, 1, \hat{1})$	$q_{64}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_6 = (3, 2, 2, 1, 1, \hat{1})$	$q_{65}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
	$w_7 = (3, 2, 2, 2, 1, \hat{1})$	$q_{66}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$
$p_{32}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_5x_6$	$w_1 = (1, 1, 1, 1, 1, \hat{1})$	$q_{67}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_4 - x_1x_5 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_6x_7$
	$w_2 = (2, 1, 1, 1, 1, \hat{1})$	$q_{68}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_3 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_6x_7$
	$w_3 = (2, 2, 1, 1, 1, \hat{1})$	$q_{69}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 + x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_6x_7$
	$w_4 = (2, 2, 1, 1, 2, \hat{1})$	$q_{70}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_6x_7$
$p_{33}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_5 + x_4x_6$	$w = (1, 1, 1, 1, 1, \hat{1})$	$q_{71}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_3x_5 - x_4x_6 + x_5x_7$

TABLE 5.8. P-critical unit forms constructed by ind: $\mathcal{Z}_6 \longrightarrow P\text{-}\mathbf{crit}_7$, for $n = 7$

$p \in \mathbf{posit}_6^\bullet$	$w \in \mathcal{SR}_p$	$q = \mathrm{ind}(p, w) \in P\text{-}\mathbf{crit}_7$
$p_{34}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_6 + x_3x_5 + x_4x_5$	$w = (1,1,1,1,\hat{1},\hat{1})$	$q_{72}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_4x_6 + x_5x_6$
$p_{16}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_2x_6 + x_3x_4 + x_5x_6$	$w = (2,1,1,1,1,\hat{1})$	$q_{27}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_4 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_6x_7$
$p_{18}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_3x_5 + x_4x_5 + x_5x_6$	$w = (1,1,1,1,\hat{1},1)$	$q_{32}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_5x_6 + x_6x_7$
$p_{30}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_2x_6 + x_3x_4$	$w = (2,2,1,1,1,\hat{1})$	$q_{59}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_4 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_3x_7 + x_4x_5$
$p_2(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 - x_3x_6 + x_4x_5$	$w = (1,\hat{1},\hat{1},1,1,\hat{1})$	$q_3(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 - x_4x_7 + x_5x_6$
$p_{19}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_1x_6 + x_2x_5 + x_3x_4$	$w_1 = (2,1,1,1,1,\hat{1})$	$q_{33}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5$
	$w_2 = (3,1,1,1,1,\hat{2})$	$q_{34}(x) = \sum_{i=1}^{7} x_i^2 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5$
	$w_3 = (3,1,1,1,1,\hat{1})$	$q_{35}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5$
$p_{26}(x) = \sum_{i=1}^{6} x_i^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 + x_3x_4 + x_3x_6 - x_5x_6$	$w = (1,1,\hat{1},1,1,1)$	$q_{52}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_5 + x_4x_7 - x_6x_7$

In the following Table 5.9, the unit forms p_j, p_{ij}, and the vectors w and w_j are the same as presented in Table 5.8.

TABLE 5.9. A set of representatives of $\mathrm{O}(n,\mathbb{Z})$-orbits in $P\text{-}\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n = 7$

(p, w)	$\mathrm{ind}(p, w) \in \mathrm{O}(7,\mathbb{Z})\text{-}\mathcal{O}rb(q) \subseteq P\text{-}\mathbf{crit}_7^\bullet$	$\mathrm{cox}_q(t)$ and E-type
(p_{33}, w)	$q_{71}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_6 + x_1x_7 - x_2x_3 - x_2x_4 - x_3x_5 - x_4x_6 + x_5x_7$	$t^7 - t^4 - t^3 + 1 = F^{(3)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $= \widetilde{\mathbb{A}}_6$
(p_1, w_2) (p_3, w_2)	$q_2(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 + x_1x_5 + x_1x_6 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 - x_5x_7 - x_6x_7$	$t^7 - t^5 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $= \widetilde{\mathbb{E}}_6$

TABLE 5.9. A set of representatives of $\mathrm{O}(n,\mathbb{Z})$-orbits in P-**crit**$(\mathbb{Z}^n,\mathbb{Z})$, for $n=7$

(p,w)	$\mathrm{ind}(p,w)\in \mathrm{O}(7,\mathbb{Z})\text{-}\mathcal{O}rb(q)\subseteq P\text{-}\mathbf{crit}_7^\bullet$	$\mathrm{cox}_q(t)$ and E-type
(p_1,w_1) (p_5,w_1) (p_7,w_3)	$q_1(x)=\sum_{i=1}^7 x_i^2+x_1x_7-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_2x_7+x_3x_4+x_3x_5+x_3x_6+x_4x_5+x_4x_6+x_5x_6-x_5x_7-x_6x_7$	$t^7-t^5-t^2+1=F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_4,w) (p_7,w_1) (p_{13},w_2)	$q_7(x)=\sum_{i=1}^7 x_i^2-x_1x_3-x_1x_7-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_3x_6+x_3x_7+x_4x_5+x_4x_6-x_4x_7+x_5x_6-x_5x_7$	$t^7-t^5-t^2+1=F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{20},w) (p_{25},w) (p_{31},w_1)	$q_{36}(x)=\sum_{i=1}^7 x_i^2+x_1x_5+x_1x_7-x_2x_3-x_2x_4-x_2x_5+x_2x_7-x_3x_6+x_4x_5+x_5x_6$	$t^7-t^6-t+1=F^{(1)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{22},w_2) (p_{28},w) (p_{32},w_2)	$q_{41}(x)=\sum_{i=1}^7 x_i^2-x_1x_4-x_2x_3-x_2x_4-x_2x_5-x_3x_6+x_4x_5+x_4x_7+x_5x_6$	$t^7-t^4-t^3+1=F^{(3)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{D}}_6$
(p_3,w_1) (p_3,w_3) (p_6,w_2) (p_{19},w_1)	$q_4(x)=\sum_{i=1}^7 x_i^2+x_1x_6-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_2x_7+x_3x_6-x_3x_7+x_4x_5+x_4x_6+x_5x_6-x_6x_7$	$t^7-t^5-t^2+1=F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_8,w) (p_9,w) (p_{22},w_1) (p_{23},w_1)	$q_{17}(x)=\sum_{i=1}^7 x_i^2-x_1x_5+x_1x_7-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_3x_6+x_4x_5+x_4x_6+x_4x_7+x_5x_6$	$t^7-t^4-t^3+1=F^{(3)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{D}}_6$
(p_{10},w_2) (p_{14},w) (p_{17},w_2) (p_{31},w_2)	$q_{20}(x)=\sum_{i=1}^7 x_i^2-x_1x_5-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_2x_7+x_3x_6+x_4x_5-x_4x_7+x_5x_6$	$t^7-t^5-t^2+1=F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{11},w) (p_{12},w) (p_{17},w_1) (p_{21},w_1)	$q_{21}(x)=\sum_{i=1}^7 x_i^2-x_1x_3+x_1x_5-x_2x_3-x_2x_4-x_2x_5-x_2x_6+x_3x_6+x_3x_7+x_4x_5+x_5x_6+x_6x_7$	$t^7-t^6-t+1=F^{(1)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{21},w_3) (p_{29},w) (p_{31},w_3) (p_{31},w_4)	$q_{39}(x)=\sum_{i=1}^7 x_i^2-x_1x_6-x_2x_3-x_2x_4-x_2x_5-x_3x_6+x_3x_7+x_4x_5+x_5x_6$	$t^7-t^5-t^2+1=F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$

TABLE 5.9. A set of representatives of $\mathrm{O}(n,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n=7$

(p,w)	$\mathrm{ind}(p,w) \in \mathrm{O}(7,\mathbb{Z})$-$\mathcal{O}rb(q) \subseteq P$-$\mathbf{crit}_7^\bullet$	$\mathrm{cox}_q(t)$ and E-type
(p_{13},w_1) (p_{15},w) (p_{21},w_2) (p_{24},w_2) (p_{24},w_3) (p_{27},w_1)	$q_{23}(x) = \sum_{i=1}^7 x_i^2 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 - x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$	$t^7 - t^5 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{32},w_4)	$q_{70}(x) = \sum_{i=1}^7 x_i^2 - x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_6x_7$	$t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_6}(t)$, E-type $=\widetilde{\mathbb{D}}_6$
(p_{16},w) (p_{23},w_2)	$q_{27}(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_6 + x_3x_7 + x_4x_5 + x_6x_7$	$t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_6}(t)$, E-type $=\widetilde{\mathbb{D}}_6$
(p_{23},w_3) (p_{30},w) (p_{32},w_3)	$q_{44}(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 + x_3x_7 + x_4x_5 - x_4x_6 - x_5x_6$	$t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1 = F_{\widetilde{\mathbb{D}}_6}(t)$, E-type $=\widetilde{\mathbb{D}}_6$
(p_{18},w) (p_{26},w) (p_{32},w_1) (p_{34},w)	$q_{32}(x) = \sum_{i=1}^7 x_i^2 - x_1x_3 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_5x_6 + x_6x_7$	$t^7 - 2t^5 + t^4 + t^3 - 2t^2 + 1$, E-type $=\widetilde{\mathbb{D}}_6$
(p_{19},w_3)	$q_{35}(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 - x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_5,w_2) (p_{24},w_4)	$q_9(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_5 + x_3x_6 + x_3x_7 + x_4x_5 + x_4x_6 + x_5x_6$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{19},w_2) (p_{27},w_3)	$q_{34}(x) = \sum_{i=1}^7 x_i^2 + x_1x_7 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_{27},w_4) (p_{31},w_6)	$q_{56}(x) = \sum_{i=1}^7 x_i^2 - x_1x_4 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3x_7 + x_5x_6$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$
(p_2,w) (p_6,w_1) (p_{10},w_1) (p_{24},w_1)	$q_3(x) = \sum_{i=1}^7 x_i^2 - x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 - x_4x_7 + x_5x_6$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $=\widetilde{\mathbb{E}}_6$

Table 5.9. A set of representatives of $\mathrm{O}(n,\mathbb{Z})$-orbits in P-$\mathbf{crit}(\mathbb{Z}^n,\mathbb{Z})$, for $n = 7$

(p, w)	$\mathrm{ind}(p, w) \in \mathrm{O}(7,\mathbb{Z})$-$\mathcal{O}rb(q) \subseteq P$-$\mathbf{crit}_7^\bullet$	$\mathrm{cox}_q(t)$ and E-type
(p_{17}, w_4) (p_{24}, w_5) (p_{24}, w_6) (p_{31}, w_5)	$q_{31}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_4x_6 + x_4x_7 + x_5x_6$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $= \widetilde{\mathbb{E}}_6$
(p_6, w_3) (p_7, w_2) (p_7, w_4) (p_{17}, w_3) (p_{27}, w_2)	$q_{12}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_3 - x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $= \widetilde{\mathbb{E}}_6$
(p_{31}, w_7)	$q_{66}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_5 - x_2x_3 - x_2x_4 - x_2x_5 - x_3x_6 + x_4x_7$	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = F_{\widetilde{\mathbb{E}}_6}(t)$, E-type $= \widetilde{\mathbb{E}}_6$

Remark 5.10. It follows from Tables 5.6 and 5.7 that the surjective correspondence

$$\mathrm{ind} : \mathrm{O}(5,\mathbb{Z})\text{-}\mathcal{O}rb(\mathcal{Z}_5) \longrightarrow \mathrm{O}(6,\mathbb{Z})\text{-}\mathcal{O}rb(P\text{-}\mathbf{crit}(\mathbb{Z}^6,\mathbb{Z}))$$

(3.3) induced by the map $(p, w) \mapsto \mathrm{ind}_{s,\varepsilon_s}(p, w) := q_{p,s,w,\varepsilon_s}$, is far from being injective. Indeed, the pairs (p_2, w), (p_4, w), (p_7, w_1) lie in different $\mathrm{O}(5,\mathbb{Z})$-orbits in $\mathcal{Z}_5$, but the P-critical unit forms $\mathrm{ind}(p_2, w) = q_2$, $\mathrm{ind}_{s,\varepsilon_s}(p_4, w) = q_5$ and $\mathrm{ind}_{s,\varepsilon_s}(p_7, w_1) = q_8$ presented in Table 5.7 lie in the $\mathrm{O}(6,\mathbb{Z})$-orbit of the unit form

$$q_2(x) = \sum_{i=1}^{6} x_i^2 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5 + x_2x_6 + x_3x_5 + x_4x_5,$$

because we have $q_5 * B_1 = q_2$, $q_8 * B_2 = q_2$, and $q_8 * B_3 = q_5$, where

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Now we present a complete list of the Coxeter polynomials $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$ of P-critical bigraphs Δ, with at most 10 vertices. Together with the following corollary, the list is obtained by a computer search.

Corollary 5.11. *Assume that Δ is a P-critical bigraph, with at most* 10 *vertices.*

(a) *If Δ is of the Euclidean type $D\Delta = \widetilde{\mathbb{A}}_n$, $n \le 9$, then $\mathrm{cox}_\Delta(t)$ is one of the polynomials listed in the following table, see* [25].

TABLE 5.12. Coxeter polynomials $\mathrm{cox}_\Delta(t)$ of P-critical bigraphs Δ of E-type $\widetilde{\mathbb{A}}_n$, $n \le 9$ and their reduced Coxeter numbers $\check{\mathbf{c}}_\Delta$

j	$F^{(j)}_\Delta(t)$	$\check{\mathbf{c}}_\Delta$
$j = 1$	$t^{n+1} - t^n - t + 1$	n
$j \in \{2, \dots, m_n\}$	$t^{n+1} - t^{n+1-j} - t^j + 1$	$\mathbf{lcm}(n-j+1, j)$

$$\text{where } m_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n+1 \text{ is even.} \end{cases}$$

(b) *If Δ is of the Euclidean type $D\Delta \in \{\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$, $n \le 9$, then $\mathrm{cox}_\Delta(t)$ is one of the polynomials presented in Table* 5.13.

TABLE 5.13. Coxeter polynomials $\mathrm{cox}_\Delta(t)$ of P-critical bigraphs Δ

$D\Delta$	$\mathcal{CG}pol^P_{\mathrm{Eucl}} = \{\mathrm{cox}_\Delta(t)\}_{D\Delta=\mathrm{Eucl}} = \left\{F^{(j)}_{D\Delta}(t)\right\}$	$\check{\mathbf{c}}_\Delta$
$\widetilde{\mathbb{D}}_4$	$F^{(1)}_{\widetilde{\mathbb{D}}_4}(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 2$
	$F^{(2)}_{\widetilde{\mathbb{D}}_4}(t) = t^5 - t^4 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_4}(t)$	$\check{\mathbf{c}}_\Delta = 4$
$\widetilde{\mathbb{D}}_5$	$F^{(1)}_{\widetilde{\mathbb{D}}_5}(t) = t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(2)}_{\widetilde{\mathbb{D}}_5}(t) = t^6 - t^5 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_5}(t)$	$\check{\mathbf{c}}_\Delta = 5$
	$F^{(3)}_{\widetilde{\mathbb{D}}_5}(t) = t^6 - t^4 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_5}(t)$	$\check{\mathbf{c}}_\Delta = 4$
	$F^{(4)}_{\widetilde{\mathbb{D}}_5}(t) = t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(5)}_{\widetilde{\mathbb{D}}_5}(t) = t^6 - 2t^5 + 3t^4 - 4t^3 + 3t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 4$

Table 5.13. Coxeter polynomials $\mathrm{cox}_\Delta(t)$ of P-critical bigraphs Δ

$D\Delta$	$\mathcal{CG}pol^P_{\mathrm{Eucl}} = \{\mathrm{cox}_\Delta(t)\}_{D\Delta=\mathrm{Eucl}} = \left\{F^{(j)}_{D\Delta}(t)\right\}$	$\check{\mathbf{c}}_\Delta$
$\widetilde{\mathbb{D}}_6$	$F^{(1)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 4$
	$F^{(2)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - t^6 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_6}(t)$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(3)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - t^4 - t^3 + 1 = F^{(3)}_{\widetilde{\mathbb{A}}_6}(t)$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(4)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - t^6 + t^5 - t^4 - t^3 + t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 4$
	$F^{(5)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - t^6 - t^5 + t^4 + t^3 - t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(6)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - 2t^6 + 2t^5 - t^4 - t^3 + 2t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(7)}_{\widetilde{\mathbb{D}}_6}(t) = t^7 - 2t^5 + t^4 + t^3 - 2t^2 + 1$	$\check{\mathbf{c}}_\Delta = 6$
$\widetilde{\mathbb{D}}_7$	$F^{(1)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 + t^7 - t^6 - t^5 - t^3 - t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 10$
	$F^{(2)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - t^7 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 7$
	$F^{(3)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - t^6 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(4)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - 2t^4 + 1 = F^{(4)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 4$
	$F^{(5)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - t^7 + t^5 - 2t^4 + t^3 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(6)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - 2t^6 + 2t^4 - 2t^2 + 1$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(7)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - 2t^7 + 2t^6 - 2t^5 + 2t^4 - 2t^3 + 2t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(8)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - 2t^7 + t^6 + 2t^5 - 4t^4 + 2t^3 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(9)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - t^7 + 2t^6 - 3t^5 + 2t^4 - 3t^3 + 2t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(10)}_{\widetilde{\mathbb{D}}_7}(t) = t^8 - t^7 - t^6 + t^5 + t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 10$
$\widetilde{\mathbb{D}}_8$	$F^{(1)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 + t^8 - t^7 - t^6 - t^3 - t^2 + t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(2)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^8 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_8}(t)$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(3)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^5 - t^4 + 1 = F^{(4)}_{\widetilde{\mathbb{A}}_8}(t)$	$\check{\mathbf{c}}_\Delta = 20$
	$F^{(4)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^8 + 2t^7 - 2t^6 - 2t^3 + 2t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 4$
	$F^{(5)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^8 + t^7 - t^6 - t^3 + t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(6)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^8 - t^7 + t^6 + t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(7)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^8 - t^7 + 3t^6 - 2t^5 - 2t^4 + 3t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(8)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^7 + t^6 - t^5 - t^4 + t^3 - t^2 + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(9)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - t^7 - t^6 + t^5 + t^4 - t^3 - t^2 + 1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(10)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - 2t^7 + t^5 + t^4 - 2t^2 + 1$	$\check{\mathbf{c}}_\Delta = 10$
	$F^{(11)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - 2t^8 + t^7 + t^6 - t^5 - t^4 + t^3 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(12)}_{\widetilde{\mathbb{D}}_8}(t) = t^9 - 2t^8 + 2t^7 - 2t^6 + t^5 + t^4 - 2t^3 + 2t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 20$

TABLE 5.13. Coxeter polynomials $\mathrm{cox}_\Delta(t)$ of P-critical bigraphs Δ

$D\Delta$	$\mathcal{CG}pol^P_{\mathrm{Eucl}} = \{\mathrm{cox}_\Delta(t)\}_{D\Delta=\mathrm{Eucl}} = \left\{F^{(j)}_{D\Delta}(t)\right\}$	$\check{\mathbf{c}}_\Delta$
$\widetilde{\mathbb{D}}_9$	$F^{(1)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}+t^9-t^8-t^7-t^3-t^2+t+1$	$\check{\mathbf{c}}_\Delta = 14$
	$F^{(2)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9-t+1 = F^{(1)}_{\widetilde{\mathbb{A}}_9}(t)$	$\check{\mathbf{c}}_\Delta = 9$
	$F^{(3)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^8-t^2+1 = F^{(2)}_{\widetilde{\mathbb{A}}_9}(t)$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(4)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^6-t^4+1 = F^{(4)}_{\widetilde{\mathbb{A}}_9}(t)$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(5)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9+t^8-2t^7+2t^6-2t^5+2t^4-2t^3+t^2-t+1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(6)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9+2t^8-2t^7+t^6-2t^5+t^4-2t^3+2t^2-t+1$	$\check{\mathbf{c}}_\Delta = 20$
	$F^{(7)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9+t^8-2t^6+2t^5-2t^4+t^2-t+1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(8)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9-t^6+2t^5-t^4-t+1$	$\check{\mathbf{c}}_\Delta = 20$
	$F^{(9)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-2t^9+2t^8-2t^7+t^6+t^4-2t^3+2t^2-2t+1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(10)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-2t^9+t^8+t^7-2t^6+2t^5-2t^4+t^3+t^2-2t+1$	$\check{\mathbf{c}}_\Delta = 30$
	$F^{(11)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9-t^8+2t^7-2t^5+2t^3-t^2-t+1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(12)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9+t^7-t^6-t^4+t^3-t+1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(13)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-2t^9+t^8+2t^6-4t^5+2t^4+t^2-2t+1$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(14)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^9-t^8+t^7+t^3-t^2-t+1$	$\check{\mathbf{c}}_\Delta = 14$
	$F^{(15)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^8+t^7-2t^5+t^3-t^2+1$	$\check{\mathbf{c}}_\Delta = 30$
	$F^{(16)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-t^8-t^7+2t^5-t^3-t^2+1$	$\check{\mathbf{c}}_\Delta = 30$
	$F^{(17)}_{\widetilde{\mathbb{D}}_9}(t) = t^{10}-2t^8+t^6+t^4-2t^2+1$	$\check{\mathbf{c}}_\Delta = 12$
$\widetilde{\mathbb{E}}_6$	$F^{(1)}_{\widetilde{\mathbb{E}}_6}(t) = t^7+t^6-2t^4-2t^3+t+1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(2)}_{\widetilde{\mathbb{E}}_6}(t) = t^7-t^6-t+1 = F^{(1)}_{\widetilde{\mathbb{A}}_6}(t)$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(3)}_{\widetilde{\mathbb{E}}_6}(t) = t^7-t^5-t^2+1 = F^{(2)}_{\widetilde{\mathbb{A}}_6}(t)$	$\check{\mathbf{c}}_\Delta = 10$
	$F^{(4)}_{\widetilde{\mathbb{E}}_6}(t) = t^7-2t^6+2t^5-t^4-t^3+2t^2-2t+1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(5)}_{\widetilde{\mathbb{E}}_6}(t) = t^7-t^6-t^5+t^4+t^3-t^2-t+1$	$\check{\mathbf{c}}_\Delta = 8$
$\widetilde{\mathbb{E}}_7$	$F^{(1)}_{\widetilde{\mathbb{E}}_7}(t) = t^8+t^7-t^5-2t^4-t^3+t+1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(2)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-t^7-t+1 = F^{(1)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 7$
	$F^{(3)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-t^6-t^2+1 = F^{(2)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(4)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-t^5-t^3+1 = F^{(3)}_{\widetilde{\mathbb{A}}_7}(t)$	$\check{\mathbf{c}}_\Delta = 15$
	$F^{(5)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-t^7+t^5-2t^4+t^3-t+1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(6)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-2t^6+2t^4-2t^2+1$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(7)}_{\widetilde{\mathbb{E}}_7}(t) = t^8-2t^7+2t^6-2t^5+2t^4-2t^3+2t^2-2t+1$	$\check{\mathbf{c}}_\Delta = 8$

Table 5.13. Coxeter polynomials $\mathrm{cox}_\Delta(t)$ of P-critical bigraphs Δ

$D\Delta$	$\mathcal{CG}pol^P_{\mathrm{Eucl}} = \{\mathrm{cox}_\Delta(t)\}_{D\Delta=\mathrm{Eucl}} = \left\{F^{(j)}_{D\Delta}(t)\right\}$	$\check{\mathbf{c}}_\Delta$
	$F^{(8)}_{\widetilde{\mathbb{E}}_7}(t) = t^8 - 2t^7 + t^6 + 2t^5 - 4t^4 + 2t^3 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(9)}_{\widetilde{\mathbb{E}}_7}(t) = t^8 - 3t^7 + 5t^6 - 6t^5 + 6t^4 - 6t^3 + 5t^2 - 3t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(10)}_{\widetilde{\mathbb{E}}_7}(t) = t^8 - 2t^7 + t^6 + t^5 - 2t^4 + t^3 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 9$
	$F^{(11)}_{\widetilde{\mathbb{E}}_7}(t) = t^8 - t^7 - t^6 + t^5 + t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 10$
	$F^{(12)}_{\widetilde{\mathbb{E}}_7}(t) = t^8 - t^7 - t^6 + 2t^4 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
$\widetilde{\mathbb{E}}_8$	$F^{(1)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1$	$\check{\mathbf{c}}_\Delta = 30$
	$F^{(2)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^8 - t + 1 = F^{(1)}_{\widetilde{\mathbb{A}}_8}(t)$	$\check{\mathbf{c}}_\Delta = 8$
	$F^{(3)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^7 - t^2 + 1 = F^{(2)}_{\widetilde{\mathbb{A}}_8}(t)$	$\check{\mathbf{c}}_\Delta = 14$
	$F^{(4)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^5 - t^4 + 1 = F^{(4)}_{\widetilde{\mathbb{A}}_8}(t)$	$\check{\mathbf{c}}_\Delta = 20$
	$F^{(5)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^8 + t^7 - t^6 - t^3 + t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(6)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^8 - t^7 + 2t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 18$
	$F^{(7)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^8 - t^7 + t^6 + t^3 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(8)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^8 - t^7 + t^5 + t^4 - t^2 - t + 1$	$\check{\mathbf{c}}_\Delta = 18$
	$F^{(9)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^8 + 2t^7 - t^6 - t^3 + 2t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(10)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^8 + 2t^7 - 2t^6 + t^5 + t^4 - 2t^3 + 2t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 20$
	$F^{(11)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^8 + t^7 + t^6 - t^5 - t^4 + t^3 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(12)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^8 + t^7 + t^2 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 14$
	$F^{(13)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^8 + 3t^6 - 2t^5 - 2t^4 + 3t^3 - 2t + 1$	$\check{\mathbf{c}}_\Delta = 12$
	$F^{(14)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 3t^8 + 4t^7 - 3t^6 + t^5 + t^4 - 3t^3 + 4t^2 - 3t + 1$	$\check{\mathbf{c}}_\Delta = 30$
	$F^{(15)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 4t^8 + 8t^7 - 9t^6 + 4t^5 + 4t^4 - 9t^3 + 8t^2 - 4t + 1$	$\check{\mathbf{c}}_\Delta = 6$
	$F^{(16)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - t^7 - t^6 + t^5 + t^4 - t^3 - t^2 + 1$	$\check{\mathbf{c}}_\Delta = 24$
	$F^{(17)}_{\widetilde{\mathbb{E}}_8}(t) = t^9 - 2t^7 - t^6 + 2t^5 + 2t^4 - t^3 - 2t^2 + 1$	$\check{\mathbf{c}}_\Delta = 12$

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