Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **18** (2014). Number 1, pp. 27 – 41
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$\mathscr{L}$-cross-sections of the finite symmetric semigroup

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Communicated by V. Mazorchuk

Abstract. In the present paper we give the description of all $\mathscr{L}$-cross-sections of the finite symmetric semigroup.

Introduction

Let ρ be an equivalence relation on a semigroup S. A subsemigroup S' of S is called a ρ-cross-section of S, provided that S' contains exactly one representative from each equivalence class. It is natural to investigate the cross-sections with respect to those equivalences related somehow to the semigroup operation: Green's relations, conjugacy and various congruences. The transformation semigroups are the classical objects for investigation in semigroup theory (see [1]). This is explained by the well-known fact that every (inverse) semigroup is embedded into some symmetric (inverse) semigroup up to an isomorphism (see e.g. [2]).

In fact, the full transformation semigroup, the symmetric inverse semigroup, and the semigroup of all partial transformations of n-element set contain exactly one cross-section with respect to the relation of conjugation if $n = 1$, and do not contain any cross-section with respect to the conjugacy if $n > 1$ (see [1]). The cross-sections of the mentioned semigroups with respect to the congruences exist only for three types of congruences [2].

[1]The author expresses gratitude to the referee for the number of helpful improving proposals of the article.

2010 MSC: 20M20.

Key words and phrases: symmetric semigroup, cross-section, Green's relations.

The description of cross-sections with respect to Green's relations in transformation semigroups and some others is much richer and more varied. In [3] L. Renner has used so called order-preserving elements for the investigation of reductive monoids. It turned out, that the set of order-preserving elements forms an inverse monoid and is an $\mathscr{H}$-cross-section. In [4] D. Cowan and N. Reilly have proved that all $\mathscr{H}$-cross-sections of the full symmetric inverse semigroup on a set X ($|X| \neq 3$) with fixed linear order consist exactly of order-preserving transformations on X. Later, $\mathscr{R}$- and, in a dual manner, $\mathscr{L}$-cross-sections of the finite symmetric inverse semigroup have been described by O. Ganyushkin and V. Mazorchuk [5]. It transpires, that in contrast with $\mathscr{H}$-cross-sections, different $\mathscr{R}$-($\mathscr{L}$-) cross-sections are not isomorphic in general, and the number of mutually non-isomorphic $\mathscr{R}$-($\mathscr{L}$-)cross-sections is equal to the number of different partitions of the positive integer n into a sum of positive integers. V. Pekhterev [6] shows that every finite poset can be embedded into some idempotent $\mathscr{D}$-cross-section of the finite symmetric inverse semigroup. All $\mathscr{H}$-, $\mathscr{R}(\mathscr{L})$-cross-sections of the infinite symmetric inverse semigroup have been described in [7].

The cross-sections with respect to Green's relations have also been investigated on other transformation semigroups. The complete description of all cross-sections with respect to Green's relations on the variant of symmetric inverse semigroup represented by the symmetric inverse 0-category is given in [8]. The description and classification of all $\mathscr{R}$-($\mathscr{L}$-), $\mathscr{H}$-cross-sections of the Brauer semigroup can be found in [9].

For the full finite symmetric semigroup $\mathscr{T}_n$ all $\mathscr{H}$- and $\mathscr{R}$-cross-sections have been described in [10]. It has been proved, that there exists a unique $\mathscr{R}$-cross-section up to an isomorphism. At the same time, a pair of non-isomorphic $\mathscr{L}$-cross-sections of $\mathscr{T}_4$ has been constructed in [11]. Thus, in case of the symmetric semigroup the dual description of $\mathscr{L}$-cross-sections is not applicable and the question of the description of $\mathscr{L}$-cross-sections of $\mathscr{T}_n$ remains open-ended [1]. In addition, the description of $\mathscr{H}$-, $\mathscr{R}$- and $\mathscr{L}$-cross-sections of the strong endomorphism monoid of finite graphs without multiply edges is reduced to the problem mentioned above [12].

The present paper contains the description of all $\mathscr{L}$-cross-sections of the full finite symmetric semigroup. The paper is organized as follows. All the necessary preliminaries are contained in Section 1. In Section 2 the notion of an L-family is introduced. We have also constructed a semigroup L_X^{Γ} and formulate the main result of the paper in Theorem 1, namely, that L_X^{Γ} is an $\mathscr{L}$-cross-section of the semigroup $\mathscr{T}_n$, and conversely, any

$\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$ is given by L_X^Γ for a suitable L-family on X. The proof of Theorem 1 is given in Section 3.

1. Preliminaries

For any nonempty set X, the set of all transformations of X into itself, written on the right, is a semigroup under the composition $x(\alpha\beta) = (x\alpha)\beta$, $x \in X$ to be denoted by $\mathscr{T}(X)$. The semigroup $\mathscr{T}(X)$ is called a symmetric semigroup. If $|X| = n$, then the symmetric semigroup $\mathscr{T}(X)$ will also be denoted by $\mathscr{T}_n$. We will write id_X for the identity transformation on X and just x if an image of transformation is $\{x\}$, $x \in X$. The image and the rank of transformation $\alpha \in \mathscr{T}_n$ is denoted by $\mathrm{im}(\alpha)$ and $\mathrm{rk}(\alpha)$ respectively. If X' is a subset of X, then $\alpha|_{X'}$ is the restriction α to X'. We will assume X is finite. As the nature of elements of X is not important for us, suppose further $X = \{1, 2, \ldots, n\}$.

Let U be a nonempty subset of X. For a given family of subsets of X denote by $\overline{U}$ the intersection of all the sets of this family that contain U.

It is well known that there always exist five equivalence relations $\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}$ on any semigroup, called Green's relations. We recall, that two elements in a semigroup S are called $\mathscr{L}$-($\mathscr{R}$-)-equivalent, provided that they generate the same principal left (right) ideal in S, and $\mathscr{J}$-equivalent, provided that they generate the same principal two-sided ideal. The product of equivalence relations $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathscr{D}$, the $\mathscr{H}$-relation is defined as the intersection of $\mathscr{L}$ and $\mathscr{R}$.

The description of Green's relations on $\mathscr{T}_n$ is well known (see e.g. [2]). We recall only that two transformations $\alpha, \beta \in \mathscr{T}_n$ are $\mathscr{L}$-equivalent if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$, and $\mathscr{D}$-equivalent if and only if $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$. We will denote the $\mathscr{D}$-class, which consists of all transformations of rank k, $1 \leqslant k \leqslant n$ by $\mathscr{D}_k$.

Suppose further that L is an $\mathscr{L}$-cross-section in $\mathscr{T}_n$. Recall that a transversal of a family of sets is a set which intersects each set of this family in a single element. If $\alpha \in \mathscr{T}_n$ is an arbitrary transformation, then the relation ρ_α defined on X by: $x\rho_\alpha y$ if and only if $x\alpha = y\alpha$ is an equivalence. The ρ_α-class containing $x \in X$ to be denoted by x_{ρ_α}, and the set of all transversals of a quotient set X/ρ_α by T_α. Besides, we set

$$X_L = \bigcup_{\alpha \in L} X/\rho_\alpha.$$

Example 1. Let L be one of the $\mathscr{L}$-cross-sections of $\mathcal{T}_4$ constructed in [11]:

$$L = \left\{ \begin{pmatrix}1234\\1111\end{pmatrix}, \begin{pmatrix}1234\\2222\end{pmatrix}, \begin{pmatrix}1234\\3333\end{pmatrix}, \begin{pmatrix}1234\\4444\end{pmatrix}, \begin{pmatrix}1234\\1133\end{pmatrix}, \begin{pmatrix}1234\\1144\end{pmatrix}, \right.$$

$$\begin{pmatrix}1234\\2233\end{pmatrix}, \begin{pmatrix}1234\\2244\end{pmatrix}, \begin{pmatrix}1234\\1122\end{pmatrix}, \begin{pmatrix}1234\\3344\end{pmatrix}, \begin{pmatrix}1234\\1233\end{pmatrix}, \begin{pmatrix}1234\\1244\end{pmatrix},$$

$$\left. \begin{pmatrix}1234\\1134\end{pmatrix}, \begin{pmatrix}1234\\2234\end{pmatrix}, \begin{pmatrix}1234\\1234\end{pmatrix} \right\}.$$

For instance, if $\alpha = \left(\begin{smallmatrix}1234\\2234\end{smallmatrix}\right)$, we have

$$X/\rho_\alpha = \{\{1,2\},\{3\},\{4\}\},$$

$$T_\alpha = \{\{1,3,4\},\{2,3,4\}\}.$$

The set X_L for the given cross-section has the form:

$$X_L = \{\{1,2,3,4\},\{1,2\},\{3,4\},\{1\},\{2\},\{3\},\{4\}\}.$$

Recall that a binary tree is a finite set of elements that is either empty or contains one element, called the root of the tree, and other elements of the set are divided into two disjoint subsets, each of which is itself a binary tree. Elements of a binary tree are called nodes. Thus, each node in a binary tree has zero or two child nodes, which are below it in the tree. A full binary tree is a tree in which every node other than a leaf has two children.

2. The description of $\mathscr{L}$-cross-sections in $\mathcal{T}_n$

First note that in the case if $n \in \{1,2\}$ the symmetric semigroup $\mathcal{T}_n$ clearly has the following unique $\mathscr{L}$-cross-sections:

(i) $L = \left\{ \begin{pmatrix}1\\1\end{pmatrix} \right\}$, if $n = 1$;

(ii) $L = \left\{ \begin{pmatrix}12\\12\end{pmatrix}, \begin{pmatrix}12\\11\end{pmatrix}, \begin{pmatrix}12\\22\end{pmatrix} \right\}$, if $n = 2$.

We fix an arbitrary strict total order on the finite set X and define the strict total order $\prec$ for subsets of X as follows. Suppose A, B are two nonempty subsets from X. We will say that $A \prec B$, if every $a \in A$ less than any $b \in B$.

Let $Y = \{1,2\}$, $F(Y)$ be a free semigroup over the alphabet Y, $a \in F(Y)$. Denote the last symbol in the word a by a^0. The word of the same length as a, which is different from a by the last symbol is denoted by a^*.

Set $A_0 = X$. If $|A_0| > 1$, then define a family Γ of nonempty subsets of X using the next algorithm. We present A_0 in the form $A_0 = A_1 \cup A_2$, where $A_1 \prec A_2$. If $|A_i| > 1$ for some $i \in \{1,2\}$ then break down A_i so that $A_i = A_{i1} \cup A_{i2}$, where $A_{i1} \prec A_{i2}$. Further, we divide all the rest non-one-element subsets in this way. It is clear that such an algorithm is finite. Note, that if we consider all these subsets from $A_0, A_1, A_2, \ldots$ to one-element subsets as a vertex set, and set $\{A_a, A_b\}$ is an edge if and only if $A_a \subseteq A_b$, we get a full binary tree (see fig.1).

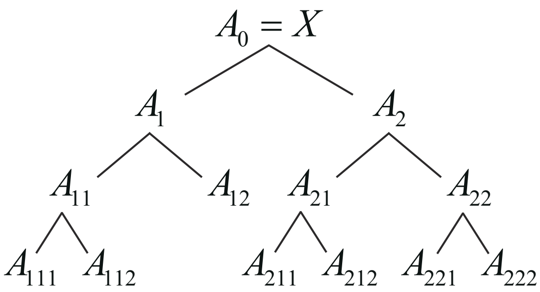

FIGURE 1. An arbitrary full binary tree

We will call a family Γ consisting of all these subsets from A_0,A_1,$A_2, \ldots$ to one-element subsets an *L-family* if for any $A_a \in \Gamma$, $a \in F(Y)$, $|A_a| > 1$ the condition $|A_{aib}| \leqslant |A_{a^*b}|$, $i \in \{1,2\}$, $i \neq a^0$ holds for all $b \in F(Y)$ such that $A_{aib}, A_{a^*b} \in \Gamma$.

We say that sets A_a and $A_{a^*} \in \Gamma$, where $a \in F(Y)$, $Y = \{1,2\}$, are *adjacent*. The sets A_1 and A_2 assumed adjacent, too.

Example 2. Let $X = \{1,2,3,4\}$ be a naturally ordered set. We can choose tree distinct L-families on X in this case:

$$\Gamma_1 = \left\{ \begin{array}{c} A_0 = \{1,2,3,4\}, A_1 = \{1\}, A_2 = \{2,3,4\}, A_{21} = \{2\}, \\ A_{22} = \{3,4\}, A_{221} = \{3\}, A_{222} = \{4\} \end{array} \right\},$$
$$\Gamma_2 = \left\{ \begin{array}{c} A_0 = \{1,2,3,4\}, A_1 = \{1,2\}, A_2 = \{3,4\}, A_{11} = \{1\}, \\ A_{12} = \{2\}, A_{21} = \{3\}, A_{22} = \{4\} \end{array} \right\},$$
$$\Gamma_3 = \left\{ \begin{array}{c} A_0 = \{1,2,3,4\}, A_1 = \{1,2,3\}, A_2 = \{4\}, A_{11} = \{1,2\}, \\ A_{12} = \{3\}, A_{111} = \{1\}, A_{112} = \{2\} \end{array} \right\}.$$

Lemma 1. *Let an L-family Γ be defined on X, V be an arbitrary subset of X, $|V| > 1$. There exists a unique adjacent pair $A_{a1}, A_{a2} \in \Gamma$, $a \in F(Y)$ such that V is a union of two nonempty subsets from A_{a1} and A_{a2} respectively.*

Proof. Clearly V is contained in some subset of Γ. Suppose $A_a \in \Gamma$ is the intersection of all elements of Γ that contain V. Since A_{a1}, A_{a2} are the proper subsets of A_a, we have $V \cap A_{a1} \neq \varnothing$, $V \cap A_{a2} \neq \varnothing$ and $V = (V \cap A_{a1}) \cup (V \cap A_{a2})$. So, V has a form of the union of two nonempty subsets of adjacent sets in Γ. Now we show that this adjacent pair is unique.

Let $A_b \in \Gamma$, $b \in F(Y)$, $b \neq a$ and $V \subseteq A_b$. Clearly, $A_a \subset A_b$. By definition of the family Γ the last condition implies either $A_a \subseteq A_{b1}$ or $A_a \subseteq A_{b2}$. Therefore V can not be represented as a union of nonempty subsets of A_{b1} and A_{b2} respectively. □

Let Γ be an L-family of subsets of X, $M \subseteq X$. Set

(a) if $M = \{m\}$, then for all $a \in F(Y)$ $\alpha_M^{A_a} = m$.

(b) if $|M| \neq 1$, then define $\alpha_M^{A_a}$ as $\alpha_M^{A_a}|_{A_{ai}} = \alpha_{A_{bi} \cap M}^{A_{ai}}$, where $A_b, b \in F(Y)$ is the intersection of all elements of an L-family Γ that contain M, $i \in \{1, 2\}$.

Remark 1. A map $\alpha_M^{A_a}$ is defined not for every pair A_a and M.

By L_X^Γ denote the set of all transformations of the form α_M^X, where $M \subseteq X$, $M \neq \varnothing$. We will write also α_M instead of α_M^X.

The main result of this paper is the following theorem.

Theorem 1. *For any L-family Γ of X the set L_X^Γ is an $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$. Conversely, any $\mathscr{L}$-cross-section of the symmetric semigroup $\mathscr{T}_n$ is given by L_X^Γ for a suitable L-family Γ on X.*

Example 3. Let $\{1, 2, 3, 4, 5\}$ be naturally ordered. We are now going to construct an $\mathscr{L}$-cross-section L_X^Γ of $\mathscr{T}_5$ for the next L-family:

$$\Gamma = \{A_0 = \{1,2,3,4,5\}, A_1 = \{1,2\}, A_2 = \{3,4,5\}, A_{11} = \{1\},$$

$$A_{12} = \{2\}, A_{21} = \{3\}, A_{22} = \{4,5\}, A_{221} = \{4\}, A_{222} = \{5\}\}.$$

We will construct the transformation α with the image $\{1, 2, 4, 5\}$, i. e. $\alpha = \alpha_{\{1,2,4,5\}}$. Clearly, $A_0 = \overline{\mathrm{im}(\alpha)}$. Thus, $\alpha \in Map(A_0, A_0)$ and

$$\alpha|_{A_1} = \alpha_{A_1 \cap \mathrm{im}(\alpha)}^{A_1} = \alpha_{\{1,2\}}^{A_1}, \quad \alpha|_{A_2} = \alpha_{A_2 \cap \mathrm{im}(\alpha)}^{A_2} = \alpha_{\{4,5\}}^{A_2}$$

imply that $A_1\alpha \subseteq A_1$, $A_2\alpha \subseteq A_{22}$ and next mappings are defined

$$\alpha|_{A_{11}} = \alpha^{A_{11}}_{A_{11}\cap \mathrm{im}(\alpha)} = \alpha^{A_{11}}_{\{1\}}, \quad \alpha|_{A_{12}} = \alpha^{A_{12}}_{A_{12}\cap \mathrm{im}(\alpha)} = \alpha^{A_{12}}_{\{2\}},$$

$$\alpha|_{A_{21}} = \alpha^{A_{21}}_{A_{221}\cap \mathrm{im}(\alpha)} = \alpha^{A_{21}}_{\{4\}}, \alpha|_{A_{22}} = \alpha^{A_{22}}_{A_{222}\cap \mathrm{im}(\alpha)} = \alpha^{A_{22}}_{\{5\}}.$$

Thus,

$$\alpha = \begin{pmatrix}12345\\12455\end{pmatrix}.$$

Similarly we obtain all the rest transformations α with $\overline{\mathrm{im}(\alpha)} = A_0$:

$$\begin{pmatrix}12345\\12345\end{pmatrix}, \begin{pmatrix}12345\\12344\end{pmatrix}, \begin{pmatrix}12345\\12355\end{pmatrix}, \begin{pmatrix}12345\\11345\end{pmatrix}, \begin{pmatrix}12345\\22345\end{pmatrix},$$

$$\begin{pmatrix}12345\\11355\end{pmatrix}, \begin{pmatrix}12345\\11344\end{pmatrix}, \begin{pmatrix}12345\\11455\end{pmatrix}, \begin{pmatrix}12345\\22355\end{pmatrix}, \begin{pmatrix}12345\\22344\end{pmatrix},$$

$$\begin{pmatrix}12345\\22455\end{pmatrix}, \begin{pmatrix}12345\\12333\end{pmatrix}, \begin{pmatrix}12345\\12444\end{pmatrix}, \begin{pmatrix}12345\\12555\end{pmatrix}, \begin{pmatrix}12345\\11333\end{pmatrix},$$

$$\begin{pmatrix}12345\\11444\end{pmatrix}, \begin{pmatrix}12345\\11555\end{pmatrix}, \begin{pmatrix}12345\\22333\end{pmatrix}, \begin{pmatrix}12345\\22444\end{pmatrix}, \begin{pmatrix}12345\\22555\end{pmatrix}.$$

Now we define α_{A_2}. In this case $\alpha_{\{3,4,5\}} \in Map(A_0, A_2)$ and we have

$$\alpha|_{A_1} = \alpha^{A_1}_{A_{21}\cap \mathrm{im}(\alpha)} = \alpha^{A_1}_{\{3\}}, \ \alpha|_{A_2} = \alpha^{A_2}_{A_{22}\cap \mathrm{im}(\alpha)} = \alpha^{A_2}_{\{4,5\}}.$$

Since $\alpha^{A_2}_{\{4,5\}} \in Map(A_2, A_{22})$ further we have $\alpha|_{A_{21}} = \alpha^{A_{21}}_{A_{221}\cap \mathrm{im}(\alpha)} = \alpha^{A_{21}}_{\{4\}}$, $\alpha|_{A_{22}} = \alpha^{A_{22}}_{A_{222}\cap \mathrm{im}(\alpha)} = \alpha^{A_{22}}_{\{5\}}$. Consequently,

$$\alpha = \begin{pmatrix}12345\\33455\end{pmatrix}.$$

Identically, we obtain two more transformations with the images in A_2:

$$\begin{pmatrix}12345\\33555\end{pmatrix}, \begin{pmatrix}12345\\33444\end{pmatrix}.$$

Let $\mathrm{im}(\alpha)$ coincide with A_1 (A_{22}). Then

$$\alpha|_{A_1} = \alpha^{A_1}_{A_{11}\cap \mathrm{im}(\alpha)} = \alpha^{A_1}_{\{1\}}, \quad \alpha|_{A_2} = \alpha^{A_2}_{A_{12}\cap \mathrm{im}(\alpha)} = \alpha^{A_2}_{\{2\}}$$
$$\left(\alpha|_{A_1} = \alpha^{A_1}_{A_{221}\cap \mathrm{im}(\alpha)} = \alpha^{A_1}_{\{4\}}, \quad \alpha|_{A_2} = \alpha^{A_2}_{A_{222}\cap \mathrm{im}(\alpha)} = \alpha^{A_2}_{\{5\}}\right)$$

and we get

$$\begin{pmatrix}12345\\11222\end{pmatrix}, \begin{pmatrix}12345\\44555\end{pmatrix}.$$

If $\mathrm{im}(\alpha)$ coincides with one of the rest five classes, we get the following constant transformations:

$$\begin{pmatrix}12345\\11111\end{pmatrix}, \begin{pmatrix}12345\\22222\end{pmatrix}, \begin{pmatrix}12345\\33333\end{pmatrix}, \begin{pmatrix}12345\\44444\end{pmatrix}, \begin{pmatrix}12345\\55555\end{pmatrix}.$$

3. The proof of Theorem 1

First make sure that L_X^Γ is an $\mathscr{L}$-cross-section of the symmetric semigroup.

Lemma 2. *The set L_X^Γ forms a semigroup with respect to the composition of transformations.*

Proof. Let Γ be an arbitrary L-family on X. Take arbitrary elements $\varphi, \psi \in L_X^\Gamma$ and consider the composition $\varphi\psi$. Let $A_a = \overline{\mathrm{im}(\varphi)}$, $A_b = \overline{\mathrm{im}(\psi|_{A_a})}$ for some $A_a, A_b \in \Gamma$. It is clear that $\varphi\psi = \varphi\psi|_{A_a}$.

The equality $|A_a| = 1$ yields $|A_b| = 1$. In this case and in the case if $|A_a| \neq 1$, $|A_b| = 1$, it is clear that $\varphi\psi$ is a constant mapping from A_0 to A_b and consequently $\varphi\psi$ belongs to L_X^Γ.

Suppose $|A_a| \neq 1$,$|A_b| \neq 1$. Since $\varphi = \alpha_{\mathrm{im}(\varphi)}^{A_0}$, $\psi|_{A_a} = \alpha_{\mathrm{im}(\psi|_{A_a})}^{A_a}$, it goes $\varphi\psi|_{A_a} = \alpha_{\mathrm{im}(\varphi)\psi|_{A_a}}^{A_0}$. Thus, $\varphi\psi \in L_X^\Gamma$. □

Besides, Lemma 1 and the definition of elements in L_X^Γ imply the uniqueness of α_M^X for all $M \subseteq X$, $M \neq \varnothing$. Thus, L_X^Γ is an $\mathscr{L}$-cross-section in $\mathscr{T}_n$. Now we only have to prove the second part of Theorem 1.

Further, we will establish some properties of an arbitrary $\mathscr{L}$-cross-section L in semigroup $\mathscr{T}_n$.

Lemma 3. *Let $\alpha, \beta \in \mathscr{D}_k \cap L$. If $\mathrm{im}(\alpha) \in T_\beta$ ($\mathrm{im}(\beta) \in T_\alpha$), then $\rho_\alpha = \rho_\beta$.*

Proof. Indeed, if $\mathrm{im}(\alpha) \in T_\beta$, then we have $\rho_{\alpha\beta} = \rho_\alpha$ for the composition $\alpha\beta$. On the other hand, since $\alpha\beta \in \mathscr{D}_k \cap L$ and $\mathrm{im}(\alpha\beta) = \mathrm{im}(\beta)$, we have $\alpha\beta = \beta$, thus, $\rho_{\alpha\beta} = \rho_\beta$. Therefore, $\rho_\alpha = \rho_\beta$. □

Corollary 1. *For all α, $\beta \in \mathscr{D}_2 \cap L$ the condition $\rho_\alpha = \rho_\beta$ holds.*

Proof. Clearly, the unequality $T_\beta \cap T_\alpha \neq \varnothing$ is valid for all $\alpha, \beta \in \mathscr{D}_2 \cap L$. Consequently, there exists $\gamma \in L$ such that $\mathrm{im}(\gamma) \in T_\alpha \cap T_\beta$. The previous lemma yields $\rho_\alpha = \rho_\gamma = \rho_\beta$. □

Lemma 4. *Let L be an arbitrary $\mathscr{L}$-cross-section in $\mathcal{T}_n$, $n > 2$.*

(i) *If $A, B \in X_L$ are distinct elements then either $A \cap B = \varnothing$ or one of these sets contains the other one.*

(ii) *Any $A \in X_L$, $|A| > 1$, is uniquely expressible as a disjoint union of two sets (which are different from A and $\varnothing$) in X_L.*

(iii) *If $A \in X_L$ and $A = A_1 \cup A_2$ for some $A_1, A_2 \in X_L$ then there exists a mapping $\beta \in L$ such that $A_1, A_2 \in X/\rho_\beta$.*

(iv) *Let $A \in X_L$, $|A| > 2$ and $A = A_1 \cup A_2$ for some $A_1, A_2 \in X_L$. If $|A_2| > 1$, $A_2 = A_{21} \cup A_{22}$ for some $A_{21}, A_{22} \in X_L$, then at least one of the cardinalities $|A_{21}|, |A_{22}|$ is no more than $|A_1|$.*

Proof. (i) Suppose that there exist $A, B \in X_L$ such that $A \cap B = C \notin \{A, B\}, C \neq \varnothing$. Let $\alpha, \beta \in L$ be mappings such that $A \in X/\rho_\alpha, B \in X/\rho_\beta$. Take $a \in A \setminus C$, $b \in B \setminus C$, $c \in C$. Thus,

$$a_{\rho_\alpha} = c_{\rho_\alpha} \neq b_{\rho_\alpha}, \quad a_{\rho_\beta} \neq b_{\rho_\beta} = c_{\rho_\beta}.$$

Choose the element $\gamma \in L$ such that $\operatorname{im}(\gamma) = \{a, b, c\}$. Consider the products $\gamma_1 = \gamma\alpha$, $\gamma_2 = \gamma\beta$. We get $\gamma_1, \gamma_2 \in \mathscr{D}_2 \cap L$ and

$$X/\rho_{\gamma_1} = \{\{a, c\}\gamma^{-1}, \{b\}\gamma^{-1}\},$$

$$X/\rho_{\gamma_2} = \{\{b, c\}\gamma^{-1}, \{a\}\gamma^{-1}\}.$$

The condition $X/\rho_{\gamma_1} \neq X/\rho_{\gamma_2}$ contradicts Corollary 1.

(ii) The proof is by induction on the number of sets from X_L that contain A. If $A = X$, $|X| > 1$, we get the proof at once from Corollary 1: assume that for an arbitrary transformation $\varphi \in L$ of rank 2 we have $X/\rho_\varphi = \{X_1, X_1'\}$. So, $X = X_1 \cup X_1'$. Such an expansion is clearly disjoint.

Suppose that $A \neq X$ and (ii) holds for p sets that contain A, $p \in \mathbb{N}$. Without loss of generality, we put

$$A \subset X_1',\ X_1' = X_2 \cup X_2',$$

$$A \subset X_2',\ X_2' = X_3 \cup X_3',$$

$$A \subset X_3',\ X_3' = X_4 \cup X_4',\ \ldots,$$

$$A \subset X_{p-1}',\ X_{p-1}' = X_p \cup X_p',$$

$$A = X_p'.$$

To show $A = A_1 \cup A_2$, $A_1, A_2 \in X_L$ we establish at first that for any $k \in \mathbb{N}$, $1 \leqslant k \leqslant p$ we have

$$\sigma_{k+1} = \begin{pmatrix} X_1 & X_2 & \dots & X_{k+1} & X'_{k+1} \\ x_1 & x_2 & \dots & x_{k+1} & x'_{k+1} \end{pmatrix} \in L,$$

where $x'_{k+1} \in X'_{k+1}$, $x_j \in X_j$, $1 \leqslant j \leqslant k+1$.

Indeed, denote the transformation such that $\operatorname{im}(\sigma_1) = \{x_1, x'_1\}$ for fixed $x_1 \in X_1$, $x'_1 \in X'_1$ by $\sigma_1 \in L$. It is easy to see that $x\sigma_1 = x_1$, $x \in X_1$, $x\sigma_1 = x'_1$, $x \in X'_1$ since $\sigma_1^2 \in L$.

Suppose that

$$\sigma_k = \begin{pmatrix} X_1 & X_2 & \dots & X_k & X'_k \\ x_1 & x_2 & \dots & x_k & x'_k \end{pmatrix} \in L,$$

where $x'_k \in X'_k$, $x_j \in X_j$, $1 \leqslant j \leqslant k$.

Let $y'_k \in X'_k$, $y'_k \neq x'_k$ be an arbitrary fixed element, $\mu_k \in L$ be the transformation such that $\operatorname{im}(\mu_k) = \{x_1, x_2, \dots, \ x_k, x'_k, y'_k\}$. Since $\mu_k\sigma_k, \sigma_k \in L$ and $\operatorname{im}(\mu_k\sigma_k) = \operatorname{im}(\sigma_k)$, we have $\mu_k\sigma_k = \sigma_k$. The latter is possible whenever

$$\operatorname{im}(\mu_k|_{X_1}) = \{x_1\}, \ \ \operatorname{im}(\mu_k|_{X_2}) = \{x_2\}, \ \dots,$$

$$\operatorname{im}(\mu_k|_{X_k}) = \{x_k\}, \ \ \operatorname{im}(\mu_k|_{X'_k}) = \{x'_k, y'_k\}.$$

As $x'_k\mu_k^{-1}, y'_k\mu_k^{-1} \in X_L$ and item (i) holds, we have $\{x'_k\mu_k^{-1}, y'_k\mu_k^{-1}\} = \{X_{k+1}, X'_{k+1}\}$. Assume $x_{k+1} \in X_{k+1}$, $x'_{k+1} \in X'_{k+1}$ are arbitrary elements and $\varepsilon \in L$ is the transformation such that $\operatorname{im}(\varepsilon) = \{x_1, x_2, \dots, x_k, x_{k+1}, x'_{k+1}\}$. As $\operatorname{im}(\varepsilon) \in T_{\mu_k}$, we have $\rho_\varepsilon = \rho_{\mu_k}$ (according to Lemma 3). Since conditions $\varepsilon^2 \in L$, $\operatorname{im}(\varepsilon^2) = \operatorname{im}(\varepsilon)$ hold, we get $\varepsilon = \sigma_{k+1}$.

Now take $a_1, a_2 \in A$, $a_1 \neq a_2$ and $\sigma_p \in L$. If

$$\mu_p \in L \ \text{ such that } \ \operatorname{im}(\mu_p) = \{x_1, x_2, \dots, x_p, a_1, a_2\},$$

then as was shown above we get $\operatorname{im}(\mu_p|_A) = \{a_1, a_2\}$. Set $A_1 = a_1\mu_p^{-1}$, $A_2 = a_2\mu_p^{-1}$. It is clear that $A = A_1 \cup A_2$, where $A_1, A_2 \in X_L$, $A_1 \cap A_2 = \varnothing$.

Suppose that $A = B_1 \cup B_2$ holds for some other $B_1, B_2 \in X_L$ such that $B_1, B_2 \subset A$. The condition $B_1 \cap B_2 \neq \varnothing$ contradicts item (i). If $A_1 \subset B_1$, then $B_2 \cap A_1 \neq \varnothing$. Similar considerations are valid in other cases. Thus, the coexistence of different pairs A_i and B_j, $i, j \in \{1, 2\}$ contradicts item (i).

(iii) Since $A_1 \in X_L$, there exists a transformation $\beta_1 \in L$ such that $A_1 \in X/\rho_{\beta_1}$. If $A_2 \in X/\rho_{\beta_1}$, then there is nothing to prove. Let $A_2 \notin$

X/ρ_{β_1}. In this case any $B \in X/\rho_{\beta_1}$ contains neither A_2, nor part of A_2 as a proper subset: the conditions $B \cap A \neq \varnothing$, $B \cap A \notin \{B, A\}$ contradict item (i). Hence, for every $B \in X/\rho_{\beta_1}$, $B \cap A_2 \neq \varnothing$ the inclusion $B \subset A_2$ holds. By $\beta' \in L$ denote a transformation of X such that $\operatorname{im}(\beta') \in T_{\beta_1}$. For β' the following is true. Firstly, $\rho_{\beta_1} = \rho_{\beta'}$ (see Lemma 3). Secondly, β' is idempotent since $\beta'^2 \in L$ and $\operatorname{im}(\beta'^2) = \operatorname{im}(\beta')$. Therefore, $a_2\beta' \in A_2$ for all $a_2 \in A_2$.

Suppose now $\beta_2 \in L$ such that $A_2 \in X/\rho_{\beta_2}$. If $A_1 \in X/\rho_{\beta_2}$, then there is nothing to prove. Let $A_1 \notin X/\rho_{\beta_2}$. Just as in the previous case the condition $C \in X/\rho_{\beta_2}$, $C \cap A_1 \neq \varnothing$ is possible whenever $C \subset A_1$. Similarly, let $\beta'' \in L$ be an idempotent transformation such that $\operatorname{im}(\beta'') \in T_{\beta_2}$.

Consider now the transformation $\beta = \beta'\beta'' \in L$. On the one hand, $A_1 \in X/\rho_{\beta'}$, so it is easy to see that $A_1 \in X/\rho_\beta$. On the other hand, the conditions $a_2\beta' \in A_2$ for all $a_2 \in A_2$ and $A_2 \in X/\rho_{\beta''}$ imply $A_2 \in X/\rho_\beta$.

(iv) Let the assumption of item (iv) be fulfilled. If $|A_1| > |A_2|$ then there is nothing to prove. Suppose that $|A_1| \leqslant |A_2|$. To be definite, assume that $|A_{21}| \leqslant |A_{22}|$. We shall prove $|A_1| \geqslant |A_{21}|$.

Assume the converse. Then $|A_1| < |A_{21}| \leqslant |A_{22}| < |A_2|$.

If $A = X$, then take $\alpha, \beta \in L$ such that $\operatorname{im}(\alpha) = A_{21} \cup A_{22}$ and $A_{21}, A_{22} \in X/\rho_\beta$ (see the previous item). Set $A'_i = A_{21}\alpha^{-1} \cap A_i$ and $A''_i = A_{22}\alpha^{-1} \cap A_i$, $i \in \{1, 2\}$. Having regard to $|A_{2i}\alpha^{-1}| > |A_1|$, the sets A'_1, A'_2 or A''_1, A''_2 at least in one pair are both not empty. Consider the product $\alpha\beta \in L$. We have $X/\rho_{\alpha\beta} = \{A'_1 \cup A'_2, A''_1 \cup A''_2\}$, which is impossible since at least one class has a nonempty intersection with $A_1, A_2 \in X_L$ at the same time. Thus, $|A_1| \geqslant |A_{21}|$.

Let $A \neq X$, denote a class from X_L that does not contain A by S_1, and the other one by S'_1, i.e. $A \subseteq S'_1$. If $A \subset S'_1$ then denote the classes in X_L such that $S'_1 = S_2 \cup S'_2$ and $A \subseteq S'_2$ by S_2 and S'_2. We will redesignate elements in X_L until $A = S'_t$ for a natural t. Consider the system of subsets $\{S_1, S_2, \ldots, S_t\}$. Note that $S_i \cap S_j = \varnothing = S_i \cap A$, $S_j \cup S_i \notin X_L$ for all $1 \leqslant i \neq j \leqslant t$.

As in the case $A = X$ we will construct a transformation $\gamma \in L$ such that $\operatorname{im}(\gamma|_A) = \{A_{21}, A_{22}\}$.

Let $s_1, s_2, \ldots, s'_t$ be arbitrary fixed elements from $S_1, S_2, \ldots, S'_t$ respectively. Just as in the proof of item (ii) one can prove by induction that $\sigma_t = \begin{pmatrix} S_1 & S_2 & S_3 & \ldots & S_t & S'_t \\ s_1 & s_2 & s_3 & \ldots & s_t & s'_t \end{pmatrix} \in L$. Set $\gamma \in L$, $\operatorname{im}(\gamma) = \{s_1, s_2, \ldots, s_t\} \cup A_{21} \cup A_{22}$. Recall that $A_{21} \cup A_{22} \subset S'_t = A$. Obviously, we get $\gamma\sigma_t = \sigma_t$, whence $\gamma = \begin{pmatrix} S_1 & S_2 & S_3 & \ldots & S_t & S'_t \\ s_1 & s_2 & s_3 & \ldots & s_t & A_{21} \cup A_{22} \end{pmatrix}$. Thus, $\operatorname{im}(\gamma|_A) = \{A_{21}, A_{22}\}$. Set $A'_i = A_{21}\alpha^{-1} \cap A_i$ and $A''_i = A_{22}\alpha^{-1} \cap A_i$, $i \in \{1, 2\}$. Having regard to $|A_{2i}\gamma^{-1}| > |A_1|$, the subsets at least in one pair A'_1, A'_2

or A_1'', A_2'' are both not empty. Consider the product $\alpha\beta \in L$. As was shown above $\gamma\beta \in L$ contradicts item (i) since $X/\rho_{\gamma\beta} = \{A_1' \cup A_2', A_1'' \cup A_2''\}$. So, we get $|A_1| \geqslant |A_{21}|$.

It can be shown dually that if $|A_1| > 1$, $A_1 = A_{11} \cup A_{12}$ for some $A_{11}, A_{12} \in X_L$, then at least one of the cardinalities $|A_{11}|, |A_{12}|$ is no more than $|A_2|$. □

Corollary 2. *For any $\mathscr{L}$-cross-section L in the semigroup $\mathcal{T}_n$, $n > 2$ the collection of sets X_L is a full binary tree.*

Proof. According to Lemma 4, item (ii), X is uniquely expressible as a disjoint union of two classes from X_L. We set $B_0 = X$ and denote classes in expansion of B_0 by B_1 and B_2. Any non-one-element class we have obtained, is expressible as a union of two other classes from X_L. Without loss of generality, we may assume that $|B_1| > 1, B_1 = B_{11} \cup B_{12}, B_{11}, B_{12} \in X_L$. Let $\alpha, \beta \in L$ be the transformations such that $\operatorname{im}(\alpha) = B_{11} \cup B_{12}$, $B_{11}, B_{12} \in X/\rho_\beta$. Since $\operatorname{rk}(\alpha\beta) = 2$, according to Corollary 1 we get $X/\rho_{\alpha\beta} = \{B_1, B_2\}$, whence $\{B_{11}\alpha^{-1}, B_{12}\alpha^{-1}\} = \{B_1, B_2\}$. Redesignate sets B_{11}, B_{12} so that equalities $B_1\alpha = B_{11}$, $B_2\alpha = B_{12}$ hold. Do the same with every obtained non-one-element class. It is clear that we get a full binary tree. □

We will denote new classes by B_{aj}, where $j \in \{1, 2\}$, a is an index of the initial set ($B_a = B_{a1} \cup B_{a2}$) and if $\alpha' \in L$ is the transformation such that $\operatorname{im}(\alpha') = B_{a1} \cup B_{a2}$ then $B_1\alpha' = B_{a1}$, $B_2\alpha' = B_{a2}$.

Hereinbelow X_L means the family of sets described in previous Lemma.

Suppose $\alpha \in L$, $B_a \in X_L$, $a \in F(Y)$. It is clear that $B_a\alpha$ can be embedded into some class from X_L. We may ask the following question: does there exist another class $B_c \in X_L$, $c \in F(Y)$, $B_c \neq B_a$ such that $\overline{B_a\alpha} \cap B_c\alpha \neq \varnothing$? Next Lemma gives a negative answer to this question.

Lemma 5. *Let $\alpha \in L$ and $B_a, B_c \in X_L$ be arbitrary classes. If $|\overline{B_a\alpha}| > 1$ and $B_a \cap B_c = \varnothing$ then $\overline{B_a\alpha} \cap B_c\alpha = \varnothing$.*

Proof. Let $B_a \cap B_c = \varnothing$. Denote the class of X_L such that $B_b = \overline{B_a\alpha}$ by B_b, $b \in F(Y)$.

Suppose $B_a\alpha \cap B_c\alpha \subseteq B_b$. In this case for any $x \in B_a\alpha \cap B_c\alpha$ we get $x\alpha^{-1} \in X/\rho_\alpha$ and $x\alpha^{-1} \cap B_a \neq \varnothing \neq x\alpha^{-1} \cap B_c$. By Lemma 4, item (i) we have $B_a \cap B_c \neq \varnothing$. Therefore $B_a\alpha \cap B_c\alpha = \varnothing$.

Suppose now that $B_a\alpha \cap B_c\alpha = \varnothing$ and $B_a\alpha \cup B_c\alpha \subseteq B_b$. Since $B_b = \overline{B_a\alpha}$, $B_a\alpha$ has a nonempty intersection with B_{b1} and B_{b2} both. So without loss of generality, we can assume that $B_c\alpha \cap B_{b1} \neq \varnothing$. Let

$B'_a \subset B_a, B'_c \subseteq B_c$ be the maximal subsets such that $B'_a\alpha \subset B_{b1}$, $B'_c\alpha \subseteq B_{b1}$. Denote a transformation from L such that $B_{b1} \in X/\rho_\beta$ by β. It is easy to see that $B'_a \cup B'_c \in X/\rho_{\alpha\beta}$, i.e. belongs to X_L, since $\alpha\beta \in L$. Thus, the class $B'_a \cup B'_c$ has a proper nonempty intersection with B_a. This contradicts Lemma 4, item (i).

Thus, $\overline{B_a\alpha} \cap B_c\alpha = \varnothing$. □

Lemma 6. *Let $\mu \in L$ be an arbitrary transformation, $B_s \in X_L$ be an arbitrary element, $s \in F(Y)$ with $|B_s| > 1$. Let $B_p = \overline{B_s\mu} \in X_L, p \in F(Y)$. If $|B_p| > 1$, then $B_{si}\mu \subseteq B_{pi}$, $i \in \{1, 2\}$.*

Proof. Suppose the assumption of this lemma is fulfilled. Since $|B_p| > 1$, we have $B_{pi} \cap \operatorname{im}(\mu) \neq \varnothing$, $i \in \{1, 2\}$. According to Lemma 5 for all $B_a \in X_L$ such that $B_a \cap B_s = \varnothing$ the condition $B_p \cap B_a\mu = \varnothing$ holds true. Hence, $(\operatorname{im}(\mu)\cap B_{pi})\mu^{-1} \subset B_s$, $i \in \{1, 2\}$. As there exists a transformation β in L such that $B_{p1}, B_{p2} \in X/\rho_\beta$ (Lemma 4, item (iii)), we get $(\operatorname{im}(\mu) \cap B_{pi})\mu^{-1} \in X/\rho_{\mu\beta}$, $i \in \{1, 2\}$. Having regard to

$$(\operatorname{im}(\mu) \cap B_{p1})\mu^{-1} \cup (\operatorname{im}(\mu) \cap B_{p2})\mu^{-1} = B_s, \quad \text{we obtain}$$

$$\{(\operatorname{im}(\mu) \cap B_{p1})\mu^{-1}, (\operatorname{im}(\mu) \cap B_{p2})\mu^{-1}\} = \{B_{s1}, B_{s2}\}.$$

So, for a fixed $i \in \{1, 2\}$ there exists $k_i \in \{1, 2\}$ such that $B_{si}\mu \subseteq B_{pk_i}$.

The case $B_p = B_0$ implies $B_s = B_0$. Let $b_1 \in B_1$, $b_2 \in B_2$ be arbitrary fixed elements, $\sigma = \begin{pmatrix} B_1 & B_2 \\ b_1 & b_2 \end{pmatrix}$. It was shown in Lemma 4, (ii) that $\sigma \in L$. Since $\operatorname{im}(\sigma) = \operatorname{im}(\mu\sigma)$, we have $\sigma = \mu\sigma$, whence we immediately get $B_{si}\mu \subseteq B_{pi}$, $i \in \{1, 2\}$.

Suppose $B_p \neq B_0$. We will first prove that for any $\eta \in L$ if $B_p = \overline{B_s\eta}$, then $B_{si}\eta$, $B_{si}\mu \subseteq B_{pk_i}$, $i \in \{1, 2\}$.

If $B_0 = B_1 \cup B_2$, then denote the set that does not contain B_p by K_1, and the set that contains B_p by K'_1. If $K'_1 \neq B_p$ then in the decomposition of K'_1 we denote the component that does not contain B_p by K_2, and the one that contains B_p by K'_2. As a continuation we obtain a system of mutually disjoint sets $K_1, K_2, \ldots, K_m \in X_L$, $m \in \mathbb{N}$ such that $K_i \cap B_p = \varnothing$, $1 \leqslant i \leqslant m$, and $K'_m = B_p$. Let $\{k_1, k_2, \ldots, k_m, k'_m\}$ be an arbitrary transversal of the family $\{K_1, K_2, \ldots, K_m, K'_m\}$. As we have seen in the proof of item (ii), Lemma 4, it can be proved by induction that $\delta_m = \begin{pmatrix} K_1 & K_2 & K_3 & \ldots & K_m & K'_m \\ k_1 & k_2 & k_3 & \ldots & k_m & k'_m \end{pmatrix} \in L$.

Let $b_1 \in B_{p1}$, $b_2 \in B_{p2}$ be arbitrary fixed elements, $\gamma \in L$ be such that $\operatorname{im}(\gamma) = \{k_1, k_2, \ldots, k_m, b_1, b_2\}$. Since $\gamma\delta_m \in L$ and $\operatorname{im}(\gamma\delta_m) = \operatorname{im}(\delta_m)$, we have $\gamma = \begin{pmatrix} K_1 & K_2 & K_3 & \ldots & K_m & B_p \\ k_1 & k_2 & k_3 & \ldots & k_m & \{b_{p1}, b_{p2}\} \end{pmatrix}$. Whence according to item (i) of

Lemma 4 we get $B_{p1}, B_{p2} \in X/\rho_\gamma$. Having regard to $\gamma^2 = \gamma \in L$, we get $\gamma = \left(\begin{smallmatrix} K_1 & K_2 & K_3 & \dots & K_m & B_{p1} & B_{p2} \\ k_1 & k_2 & k_3 & \dots & k_m & b_1 & b_2 \end{smallmatrix}\right)$.

The condition $B_s\eta$, $B_s\mu \subseteq B_p$ implies that there are representatives of the same classes from $K_1, K_2, \dots, K_m$ in $\operatorname{im}(\eta)$ and $\operatorname{im}(\mu)$. Hence, $\operatorname{im}(\eta\gamma) = \operatorname{im}(\mu\gamma)$ whence $\eta\gamma = \mu\gamma$. The last implies in particular that $B_{si}\eta$, $B_{si}\mu \subseteq B_{pk_i}$, $i \in \{1, 2\}$.

Now we prove that $B_{si}\mu \subseteq B_{pi}$, $i \in \{1, 2\}$. Assume the converse: $B_{si}\mu \subseteq B_{pj}$, $i, j \in \{1, 2\}$, $i \neq j$. Let $\psi, \psi' \in L$, $\operatorname{im}(\psi) = B_s$, $\operatorname{im}(\psi') = B_p$. By construction of X_L (Corollary 2) we have $B_i\psi = B_{si}$, $B_i\psi' = B_{pi}$, $i \in \{1, 2\}$. On the other hand, $\psi\mu \in L$ and $B_i(\psi\mu) = B_{si}\mu \subseteq B_{pj}$. The contradiction $B_i\psi' \subseteq B_{pi}$, $B_i(\psi\mu) \subseteq B_{pj}$, $i, j \in \{1, 2\}$, $i \neq j$ proves the theorem. □

Corollary 3. *For any $\mathscr{L}$-cross-section L in the semigroup $\mathscr{T}_n$, $n > 2$, the collection of sets X_L is L-family.*

Proof. Having regard to Corollary 2, the collection of sets X_L forms a full binary tree. Let B_a be an arbitrary element from X_L. In fact, it was shown in Lemma 4, item (iv) that inequality $|B_{aib}| \leqslant |B_{a^*b}|$, $i \in \{1, 2\}$, $i \neq a^0$ holds for an empty word b. Without loss of generality we assume $a = c2$, $a^* = c1$ and consider inequality $|B_{c21}| \leqslant |B_{c1}|$. Let $\gamma \in L$ be the transformation such that $\operatorname{im}(\gamma|_{B_c}) = B_{c2}$. By the previous lemma we have $\operatorname{im}(\gamma|_{B_{c1b}}) = B_{c21b}$ for all $b \in F(Y)$ such that $B_{c1b}, B_{c21b} \in X_L$. Hence, $|B_{c21b}| \leqslant |B_{c1b}|$ for all $b \in F(Y)$ such that B_{c21b}, $B_{c1b} \in X_L$. It can be shown dually that $|B_{a2b}| \leqslant |B_{a^*b}|$ for all $b \in F(Y)$ such that $B_{a2b}, B_{a^*b} \in X_L$.

Now we only have to define a strict linear order on X. For any $x, y \in X$ let x be less than y, if $x \in B_{b1}$, $y \in B_{b2}$ for some $B_{b1}, B_{b2} \in X_L$. Such a definition is correct, since firstly, B_{b1}, B_{b2} are disjoint, and secondly, X_L contains all one-element subsets of X.

Thus, the collection of sets X_L satisfies the definition of an L-family. □

Corollary 4. *Any $\mathscr{L}$-cross-section in $\mathscr{T}_n$ has a form L_X^Γ for a suitable L-family Γ on the set X.*

Proof. If $|X| = 1$, then $X_L = \{B_0\}$, and $\beta \in L$ is expressible as α_X. Let further $|X| > 1$. Suppose $\operatorname{im}(\beta) = \{m\}$, $m \in X$. Then there exists $B_w \in X_L$, $w \in F(Y)$ such that $B_w = \{m\}$ and thus $\beta = \alpha_{\{m\}} \in L_X^{X_L}$.

Let $B_a = \overline{\operatorname{im}(\beta)}$, $B_a \in X_L$, $|\operatorname{im}(\beta)| > 1$. By Lemma 6 we have $B_i\beta \subseteq B_{ai}$, consequently $\operatorname{im}(\beta|_{B_i}) = B'_{ai} \cap \operatorname{im}(\beta)$, $i \in \{1, 2\}$. An analogous reasoning can be applied to each of the transformations $\beta|_{B_i}$, $i \in \{1, 2\}$, etc., until we get $|B_{at} \cap \mathrm{im}(\beta)| = 1$ for some $B_{at} \in X_L$, $t \in F(Y)$. Consequently, any $\beta = \alpha_{\mathrm{im}(\beta)} \in L_X^{X_L}$. Since $\mathrm{im}(\beta)$, $\beta \in L$, goes through all possible subsets of X, we get $L = L_X^{X_L}$. □

Corollary 4 completes the proof of Theorem 1.

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Received by the editors: 08.05.2013
and in final form 02.04.2014.