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RESEARCH ARTICLE

On strongly almost m-ω_1-$p^{\omega+n}$-projective abelian p-groups

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Abstract. For any non-negative integers m and n we define the class of *strongly almost* m-ω_1-$p^{\omega+n}$-*projective groups* which properly encompasses the classes of *strongly* m-ω_1-$p^{\omega+n}$-*projective groups* and *strongly almost* ω_1-$p^{\omega+n}$-*projective groups*, defined by the author in Demonstr. Math. (2014) and Hacettepe J. Math. Stat. (2015), respectively. Certain results about this new group class are proved as well as it is shown that it shares many analogous basic properties as those of the aforementioned two group classes.

1. Introduction and terminology

Let all groups considered in this paper be p-primary abelian, for some arbitrary fixed prime p. Besides, everywhere in the text, m and n are arbitrary integers greater than or equal to $\{0\}$. Our notions and notations are in the most part standard and follow those from the classical books [9], [10] and [12]. The not well-known of them will be explained below in detail.

A class of groups that plays a major role in torsion abelian group theory is the one consisting of all almost direct sums of cyclic groups, introduced in [11] as follows.

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The separable group G is called *an almost direct sum of cyclic groups* if there is a collection $\mathcal{C}$ consisting of nice subgroups of G, satisfying the following three conditions:

(1) $\{0\} \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed with respect to ascending unions, i.e., if $H_i \in \mathcal{C}$ with $H_i \subseteq H_j$ whenever $i \leqslant j$ $(i, j \in I)$ then $\cup_{i\in I} H_i \in \mathcal{C}$;

(3) If K is a countable subgroup of G, then there is $L \in \mathcal{C}$ (that is, a nice subgroup L of G) such that $K \subseteq L$ and L is countable.

Furthermore, an important class of p-torsion groups is the class of all almost $p^{\omega+n}$-projective groups, where $n \geqslant 0$ is an integer, defined in [1] and [2] like this: The group G is called *almost $p^{\omega+n}$-projective* if there exists a p^n-bounded subgroup $P \leqslant G$ such that G/P is an almost direct sum of cyclic groups (note that P is necessarily nice in G because the quotient G/P is separable). It is demonstrated there that this is tantamount to the fact that G is isomorphic to S/B, where S is an almost direct sum of cyclic groups and B is p^n-bounded.

Using the specific nature of countable subgroups, we generalized in [2] the last concept to the following: A group G is said to be *almost ω_1-$p^{\omega+n}$-projective* if there is a countable subgroup $C \leqslant G$ such that G/C is almost $p^{\omega+n}$-projective. Notice that such a subgroup C can be chosen to satisfy the inequalities $p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$, and thus resultantly C is of necessity nice in G.

On the other vein, we showed in [2] also that almost ω_1-$p^{\omega+n}$-projective groups can be characterized in a different way as follows: The group G is almost ω_1-$p^{\omega+n}$-projective if there exists a p^n-bounded subgroup $H \leqslant G$ such that G/H is the sum of a countable group and an almost direct sum of cyclic groups. As observed, such a subgroup H need not always be nice in G, and so in [7] was given the following definition: A group G is called *strongly almost ω_1-$p^{\omega+n}$-projective* if there is a p^n-bounded nice subgroup $N \leqslant G$ with G/N a sum of a countable group and an almost direct sum of cyclic groups. Note that almost $p^{\omega+n}$-projective groups are obviously strongly almost ω_1-$p^{\omega+n}$-projective. Some principal results concerning certain generalizations of strongly almost ω_1-$p^{\omega+n}$-projective groups were established in [4], [5], [6] and [8], respectively.

On the other hand, in order to extend some classical sorts of groups, e.g. $p^{\omega+n}$-projective groups and ω_1-$p^{\omega+n}$-projective groups, in [3] were introduced a few classes of groups by using a single parameter m. So, the objective of the present article is to develop that idea to some new concepts which use the term “almost”, and also to find suitable relationships between them and the mentioned above group classes.

Definition 1.1. The group G is said to be *almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup A of G such that G/A is strongly almost ω_1-$p^{\omega+n}$-projective.

In particular, if A is nice in G, then G is called *strongly almost m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$ we obtain strongly almost ω_1-$p^{\omega+n}$-projective groups, while we obtain strongly almost ω_1-$p^{\omega+m}$-projective groups when $n = 0$.

Definition 1.2. The group G is said to be *weakly almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup X of G such that G/X is almost ω_1-$p^{\omega+n}$-projective.

Substituting $m = 0$ we yield almost ω_1-$p^{\omega+n}$-projective groups, while if $n = 0$ we yield strongly almost ω_1-$p^{\omega+m}$-projective groups. In fact, the first fact is trivial, while for the second one we have the following arguments: in view of Lemma 2.16 of [1] an almost ω_1-p^{ω}-projective group is actually a sum of a countable group and an almost direct sum of cyclic groups. Hence the definition of a strongly almost ω_1-$p^{\omega+m}$-projective group is directly applicable, and we are set.

Definition 1.3. The group G is said to be *decomposably almost m-ω_1-$p^{\omega+n}$-projective* if there is a p^m-bounded subgroup S of G with the property that G/S is a sum of a countable group and an almost $p^{\omega+n}$-projective group.

In particular, if S is nice in G, then G is called *nice decomposably almost m-ω_1-$p^{\omega+n}$-projective.* In addition, if the sum above is direct, we shall say that G is *(nice) direct decomposably almost m-ω_1-$p^{\omega+n}$-projective.*

If $m = 0$ we identify the sums of countable groups and almost $p^{\omega+n}$-projective groups. If $n = 0$ we unify all almost ω_1-$p^{\omega+m}$-projective groups.

As for the second part, choosing $m = 0$ we will again obtain the sums of countable groups and almost $p^{\omega+n}$-projective groups, but choosing $n = 0$ we will obtain strongly almost ω_1-$p^{\omega+m}$-projective groups.

Definition 1.4. The group G is called *nicely almost m-$p^{\omega+n}$-projective* if there is a p^m-bounded nice subgroup Y of G such that G/Y is almost $p^{\omega+n}$-projective.

Putting $m = 0$ we get almost $p^{\omega+n}$-projective groups, and putting $n = 0$ we get almost $p^{\omega+m}$-projective groups. Likewise, nicely almost

m-$p^{\omega+n}$-projective groups are both nice decomposably almost m-ω_1-$p^{\omega+n}$-projective and almost $p^{\omega+m+n}$-projective. Actually, almost $p^{\omega+m+n}$-projective groups are groups for which there is (not necessarily nice) a p^m-bounded subgroup M and, respectively, a p^n-bounded subgroup N, such that G/M is almost $p^{\omega+n}$-projective, respectively, G/N is almost $p^{\omega+m}$-projective.

Generally, the following self containments are fulfilled (this manifestly visualizes some immediate relationships between the new group classes):

- {strongly almost ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {decomposably almost n-ω_1-$p^{\omega+n}$-projective groups}.
- {strongly almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {weakly almost m-ω_1-$p^{\omega+n}$-projective groups}.
- {nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {weakly almost m-ω_1-$p^{\omega+n}$-projective groups}.
- {nicely almost m-ω_1-$p^{\omega+n}$-projective groups} $\subseteq$ {nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups}.

2. Some more relationships

In this section we will prove certain basic relation properties of the groups from the above definitions. Throughout the rest of the paper, we once again recollect that m and n are arbitrary fixed naturals or zero.

We start with the following:

Theorem 2.1. *For any group G there exists a p^m-bounded subgroup K such that G/K is almost ω_1-$p^{\omega+n}$-projective if and only if G is almost ω_1-$p^{\omega+m+n}$-projective.*

Proof. We shall first show that G is as in the necessity of the theorem $\iff$ there exists $C \leqslant G$ such that $p^m C$ is countable with $p^m C \subseteq p^\omega G$ and G/C is almost $p^{\omega+n}$-projective $\iff$ there exists $L \leqslant G$ with $p^{m+n}L$ countable and G/L is an almost direct sum of cyclic groups.

Since the second equivalence follows directly by the methods used in the proof of Theorem 2.21 from [2] or by an immediate application of the corresponding definitions, we will be concentrated on the first one. In fact, if $p^{m+n}L$ is countable, then $L = R \oplus T$, where R is countable and $p^{m+n}T = \{0\}$. Thus

$$G/L = G/(R \oplus T) \cong [G/(R \oplus p^n T)]/[(R \oplus T)/(R \oplus p^n T)]$$

being an almost direct sum of cyclic groups implies that $G/(R \oplus p^nT)$ is almost $p^{\omega+n}$-projective with $C = R \oplus p^nT$, so $p^mC = p^mR$ is countable, as asked for.

"$\Rightarrow$". Suppose by assumption that there is a p^m-bounded subgroup $K \leqslant G$ such that G/K is almost ω_1-$p^{\omega+n}$-projective. Owing to Theorem 2.25 of [2], there exists a countable (nice) subgroup C/K of G/K such that $(G/K)/(C/K) \cong G/C$ is almost $p^{\omega+n}$-projective and $C/K \subseteq p^{\omega}(G/K) = [\cap_{i<\omega}(p^iG + K)]/K$. Therefore, $C \leqslant G$, $C = K + L$ for some countable $L \leqslant C$ and $C \subseteq \cap_{i<\omega}(p^iG + K)$. These conditions together imply that $p^mC \subseteq L$ is countable and $p^mC \subseteq \cap_{i<\omega}p^{i+m}G = p^{\omega}G$, as required.

"$\Leftarrow$". Write $C = X \oplus V$, where X is countable and V is p^m-bounded. Hence $G/C = G/(X \oplus V) \cong [G/V]/(X \oplus V)/V$ is almost $p^{\omega+n}$-projective, where $(X \oplus V)/V \cong X$ is countable. Thus, in accordance with [2], G/V is almost ω_1-$p^{\omega+n}$-projective, as desired. Moreover, $(X \oplus V)/V$ can be chosen so that

$$\begin{aligned} p^m[(X \oplus V)/V] &= (p^mX \oplus V)/V \\ &= (p^mC \oplus V)/V \subseteq (p^{\omega}G + V)/V \subseteq p^{\omega}(G/V). \end{aligned}$$

This proves the preliminary claim.

Now, we have all the information necessary to prove the full assertion. To that aim we just will show that G is almost ω_1-$p^{\omega+m+n}$-projective $\iff$ there is $S \leqslant G$ such that $p^{m+n}S$ is countable and G/S is an almost direct sum of cyclic groups, which is precisely the stated above equivalence (compare with points (1) and (4) in Theorem 2.21 from [2]).

Necessity. Appealing to [2], G is almost ω_1-$p^{\omega+m+n}$-projective if there is a countable subgroup K with G/K being almost $p^{\omega+m+n}$-projective. Thus, again in view of [2], there exists $S \leqslant G$ containing K such that G/S is an almost direct sum of cyclic groups and $p^{m+n}S \subseteq K$. The last yields that $p^{m+n}S$ is countable, as required.

Sufficiency. Suppose now that there exists $S \leqslant G$ such that $p^{m+n}S$ is countable and G/S is an almost direct sum of cyclic groups. Therefore, the quotient $G/S \cong (G/p^{m+n}S)/(S/p^{m+n}S)$ being an almost direct sum of cyclic groups implies with the aid of [2] that $G/p^{m+n}S$ is almost $p^{\omega+m+n}$-projective. And since $p^{m+n}S$ is countable, again the application of [2] leads to G is almost ω_1-$p^{\omega+m+n}$-projective, as desired. $\square$

Remark 1. Note that the condition $p^mC \subseteq p^{\omega}G$ stated in the proof of Theorem 2.1 was at all redundant and therefore not further used. One of the important consequences of Theorem 2.1 is that (weakly) almost

m-ω_1-$p^{\omega+n}$-projective groups are almost ω_1-$p^{\omega+m+n}$-projective. Likewise, the central role of Theorem 2.1 is to demonstrate unambiguously that the concepts in Definitions 1.1 and 1.2 are nontrivial.

Imitating Theorem 2.1, it is quite natural to ask whether or not strongly almost m-ω_1-$p^{\omega+n}$-projective groups are exactly the strongly almost ω_1-$p^{\omega+m+n}$-projective ones. Referring to the following statement, this seems to be true.

Proposition 2.2. *If G is a strongly almost m-ω_1-$p^{\omega+n}$-projective group, then G is strongly almost ω_1-$p^{\omega+m+n}$-projective.*

Proof. Assume that there exists a p^m-bounded nice subgroup T of G such that G/T is strongly almost ω_1-$p^{\omega+n}$-projective. Thus there is a nice subgroup A/T of G/T with the property that $p^n A \subseteq T$ and $(G/T)/(A/T) \cong G/A$ is the sum of a countable group and an almost direct sum of cyclic groups. Hence $p^{n+m}A = \{0\}$ and A is nice in G (cf. [9]), which conditions ensure that G is strongly almost ω_1-$p^{\omega+m+n}$-projective, as claimed. $\square$

As noted above, a question of some majority is of whether or not the converse holds, that is, whether or not every strongly almost ω_1-$p^{\omega+m+n}$-projective group is strongly almost m-ω_1-$p^{\omega+n}$-projective.

An other question of some interest, which immediately arises, is also whether or not almost $p^{\omega+m+n}$-projective groups are strongly almost m-ω_1-$p^{\omega+n}$-projective (and, in particular, weakly almost m-ω_1-$p^{\omega+n}$-projective). This is inspired by the fact that, taking $m = 0$, almost $p^{\omega+n}$-projective groups are themselves strongly almost ω_1-$p^{\omega+n}$-projective (cf. [7]).

In this way, we have the following weaker relationship:

Proposition 2.3. *If G is an almost $p^{\omega+m+n}$-projective group, then G is a (direct) decomposably almost m-ω_1-$p^{\omega+n}$-projective group.*

Proof. Let $P \leqslant G$ such that G/P is an almost direct sum of cyclic groups and $p^{m+n}P = \{0\}$. Since $G/P \cong [G/p^nP]/[P/p^nP]$, we deduce that G/p^nP is almost $p^{\omega+n}$-projective and hence it is a sum of a countable group and an almost $p^{\omega+n}$-projective group. But $p^m(p^nP) = \{0\}$ and so G is decomposably almost m-ω_1-$p^{\omega+n}$-projective, as promised. $\square$

Remark 2. The converse implication is, however, not true as simple examples show. Nevertheless, decomposably almost m-ω_1-$p^{\omega+n}$-projective groups are eventually intermediate situated between almost $p^{\omega+m+n}$-projective groups and almost m-ω_1-$p^{\omega+n}$-projective groups.

For separable groups (i.e., groups without elements of infinite height) all of the above notions are tantamount; we do not consider here concrete examples to show that these concepts are independent for lengths beyond ω, but we refer the interested reader to [4], [5] or [6] for more details when the group length is $> \omega$.

Theorem 2.4. *Suppose G is a group such that $p^{\omega}G = \{0\}$. Then all of the next points are equivalent:*

(a) *G is almost ω_1-$p^{\omega+m+n}$-projective;*

(b) *G is almost m-ω_1-$p^{\omega+n}$-projective;*

(c) *G is strongly almost m-ω_1-$p^{\omega+n}$-projective;*

(d) *G is weakly almost m-ω_1-$p^{\omega+n}$-projective;*

(e) *G is decomposably almost m-ω_1-$p^{\omega+n}$-projective;*

(f) *G is nice decomposably almost m-ω_1-$p^{\omega+n}$-projective;*

(g) *G is nicely almost m-$p^{\omega+n}$-projective;*

(h) *G is almost $p^{\omega+m+n}$-projective.*

Proof. Apparently, all of the points (b)-(h) imply (a) and, in virtue of [2], we obtain that point (a) holds provided (h) is fulfilled. Moreover, it is easy to see that clause (g) implies all other ones. So, what remains to show is the implication (h) $\Rightarrow$ (g). To this purpose, [2] helps us to write that G/Z is an almost direct sum of cyclic groups for some subgroup $Z \leqslant G$ which is bounded by p^{m+n}. Thus $(G/Z[p^m])/(Z/Z[p^m]) \cong G/Z$ being an almost direct sum of cyclic groups guarantees again by [2] that $G/Z[p^m]$ is almost $p^{\omega+n}$-projective since $Z/Z[p^m] \cong p^m Z$ is obviously bounded by p^n. But $Z[p^m] = Z \cap G[p^m]$ and both Z and $G[p^m]$ are nice in G because G/Z is p^{ω}-bounded and $G/G[p^m] \cong p^m G \subseteq G$ is p^{ω}-bounded too. So, resulting, $Z[p^m]$ must be nice in G (see, e.g., [9]), and since $Z[p^m]$ is p^m-bounded, we consequently get the desired fact that G is nicely almost m-$p^{\omega+n}$-projective. $\square$

We now proceed with two useful necessary and sufficient conditions which are needed for applicable purposes in the next section.

Proposition 2.5. *The group G is strongly almost m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup T of G such that $G/(T + p^{\omega+n}G)$ is almost $p^{\omega+n}$-projective and $p^{\omega+n}(G/T)$ is countable.*

Proof. It follows directly from [2] because the isomorphism

$$[G/T]/p^{\omega+n}(G/T) \cong G/(T + p^{\omega+n}G)$$

is fulfilled. □

Proposition 2.6. *The group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if there exists a p^m-bounded nice subgroup X of G such that $G/(X+p^{\omega+n}G)$ is almost ω_1-$p^{\omega+n}$-projective and $p^{\omega+n}(G/X)$ is countable.*

Proof. It follows immediately from [2] since the isomorphism

$$[G/X]/p^{\omega+n}(G/X) \cong G/(X + p^{\omega+n}G)$$

holds. □

3. Ulm subgroups and Ulm factors

In [7] it was proved that if the group G is strongly almost ω_1-$p^{\omega+n}$-projective, then so is $G/p^\alpha G$ for any ordinal α. Here we will give a simpler proof to the same fact devoted to almost ω_1-$p^{\omega+n}$-projective groups (see Proposition 2.13 (b) from [2], too).

Proposition 3.1. *If G is an almost ω_1-$p^{\omega+n}$-projective group, then $G/p^\alpha G$ is an almost ω_1-$p^{\omega+n}$-projective group for every ordinal α.*

Proof. For finite ordinals α, the assertion is self-evident. So, we will assume that α is infinite. By virtue of Theorem 2.21 (2) in [2], let G/A be the sum of a countable group and an almost direct sum of cyclic groups for some $A \leqslant G$ with $p^n A = \{0\}$. Thus, by utilizing the methods in [1] and [2], we deduce that $p^\alpha(G/A)$, being contained in a countable summand of G/A, remains countable and $[G/A]/p^\alpha(G/A)$ is again a sum of a countable group and an almost direct sum of cyclic groups. If $T \subseteq p^\alpha(G/A)$, the same is still true for $(G/A)/T$; we specially take $T = (p^\alpha G + A)/A$.

But the following isomorphisms hold:

$$[G/A]/(p^\alpha G + A)/A \cong G/(p^\alpha G + A) \cong [G/p^\alpha G]/(p^\alpha G + A)/p^\alpha G.$$

Observing that $p^n((p^\alpha G + A)/p^\alpha G) = \{0\}$, we are finished. □

Remark 4. Reciprocally, we showed in Theorem 2.16 of [2] that a group G is almost ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is almost ω_1-$p^{\omega+n}$-projective.

Our further work in this section will be focussed on the behavior of the new group classes about Ulm subgroups and Ulm factors. Our main results presented below settle this matter in some aspect.

The following claim on niceness is pivotal. Its proof, although not difficult, is rather technical, so that we leave it to the interested readers.

Lemma 3.2. *Suppose N is a nice subgroup of a group A and $M \subseteq p^\lambda A$ for some infinite ordinal λ where $p^\lambda A$ is bounded. Then $(N+M)/M$ is nice in A/M.*

Lemma 3.3. *Suppose that A is a group with a subgroup B such that A/B is bounded. Thenthe following are true:*

(a) *If N is nice in B, then N is nice in A.*

(b) *If M is nice in A, then $M \cap B$ is nice in B.*

Proof. Appealing to [9], note that a subgroup V of a group W is nice if, for any limit ordinal δ, the equality $\cap_{\alpha<\delta}(V + p^\alpha W) = V + p^\delta W$.

(a) Since $p^j A \subseteq B$ for some $j \in \mathbb{N}$ and hence $p^\omega A = p^\omega B$, it suffices to check the equality only for the ordinal ω. In fact,

$$\begin{aligned}\cap_{i<\omega}(N + p^i A) = \cap_{j\leqslant i<\omega}(N + p^i A) \subseteq \cap_{k<\omega}(N + p^k B)\\ = N + p^\omega B \subseteq N + p^\omega A,\end{aligned}$$

as required.

(b) We subsequently deduce that

$$\begin{aligned}\cap_{\alpha<\delta}(M \cap B + p^\alpha B) \subseteq \cap_{\alpha<\delta}(M + p^\alpha A) \cap B = (M + p^\delta A) \cap B\\ = (M + p^\delta B) \cap B = M \cap B + p^\delta B,\end{aligned}$$

as required, where the last equality follows by the modular law. □

We now proceed by proving with the next crucial statement, needed for our further application.

Proposition 3.4. *Let A be a group and $\lambda \geqslant \omega$ an ordinal.*

(i) *If A is strongly almost ω_1-$p^{\omega+n}$-projective and $Z \subseteq p^\lambda A$, where $p^\lambda A$ is bounded, then A/Z is strongly almost ω_1-$p^{\omega+n}$-projective.*

(ii) *If $X \subseteq p^{\omega+n}A$, $p^{\omega+n}A$ is countable and A/X is strongly almost ω_1-$p^{\omega+n}$-projective, then A is also strongly almost ω_1-$p^{\omega+n}$-projective.*

Proof. (i) Let Q be a nice subgroup of A with $p^nQ = \{0\}$ and suppose A/Q is the sum of a countable group and an almost direct sum of cyclic groups, say $A/Q = K + S$. It is easily seen that $Q' = (Q + Z)/Z$ is p^n-bounded and in accordance with Lemma 3.2 it is nice in $A' = A/Z$ as well. In addition, $A'/Q' \cong A/(Q + Z) \cong [A/Q]/[(Q + Z)/Q]$ and $(Q + Z)/Q \subseteq (Q + p^\lambda A)/Q = p^\lambda(A/Q)$. Since $K \cap S \subseteq S$ is countable, there exists a countable nice subgroup C of S such that $K \cap S \subseteq C$. Consequently, $(A/Q)/C = [(K+C)/C]\oplus[S/C]$. Since S/C is p^ω-bounded, we derive that

$$(p^\lambda(A/Q) + C)/C \subseteq p^\lambda((A/Q)/C) = p^\lambda((K + C)/C)$$

is countable, whence so is $p^\lambda(A/Q)$. Furthermore, in virtue of Lemma 2.16 from [1], we observe that A/Q is actually almost ω_1-p^ω-projective. Since $p^\lambda(A/Q)$ is countable, we obtain the same for $(Q + Z)/Q$ and thus in accordance with Theorem 2.23 of [2], we conclude that A'/Q' is also ω_1-p^ω-projective, as required.

(ii) With the aid of [7] we observe that the quotient

$$[A/X]/p^{\omega+n}(A/X) = [A/X]/[p^{\omega+n}A/X] \cong A/p^{\omega+n}A$$

is almost $p^{\omega+n}$-projective. We next again employ [7] to derive that A is strongly almost ω_1-$p^{\omega+n}$-projective, as asserted. □

The next statement is pivotal.

Lemma 3.5. *Suppose that A is a group with a subgroup B such that A/B is bounded. Then*

(i) *A is almost $p^{\omega+n}$-projective if and only if B is almost $p^{\omega+n}$-projective.*

(ii) *A is strongly almost ω_1-$p^{\omega+n}$-projective if and only if B is strongly almost ω_1-$p^{\omega+n}$-projective.*

(iii) *A is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if B (strongly) almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. (i) It is straightforward.

(ii) Since $p^tA \subseteq B$ for some $t \in \mathbb{N}$, we obtain that $p^\omega A = p^\omega B$ and thus $p^{\omega+n}A = p^{\omega+n}B$. Moreover, in virtue of (i), $B/p^{\omega+n}B = B/p^{\omega+n}A$ is almost $p^{\omega+n}$-projective uniquely when $A/p^{\omega+n}A$ is almost $p^{\omega+n}$-projective, because the factor-group $(A/p^{\omega+n}A)/(B/p^{\omega+n}A) \cong A/B$ remains bounded. We finally apply [7] to conclude the claim.

(iii) "$\Rightarrow$". Let A/H be strongly almost ω_1-$p^{\omega+n}$-projective for some $H \leqslant A[p^m]$ (which is nice in A). Since $[A/H]/[(B+H)/H] \cong A/(B+H)$ remains bounded as an epimorphic image of A/B, we deduce with the help of (ii) that $(B+H)/H \cong B/(B \cap H)$ is strongly almost ω_1-$p^{\omega+n}$-projective. In addition, $B \cap H \leqslant B[p^m]$ (which is nice in B), and we are finished.

"$\Leftarrow$". Let B/L be strongly ω_1-$p^{\omega+n}$-projective factor-group for some $L \leqslant B[p^m]$ (which is nice in B). Since $[A/L]/[B/L] \cong A/B$ is bounded, point (ii) is applicable to infer that A/L is strongly almost ω_1-$p^{\omega+n}$-projective. But $L \leqslant A[p^m]$ (which is nice in A), and we are done.

The niceness in both directions follows immediately from Lemma 3.3.

□

We have now at our disposal all the ingredients needed to prove the following basic assertion on both Ulm subgroups and Ulm factors pertaining to the other remaining group classes.

Proposition 3.6. *If the group G is either*

(a) *strongly almost m-ω_1-$p^{\omega+n}$-projective or*

(b) *weakly almost m-ω_1-$p^{\omega+n}$-projective or*

(c) *nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective or*

(d) *nicely almost m-$p^{\omega+n}$-projective,*

then the same are both $p^\alpha G$ and $G/p^\alpha G$ for any ordinal α.

Proof. (a) Suppose that G/T is strongly almost ω_1-$p^{\omega+n}$-projective for some nice p^m-bounded subgroup T of G. Thus

$$p^\alpha G/(p^\alpha G \cap T) \cong (p^\alpha G + T)/T = p^\alpha(G/T)$$

is also strongly almost ω_1-$p^{\omega+n}$-projective in view of [7], with $p^\alpha G \cap T$ being p^m-bounded and nice in $p^\alpha G$ (cf. [9]). Hence $p^\alpha G$ is strongly almost m-ω_1-$p^{\omega+n}$-projective as well.

To show the second part, we consequently apply again [7] to infer that

$$\begin{aligned}(G/T)/p^\alpha(G/T) &= (G/T)/(p^\alpha G + T)/T \cong G/(p^\alpha G + T)\\ &\cong (G/p^\alpha G)/(p^\alpha G + T)/p^\alpha G\end{aligned}$$

is also strongly almost ω_1-$p^{\omega+n}$-projective. Moreover, it is plainly observed that $(p^\alpha G + T)/p^\alpha G$ is bounded by p^m because so is T, and that

$(p^\alpha G + T)/p^\alpha G$ is nice in $G/p^\alpha G$ since it is well known that $p^\alpha G + T$ is nice in G - see, for example, [9].

(b) Suppose G/X is almost ω_1-$p^{\omega+n}$-projective for some nice $X \leqslant G$ with $p^m X = \{0\}$. Observe that the following relations are valid:

$$p^\alpha G/(p^\alpha G \cap X) \cong (p^\alpha G + X)/X \subseteq G/X.$$

But a subgroup of an almost ω_1-$p^{\omega+n}$-projective group is again almost ω_1-$p^{\omega+n}$-projective (cf. [2]). Thus $p^\alpha G/(p^\alpha G \cap X)$ is almost ω_1-$p^{\omega+n}$-projective as well. Moreover, $p^\alpha G \cap X$ is obviously p^m-bounded and also, in accordance with [9], it is nice in $p^\alpha G$. So, $p^\alpha G$ is weakly almost m-ω_1-$p^{\omega+n}$-projective.

Furthermore,

$$\begin{aligned}(G/X)/p^\alpha(G/X) &= (G/X)/(p^\alpha G + X)/X \cong G/(p^\alpha G + X)\\ &\cong (G/p^\alpha G)/(p^\alpha G + X)/p^\alpha G\end{aligned}$$

is almost ω_1-$p^{\omega+n}$-projective too, owing to Proposition 3.1.

Besides, it is obviously seen that

$$p^m((p^\alpha G + X)/p^\alpha G) = (p^{\alpha+m} G + p^\alpha G)/p^\alpha G = \{0\},$$

and in the case of niceness that $(p^\alpha G + X)/p^\alpha G$ is nice in $G/p^\alpha G$ because it is well known that $p^\alpha G + X$ is nice in G, see [9], for instance.

(c) Accordingly, write $G/H = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. But

$$p^\alpha G/(p^\alpha G \cap H) \cong (p^\alpha G + H)/H = p^\alpha(G/H) = p^\alpha B \oplus p^\alpha R,$$

where $p^\alpha B$ is obviously countable and $p^\alpha R$ is by [2] almost $p^{\omega+n}$-projective. Since $p^\alpha G \cap H$ is p^m-bounded and nice in $p^\alpha G$ (see [9]), we derive that $p^\alpha G$ is nice direct decomposably almost m-ω_1-$p^{\omega+m}$-projective, as stated.

Concerning the other part, the direct sum

$$\begin{aligned}(B/p^\alpha B) \oplus (R/p^\alpha R) &\cong [G/H]/p^\alpha(G/H)\\ &\cong G/(p^\alpha G + H) \cong [G/p^\alpha G]/(p^\alpha G + H)/p^\alpha G\end{aligned}$$

is again a direct sum of a countable group and an almost $p^{\omega+n}$-projective group, because of the obvious facts that $B/p^\alpha B$ is countable and $R/p^\alpha R$ is almost $p^{\omega+n}$-projective, where the later one exploits [2]. In this vein, it is self-evident that $(p^\alpha G + H)/p^\alpha G$ is bounded by p^m and, in conjunction with [9], that $(p^\alpha G + H)/p^\alpha G$ is nice in $G/p^\alpha G$, as required.

(d) Given a p^m-bounded nice subgroup Y of G such that G/Y is almost $p^{\omega+n}$-projective. Hence, in view of [2], $p^\alpha G/(p^\alpha G \cap Y) \cong (p^\alpha G + Y)/Y \subseteq G/Y$ is almost $p^{\omega+n}$-projective as well, with $p^\alpha G \cap Y$ being p^m-bounded and nice in $p^\alpha G$ (cf. [9]).

On the other hand,

$$\begin{aligned}(G/p^\alpha G)/(Y+p^\alpha G)/p^\alpha G &\cong G/(Y+p^\alpha G)\\ &\cong (G/Y)/(Y+p^\alpha G)/Y = (G/Y)/p^\alpha(G/Y)\end{aligned}$$

is almost $p^{\omega+n}$-projective by exploiting [2]. Since $(Y + p^\alpha G)/p^\alpha G \cong Y/(Y \cap p^\alpha G)$ is p^m-bounded and nice in $G/p^\alpha G$ (see [9]), the assertion follows. □

Under some extra restrictions on α, we can say even a little more:

Proposition 3.7. *If G is a nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $G/p^{\alpha+m}G$ is nicely almost m-$p^{\omega+n}$-projective for every ordinal $\alpha \leqslant \omega + n$. In particular, $G/p^{\omega+m+n}G$ is nicely almost m-$p^{\omega+n}$-projective.*

Proof. By Definition 1.3, we write that $G/H = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded nice subgroup H of G. An appeal to the proof of Proposition 3.6 (c) gives that

$$[G/H]/p^\alpha(G/H) \cong G/(p^\alpha G + H) \cong [G/p^{\alpha+m}G]/(p^\alpha G + H)/p^{\alpha+m}G.$$

is almost $p^{\omega+n}$-projective with

$$p^m((p^\alpha G + S)/p^{\alpha+m}G) = p^{\alpha+m}G/p^{\alpha+m}G = \{0\},$$

so that the claim follows. The final part is an immediate consequence by taking $\alpha = \omega + n$. □

The following somewhat supplies Proposition 3.5 listed above.

Proposition 3.8. *If G is a direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective for all ordinals α. In particular, if $\alpha \geqslant \omega$, then $p^\alpha G$ is almost ω_1-$p^{\omega+m}$-projective.*

In addition, if G is a nice direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group and $\alpha \geqslant \omega$, then $p^\alpha G$ is strongly almost ω_1-$p^{\omega+m}$-projective.

Proof. Using Definition 1.3, let $S \leqslant G[p^m]$ such that $G/S = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective. If $\alpha \geqslant \omega$, then one sees that $p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S \subseteq p^\alpha(G/S) = K \oplus P$ where K is countable and P is p^n-bounded. Hence $p^\alpha G/(p^\alpha G \cap S)$ is also such a direct sum of a countable group and a p^n-bounded group (which itself is a direct sum of cyclic groups) with p^m-bounded intersection $S \cap p^\alpha G$, so that $p^\alpha G$ is almost ω_1-$p^{\omega+m}$-projective.

If now $\alpha < \omega$ is finite, then in virtue of [2] the quotient

$$p^\alpha G/(p^\alpha G \cap S) \cong (p^\alpha G + S)/S = p^\alpha(G/S) = p^\alpha B \oplus p^\alpha R$$

is again a direct sum of the countable group $p^\alpha B$ and the almost $p^{\omega+n}$-projective group $p^\alpha R$, as needed. That is why, in both cases, $p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective.

The final part follows easily since S being nice in G yields that $S \cap p^\alpha G$ is nice in $p^\alpha G$ (cf. [9]). $\square$

We now strengthen the idea in the proof of Proposition 3.1 by the following statement; however we cannot yet establish that, for all ordinals α, the Ulm factor $G/p^\alpha G$ possesses the direct decomposable almost m-ω_1-$p^{\omega+n}$-projective property provided that the same holds for G.

Proposition 3.9. *If G is a direct decomposably almost m-ω_1-$p^{\omega+n}$-projective group, then $G/p^\alpha G$ is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective for all ordinals $\alpha \geqslant \omega + n$.*

Proof. Utilizing Definition 1.3, write that $G/S = B \oplus R$ where B is countable and R is almost $p^{\omega+n}$-projective for some p^m-bounded subgroup S of G.

Standardly, the following isomorphisms are true:

$$(G/p^\alpha G)/[(S + p^\alpha G)/p^\alpha G] \cong G/(S + p^\alpha G) \cong (G/S)/[(S + p^\alpha G)/S].$$

Moreover, $(S + p^\alpha G)/S \subseteq p^\alpha(G/S) = p^\alpha B$. Therefore, setting $T = (S + p^\alpha G)/S$, we deduce that

$$(G/S)/T = (B \oplus R)/T \cong (B/T) \oplus R$$

is again a direct sum of a countable group and an almost $p^{\omega+n}$-projective group. And since $p^m((p^\alpha G + S)/p^\alpha G) = \{0\}$, we are finished. $\square$

Remark 5. When $\alpha = \omega$, we know by [2] or by Theorem 2.4 that $G/p^{\omega}G$ must be almost $p^{\omega+m+n}$-projective and thus in virtue of Proposition 2.3 it is direct decomposably almost m-ω_1-$p^{\omega+n}$-projective. However, the unsettled situation is when $\omega < \alpha < \omega + n$.

Now, we are ready to establish the following:

Theorem 3.10 (First Reduction Criterion)**.** *The group G is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if the following two conditions are fulfilled:*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is (strongly) almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". As observed before, G is almost ω_1-$p^{\omega+m+n}$-projective, so point (1) follows automatically appealing to [2]. Concerning point (2), it follows immediately from Proposition 3.6(a).

"$\Leftarrow$". Assume now that clauses (1) and (2) are valid. For convenience put $k = m + n$. By definition, let $L/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ be a p^m-bounded subgroup such that $(G/p^{\omega+k}G)/(L/p^{\omega+k}G) \cong G/L$ is strongly almost ω_1-$p^{\omega+n}$-projective. Thus $p^m L \subseteq p^{\omega+k}G$. Since G/L is $p^{\omega+k+m}$-bounded, we see that $p^{\omega+n}(G/L)$ is bounded (by p^{2m}), and applying Proposition 3.4 (i) to G/L, we deduce that

$$(G/L)/(p^{\omega+n}G + L)/L \cong G/(p^{\omega+n}G + L)$$

is strongly almost ω_1-$p^{\omega+n}$-projective, because $(p^{\omega+n}G+L)/L \subseteq p^{\omega+n}(G/L)$. Putting $M = p^{\omega+n}G + L$, it is obvious that $p^{\omega+n}G \subseteq M$ and $p^m M = p^{\omega+k}G$. That is why, G/M is strongly almost ω_1-$p^{\omega+n}$-projective with $M \leqslant G$ satisfying the above two relations.

Furthermore, supposing that Y is a maximal p^m-bounded summand of $p^{\omega+n}G$, so there is a direct decomposition $p^{\omega+n}G = X \oplus Y$ and, by what we have just shown above, the inclusions $X \subseteq p^{\omega+n}G \subseteq M$ are true. We can without loss of generality assume that X is countable because of the following reasons: Since $p^{\omega+k}G = p^m X$ is countable, it follows that $X = K \oplus Z$ where K is countable and Z is p^m-bounded. Therefore, $p^{\omega+n}G = K \oplus Z \oplus Y = K \oplus Y'$ where $Y' = Z \oplus Y$, as needed.

We next routinely verify that $X[p] = (p^{\omega+k}G)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$. So, suppose H is a $p^{\omega+k}$-high subgroup of G such that $H \supseteq Y$. Now, $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ together with H being pure in G (cf. [9]) readily force that $G[p^m] = X[p^m] \oplus H[p^m]$ whenever $m \geqslant 1$. In fact, given $g \in G$ with $p^m g \in p^{\omega+k}G$, we write $p^m g = p^m a$

where $a \in p^{\omega+n}G = X \oplus Y$. Then $p^m g = p^m x$ for some $x \in X$, whence $g \in x + G[p^m] \subseteq X + H[p^m]$, as required.

Besides, $X \cap H[p^m] \subseteq X \cap H = \{0\}$ and consequently $(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$ because $p^{\omega+k}G = p^m X \subseteq X$. Since $M/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m]$, it follows that $M \subseteq X \oplus H[p^m]$ and hence

$$M = (X \oplus H[p^m]) \cap M = X + H[p^m] \cap M$$

by virtue of the modular law. Substituting $P = H[p^m] \cap M$, we derive that $p^m P = \{0\}$ and that $M = X + P$. In addition, $M = M + p^{\omega+n}G = P + p^{\omega+n}G$ and so $G/(p^{\omega+n}G + P) \cong (G/P)/(p^{\omega+n}G + P)/P$ is strongly almost ω_1-$p^{\omega+n}$-projective.

We now claim that $p^{\omega+n}(G/P)$ is countable. In fact, $p^{\omega+n}(G/M)$ is countable because G/M is strongly almost ω_1-$p^{\omega+n}$-projective (see [7]). But we subsequently have that

$$\begin{aligned} p^{\omega+n}(G/M) &= p^n(p^\omega(G/M)) = p^n(\cap_{i<\omega}(p^iG + M)/M) \\ &= p^n(\cap_{i<\omega}(p^iG + P)/M) \cong p^n(\cap_{i<\omega}[(p^iG + P)/P]/[M/P]) \\ &= p^n(p^\omega(G/P)/[M/P]) = [p^{\omega+n}(G/P) + (M/P)]/[M/P] \\ &= p^{\omega+n}(G/P)/[M/P] \end{aligned}$$

since $M/P = (p^{\omega+n}G + P)/P \subseteq p^{\omega+n}(G/P)$. Moreover,

$$\begin{aligned} M/P &= M/(M \cap H[p^m]) \cong (M + H[p^m])/H[p^m] \\ &= (X + H[p^m])/H[p^m] \cong X/(X \cap H[p^m]) \cong X \end{aligned}$$

is countable. Finally, $p^{\omega+n}(G/P)$ is countable as well, as claimed.

Also, because $(p^{\omega+n}G + P)/P \leqslant p^{\omega+n}(G/P)$, Proposition 3.4 (ii) applied to G/P shows that G/P is strongly almost ω_1-$p^{\omega+n}$-projective with $p^m P = \{0\}$, as required.

As for the “niceness” property, it can be established as Theorem 3.12 quoted below. □

Now, with Proposition 2.5 at hand, we deduce the following consequence.

Corollary 3.11. *Suppose that $p^\lambda G$ is countable for some ordinal $\lambda \geqslant \omega$. Then the group G is (strongly) almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We henceforth have all the information to prove our next basic result.

Theorem 3.12 (Second Reduction Criterion)**.** *The group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is weakly almost m-ω_1-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". It follows directly from [2] together with Proposition 3.6 (b).

"$\Leftarrow$". For our convenience, set $k = m+n$. By definition, let $T/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ be a p^m-bounded nice subgroup such that

$$(G/p^{\omega+k}G)/(T/p^{\omega+k}G) \cong G/T$$

is almost ω_1-$p^{\omega+n}$-projective. Thus T is nice in G (see, e.g., [9]), and $p^mT \subseteq p^{\omega+k}G$. Applying Proposition 3.1 or Proposition 2.13 (b) in [2],

$$G/(T+p^{\omega+n}G) \cong [G/T]/(T+p^{\omega+n}G)/T = [G/T]/p^{\omega+n}(G/T)$$

is also almost ω_1-$p^{\omega+n}$-projective. Putting $T' = T + p^{\omega+n}G$, we see that G/T' is almost ω_1-$p^{\omega+n}$-projective and that $T' \supseteq p^{\omega+n}G$ remains nice in G and $p^mT' = p^mT + p^{\omega+k}G = p^{\omega+k}G$. So, replacing hereafter T' with T, we may without loss of generality assume that $p^{\omega+n}G \leqslant T$.

Suppose now Y is a maximal p^m-bounded summand of $p^{\omega+n}G$; so there exists a direct decomposition $p^{\omega+n}G = X \oplus Y$ and thus the inclusions $X \subseteq p^{\omega+n}G \subseteq T$ hold. We may also assume with no harm of generality that X is countable; in fact, $p^{\omega+k}G = p^mX$ is countable and therefore we can decompose $X = K \oplus Z$, where K is countable and Z is p^m-bounded (whence Z is a p^m-bounded summand of $p^{\omega+n}G$ and so $Z \subseteq Y$). Consequently, it is readily checked that $p^{\omega+n}G = K \oplus Y$ with countable summand K, as wanted.

Next, a straightforward check shows that $X[p] = (p^{\omega+k}G)[p] = (p^mX)[p]$ and thus $Y \cap p^{\omega+k}G = \{0\}$ because

$$(Y \cap p^{\omega+k}G)[p] = Y \cap (p^{\omega+k}G)[p] = Y \cap X[p] = \{0\}.$$

Let us now H be a $p^{\omega+k}$-high subgroup of G containing Y (thus H is maximal with respect to $H \cap p^{\omega+k}G = \{0\}$ with $H \supseteq Y$). We now assert that

$$(G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G.$$

In fact, as noted above, $X[p] = (p^{\omega+k}G)[p]$ and thereby $X \cap H = \{0\}$ because

$$(X \cap H)[p] = X[p] \cap H = (p^{\omega+k}G)[p] \cap H = \{0\}.$$

Since $G[p] = (p^{\omega+k}G)[p] \oplus H[p] = X[p] \oplus H[p]$ and H is pure in G (see [9]), it plainly follows that $G[p^m] = X[p^m] \oplus H[p^m]$. To prove this, given $v \in G$ with $p^m v \in p^{\omega+k}G$, it suffices to show that $v \in X \oplus H[p^m]$. In fact, $p^m v = p^m d$ where $d \in p^{\omega+n}G = X \oplus Y$. Then $p^m d = p^m x$ for some $x \in X$ and so $p^m v = p^m x$. Therefore,

$$v \in x + G[p^m] = x + X[p^m] + H[p^m] \subseteq X + H[p^m],$$

as required. So, the assertion is sustained.

Furthermore, by what we have obtained above,

$$T/p^{\omega+k}G \subseteq (G/p^{\omega+k}G)[p^m] = (X \oplus H[p^m])/p^{\omega+k}G$$

implies that $T \subseteq X \oplus H[p^m]$; note also that $X \subseteq T$. Put $L = T \cap H[p^m] \subseteq H$, so that it is clear that $L \cap p^{\omega+k}G = \{0\}$. Moreover, the modular law ensures that

$$T = (X \oplus H[p^m]) \cap T = X \oplus (T \cap H[p^m]) = X \oplus L.$$

We consequently conclude that $T = p^{\omega+n}G + T = p^{\omega+n}G + L$ and $G/T = G/(p^{\omega+n}G + L)$ is almost ω_1-$p^{\omega+n}$-projective. Observe also that L is p^m-bounded, and that L is nice in G. The first fact is trivial, as for the second one $L \cap p^{\omega+k}G = \{0\}$ easily forces that $L \cap p^{\omega+n}G$ is nice in $p^{\omega+n}G$ and thus it is nice in G. On the other hand, as noticed above, $p^{\omega+n}G + L = T$ is also nice in G. According to [9], these two conditions together imply that L is nice in G, as expected.

What remains to illustrate is that $p^{\omega+n}(G/L)$ is countable. Indeed, we have $p^{\omega+n}(G/L) = (p^{\omega+n}G + L)/L = T/L$. Also,

$$\begin{aligned} T/L &= T/(T \cap H[p^m]) \cong (T + H[p^m])/H[p^m] \\ &= (p^{\omega+n}G + H[p^m])/H[p^m] \cong p^{\omega+n}G/(p^{\omega+n}G \cap H[p^m]). \end{aligned}$$

But as obtained above, $p^{\omega+n}G = X \oplus Y$ and since $Y \subseteq H$, we have with the aid of the modular law that $p^{\omega+n}G \cap H = (X \oplus Y) \cap H = (X \cap H) \oplus Y = Y$, whence $p^{\omega+n}G \cap H[p^m] = Y[p^m]$. We therefore establish that

$$T/L \cong (X \oplus Y)/Y[p^m] \cong X \oplus (Y/Y[p^m]) \cong X \oplus p^m Y = X.$$

Since X is shown above to be countable, so does $T/L = p^{\omega+n}(G/L)$. We finally apply Proposition 2.6 to get the desired claim. $\square$

Mimicking the method demonstrated above, with Proposition 2.6 in hand we can state:

Corollary 3.13. *Let $\lambda \geqslant \omega$ be an ordinal such that $p^\lambda G$ is countable. Then the group G is weakly almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We now ready to establish our next reduction theorem.

Theorem 3.14 (Third Reduction Criterion)**.** *The group G is nicely almost m-$p^{\omega+n}$-projective if and only if*

(1) *$p^{\omega+m+n}G$ is countable;*

(2) *$G/p^{\omega+m+n}G$ is nicely almost m-$p^{\omega+n}$-projective.*

Proof. "$\Rightarrow$". Clause (1) follows immediately as above.

As for clause (2), it follows directly by Proposition 3.6 (d).

"$\Leftarrow$". Assume that (1) and (2) are fulfilled, so that let there exist a nice p^m-bounded subgroup $A/p^{\omega+m+n}G$ of $G/p^{\omega+m+n}G$ with $A \leqslant G$ such that G/A is almost $p^{\omega+n}$-projective. Thus, as we have seen before, $p^m A \subseteq p^{\omega+k}G$ for $k = m + n$, and A is nice in G. Imitating the same technique as in Theorems 3.10 and 3.12, we can find a p^m-bounded nice subgroup N of G such that G/N is almost $p^{\omega+n}$-projective, and so we complete the arguments. □

Same as above, we derive:

Corollary 3.15. *Let $\lambda \geqslant \omega$ be an ordinal for which $p^\lambda G$ is countable. Then the group G is nicely almost m-ω_1-$p^{\omega+n}$-projective if and only if $G/p^\lambda G$ is.*

We will be now concentrated on nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups, which are somewhat difficult to handle. So, we will restrict our attention on the ideal case $n = 1$ by showing that the investigation of nice decomposably almost m-ω_1-$p^{\omega+1}$-projective groups can be reduced to these of length not exceeding $\omega + m + 1$. Specifically, the following holds:

Theorem 3.16 (Fourth Reduction Criterion)**.** *The group G is nice direct decomposably almost m-ω_1-$p^{\omega+1}$-projective if and only if*

(1) *$p^{\omega+m+1}G$ is countable;*

(2) *$G/p^{\omega+m+1}G$ is nice direct decomposably almost m-ω_1-$p^{\omega+m+1}$-projective.*

Proof. The "and only if" part follows directly as above in a combination with Proposition 3.6 (c), respectively.

Concerning the "if" part, we set for simpleness $k = m + 1$. Using the corresponding definition, suppose $T/p^{\omega+k}G \leqslant G/p^{\omega+k}G$ is a p^m-bounded nice subgroup such that $[G/p^{\omega+k}G]/[T/p^{\omega+k}G] \cong G/T$ is a direct sum of a countable group and an almost $p^{\omega+1}$-projective group. Hence T is nice in G (see, e.g., [9]), and $p^mT \subseteq p^{\omega+k}G$. Also, it is routinely checked that $[G/T]/p^{\omega+1}(G/T) \cong G/(T + p^{\omega+1}G)$ is almost $p^{\omega+1}$-projective. Henceforth, the proof goes on imitating the same scheme of proof as that in Theorems 3.10 and 3.12 to infer the wanted statement. $\square$

Remark 6. As observed in Proposition 3.6 (c), the necessity in Theorem 3.16 is valid for any natural n. However, the sufficiency probably fails for each other $n > 1$.

4. Open questions

We close the work with certain challenging problems which are worthwhile for a further study.

Problem 1. Is it true that weakly almost n-ω_1-$p^{\omega+m}$-projective groups are almost m-ω_1-$p^{\omega+n}$-projective?

Problem 2. Are (strongly) almost m-ω_1-$p^{\omega+n}$-projective groups strongly almost ω_1-$p^{\omega+m+n}$-projective?

Problem 3. Does it follow that nice decomposably almost m-ω_1-$p^{\omega+n}$-projective groups are strongly almost m-ω_1-$p^{\omega+n}$-projective?

Correction. In the proof of Theorem 2.23 from [2], on lines 4 and 6 the phrase *"almost $p^{\omega+n}$-projective"* should be stated as *"almost ω_1-$p^{\omega+n}$-projective"*. The omission "ω_1" was involuntarily.

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