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RESEARCH ARTICLE

The endomorphism monoids of $(n-3)$-regular graphs of order n

Nirutt Pipattanajinda, Ulrich Knauer, Boyko Gyurov and Sayan Panma

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Abstract. This paper is motivated by the result of W. Li, that presents an infinite family of graphs - complements of cycles — which possess a regular monoid. We show that these regular monoids are completely regular. Furthermore, we characterize the regular, orthodox and completely regular endomorphisms of the join of complements of cycles, i.e. $(n-3)$-regular graphs of order n.

Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The techniques that are used in those studies connect semigroup theory to graph theory and establish relationships between graphs and semigroups.

We review the regularity of endomorphism monoids of graphs. A characterization of regular elements in $\mathrm{End}(G)$ using endomorphic image and kernel was given by Li, in 1994, see [7]. In 1996, the connected bipartite graphs whose endomorphism monoids are regular and orthodox were explicitly found by Wilkeit [12] and Fan [2], respectively. In 2003, see [8], Li showed that the endomorphism monoids of $\overline{C}_{2n+1}$ $(n>1)$ are

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groups and the endomorphism monoids of $\overline{C}_{2n}$ are regular, where $\overline{C}_n$ denotes the complement of a cycle C_n.

Characterization of the join product of graphs whose endomorphism monoids are regular or orthodox were studied extensively throughout the years, with Knauer discovering in 1987, that the endomorphism monoid of complete r-partite graph - the join of totally disconnected graphs $\overline{K}_{n_1} + \ldots + \overline{K}_{n_r}, r \in \mathbb{Z}^+$, is regular, see [5]. In 2003, Li showed that if the endomorphism monoid of the join of graphs is regular, then each endomorphism of the graphs is regular, and for any graph G, the endomorphism monoid of G is regular if and only if the endomorphism monoid of the join of G and a complete graph is also regular, see [8]. In 2008, Hou and Luo characterized graphs with regular endomorphism monoids and graphs with orthodox endomorphism monoids studying joins of two bipartite graphs, see [4].

Consider *finite simple graphs* G with *vertex set* $V(G)$ and *edge set* $E(G)$. The number of vertices of G is often called the *order* of G. The *degree* of a vertex u in a graph G is the number of vertices adjacent to u and is denoted by $d_G(u)$ or simply by $d(u)$ if the graph G is clear from the context. If $d(u) = r$ for every vertex u of G, where $0 \leqslant r \leqslant n-1$, then G is called a *r-regular*. The complement (graph) $\overline{G}$ of G is a graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in (\overline{G})$ if and only if $\{u, v\} \notin E(G)$ for any $a, b \in V(G), a \neq b$. A subgraph H of G is called an *induced subgraph*, if for any $u, v \in V(H), \{u, v\} \in E(G)$ implies $\{u, v\} \in E(H)$. The induced subgraph H with $V(H) = S$ is also denoted by $\langle S \rangle$. A *clique* of graph G is a maximal complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is the maximal order among the cliques of G. Let G and H be two graphs. The *join* of G and H, denoted by $G+H$, is a graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} | u \in V(G), v \in V(H)\}$. The graph with vertex set $\{1, \ldots, n\}$, such that $n \geqslant 3$, and edge set $\{\{i, i+1\} | i = 1, \ldots, n\} \cup \{1, n\}$ is called a cycle C_n.

Let G and H be graphs. A (graph) *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e. $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. By $G \to H$ we denote that there exists a homomorphism from G to H. A homomorphism f is called an (graph) *isomorphism* if f is bijective and f^{-1} is a homomorphism. We call G *isomorphic* to H and denote by $G \cong H$, if there exists an isomorphism f from G onto H. A homomorphism from G into itself is called an (graph) *endomorphism* of G. An endomorphism f is said to be *locally strong* if $\{f(u), f(v)\} \in E(G)$ implies that for every preimage x of $f(u)$ there exists a preimage y of $f(v)$ such that $\{x, y\} \in E(G)$ and analogously for every preimage of $f(v)$. An isomorphism from G into itself is called an *automorphism*. In this paper we use the following notations:

- $\mathrm{End}(G)$, the set of all endomorphisms of G,
- $\mathrm{End}'(G)$, the set of all non-injective endomorphisms of G,
- $LEnd(G)$, the set of all locally strong endomorphisms of G, and
- $Aut(G)$, the set of all automorphisms of G.

A factor graph I_f of G under f which is a subgraph of G is called the *endomorphic image* of G under f. This means, $V(I_f) = f(V(G))$ and $\{f(u), f(v)\} \in E(I_f)$ if and only if there exist $u' \in f^{-1}f(u)$ and $v' \in f^{-1}f(v)$ such that $\{u', v'\} \in E(G)$, where $f^{-1}(t)$ denotes the set of *preimages* of some vertex t of G under the mapping f. By ρ_f, we denote the *equivalence relation* on $V(G)$ induced by f, i. e. for any $u, v \in V(G)$, $(u, v) \in \rho_f$ if and only if $f(u) = f(v)$.

Let S be a semigroup (monoid respectively). An element a of S is called an *idempotent* if $a^2 = a$. An element a of S is called a *regular* if $a = aa'a$ for some $a' \in S$, such a' is called a *pseudo inverse* to a. The semigroup S is called *regular* if every element of S is regular. A regular element a of S is called *completely regular* if there exists a pseudo inverse a' to a such that $aa' = a'a$. In this case we call a' a *commuting pseudo inverse* to a. The semigroup S is called *completely regular* if every element of S is completely regular. A regular semigroup S is called *orthodox* if the set of all idempotent elements of S (denoted by $Idpt(S)$) forms a semigroup under the operation of S. The Green's relations $\mathcal{H}$ on S are defined by $a\mathcal{H}b \Leftrightarrow S^1a = S^1b$ and $aS^1 = bS^1$. We denote the equivalence class $\mathcal{H}$ of S containing element a by H_a.

Lemma 1 ([10])**.** *Let S be a semigroup and e be an idempotent of S. Then H_e is a subgroup of S.*

Lemma 2 ([10])**.** *A semigroup S is completely regular if and only if S is a union of (disjoint) groups.*

We recall, finally, that a *left group* and a *right group* are semigroups of the form $L_l \times G$ and $G \times R_r$ where G is a group and L_l and R_r is a *left zero semigroup* and a *right zero semigroup* of order r, respectively, i.e., L_l is a semigroup with multiplication $l_i l_j = l_i$ for all $l_i, l_j \in L_l$ and R_r is a semigroup with the multiplication $r_i r_j = r_j$ for all $r_i, r_j \in R_r$.

Note that every left group and every right group is a union of groups.

A graph G is *endo-regular* (*endo-orthodox*, *endo-completely-regular*) if the monoid of all endomorphisms on G is regular (orthodox, completely regular respectively). Further, a graph G is *unretractive* if $\mathrm{End}(G) = Aut(G)$.

The following results have been proved:

Lemma 3 ([9]). *Let G be a graph. Suppose $f, g \in \mathrm{End}(G)$ are regular. Then $f\mathcal{H}g$ if and only if $\rho_f = \rho_g$ and $I_f = I_g$.*

Lemma 4 ([7]). *Let G be a graph and let $f \in \mathrm{End}(G)$. Then f is regular if and only if there exist idempotents $g, h \in \mathrm{End}(G)$ such that $\rho_g = \rho_f$ and $I_f = I_h$.*

Lemma 5 ([8]). *Let G be a graph. Then G is endo-regular if and only if $G + K_n$ is endo-regular for any $n \geqslant 1$.*

Lemma 6 ([4]). *Let G_1 and G_2 be two graphs. Then $G_1 + G_2$ is endo-orthodox if and only if*
(1) *$G_1 + G_2$ is endo-regular, and*
(2) *Both of G_1 and G_2 are endo-orthodox.*

Lemma 7 ([6]). *Let G_1 and G_2 be graphs. The join $G_1 + G_2$ is unretractive if and only if G_1 and G_2 are unretractive.*

Lemma 8 ([3]). (1) *$Aut(C_n) \cong D_n$, where D_n denotes the dihedral group of degree n.*
(2) *$Aut(\overline{C}_n) = Aut(C_n)$.*

1. Endo-regularity of complements of cycles

Recall that the complement of cycle C_n with vertex set $V(C_n) = \{1, 2, \ldots, n\}$ is a graph $\overline{C}_n$ with the same vertex set and $E(\overline{C}_n) = \{\{i, j\} \mid n - 2 \geqslant |i - j| \geqslant 2, \forall i, j \in \{1, 2, \ldots, n\}\}$.

In [8] W. Li, characterized the monoid of $\mathrm{End}(\overline{C}_n)$. There he showed that the $\mathrm{End}(\overline{C}_{2m+1}) \cong D_{2m+1}$ $(m \geqslant 2)$ where D_n denotes the dihedral group of degree n, i.e. $\overline{C}_{2m+1}$ is unretractive, and the endomorphism monoid of $\overline{C}_{2m}$ $(m \geqslant 2)$ is regular.

Lemma 9 ([8]). (1) *$\overline{C}_{2n+1}$ $(n > 1)$ is unretractive.*
(2) *$\overline{C}_{2n}$ $(n > 1)$ is endo-regular.*

Furthermore, he showed that

Lemma 10 ([8]). (1) *The cliques of $\overline{C}_{2m}$ or $\overline{C}_{2m+1}$ are isomorphic to K_m.*
(2) *There exist exactly two cliques in $\overline{C}_{2m}$ namely $\langle\{1, 3, \ldots, 2m - 1\}\rangle$ and $\langle\{2, 4, \ldots, 2m\}\rangle$.*

A *full transformation semigroup* T_X on a set X is the set X^X of all transformations (i.e., self-maps) $X \to X$ of X with composition of transformations as multiplication. We can assume that $X = \{1, 2, \ldots, n\}$ and write T_n instead of T_X, see [11]. For $\overline{C}_3$, it is easy to see that,

Lemma 11. $\mathrm{End}(\overline{C}_3) = \mathrm{End}(\overline{K}_3) \cong T_3$.

Let $f \in \mathrm{End}'(\overline{C}_{2m})$. Next, we use the congruence relation by the endomorphism f to show the algebraic structure of the monoid of $\mathrm{End}'(\overline{C}_{2m})$.

Lemma 12. *Let $f \in \mathrm{End}(\overline{C}_{2m})$. If f is non-injective, then either $f(\overline{C}_{2m}) = \{1, 3, \ldots, 2m-1\}$ or $f(\overline{C}_{2m}) = \{2, 4, \ldots, 2m\}$.*

Proof. Clearly, for the graph $\overline{C}_{2m}$, there exist two unique complete subgraphs K_m of $\overline{C}_{2m}$ where $V(K_m) = \{1, 3, \ldots, 2m-1\}$ or $V(K_m) = \{2, 4, \ldots, 2m\}$. Now $\{x, y\} \notin E(\overline{C}_{2m}) \Leftrightarrow y = x + 1$ or $y = x - 1$. Let $f \in \mathrm{End}'(\overline{C}_{2m})$. Then $f(x) = f(y) \Leftrightarrow y = x + 1$ or $y = x - 1$. Therefore, $f(\{1, 3, \ldots, 2m-1\}) = f(\{2, 4, \ldots, 2m\}) = \{1, 3, \ldots, 2m-1\}$ or $f(\{1, 3, \ldots, 2m-1\}) = f(\{2, 4, \ldots, 2m\}) = \{2, 4, \ldots, 2m\}$. □

Example 1. Let $f : V(\overline{C}_6) \to V(\overline{C}_6)$ be defined by $f = \binom{1\,2\,3\,4\,5\,6}{1\,1\,3\,3\,5\,5} \in \mathrm{End}'(\overline{C}_6)$. The defining congruence relation ϱ_f can be exemplified as Figure 1a. There is exactly one other congruence relation which is exemplified by Figure 1b. By the homomorphism theorem each endomorphism is specified by one of these 2 congruence relations and the possible embedding of the K_m into $\overline{C}_{2m}$.

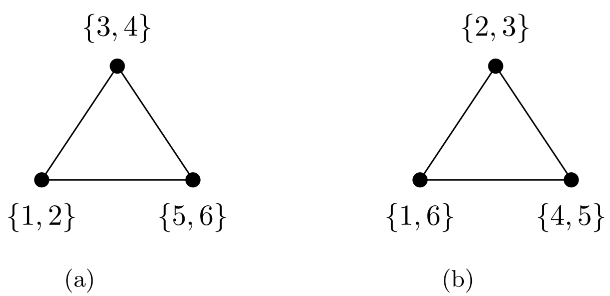


FIGURE 1. Congruence relations on $\overline{C}_6$ induced by f

Lemma 13. $\mathrm{End}(\overline{C}_{2m}) = LEnd(\overline{C}_{2m})$.

Proof. Obvious from Example 1. □

Lemma 14. *Consider the graph $\overline{C}_{2m}$, and $f \in \mathrm{End}'(\overline{C}_{2m})$. Let ϱ_f be the congruence of the graph $\overline{C}_{2m}$ when defining $x\varrho_f y \Leftrightarrow f(x) = f(y)$ which here means $y = x+1$ or $y = x-1$. Denote by $(\overline{C}_{2m})_{\varrho_f}$ the factor graph. Then either $V((\overline{C}_{2m})_{\varrho_f}) = \{\{1,2\},\ldots,\{2m-1,2m\}\}$ or $V((\overline{C}_{2m})_{\varrho_f}) = \{\{2m,1\},\ldots,\{2(m-1),2m-1\}\}$.*

Lemma 15. *Let $f \in \mathrm{End}'(\overline{C}_{2m})$ and $\hat{f} : V((\overline{C}_{2m})_{\varrho_f}) \to V(\overline{C}_{2m})$ be defined by $\hat{f}(x_{\varrho_f}) = y$ if $f(x) = y$. Then either $\hat{f}(\overline{C}_{2m}) = \{1,3,\ldots,2m-1\}$ or $\hat{f}(\overline{C}_{2m}) = \{2,4,\ldots,2m\}$.*

From Lemmas 14 and 15, we can define the following classes of non-injective endomorphisms on $\overline{C}_{2m}$ by ϱ_f and element $f(1) \in V(\overline{C}_{2m})$.

1) S_m^{or}, the class of all endomorphisms f of $\overline{C}_{2m}$ where $\hat{f}(\overline{C}_{2m})$ are the odd integers and $f(1) = f(2)$, i.e. $\{1,2\} \in \varrho_f$,
2) S_m^{er}, the class of all endomorphisms f of $\overline{C}_{2m}$ where $\hat{f}(\overline{C}_{2m})$ are the even integers and $f(1) = f(2)$, i.e. $\{1,2\} \in \varrho_f$,
3) S_m^{ol}, the class of all endomorphisms f of $\overline{C}_{2m}$ where $\hat{f}(\overline{C}_{2m})$ are the odd integers and $f(2m) = f(1)$, i.e. $\{2m,1\} \in \varrho_f$, and
4) S_m^{el}, the class of all endomorphisms f of $\overline{C}_{2m}$ where $\hat{f}(\overline{C}_{2m})$ are the even integers and $f(2m) = f(1)$, i.e. $\{2m,1\} \in \varrho_f$.

Example 2. For the semigroup $\mathrm{End}'(\overline{C}_6)$, we choose the following notations

$$S_3^{or} = \left\{ id = s_{3_1}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,1\,3\,3\,5\,5 \end{pmatrix}, s_{3_2}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,5\,3\,3\,1\,1 \end{pmatrix}, \right.$$

$$s_{3_3}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,1\,5\,5\,3\,3 \end{pmatrix}, s_{3_4}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 3\,3\,1\,1\,5\,5 \end{pmatrix},$$

$$\left. s_{3_5}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 3\,3\,5\,5\,1\,1 \end{pmatrix}, s_{3_6}^{or} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,5\,1\,1\,3\,3 \end{pmatrix} \right\},$$

$$S_3^{er} = \left\{ id = s_{3_1}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,2\,4\,4\,6\,6 \end{pmatrix}, s_{3_2}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,6\,4\,4\,2\,2 \end{pmatrix}, \right.$$

$$s_{3_3}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,2\,6\,6\,4\,4 \end{pmatrix}, s_{3_4}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,4\,6\,6\,2\,2 \end{pmatrix},$$

$$\left. s_{3_5}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,6\,2\,2\,4\,4 \end{pmatrix}, s_{3_6}^{er} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,6\,4\,4\,2\,2 \end{pmatrix} \right\},$$

and

$$S_3^{ol} = \Big\{ id = s_{3_1}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,3\,3\,5\,5\,1 \end{pmatrix}, s_{3_2}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,3\,3\,1\,1\,5 \end{pmatrix},$$

$$s_{3_3}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,5\,5\,3\,3\,1 \end{pmatrix}, s_{3_4}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 3\,5\,5\,1\,1\,3 \end{pmatrix},$$

$$s_{3_5}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,1\,1\,3\,3\,5 \end{pmatrix}, s_{3_6}^{ol} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 5\,3\,3\,1\,1\,5 \end{pmatrix} \Big\},$$

$$S_3^{el} = \Big\{ id = s_{3_1}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,2\,2\,4\,4\,6 \end{pmatrix}, s_{3_2}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,2\,6\,6\,4 \end{pmatrix},$$

$$s_{3_3}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,4\,4\,2\,2\,6 \end{pmatrix}, s_{3_4}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,4\,4\,6\,6\,2 \end{pmatrix},$$

$$s_{3_5}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,6\,6\,2\,2\,4 \end{pmatrix}, s_{3_6}^{el} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,2\,6\,6\,4 \end{pmatrix} \Big\}.$$

Proposition 1. *The sets $S_m^{or}, S_m^{er}, S_m^{ol}$ and S_m^{el} form groups isomorphic to S_m.*

Theorem 1. (1) *$\overline{C}_{2m}$ is endo-completely-regular, and*
(2) $|\operatorname{End}'(\overline{C}_{2m})| = 4|S_m| = 4m!$.

Proof. (1) Since $\operatorname{End}'(\overline{C}_{2m})$ is a union of groups $S_m^{or}, S_m^{er}, S_m^{ol}$ and S_m^{el}, $\operatorname{End}'(\overline{C}_{2m})$ is completely regular by Lemma 2.

(2) Obvious, since the cardinality of each $S_m^{or}, S_m^{er}, S_m^{ol}$ and S_m^{el} is $m!$. □

Now we analyze the algebraic structure of $\operatorname{End}'(\overline{C}_{2m})$.

Lemma 16. *Let $x \in \{o, e\}$, $y \in \{r, l\}$ and $X \in \{S_m^{or}, S_m^{er}, S_m^{ol}, S_m^{el}\}$. Then*
(1) *$XS_m^{xr} \subseteq S_m^{x'r}$ and also $XS_m^{xl} \subseteq S_m^{x'l}$ for $x' \in \{o, e\}$,*
(2) *$S_m^{oy}X \subseteq S_m^{oy'}$ and also $S_m^{ey}X \subseteq S_m^{ey'}$ for $y' \in \{r, l\}$.*

Proof. If $f(1) = f(2)$ is odd or even, then $gf(1) = gf(2)$ is also odd or even. Similarly, if $f(1) = f(2m)$ is odd or even, then $gf(1) = gf(2m)$ is also odd or even. □

Proposition 2. *Let $x, x' \in \{o, e\}$ and $y, y' \in \{r, l\}$. Then $S_m^{xy'} S_m^{x'y} = S_m^{xy}$.*

From Proposition 2, we get the algebraic structure of $\operatorname{End}'(\overline{C}_{2m})$, see Table 1, and Theorem 2.

$\circ$	$s^{or}_{m_1} \cdots s^{or}_{m_{m!}}$	$s^{er}_{m_1} \cdots s^{er}_{m_{m!}}$	$s^{ol}_{m_1} \cdots s^{ol}_{m_{m!}}$	$s^{el}_{m_1} \cdots s^{el}_{m_{m!}}$
$s^{or}_{m_1}$ $\vdots$ $s^{or}_{m_{m!}}$	S^{or}_m	S^{or}_m	S^{ol}_m	S^{ol}_m
$s^{er}_{m_1}$ $\vdots$ $s^{er}_{m_{m!}}$	S^{er}_m	S^{er}_m	S^{el}_m	S^{el}_m
$s^{ol}_{m_1}$ $\vdots$ $s^{ol}_{m_{m!}}$	S^{or}_m	S^{or}_m	S^{ol}_m	S^{ol}_m
$s^{el}_{m_1}$ $\vdots$ $s^{el}_{m_{m!}}$	S^{er}_m	S^{er}_m	S^{el}_m	S^{el}_m

TABLE 1. The form of the semigroup of $\operatorname{End}'(\overline{C}_{2m})$

Theorem 2. *The following statements are true:*

(1) *The sets $S^{or}_m \cup S^{er}_m$ and $S^{ol}_m \cup S^{el}_m$ form left groups isomorphic to $L_2 \times S_m$.*

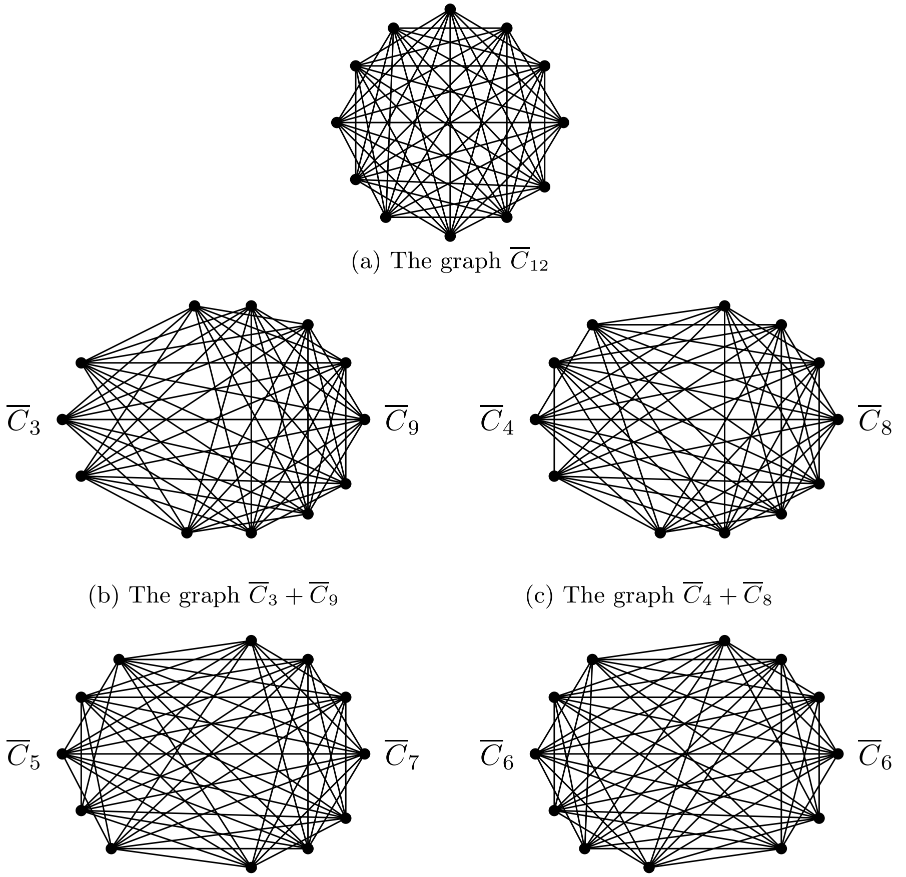
(2) *The sets $S^{or}_m \cup S^{ol}_m$ and $S^{er}_m \cup S^{el}_m$ form right groups isomorphic to $S_m \times R_2$.*

2. Endo-regularity of $(n-3)$-regular graphs of order n

In this section, we consider the endo-regularity of an $(n-3)$-regular graph of order n. Let $\overset{r}{\underset{i=1}{+}} \overline{C}_{n_i} = \overline{C}_{n_1} + \ldots + \overline{C}_{n_r}$. In [1], Amar shows that if G is an $(n-3)$-regular graph of order $n \geqslant 7$, then $\overline{G}$ is 2-regular and, thus, is a disjoint union of cycles. We rewrite this as

Lemma 17. *Let G be a graph of order $n \geqslant 3$. Then G is $(n-3)$-regular graph if and only if $G = \overset{r}{\underset{i=1}{+}} \overline{C}_{n_i}$ where $n = n_1 + \ldots + n_r$ and $r \geqslant 1$. In particular $r > 1$ implies $n \geqslant 6$.*

Example 3. The 5 types of non-isomorphic 9-regular graphs of order 12, are shown in Figure 2.



(a) The graph $\overline{C}_{12}$

(b) The graph $\overline{C}_3 + \overline{C}_9$

(c) The graph $\overline{C}_4 + \overline{C}_8$

(d) The graph $\overline{C}_5 + \overline{C}_7$

(e) The graph $\overline{C}_6 + \overline{C}_6$

Figure 2. The 9-regular graphs of order 12 which are joins of one or two complements of cycles

Take $G = \overset{r}{\underset{i=1}{+}} \overline{C}_{n_i}$, the $(n-3)$-regular graph of order n, where $n = n_1 + \ldots + n_r$ and $r \geqslant 2$. Set $V(\overline{C}_{n_i}) = \{1_i, \ldots, n_i\}$. It is clear from Lemma 9(1), that $\overline{C}_{n_i}$ is unretractive if and only if n_i is odd and $n_i \geqslant 5$. Then by Lemma 7, an $(n-3)$-regular graph G of order n is unretractive if and only if $G = \overset{r}{\underset{i=1}{+}} \overline{C}_{n_i}$ where n_i is odd and $n_i \geqslant 5$, for all $i = 1, \ldots, r$. Consider the retractive $(n-3)$-regular graph of order n.

Lemma 18. *Let G be an $(n-3)$-regular graph of order n and $f \in \mathrm{End}'(G)$. If $f(x_i) = f(y_i)$ for two different elements $x_i, y_i \in V(\overline{C}_{n_i})$, then $y_i = (x-1)_i$ or $y_i = (x+1)_i$.*

Lemma 19. *There exists a homomorphism $\overline{C}_m \to \overline{C}_{m'}$ if and only if $m < m'$.*

Proof. Necessity. First, let m, m' be integers of the same type, or m even and m' odd. Then there exist k and k', $k > k'$ such that (i) $m = 2k$, $m' = 2k'$, (ii) $m = 2k+1$, $m' = 2k'+1$, or (iii) $m = 2k$, $m' = 2k'+1$, respectively. Thus the cliques of $\overline{C}_m$ and $\overline{C}_{m'}$ are isomorphic to K_k and $K_{k'}$, respectively. Therefore, $\overline{C}_m \not\to \overline{C}_{m'}$.

Suppose now that $m = 2k+1$, $m' = 2k'$, and the cliques of $\overline{C}_m$ and $\overline{C}_{m'}$ are isomorphic to K_k and $K_{k'}$, respectively. For each $k > k'$, it is easy to see that $\overline{C}_m \not\to \overline{C}_{m'}$. Case $k = k'$. Now the subgraphs $\langle\{1, 3, \ldots, m-2\}\rangle$ and $\langle\{2, 4, \ldots, m-1\}\rangle$ of $\overline{C}_m$ are isomorphic to K_k, the cliques of $\overline{C}_{m'}$ are only $\langle\{1, 3, \ldots, m'-1\}\rangle$ and $\langle\{2, 4, \ldots, m'\}\rangle$ are also isomorphic to K_k and $V(\overline{C}_{m'}) = \{1, 3, \ldots, m'-1\} \cup \{2, 4, \ldots, m'\}$. Let $f \in Hom(\overline{C}_m, \overline{C}_{m'})$. Then $f(\{1, 3, \ldots, m-2\})$ and $f(\{2, 4, \ldots, m-1\})$ is $\{1, 3, \ldots, m'-1\}$ or $\{2, 4, \ldots, m'\}$.

1) If $f(\{1, 3, \ldots, m-2\}) = f(\{2, 4, \ldots, m-1\}) = \{1, 3, \ldots, m'-1\}$, then $f(1) = f(2), f(3) = f(4), \ldots, f(m-2) = f(m-1)$ and $f(m) \notin \{1, 3, \ldots, m'-1\}$. Let $f(m) = 2x$ where $2x \in \{2, 4, \ldots, m'\}$. Since $\{m, y\} \in E(\overline{C}_m)$ for all $y \in \{2, 3, \ldots, m-2\}$, we get $f(y) \notin \{2x-1, 2x+1\}$, which is a contradiction.
2) If $f(\{1, 3, \ldots, m-2\}) = \{1, 3, \ldots, m'-1\}$ and $f(\{2, 4, \ldots, m-1\}) = \{2, 4, \ldots, m'\}$, assume that $f(1) = 1$, then $f(2) = 2, f(3) = 3, \ldots, f(m-1) = m'$. But $\{1, m-1\} \in E(\overline{C}_m)$, which is a contradiction.

For the cases $f(\{1, 3, \ldots, m-2\}) = f(\{2, 4, \ldots, m-1\}) = \{2, 4, \ldots, m'\}$, and $f(\{1, 3, \ldots, m-2\}) = \{2, 4, \ldots, m'\}$ and $f(\{2, 4, \ldots, m-1\}) = \{1, 3, \ldots, m'-1\}$, the proofs are similar to (1) and (2), respectively.

Sufficiency. Let $m < m'$. Then there exists a homomorphism f from $\overline{C}_m$ which is an embedding $\overline{C}_{m'}$ by $f|_{\overline{C}_m}$ which is the identity map, where $f|_{\overline{C}_m}$ denotes the restriction of f to $\overline{C}_m$. □

Corollary 2.36. *Let G be an $(n-3)$-regular graph of order n, $f \in \mathrm{End}(G)$, and let G contain the induced subgraphs $\overline{C}_m$ and $\overline{C}_{m'}$. If $f(\overline{C}_m) \subseteq \overline{C}_{m'}$, then $m \leqslant m'$.*

Lemma 20. *Let G be an $(n-3)$-regular graph of order $n, f \in \mathrm{End}(G)$ and let G contain induced subgraph $\overline{C}_{2m+1}$. If $f(\overline{C}_{2m+1}) = X$, then $\langle X\rangle \cong \overline{C}_{2m+1}$.*

Proof. Suppose that $\omega(G) = s$. If $\langle X\rangle \ncong \overline{C}_{2m+1}$, then $\omega(\langle X\rangle) = m'$ and $m < m'$. Thus $\omega(G \setminus \overline{C}_{2m+1}) = s - m$ and $\omega(G \setminus X) = s - m'$. But since $s - m > s - m'$, it is impossible that $f|_{G\setminus\overline{C}_{2m+1}}$ is a homomorphism from $G \setminus \overline{C}_{2m+1}$ to $G \setminus X$. □

Let G be an $(n-3)$-regular graph of order n. Denote by G_x and G_E set of all induced subgraphs $\overline{C}_x$ and $\overline{C}_{2m}$ of G, respectively. Note that $G_x = \varnothing$, if G does not contain an induced subgraph $\overline{C}_x$. From Lemma 20, we get that

Proposition 3. *Let G be an $(n-3)$-regular graph of order n. Then*

(1) $|\operatorname{End}(G)| = |\operatorname{End}(G_E)| \times |\operatorname{End}(G_3)| \times |\operatorname{End}(G_5)| \times |\operatorname{End}(G_7)| \times \ldots,$ *if $G_x \neq \varnothing$, for all $x = 3, 5, 7, \ldots,$*

(2) $\operatorname{End}(G_3) \cong S_{m_1} \times T_3$, *where $|G_3| = m_1$, and*

(3) *for each odd integer $x \geqslant 5$, $\operatorname{End}(G_x) = Aut(G_x) \cong S_{m_2} \times D_x$, where $|G_x| = m_2$.*

Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ be an $(n-3)$-regular graph of order n. Set $O_i = \{1_i, 3_i, \ldots, (2m_i - 1)_i\}$ and $E_i = \{2_i, 4_i, \ldots, (2m_i)_i\}$. Denote $S_{X_1,X_2,\cdots,X_r} = X_1 \cup X_2 \cup \ldots \cup X_r$ where $X_i \in \{O_i, E_i\}$, $1 \leqslant i \leqslant r$.

Lemma 21. *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ be an $(n-3)$-regular graph of order n. There exist exactly $2r$ maximal complete subgraphs K_m of G, where $n = 2m$.*

Proof. Since $V(\overline{C}_{(2m_i)_i}) = O_i \cup E_i$ and $\langle O_i\rangle, \langle E_i\rangle$ are isomorphic to K_{m_i}, for each $X_i \in \{O_i, E_i\}$ and $S_{X_1,X_2,\cdots,X_r} = X_1 \cup X_2 \cup \ldots \cup X_r$, $\langle S\rangle$ is a complete subgraph K_m, where $m = m_1 + \ldots + m_r$. From $|\{S_{X_1,X_2,\cdots,X_r} | S_{X_1,X_2,\cdots,X_r} = X_1 \cup X_2 \cup \ldots \cup X_r\}| = 2r$, we get that there exist exactly $2r$ maximal complete subgraphs K_m of G such that $n = 2m$. □

Corollary 2.37. *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ and $f \in \operatorname{End}(G)$. Then $f(S_{O_1,\cdots,O_r}) = X_1 \cup \ldots \cup X_r$ and $f(S_{E_1,\cdots,E_r}) = Y_1 \cup \ldots \cup Y_r$ where $X_i, Y_i \in \{O_i, E_i\}$, $1 \leqslant i \leqslant r$.*

Lemma 22. *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ and $f \in \operatorname{End}(G)$. If $O_i, E_i \in f(G)$ for some $1 \leqslant i \leqslant r$, then $f(\overline{C}_{(2m_j)_j}) = \overline{C}_{(2m_i)_i}$ and $m_j = m_i$ for some $1 \leqslant j \leqslant r$.*

Proof. If $f(\overline{C}_{(2m_j)_j}) \nsubseteq \overline{C}_{(2m_i)_i}$ for all $j \in \{1, \ldots, r\}$, then there exists $m_k \neq m_l$, $x_k \in V(\overline{C}_{(2m_k)_k})$ and $y_l \in V(\overline{C}_{(2m_l)_l})$ such that $f(x_k) = (2z+1)_i \in O_i$ and $f(y_l) = (2z)_i \in E_i$. But $\{x_k, y_l\} \in E(G)$, which is a contradiction. Thus $f(\overline{C}_{(2m_j)_j}) \subseteq \overline{C}_{(2m_i)_i}$ for some $j \in \{1, \ldots, r\}$. From Corollary 2.36, we get $m_j \leqslant m_i$. Then there exists $x_i, (x+1)_i \in V(\overline{C}_{(2m_i)_i})$ such that $x_i = f(y_j) \in f(\overline{C}_{(2m_j)_j})$ but $(x+1)_i \notin f(\overline{C}_{(2m_j)_j})$. Suppose $(x+1)_i = f(x'_k) \in f(\overline{C}_{(2m_k)_k})$. Now $\{y_j, x'_k\} \in E(G)$ is contradiction to $\{x_i, (x+1)_i\} \notin E(G)$. Therefore, $m_j = m_i$. □

Lemma 23. *Let* $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ *and* $f \in \mathrm{End}(G)$. *If* $f(x_i) = f((x+1)_i)$ *for some* $x_i \in V(\overline{C}_{2m_i})$, *then*

(1) $f(1_i) = f(2_i), f(3_i) = f(4_i), \ldots, f((2m_i - 1)_i) = f((2m_i)_i)$, *if* x *is odd, or*

(2) $f((2m_i)_i) = f(1_i), f(2_i) = f(3_i), \ldots, f((2m_i - 2)_i) = f((2m_i - 1)_i)$, *if* x *is even.*

Proof. Now the cliques of G are isomorphic to K_m, where $m = m_1 + \ldots + m_r$.

(1) Let x be odd integer and $x \neq 1$. Suppose that $f(1_i) \neq f(2_i)$. If $\{f(1_i), f(2_i)\} \notin E(G)$, then $f(1_i) = y_j$ and $f(2_i) = (y+1)_j$, for some $y_j, (y+1)_j \in V(\overline{C}_{(2m_j)_j})$. But $f(x_i) = f((x+1)_i)$ is a contradiction to Lemma 22.

Then either (1) $\{x_i, 1_i\} \in E(G)$ and $\{x_i, 2_i\} \in E(G)$, or (2) $\{(x+1)_i, 1_i\} \in E(G)$ and $\{(x+1)_i, 2_i\} \in E(G)$, or both. Thus the cliques of $\langle\{f(x_i), f((x+1)_i), f(1_i), f(2_i)\}\rangle$ are isomorphic to K_3. Now the cliques of $\langle\{x_i, (x+1)_i, 1_i, 2_i\}\rangle$ are isomorphic to K_2. Therefore, $G \setminus K_2 \to G \setminus K_3$.

Consider $\omega(G \setminus K_2) = m - 2$ and $\omega(G \setminus K_3) = m - 3$, respectively. But since $m - 2 > m - 3$, it is impossible that $f|_{G \setminus K_2}$ is a homomorphism from $G \setminus K_2$ to $G \setminus K_3$.

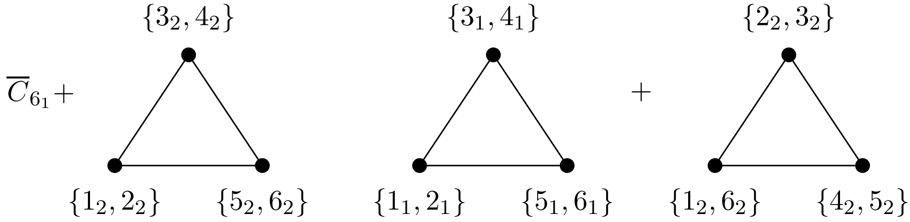
For the case $x = 1$ and the case x is even integer, the proof is similar to case (1). □

Example 4. Let again $f : V(\overline{C}_6) \to V(\overline{C}_6)$ be defined by $f = \binom{1\,2\,3\,4\,5\,6}{1\,1\,3\,3\,5\,5} \in \mathrm{End}'(\overline{C}_6)$. The congruence relation ϱ_f can be exemplified as in Figure 1a. There is exactly one other congruence relation which is exemplified in Figure 1b. By the homomorphism theorem each endomorphism in specified by one of these 2 congruence relations and the possible embedding of the K_m into $\overline{C}_{2m}$.

Then let $G \cong \overline{C}_{6_1} + \overline{C}_{6_2}$ and $f : V(G) \to V(G)$ be defined by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 1_2 & 3_2 & 3_2 & 5_2 & 5_2 \end{pmatrix} \in \mathrm{End}'(G).$$

The defining congruence relation ϱ_f can be exemplified as $\overline{C}_{6_1}+$ Figure 1a (see Figure 3a). From Lemma 22 and Lemma 23, and by the homomorphism theorem each endomorphism in specified by one of these 8 congruence relation and the possible embedding of each case into G, as follows $\overline{C}_{6_1}+$ Figure 1b, Figure 1a$+\overline{C}_{6_2}$, Figure 1b$+\overline{C}_{6_2}$, Figure 1a+Figure 1a, Figure 1a+Figure 1b (see Figure 3b), Figure 1b+Figure 1a and Figure 1b+Figure 1b.



(a) $\overline{C}_6 + K_3$

(b) $K_3 + K_3$

FIGURE 3. $\overline{C}_6 + K_3$ and $K_3 + K_3$

Lemma 24. *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ and $f : V(G) \to V(G)$. Then $f \in \mathrm{End}(G)$ if and only if f satisfies:*

(1) *If $f(x_i) = f(y_i)$ for some two different elements $x_i, y_i \in V(\overline{C}_{n_i})$, then $y_i = (x-1)_i$ or $y_i = (x+1)_i$.*

(2) *$f(S_{O_1,\cdots,O_r}) = X_1 \cup \ldots \cup X_r$ and $f(S_{E_1,\cdots,E_r}) = Y_1 \cup \ldots \cup Y_r$ where $X_i, Y_i \in \{O_i, E_i\}$, $1 \leqslant i \leqslant r$.*

(3) *If $O_i, E_i \in f(G)$ for some $1 \leqslant i \leqslant r$, then $f(\overline{C}_{(2m_j)_j}) = \overline{C}_{(2m_i)_i}$ and $m_j = m_i$ for some $1 \leqslant j \leqslant r$.*

(4) *If $f(x_i) = f((x+1)_i)$ for some $x_i \in V(\overline{C}_{2m_i})$, then*

(4.1) *$f(1_i) = f(2_i), f(3_i) = f(4_i), \ldots, f((2m_i-1)_i) = f((2m_i)_i)$, if x is odd, or*

(4.2) *$f((2m_i)_i) = f(1_i), f(2_i) = f(3_i), \ldots, f((2m_i-2)_i) = f((2m_i-1)_i)$, if x is even.*

Proof. Necessity. Obvious from Lemma 18, Corollary 2.37, Lemma 22 and Lemma 23.

Sufficiency. Obvious from Example 4. □

Lemma 25. *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ and $f \in \mathrm{End}(G)$. Then f is an idempotent if and only if $f|_{\overline{C}_{(2m_i)_i}}$ is also idempotent in $\overline{C}_{(2m_i)_i}$, for all $i = 1, \ldots, r$.*

Example 5. For the monoid $\mathrm{End}(\overline{C}_{6_1} + \overline{C}_{6_2})$, we choose following

$$Idpt(\mathrm{End}(\overline{C}_{6_1} + \overline{C}_{6_2})) =$$

$$\Big\{ id_1 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \end{pmatrix} \in D_{6_1} \times D_{6_2},$$

$$id_2 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 1_2 & 3_2 & 3_2 & 5_2 & 5_2 \end{pmatrix} \in D_{6_1} \times S^{or}_{3_2},$$

$$id_3 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 2_2 & 2_2 & 4_2 & 4_2 & 6_2 & 6_2 \end{pmatrix} \in D_{6_1} \times S^{er}_{3_2},$$

$$id_4 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 3_2 & 3_2 & 5_2 & 5_2 & 1_2 \end{pmatrix} \in D_{6_1} \times S^{ol}_{3_2},$$

$$id_5 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 6_2 & 2_2 & 2_2 & 4_2 & 4_2 & 6_2 \end{pmatrix} \in D_{6_1} \times S^{el}_{3_2},$$

$$id_6 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 1_1 & 3_1 & 3_1 & 5_1 & 5_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \end{pmatrix} \in S^{or}_{3_1} \times D_{6_2},$$

$$id_7 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 2_1 & 2_1 & 4_1 & 4_1 & 6_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \end{pmatrix} \in S^{er}_{3_1} \times D_{6_2},$$

$$id_8 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 3_1 & 3_1 & 6_1 & 5_1 & 1_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \end{pmatrix} \in S^{ol}_{3_1} \times D_{6_2},$$

$$id_9 = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 6_1 & 2_1 & 2_1 & 4_1 & 4_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \end{pmatrix} \in S^{el}_{3_1} \times D_{6_2},$$

$$id_{10} = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 1_1 & 3_1 & 3_1 & 5_1 & 5_1 & 1_2 & 1_2 & 3_2 & 3_2 & 5_2 & 5_2 \end{pmatrix} \in S^{or}_{3_1} \times S^{or}_{3_2}, \ldots,$$

$$id_{25} = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 6_1 & 2_1 & 2_1 & 4_1 & 4_1 & 6_1 & 6_2 & 2_2 & 2_2 & 4_2 & 4_2 & 6_2 \end{pmatrix} \in S^{el}_{3_1} \times S^{el}_{3_2} \Big\}.$$

Let $f \in \mathrm{End}'(\overline{C}_{6_1} + \overline{C}_{6_2})$ be define by

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 & 2_1 & 2_1 & 4_1 & 4_1 & 6_1 & 6_1 \end{pmatrix}.$$

Then there exist unique idempotents id_3 and id_7 in $\mathrm{End}(\overline{C}_{6_1} + \overline{C}_{6_2})$ such that $\rho_f = \rho_{id_3}$ and $I_f = I_{id_7}$, respectively. So, f satisfies Lemma 4, i.e. f is regular, but for each idempotent $id_k \in E(\mathrm{End}(\overline{C}_{6_1} + \overline{C}_{6_2}))$, $k = 1, \ldots, 25$, has no both $\rho_f = \rho_{id_k}$ and $I_f = I_{id_k}$. Thus f does not satisfy Lemma 3, i.e. $f \notin H_{id_k}$.

Proposition 4. *Let* $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$. *Then* End$(G)$ *is regular.*

Proof. Let $f \in \mathrm{End}(G)$. Then f satisfies (1)–(4) in Lemma 24.

Define $e_1 : V(G) \to V(G)$ by $e_1(\overline{C}_{(2m_i)_i}) = X_i \cup Y_i$ such that

1) if $X_i = Y_i = O_i$, then $e_1(1_i) = e_1(2_i) = 1_i, e_1(3_i) = e_1(4_i) = 3_i, \ldots, e_1((2m_i - 1)_i) = e_1((2m_i)_i) = (2m_i - 1)_i$,
2) if $X_i = Y_i = E_i$, then $e_1(1_i) = e_1(2_i) = 2_i, e_1(3_i) = e_1(4_i) = 4_i, \ldots, e_1((2m_i - 1)_i) = e_1((2m_i)_i) = (2m_i)_i$, and
3) if $X_i \neq Y_i$, then $e_1|_{\overline{C}_{(2m_i)_i}}$ is the identity map.

Thus e_1 is idempotent in End(G) and $I_f = I_{e_1}$.

Define $e_2 : V(G) \to V(G)$ by

1) $e_2|_{\overline{C}_{(2m_j)_j}}$ is the identity map, if $O_i, E_i \in f(G)$ for some $1 \leqslant i \leqslant r$, then $f(\overline{C}_{(2m_j)_j}) = \overline{C}_{(2m_i)_i}$ and $m_j = m_i$ for some $1 \leqslant j \leqslant r$, and
2) $e_2(x_i) = e_2((x+1)_i) = x_i, e_2((x+2)_i) = e_2((x+3)_i) = (x+2)_i, \ldots, e_2((x-2)_i) = e_2((x-1)_i) = (x-2)_i$ (with addition modulo $2m_i$), if $f(x_i) = f((x+1)_i)$ for some $x_i \in V(\overline{C}_{2m_i})$.

Thus e_2 is idempotent in End(G) and $\rho_f = \rho_{e_1}$.

Thus, $I_f = I_{e_1}$ and $\rho_f = \rho_{e_2}$. From Lemma 4, we get that f is regular. □

Lemma 26. *Let* $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$. *If* $m_i = m_j$ *for some* $i, j \in \{1, \ldots, r\}$, *then there exist* $f \in \mathrm{End}(G)$ *such that* $f \notin H_e$ *for all* $e \in E(\mathrm{End}(G))$.

Proof. Suppose $m_1 = m_2$ are even. Let $f \in \mathrm{End}(\overline{C}_{m_1} + \overline{C}_{m_2})$ be defined by

$$f(x_i) = \begin{cases} x_2, & i = 1; \\ x_1, & x \text{ is odd and } i = 2; \\ (x-1)_1, & x \text{ is even and } i = 2. \end{cases}$$

Then $I_f \cong \overline{C}_{m_2} + K_m$, where $V(K_m) = \{1_1, 3_1, \ldots, (m_1 - 1)_1\}$, and for each $x \neq y$, $(x_i, y_i) \in \rho_f \Leftrightarrow i = 2$ and $x_2 = (y-1)_2$. Let e be the idempotent in End$(\overline{C}_{m_1} + \overline{C}_{m_2})$ such that $I_f = I_e$ and $\rho_f = \rho_e$. From Lemma 25, we get that $e|_{\overline{C}_{m_2}}$ is the identity map. Since $(x_2, (x+1)_2) \in \rho_e$ for all $x_2, (x+1)_2 \in V(\overline{C}_{m_2})$, this is a contradiction. □

Lemma 27. *Let* $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_r)_r}$ *and* $f \in \mathrm{End}(G)$. *If* $m_i \neq m_j$ *for all* $i, j \in \{1, \ldots, r\}$, *then* $f \in H_e$ *for some* $e \in E(\mathrm{End}(G))$.

Proof. Let $f \in \mathrm{End}(G)$. From Proposition 4, f is regular.

Define $e : V(G) \to V(G)$ by $e(\overline{C}_{(2m_i)_i}) = X_i \cup Y_i$ such that

1) if $X_i \neq Y_i$, then $e|_{\overline{C}_{(2m_i)_i}}$ is the identity map,

2) if $X_i = Y_i = O_i$, then

(2.1) $e(x_i) = e((x+1)_i) = x_i, e((x+2)_i) = e((x+3)_i) = (x+2)_i, \ldots, e((x-2)_i) = e((x-1)_i) = (x-2)_i$ (with addition modulo $2m_i$), if $f(x_i) = f((x+1)_i)$ for some odd integer $x_i \in V(\overline{C}_{2m_i})$, or

(2.2) $e(x_i) = e((x+1)_i) = (x+1)_i, e((x+2)_i) = e((x+3)_i) = (x+3)_i, \ldots, e((x-2)_i) = e((x-1)_i) = (x-1)_i$ (with addition modulo $2m_i$), if $f(x_i) = f((x+1)_i)$ for some even integer $x_i \in V(\overline{C}_{2m_i})$, and

3) if $X_i = Y_i = E_i$, then

(3.1) $e(x_i) = e((x+1)_i) = x_i, e((x+2)_i) = e((x+3)_i) = (x+2)_i, \ldots, e((x-2)_i) = e((x-1)_i) = (x-2)_i$ (with addition modulo $2m_i$), if $f(x_i) = f((x+1)_i)$ for some even integer $x_i \in V(\overline{C}_{2m_i})$, or

(3.2) $e(x_i) = e((x+1)_i) = (x+1)_i, e((x+2)_i) = e((x+3)_i) = (x+3)_i, \ldots, e((x-2)_i) = e((x-1)_i) = (x-1)_i$ (with addition modulo $2m_i$), if $f(x_i) = f((x+1)_i)$ for some odd integer $x_i \in V(\overline{C}_{2m_i})$.

Thus, $I_f = I_e$ and $\rho_f = \rho_e$. From Lemma 3, we get $f \in H_e$. □

Theorem 3. *Let G be an $(n-3)$-regular graph of order n. Then*

(1) *G is endo-regular, and*

(2) *G is endo-completely-regular if and only if $|G_3| = 0$ and $|G_{2m}| = 1$ for all induced subgraphs $\overline{C}_{2m}$ of G.*

Proof. (1) Since $\mathrm{End}(G_3)$ is regular and for each odd integer $x \geqslant 5$, $\mathrm{End}(G_x) = Aut(G_x)$ - form group - and from Proposition 3, $\mathrm{End}(G)$ is regular, if $\mathrm{End}(G_E)$ is also regular. Since $\mathrm{End}(G_E)$ is always regular, from Proposition 4, we get that G is endo-regular.

(2) Since $\mathrm{End}(G_3)$ is not completely regular and for each odd integer $x \geqslant 5, \mathrm{End}(G_x) = Aut(G_x)$ - form group - and from Proposition 3, $\mathrm{End}(G)$ is completely regular if and only if $|G_3| = 0$ and $\mathrm{End}(G_E)$ is completely regular. From Lemma 26 and Lemma 27, G_E is endo-completely-regular if and only if $|G_{2m}| = 1$ for all induced subgraphs $\overline{C}_{2m}$ of G. □

Remark 2.38. A regular monoid S is called orthodox, if the set of all idempotents from a submonoid of S. Since T_3 is not orthodox, and from [8], $\overline{C}_{2m}$ also is not endo-orthodox, we get that the retractive $(n-3)$-regular graph of order n is not endo-orthodox.

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Contact information

N. Pipattanajinda Program of Mathematics, Faculty of Science and Technology, Kamphaeng Phet Rajabhat University, Kamphaeng Phet 62000, THAILAND
E-Mail(s): nirutt.p@gmail.com

U. Knauer Institut für Mathematik, Carl von Ossietzky Universität, D-26111 Oldenburg, GERMANY
E-Mail(s): ulrich.knauer@uni-oldenburg.de

B. Gyurov School of Science and Technology, Georgia Gwinnett College, University System of Georgia, Lawrenceville, GA 30043, USA
E-Mail(s): bgyurov@ggc.edu

S. Panma Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai 50200, THAILAND
E-Mail(s): panmayan@yahoo.com

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