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On size effect of elastic modules in thin fibres

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In the paper the dependence of elasticity modules in thin fibres on the fibre radius is investigated. The relations of local gradient approach in thermomechanics are taken as a starting point. The approach is based on including of a chemical potential gradient in the space of state parameters along with the stress and strain tensors. The chemical potential disturbance is identified with disturbance of the bonding energy. It is shown that the dependence of elasticity modules on the fibre radius is essential in the fibres of radius less than ten times of the size of nearsurface nonhomogeneity region. Using asymptotic expansions of modified Bessel functions the Young's modulus, Poisson's ratio, Lamé's parameters, and bulk modulus at large fibre radius are studied. The agreement of the obtained results with known in the literature theoretical and experimental data is shown.

Keywords: local gradient approach, thin fibres, size effect of elasticity modules.

Introduction. Recently the scientific literature pays a considerable attention to the modelling, description and study of the properties of solids, distinguished by various size effects. Such solids feature comparable contributions of surface and volume factors to internal energy and one of their geometrical sizes (further — characteristic size) is comparable to the size of the region of nearsurface nonhomogeneity. The properties of the solids essentially differ from the properties of solids without such effects. Such solids include nanosized elements, which are bases for construction of the structures (nanomaterials) allowing to build enhanced devices and substances of improved properties [1, 2]. At present nanoelements and nanomaterials are widely used in electronics and nanobiotechnology.

One of the effective approaches to investigation of the stressed-strained state in solids with significant interface phenomena, is the local gradient approach in thermomechanics [3, 4]. In this paper the equation set of the model built at such an approach is used to describe and study the elasticity modules in thin fibres. Under thin fibres we understand fibres the radius of which is comparable with the size of the nearsurface heterogeneity region.

1. The stressed-strained state of the cylinder

Choosing as key functions the stress tensor $\hat{\sigma}$ and chemical potential disturbance η, the solving equation set of the model for description of the steady state of a solid for local gradient approach can be written in the form [4, 5]

$$\nabla^2\eta-\kappa_\eta^2\eta-\kappa_\sigma^2\sigma=0\,,\qquad \vec{\nabla}\cdot\hat{\sigma}=0\,,$$
$$\vec{\nabla}\times\left[\left(3a_\lambda+2a_\mu\right)\hat{\sigma}-\left(a_\lambda\sigma+2a_\mu a_{eh}\eta\right)\hat{I}\right]\times\vec{\nabla}=0\,. \qquad (1)$$

Here $a_\lambda, a_\mu, a_{eh}, \kappa_\eta, \kappa_\sigma$ are constants, $\hat{I}$ is identity tensor, $\sigma=\hat{\sigma}:\hat{I}$.

This set of equations is applied to study of the stressed-strained state of a cylinder under action of the stretching loading. We consider the infinite circular cylinder of radius R, which in the cylindrical system of coordinates $\{r,\phi,z\}$ occupies a region $r\le R$ and is loaded at infinity by stretching force intensity p in direction of axis *Oz*. We consider that lateral surface of the cylinder is free of a force loading and the value of chemical potential at the surface is constant and equal $\eta_a\ne 0$.

Writing equations (1) in cylindrical coordinates for nonzero components of stress tensor and chemical potential disturbance with account for the problem geometry we obtain

$$\frac{d\sigma_z}{dr}=\frac{d}{dr}\left(\frac{a_\lambda\sigma+2a_\mu a_{eh}\eta}{3a_\lambda+2a_\mu}\right),\; r\frac{d^2\sigma_\phi}{dr^2}+2\frac{d\sigma_\phi}{dr}-\frac{d\sigma_r}{dr}=\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{a_\lambda\sigma+2a_\mu a_{eh}\eta}{3a_\lambda+2a_\mu}\right)\right),$$
$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\phi}{r}=0\,,\qquad \frac{d^2\eta}{dr^2}+\frac{1}{r}\frac{d\eta}{dr}-\kappa_\eta^2\eta-\kappa_\sigma^2\sigma=0\,. \qquad (2)$$

The necessary conditions are

$$\eta\big|_{r=R}=\eta_a\,,\qquad \sigma_r\big|_{r=R}=0\,,\qquad \frac{1}{\pi R^2}\int_0^{2\pi}d\phi\int_0^R r\sigma_z dr=p\,, \qquad (3)$$

and the conditions of solution finiteness in the region of a body are written as

$$|\,\eta,\sigma_r,\sigma_\phi,\sigma_z\,|_{r\le R}<+\infty\,.$$

The problem (2), (3) solution takes the form

$$\eta(r)=\eta_a+\frac{\chi}{\zeta_c}\left(\frac{I_0(\xi r)}{I_0(\xi R)}-1\right),\qquad \sigma_r(r)=\frac{b_m}{2}\frac{\chi}{\zeta_c}\left(\frac{I_1(\xi r)}{\xi r\,I_0(\xi R)}-\frac{I_1(\xi R)}{\xi R\,I_0(\xi R)}\right),$$
$$\sigma_\phi(r)=\frac{b_m}{2}\frac{\chi}{\zeta_c}\left(\frac{I_0(\xi r)}{I_0(\xi R)}-\frac{I_1(\xi r)}{\xi r\,I_0(\xi R)}-\frac{I_1(\xi R)}{\xi R\,I_0(\xi R)}\right),$$
$$\sigma_z(r)=p+\frac{b_m}{2}\frac{\chi}{\zeta_c}\left(\frac{I_0(\xi r)}{I_0(\xi R)}-\frac{2I_1(\xi R)}{\xi R\,I_0(\xi R)}\right). \qquad (4)$$

Here I_0, I_1 are modified Bessel functions, $\zeta_c=1-D\left(1-\frac{2I_1(\xi R)}{\xi R\,I_0(\xi R)}\right)$, $D=b_m\frac{\kappa_\sigma^2}{\xi^2}$,

$\xi^2=\kappa_\eta^2+b_m\kappa_\sigma^2$, $b_m=\frac{4a_\mu a_{eh}}{a_\lambda+2a_\mu}$, $\chi=\frac{\kappa_\sigma^2}{\xi^2}p+\frac{\kappa_\eta^2}{\xi^2}\eta_a$.

From the solution it is seen that in the case of the force load absence ($p = 0$) the stressed-strained state of the cylinder is conditioned by the nonzero value of chemical potential at the body surface. It is shown in [4] that disturbance of chemical potential can be identified with disturbance of bonding energy. For the cylinders of large radius ($\xi R >> 1$) stresses are localized in a narrow nearsurface region, while in the thin fibres stresses are substantial in the whole region of the body.

2. Size effect of elastic modules

To find the Young's modulus and Poisson's ratio the definition is used which in the considered case can be written as

$$E = \frac{p}{e_z}, \qquad \nu = -\frac{e_{trans}}{e_{axial}}, \tag{5}$$

where e_{trans} is deformation in longitudinal direction, e_{axial} is deformation in radial direction.

Within the framework of the considered model for the strain tensor the following formula is held

$$\hat{e} = \frac{\hat{\sigma}}{2a_\mu} - \frac{a_\lambda \sigma + 2a_\mu a_{eh} \eta}{2a_\mu \left(3a_\lambda + 2a_\mu\right)} \hat{I} .$$

On this base for e_z it is written

$$e_z = \frac{a_\lambda + a_\mu}{a_\mu \left(3a_\lambda + 2a_\mu\right)} p - \frac{a_{eh}}{3a_\lambda + 2a_\mu} \eta_a + \frac{b_m \left(a_\lambda + 2a_\mu\right)}{4a_\mu \left(3a_\lambda + 2a_\mu\right)} \frac{\chi}{\zeta_c} \left(1 - \frac{2I_1(\xi R)}{\xi R I_0(\xi R)}\right). \tag{6}$$

Taking into account the state of body in the case of the force load absence for the modules E, ν is obtained from (5), (6)

$$E = \left(3a_\lambda + 2a_\mu\right) \Big/ \left[\frac{a_\lambda + a_\mu}{a_\mu} + \Psi(\xi R)\right],$$

$$\nu = \left(\frac{a_\lambda}{2a_\mu} - \Psi(\xi R)\right) \Big/ \left[\frac{a_\lambda + a_\mu}{a_\mu} + \Psi(\xi R)\right], \tag{7}$$

where $\Psi(\xi R) = \frac{a_\lambda + 2a_\mu}{4a_\mu} D \left(1 - \frac{2I_1(\xi R)}{\xi R I_0(\xi R)}\right) \Big/ \left[1 - D\left(1 - \frac{2I_1(\xi R)}{\xi R I_0(\xi R)}\right)\right]$.

The reduced Young's modulus E_{pr}

$$E_{pr} = \frac{E}{E_\infty}, \qquad E_\infty = \left(3a_\lambda + 2a_\mu\right) \Big/ \left[\frac{a_\lambda + a_\mu}{a_\mu} + \frac{a_\lambda + 2a_\mu}{4a_\mu} \frac{D}{1 - D}\right],$$

dependence on a cylinder radius are illustrated by the graphs in Fig. 1.

It is clear from the graphs, that module E dependence on ξR can be substantial in thin fibres. The obtained dependence is in accordance with known results of experimental investigation of the size effect of the elastic modulus [6, 7].

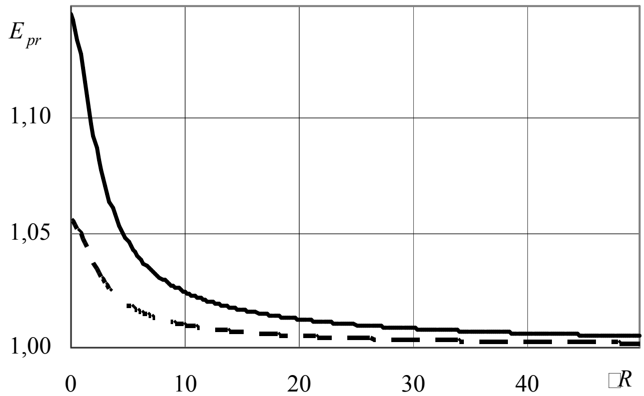

Fig. 1. Young modulus E_{pr} dependence on a cylinder radius for a_λ / a_μ = 1,25; D = 0,1 (dashed line); D = 0,2 (solid line)

In scientific literature next to the modules E, ν the Lamé's parameters λ, μ, shear G and bulk K modules are used. These parameters are related to E, ν by formulas

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}. \tag{8}$$

From (7), (8), for these modules we get

$$\mu = a_\mu, \quad \lambda = \frac{a_\lambda - 2a_\mu \Psi(\xi R)}{1+3\Psi(\xi R)},$$

$$G = a_\mu, \quad K = \left(a_\lambda + \frac{2}{3}a_\mu\right) \Big/ \left[1+3\Psi(\xi R)\right]. \tag{9}$$

Note that within the accepted approximation (particularly linear state equation and geometric linearity) of the model the elasticity modules do not rely on the surface value of chemical potential η_a.

3. Approximation for «thick» fibres

Using asymptotic presentation of the modified Bessel functions [8]

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{k-1}{8z} + \frac{(k-1)(k-9)(k-25)}{3!(8z)^3} + \ldots\right], \quad k = 4n^2, \quad |z| >> 1,$$

for fibres of the radius satisfying inequality $\xi R >> 1$, keeping the first terms in expansions over parameter $(\xi R)^{-1}$ in formulas (7), (9) we get

$$E(\xi R) = E_\infty\left(1 + \frac{d_E}{\xi R}\right), \quad \nu(\xi R) = \nu_\infty\left(1 + \frac{d_\nu}{\xi R}\right),$$

$$\lambda(\xi R) = \lambda_\infty\left(1 + \frac{d_\lambda}{\xi R}\right), \quad K(\xi R) = K_\infty\left(1 + \frac{d_K}{\xi R}\right), \tag{10}$$

where

$$E_\infty = \left(3a_\lambda + 2a_\mu\right)\left(\frac{a_\lambda + a_\mu}{a_\mu} + \Psi_\infty\right)^{-1}, \quad \nu_\infty = \left(\frac{a_\lambda}{2a_\mu} - \Psi_\infty\right)\left(\frac{a_\lambda + a_\mu}{a_\mu} + \Psi_\infty\right)^{-1},$$

$$\lambda_\infty = \frac{a_\lambda - 2a_\mu \Psi_\infty}{1 + 3\Psi_\infty}, \quad K_\infty = \frac{a_\lambda + 2a_\mu/3}{1 + 3\Psi_\infty}, \quad \Psi_\infty = \frac{a_\lambda + 2a_\mu}{4a_\mu}\frac{D}{1-D},$$

$$d_E^{-1} = \frac{1-D}{2}\left(1 + \frac{a_\lambda + a_\mu}{a_\mu \Psi_\infty}\right), \quad d_\nu = \frac{2\Psi_\infty}{1-D}\left[\left(\frac{a_\lambda}{2a_\mu} - \Psi_\infty\right)^{-1} + \left(\frac{a_\lambda + a_\mu}{a_\mu} + \Psi_\infty\right)^{-1}\right],$$

$$d_\lambda = \frac{2\left(3a_\lambda + 2a_\mu\right)\Psi_\infty}{(1-D)\left(a_\lambda - 2a_\mu\Psi_\infty\right)\left(1 + 3\Psi_\infty\right)}, \quad d_K = \frac{6\Psi_\infty}{(1-D)\left(1+3\Psi_\infty\right)}.$$

With the growth of the cylinder radius, as easily seen from (10), the value E, ν, λ, K tends to $E_\infty, \nu_\infty, \lambda_\infty, K_\infty$ accordingly, and these last ones are possible to be interpreted as elasticity modules for massive bodies.

The obtained dependences of the elasticity modules on the characteristic size of the body (radius of fibre) conform to the known in literature results of theoretical and experimental researches [6, 9, 10].

Conclusions. Local gradient approach in thermomechanics allows describing the size effect of the elasticity modules. Within the framework of accepted in the model approximation the elasticity modules do not rely on the surface value of chemical potential. For thin fibres which characteristic radius does not exceed ten sizes of the region of nearsurface heterogeneity the dependence of the elasticity modules on the radius of fibre is substantial. It is indicated that the obtained results agree well with those known in literature.

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До опису розмірного ефекту пружних модулів у тонких волокнах

Тарас Нагірний, Костянтин Червінка

У роботі вивчено залежність модулів пружності тонких волокон від їх радіуса. За основу прийнято співвідношення локально градієнтного підходу у термомеханіці. Цей підхід базується на впровадженні у простір параметрів стану поряд із тензорами напружень та деформацій також градієнта хімічного потенціалу. При цьому збурення хімічного потенціалу ототожнюється зі збуренням енергії взаємодії. Показано, що залежність модулів пружності від радіуса волокна є суттєва у волокнах, радіус яких є менший від десяти розмірів області приповерхневої неоднорідності. Використовуючи асимптотичні розвинення модифікованих функцій Беселя, досліджено поведінку модуля Юнга, коефіцієнтів Пуассона, параметрів Ляме та модуля об'ємного стиску за зростання радіуса волокна. Вказано на узгодженість одержаних результатів із відомими у літературі теоретичними й експериментальними даними.

О размерном эффекте упругих модулей в тонких волокнах

Тарас Нагирный, Константин Червинка

В работе изучена зависимость модулей упругости тонких волокон от их радиуса. За основу принято соотношения локально градиентного подхода в термомеханике. Этот подход базируется на введении в пространство параметров состояния рядом с тензорами напряжений и деформаций также градиента химического потенциала. При этом возмущение химического потенциала отождествляется с возмущением энергии взаимодействия. Показано, что зависимость модулей упругости от радиуса волокна является существенной в волокнах, радиус которых меньше десяти размеров области приповерхностной неоднородности. Используя асимптотические представления модифицированных функций Бесселя, исследовано поведение модуля Юнга, коэффициентов Пуассона, параметров Ляме и модуля объемного сжатия при больших радиусах волокна. Указано на согласованность полученных результатов с известными в литературе теоретическими и экспериментальными данными.

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