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ON THE STRONG AND Δ-CONVERGENCE FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS ON A $CAT(0)$ SPACE

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In this paper we give the strong and Δ-convergence theorems of the modified S-iteration and the modified two-step iteration processes for total asymptotically nonexpansive mappings on a $CAT(0)$ space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

Key words and phrases: $CAT(0)$ space, total asymptotically nonexpansive mapping, strong convergence, Δ-convergence, iterative process, fixed point.

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INTRODUCTION

A metric space X is a $CAT(0)$ *space* if it is geodesically connected and every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [9, 10]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory for various mappings in a $CAT(0)$ space has been rapidly developed and a lot of papers have appeared (see [4, 5, 6, 17, 18]).

Nanjaras and Panyanak [13] proved the demiclosedness principle for asymptotically nonexpansive mappings and gave the Δ-convergence theorem of the modified Mann iteration process for mappings of this type in a $CAT(0)$ space. Recently, Chang et. al. [3] introduced total asymptotically nonexpansive mappings and proved the demiclosedness principle for mappings of this type in a $CAT(0)$ space. Also, they presented the Δ-convergence theorem of the modified Mann iteration process for total asymptotically nonexpansive mappings in a $CAT(0)$ space.

In this paper, motivated by the above results, we get some results which are related to the strong and Δ-convergence of the modified S-iteration and the modified two-step iteration processes for total asymptotically nonexpansive mappings on a $CAT(0)$ space. Our results extend and improve the corresponding ones announced by Chang et. al. [3], Khan and Abbas [8], Nanjaras and Panyanak [13] and many others.

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1 Preliminaries and lemmas

Let (X, d) be a metric space, K be a nonempty subset of X and let $T : K \to K$ be a mapping. Recall that T is said to be a nonexpansive mapping if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in K.$$

The map T is said to be an asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall n \in \mathbb{N},\ x, y \in K.$$

The map T is said to be a uniformly L-Lipschitzian mapping if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad \forall n \in \mathbb{N},\ x, y \in K.$$

Chang et. al. [3] defined the concept of total asymptotically nonexpansive mapping as follows.

Definition 1 ([3, Definition 2.1]). *Let (X, d) be a metric space, K be a nonempty subset of X and let $T : K \to K$ be a mapping. T is said to be a total asymptotically nonexpansive mapping if there exist non-negative real sequences $\{\mu_n\}, \{v_n\}$ with $\mu_n \to 0, v_n \to 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that*

$$d(T^n x, T^n y) \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n$$

for all $n \in \mathbb{N}$ and $x, y \in K$.

Remark 1. *From the definitions, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1$, $\forall n \in \mathbb{N}$, each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $v_n = k_n - 1$, $\forall n \in \mathbb{N}$, $\zeta(t) = t, t \geq 0$, and each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n \in \mathbb{N}} \{k_n\}$.*

We now give the definition and collect some basic properties of the $CAT(0)$ space.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ and $y \in X$ (or more briefly, a *geodesic* from x to y) is a map $c : [0, l] \subset R \to X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be a *uniquely geodesic space* if there is exactly one geodesic joining x and y for all $x, y \in X$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consist of three points in X (the vertices of Δ) and three geodesic segments joining each pair of vertices (the edges of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [2]).

A geodesic space is said to be a $CAT(0)$ space [2] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

Let $x, y \in X$, and by Lemma 2.1(iv) of [6] for each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y). \tag{1}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (1). By using this notation, Dhompongsa and Panyanak [6] obtained the following lemma which will be used frequently in the proof of our main results.

Lemma 1 ([6, Lemma 2.4]). *Let X be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$$

for all $t \in [0,1]$ and $x, y, z \in X$.

In 1976 Lim [12] introduced the concept of Δ-convergence in a general metric space. In 2008 Kirk and Panyanak [11] specialized Lim's concept to the $CAT(0)$ space and proved that it is very similar to the weak convergence in a Banach space. Also, Dhompongsa and Panyanak [6] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space for nonexpansive mappings.

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point (see [5, Proposition 7]).

Definition 2 ([11, 12]). *A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write Δ-$\lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.*

Lemma 2. *i) Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence (see [11, p. 3690]).*

ii) Let K be a nonempty closed convex subset of a complete $CAT(0)$ space and let $\{x_n\}$ be a bounded sequence in K. Then the asymptotic center of $\{x_n\}$ is in K (see [4, Proposition 2.1]).

Lemma 3 ([6, Lemma 2.8]). *If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

In [3] it is proved demiclosedness principle for total asymptotically nonexpansive mappings in a $CAT(0)$ space as follows.

Lemma 4 ([3, Theorem 2.8]). *Let K be a closed convex subset of a complete $CAT(0)$ space X and let $T: K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = p$. Then $Tp = p$.*

The following lemma is crucial in the study of iteration processes in metric spaces.

Lemma 5 ([14, Lemma 2]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 6 ([13, Lemma 4.5]). *Let X be a $CAT(0)$ space, $x \in X$ be a given point and let $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \qquad \limsup_{n\to\infty} d(y_n, x) \leq r, \qquad \lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process which is independent of those of the modified Mann iteration [16] and the modified Ishikawa iteration [19]. We apply this iteration process into a $CAT(0)$ space as

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-a_n)T^n x_n \oplus a_n T^n y_n \\ y_n = (1-b_n)x_n \oplus b_n T^n x_n, \quad n \in \mathbb{N}. \end{cases} \tag{2}$$

By taking $T^n = T$ for all $n \in \mathbb{N}$ in (2), we obtain the S-iteration process which is introduced in [1].

Thianwan [20] introduced the two-step iteration process in a Banach space. We give *the modified two-step iteration process* in a $CAT(0)$ space as follows

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-a_n)y_n \oplus a_n T^n y_n \\ y_n = (1-b_n)\, x_n \oplus b_n T^n x_n, \quad n \in \mathbb{N}. \end{cases} \tag{3}$$

If $b_n = 0$ for each $n \in \mathbb{N}$, then (3) reduces to the modified Mann iteration process. By taking $T^n = T$ for all $n \in \mathbb{N}$ in (3), we obtain the two-step iteration process.

Our purpose in this paper is to get some results on the strong and Δ-convergence of the modified S-iteration and the modified two-step iteration processes for total asymptotically nonexpansive mappings in a $CAT(0)$ space. It is worth mentioning that our results in a $CAT(0)$ space can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(m)$ space for every $m \geq k$ (see [2, p. 165]).

2 Main results

We will denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in X : Tx = x\}$. Firstly, we prove the Δ-convergence theorem of the modified S-iteration process in a $CAT(0)$ space.

Theorem 1. *Let K be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X, $T : K \to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping with $F(T) \neq \varnothing$ and let $\{x_n\}$ be a sequence defined by (2). If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} a_n < \infty$;

(ii) there exists a constant $M^* > 0$ such that $\zeta(r) \leq M^* r, r \geq 0$;

(iii) $\{b_n\}$ is the sequence in $[0,1]$;

(iv) $\sum_{n=1}^{\infty} \sup\{d(z, T^n z) : z \in B\} < \infty$ for each bounded subset B of K;

(v) there exist constants $b, c \in (0,1)$ with $0 < b(1-c) \leq \frac{1}{2}$ such that $\{a_n\} \subset [b,c]$,

then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. We divide the proof of Theorem 1 into three steps.

Step I. First we prove that for each $p \in F(T)$ the following limit $\lim_{n\to\infty} d(x_n, p)$ exists. In fact for each $p \in F(T)$, by Lemma 1, we have

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n T^n x_n, p) \leq (1-b_n)d(x_n, p) + b_n d(T^n x_n, p) \\ &\leq (1-b_n)d(x_n, p) + b_n \{d(x_n, p) + v_n \zeta(d(x_n, p)) + \mu_n\} \\ &\leq (1 + b_n v_n M^*) d(x_n, p) + b_n \mu_n \leq (1 + v_n M^*) d(x_n, p) + \mu_n. \end{aligned}$$

Also, we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)T^n x_n \oplus a_n T^n y_n, p) \leq (1-a_n)d(T^n x_n, p) + a_n d(T^n y_n, p) \\ &\leq (1-a_n)\{d(x_n, p) + v_n \zeta(d(x_n, p)) + \mu_n\} + a_n L d(y_n, p) \\ &\leq (1-a_n)\{(1 + v_n M^*) d(x_n, p) + \mu_n\} + a_n L \{(1 + v_n M^*) d(x_n, p) + \mu_n\} \\ &= \{(1-a_n)(1 + v_n M^*) + a_n L (1 + v_n M^*)\} d(x_n, p) + (1 + a_n(L-1))\mu_n \\ &= \{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\} d(x_n, p) + (1 + a_n(L-1))\mu_n. \end{aligned}$$

It follows from condition (i) and Lemma 5 that $\lim_{n\to\infty} d(x_n, p)$ exists.

Step II. Next we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{4}$$

In fact, it follows from Step I that for all $p \in F(T)$, $\lim_{n\to\infty} d(x_n, p)$ exists, so we can assume that $\lim_{n\to\infty} d(x_n, p) = r$. Since

$$\begin{aligned} d(T^n y_n, p) &= d(T^n y_n, T^n p) \leq d(y_n, p) + v_n \zeta(d(y_n, p)) + \mu_n \leq (1 + v_n M^*) d(y_n, p) + \mu_n \\ &\leq (1 + v_n M^*)\{(1 + v_n M^*) d(x_n, p) + \mu_n\} + \mu_n \\ &= (1 + v_n M^*)(1 + v_n M^*) d(x_n, p) + (2 + v_n M^*)\mu_n, \end{aligned}$$

we have $\limsup_{n\to\infty} d(T^n y_n, p) \le r$. Similarly, we obtain $\limsup_{n\to\infty} d(T^n x_n, p) \le r$. On the other hand, since $\lim\limits_{n\to\infty} d((1-a_n)T^n x_n \oplus a_n T^n y_n, p) = \lim\limits_{n\to\infty} d(x_{n+1}, p) = r$, by Lemma 6, we have

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = 0. \tag{5}$$

Since $d(x_{n+1}, T^n x_n) \le d((1-a_n)T^n x_n \oplus a_n T^n y_n, T^n x_n) \le a_n d(T^n y_n, T^n x_n)$ from (5), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_n) = 0. \tag{6}$$

From condition (iv), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{7}$$

Hence from (6) and (7) we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{8}$$

Since T is a uniformly L-Lipschitzian mapping, from (7) and (8) we have that

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(x_{n+1}, x_n) + Ld(T^n x_n, x_n) \\ &= (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(T^n x_n, x_n) \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

The equation (4) is proved.

Step III. To show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we prove that

$$W_\Delta(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subseteq F(T)$$

and $W_\Delta(x_n)$ consists of exactly one point. Let $u \in W_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v \in K$. By Lemma 4, $v \in F(T)$. Since $\{d(u_n, v)\}$ converges, by Lemma 3, $u = v$. This shows that $W_\Delta(x_n) \subseteq F(T)$. Now we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ converges, by Lemma 3, $x = v \in F(T)$. This shows that $W_\Delta(x_n) = \{x\}$. □

Now we give an example of such mappings which are total asymptotically nonexpansive and uniformly L-Lipschitzian as in Theorem 1.

Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $T, S : K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1] \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0) \end{cases} \quad \text{and} \quad S(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

It is proved in [7, Example 3.1] that both T and S are asymptotically nonexpansive mappings. Therefore they are total asymptotically nonexpansive and uniformly L-Lipschitzian mappings. Additionally, $F(T) = \{0\}$ and $F(S) = \{x \in K; 0 \le x \le 1\}$.

We give the characterization of strong convergence for the modified S-iteration process on a $CAT(0)$ space as follows.

Theorem 2. *Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 1. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Theorem 1 (Step I), for all $p \in F(T)$,

$$d(x_{n+1}, p) \leq \{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\} d(x_n, p) + (1 + a_n(L-1))\mu_n.$$

This implies that

$$d(x_{n+1}, F(T)) \leq \{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\} d(x_n, F(T)) + (1 + a_n(L-1))\mu_n.$$

By Lemma 5, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4}$$

for all $n \geq n_0$. In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus, there exists $p^\star \in F(T)$ such that

$$d(x_{n_0}, p^\star) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^\star) + d(x_n, p^\star) \leq 2d(x_{n_0}, p^\star) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in the closed subset K of a complete $CAT(0)$ space and so it must be convergent to a point q in K. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of $F(T)$ forces q to be in $F(T)$. This completes the proof. □

Senter and Dotson [15] introduced the concept of *Condition* (I) as follows.

Definition 3 ([15, p. 375])**.** *A mapping $T : K \to K$ is said to satisfy Condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

$$d(x, Tx) \geq f(d(x, F(T))), \quad \forall\, x \in K.$$

With respect to the above definition, we have the following strong convergence theorem.

Theorem 3. *Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 1 and let T be a mapping satisfying Condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As proved in Theorem 2, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Theorem 1 (Step II), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from *Condition* (I) that

$$\lim_{n\to\infty} f\left(d\left(x_n, F(T)\right)\right) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f\left(d\left(x_n, F(T)\right)\right) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, we obtain

$$\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0.$$

The conclusion now follows from Theorem 2. □

Remark 2. *Theorems 1–3 contain some results of Khan and Abbas [8, Theorems 1–3] since each nonexpansive mapping is a total asymptotically nonexpansive mapping.*

Now, we give the Δ-convergence theorem of the modified two-step iteration process in a $CAT(0)$ space.

Theorem 4. *Let $X, K, T, \{b_n\}$ satisfy the hypotheses of Theorem 1, $\{a_n\}$ be a sequence in $[0,1]$ and let $\{x_n\}$ be a sequence defined by (3). If the conditions (i)–(iv) in Theorem 1 are satisfied, then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. First we prove that for all $p \in F(T)$ the following limit $\lim_{n\to\infty} d(x_n, p)$ exists. As proved in Theorem 1, we have

$$d(y_n, p) \leq (1 + v_n M^*)\, d(x_n, p) + \mu_n. \tag{9}$$

Since T is a uniformly L-Lipschitzian mapping, from (9) we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)y_n \oplus a_n T^n y_n, p) \\ &\leq (1-a_n)d(y_n, p) + a_n d(T^n y_n, p) \\ &\leq (1-a_n)d(y_n, p) + a_n L d(y_n, p) \\ &= (1 + a_n(L-1))\, d(y_n, p) \\ &\leq (1 + a_n(L-1))\left\{(1 + v_n M^*)\, d(x_n, p) + \mu_n\right\} \\ &= \left\{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\right\} d(x_n, p) + (1 + a_n(L-1))\mu_n. \end{aligned}$$

It follows from Lemma 5 that $\lim_{n\to\infty} d(x_n, p)$ exists. Next we prove that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From condition (iv), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = \lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{10}$$

By the above equality, we get

$$d(T^n x_n, T^n y_n) \leq L d(x_n, y_n) \leq L b_n d(x_n, T^n x_n) \leq L d(x_n, T^n x_n) \to 0 \quad (\text{as } n \to \infty). \tag{11}$$

Since

$$d(x_{n+1}, T^n y_n) \leq d((1-a_n)y_n \oplus a_n T^n y_n, T^n y_n) \leq (1-a_n)d(y_n, T^n y_n)$$

from (10), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, T^n y_n) = 0. \tag{12}$$

From (10), (11) and (12) we have that

$$d(x_n, x_{n+1}) \leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n) + d(T^n y_n, x_{n+1}) \to 0 \quad (\text{as } n \to \infty).$$

The rest of the proof follows the pattern of the Theorem 1 and is therefore omitted. □

Remark 3. *Theorem 4 contains the main result of Chang et. al. [3, Theorem 3.5] since the modified two-step iteration reduces to the modified Mann iteration. Also, Theorem 4 contains the main result of Nanjaras and Panyanak [13, Theorem 5.7] since each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping.*

Finally, we give following theorems related to the strong convergence of the modified two-step iteration process which their proofs are similar arguments of Theorem 2 and Theorem 3, respectively.

Theorem 5. *Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 4. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

Theorem 6. *Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 4 and let T be a mapping satisfying Condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

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Башарір М., Шагін А. *Про сильну і Δ-збіжність для тотальних асимптотично нерозширюваних відображень на $CAT(0)$ простір* // Карпатські математичні публікації. — 2013. — Т.5, №2. — C. 170–179.

В цій статті ми доводимо теореми про сильну і Δ-збіжність модифікованих S-ітерацій і модифікованих двокрокових ітераційних процесів для тотальних асимптотично нерозширюваних відображень на $CAT(0)$ простір. Наші результати розширюють і покращують відповідні недавні результати, що анонсовані багатьма авторами в літературі.

Ключові слова і фрази: $CAT(0)$ простір, тотальне асимптотично нерозширюване відображення, сильна збіжність, Δ-збіжність, ітераційний процес, нерухома точка.

Башарир М., Шагин А. *О сильной и Δ-сходимости для тотальных ассимптотически нерасширяющихся отображений на $CAT(0)$ пространство* // Карпатские математические публикации. — 2013. — Т.5, №2. — C. 170–179.

В этой статье мы доказываем теоремы о сильной и Δ-сходимости модифицированных S-итераций и модифицированных двухшаговых итерационных процессов для тотальных ассимптотически нерасширяющихся отображений на $CAT(0)$ пространство. Наши результаты расширяют и улучшают соответсвующие недавние результаты, анонсированные многими авторами в литературе.

Ключевые слова и фразы: $CAT(0)$ пространство, тотальное ассимптотически нерасширяющееся отображение, сильная сходимость, Δ-сходимость, итерационный процесс, неподвижная точка.