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ASYMPTOTICS OF A FUNDAMENTAL SOLUTION SYSTEM FOR A QUASIDIFFERENTIAL EQUATION WITH MEASURES ON THE SEMIAXIS

With the help of a conception of quasiderivatives asymptotic formulas for a fundamental solution system of a quasidifferential equation with measures on the semiaxis $[0,\infty)$ are constructed. The obtained asymptotic formulas allow to investigate asymptotics of eigenvalues and eigenfunctions of the corresponding boundary value problem.

Key words and phrases: quasidifferential equation, measure, distribution, quasiderivative, semiaxis, asymptotics of solutions.

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INTRODUCTION

Linear differential operators generated by differential expressions with smooth coefficients (including asymptotics of the eigenvalues and eigenfunctions) were studied quite comprehensively in the literature (e.g., see [7]). There are numerous recent results that generalize these operators to some extent. In particular, interesting results for functional-differential equations of the form $y^{(n)} + Fy + \rho^n y = 0$, where F is a linear operator mapping the Hölder space $C^\gamma[0,1]$, $\gamma < n-1$, into the space $L_1[0,1]$, were obtained in the papers of the Kiev mathematicians [3, 8]. The papers [5, 9, 15], as well as the present paper, aim at relaxing the conditions imposed on the coefficients of differential expressions. A wide bibliography on the theory of differential operators with singularities can be found in [1].

Real problems often lead to differential expressions that contain terms of the following form $\left(p(x)y^{(m)}\right)^{(n)}$ and cannot be reduced to conventional differential expressions by n-fold differentiation if the coefficient $p(x)$ is not sufficiently smooth. Such expressions are said to be quasidifferential. The introduction of quasiderivatives [10, 11] is one of the oldest methods for their analysis. (The quasiderivatives are the components of a vector reducing a quasidifferential equation to a system of first-order differential equations.)

In the paper [7], in particular, by using the investigation of the asymptotics of a fundamental solution system for a quasidifferential equation with integrable coefficients on the interval $[a,b]$, the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding differential operator was obtained. In the papers [2, 7] the previous results were extended to the semiaxis $[0,\infty)$.

In the present paper, by using the method of quasiderivatives, we analyze the asymptotics of a fundamental solution system for a quasidifferential equation with distributions in the coefficients on the semiaxis $[0,\infty)$. Our results generalize some of those in [2, 5, 6, 7].

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1 FORMULATION OF THE PROBLEM

Consider the quasidifferential expression

$$L_{mn}(y) \equiv \sum_{i=0}^{n}\sum_{j=0}^{m}\left(a_{ij}y^{(n-i)}\right)^{(m-j)},$$

where m and n are positive integers, a_{00} is a constant, $a_{10}=a_{01}=0$, $a_{i0}(x), a_{0j}(x) \in L_2[0,\infty)$, $a_{ij}(x) = b'_{ij}(x)$, $b_{ij}(x) \in BV^+[0,\infty)$, $i=\overline{1,n}$, $j=\overline{1,m}$. Here $BV^+[0,\infty)$ is the space of right continuous functions of bounded variation on any interval $[a,b] \subset [0,\infty)$. The prime stands for generalized differentiation, and hence the a_{ij} are measures, i.e., zero-order distributions [4, p. 160]. The functions $a_{ij}(x)$ and $b_{ij}(x)$ are assumed to be complex-valued.

The quasiderivatives of $y(x)$ corresponding to the expression $L_{mn}(y)$ are defined as the functions given by the formulas

$$\begin{cases} y^{[k]} = y^{(k)},\ k=\overline{0,n-1};\ y^{[n]} = \sum\limits_{i=0}^{n} a_{i0}y^{(n-i)}; \\ y^{[n+k]} = \left(y^{[n+k-1]}\right)' - \sum\limits_{i=0}^{n} a_{ik}y^{(n-i)},\ k=\overline{1,m}. \end{cases}$$

Let us pose the initial problem

$$L_{mn}(y) = \lambda y, \tag{1}$$

$$y^{[\nu-1]}(a) = \tilde{c}_\nu,\ \nu=\overline{1,n+m}. \tag{2}$$

It was proved in [12, 14] that there exists a unique solution of the initial problem (1), (2); moreover, the solution, together with quasiderivatives of order less than $n-1$, is absolutely continuous, and other quasiderivatives of order less than $n+m-1$ have bounded variation on any interval $[a,b] \subset [0,\infty)$.

We assume that $a_{00}=1$; otherwise we can divide equation (1) by a_{00}. For reduction we enter denotation $r=n+m$. Set $\lambda=-\rho^r$; then equation (1) can be represented in the form

$$y^{(r)} + \rho^r y = -\sum_{\substack{0\le i\le n\\ 0\le j\le m\\ i+j>1}}\left(a_{ij}y^{(n-i)}\right)^{(m-j)}. \tag{3}$$

We split the entire complex ρ-plane into $2r$ sectors S_q, $q=\overline{0,2r-1}$, where

$$S_q = \{\rho : q\pi/r \le \arg\rho \le (q+1)\pi/r\}.$$

We shall denote the domains S_q by S.

By $\omega_1, \omega_2, \ldots, \omega_r$ we denote the distinct r-th roots of -1. For each sector S_q, there exists a numbering [7, p. 55] of $\omega_1, \omega_2, \ldots, \omega_r$ such that

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \ldots \le \mathrm{Re}(\rho\omega_r), \quad \rho \in S_q. \tag{4}$$

In [2] the asymptotics of a linearly independent system of solutions of the equation

$$y^{(n)} + p_2(x)y^{(n-2)} + \ldots + p_n(x)y + \rho^n y = 0$$

with integrable coefficients on the semiaxis $[0,\infty)$ is obtained for large values of the parameter ρ. In the present paper we obtain the analogous formulas for the solutions of equation (1) with imposed conditions to the coefficients at the beginning of this section. These formulas generalize some results of the paper [2].

2 MAIN RESULTS

By using the vector $\mathbf{y} = (y, y^{[1]}, \ldots, y^{[r-1]})^T$ (where T stands for transposition), one can reduce equation (1) to the system of first-order differential equations

$$\mathbf{y}' = C'(x)\mathbf{y}, \tag{5}$$

where

$$C'(x) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -a_{n0} & -a_{n-1,0} & \cdots & -a_{10} & 1 & 0 & \cdots & 0 \\ A_{n1} & A_{n-1,1} & \cdots & A_{11} & -a_{01} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ A_{n,m-1} & A_{n-1,m-1} & \cdots & A_{1,m-1} & -a_{0,m-1} & 0 & \cdots & 1 \\ A_{nm}+\lambda & A_{n-1,m} & \cdots & A_{1m} & -a_{0m} & 0 & \cdots & 0 \end{pmatrix},$$

$A_{ij} = a_{0j}a_{i0} - a_{ij}$ $(i = \overline{1,n},\ j = \overline{1,m})$.

Obviously,

$$\Delta C(x) = C(x) - C(x-0) = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\Delta b_{n1} & \cdots & -\Delta b_{11} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\Delta b_{nm} & \cdots & -\Delta b_{1m} & 0 & \cdots & 0 \end{pmatrix}.$$

Since $[\Delta C(x)]^2 = 0$, it follows that system (5) is well posed [12].

The homogeneous equation

$$y^{(r)} + \rho^r y = 0 \tag{6}$$

has the fundamental solution system $e^{\rho\omega_1 x}$, $e^{\rho\omega_2 x}$, …, $e^{\rho\omega_r x}$. Vector equation (5) can be represented in the form $\mathbf{y}' = C_1'\mathbf{y} + C_2'\mathbf{y}$ such that the system $\mathbf{y}' = C_1'\mathbf{y}$ is equivalent to equation (6). Consequently, the matrix C_1' contains the unities above the main diagonal, $-\rho^r$ in the left below corner and zeros. If the right-hand side of relation (3) is treated as an "inhomogeneity", then, by the Cauchy formula for the inhomogeneous equation (see [14, p. 61]),

$$\mathbf{y}(x) = B(x,a)\,\mathbf{y}(a) + \int_a^x B(x,\xi)\,dC_2(\xi)\,\mathbf{y}(\xi), \tag{7}$$

where $a \geq 0$, $B(x,\xi)$ is the fundamental matrix of the "homogeneous" system $\mathbf{y}' = C_1'\mathbf{y}$; it has the structure [13]

$$B(x,\xi) = \begin{pmatrix} K^{\{r-1\}}(x,\xi) & \cdots & K^{\{1\}}(x,\xi) & K(x,\xi) \\ K^{\{r-1\}(1)}(x,\xi) & \cdots & K^{\{1\}(1)}(x,\xi) & K^{(1)}(x,\xi) \\ \cdots & \cdots & \cdots & \cdots \\ K^{\{r-1\}(r-1)}(x,\xi) & \cdots & K^{\{1\}(r-1)}(x,\xi) & K^{(r-1)}(x,\xi) \end{pmatrix}, \tag{8}$$

where $K(x,\xi)$ is the Cauchy function of equation (6). The parentheses in (8) stand for the ordinary derivatives with respect to the variable x, and the curly braces are used to denote quasiderivatives in the sense of the adjoint equation of (6); they are taken with respect to the second variable and are defined by the formulas [14, p. 122]

$$z^{\{0\}} \stackrel{def}{=} z, \quad z^{\{i\}} = -(z^{\{i-1\}})', \quad i = \overline{1, r-1}. \tag{9}$$

One can readily see that the Cauchy function for equation (6) has the form

$$K(x,\xi) = -\frac{\omega_1 e^{\rho\omega_1(x-\xi)} + \omega_2 e^{\rho\omega_2(x-\xi)} + \ldots + \omega_r e^{\rho\omega_r(x-\xi)}}{r\,\rho^{r-1}}. \tag{10}$$

Indeed, it satisfies equation (6) with respect to x, $K^{(\nu)}(\xi,\xi) = 0$, $\nu = \overline{0, r-2}$, $K^{(r-1)}(\xi,\xi) = 1$ since $\sum\limits_{j=1}^{r} \omega_j^{\nu+1} = 0$, and $\sum\limits_{j=1}^{r} \omega_j^{r} = -r$ (see [6, p. 55]).

By using relations (9) and (10), equality (8) can be represented in the form

$$B(x,\xi) = -\begin{pmatrix} \frac{1}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-\xi)} & \ldots & \frac{1}{r\rho^{r-2}}\sum\limits_{j=1}^{r}\omega_j^2 e^{\rho\omega_j(x-\xi)} & \frac{1}{r\rho^{r-1}}\sum\limits_{j=1}^{r}\omega_j e^{\rho\omega_j(x-\xi)} \\ \frac{\rho}{r}\sum\limits_{j=1}^{r}\omega_j^{r+1} e^{\rho\omega_j(x-\xi)} & \ldots & \frac{1}{r\rho^{r-3}}\sum\limits_{j=1}^{r}\omega_j^3 e^{\rho\omega_j(x-\xi)} & \frac{1}{r\rho^{r-2}}\sum\limits_{j=1}^{r}\omega_j^2 e^{\rho\omega_j(x-\xi)} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\rho^{r-1}}{r}\sum\limits_{j=1}^{r}\omega_j^{2r-1} e^{\rho\omega_j(x-\xi)} & \ldots & \frac{\rho}{r}\sum\limits_{j=1}^{r}\omega_j^{r+1} e^{\rho\omega_j(x-\xi)} & \frac{1}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-\xi)} \end{pmatrix}.$$

By denoting $\mathbf{y}(a) = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_r)^T$, from equation (7) we can obtain

$$\begin{pmatrix} y(x) \\ \ldots \\ y^{[r-1]}(x) \end{pmatrix} = \begin{pmatrix} -\frac{\tilde{c}_1}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-a)} - \ldots - \frac{\tilde{c}_r}{r\rho^{r-1}}\sum\limits_{j=1}^{r}\omega_j e^{\rho\omega_j(x-a)} \\ \ldots \\ -\frac{\tilde{c}_1\rho^{r-1}}{r}\sum\limits_{j=1}^{r}\omega_j^{2r-1} e^{\rho\omega_j(x-a)} - \ldots - \frac{\tilde{c}_r}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-a)} \end{pmatrix}$$

$$+\int\limits_a^x \begin{pmatrix} \frac{1}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-\xi)} & \ldots & \frac{1}{r\rho^{r-1}}\sum\limits_{j=1}^{r}\omega_j e^{\rho\omega_j(x-\xi)} \\ \ldots & \ldots & \ldots \\ \frac{\rho^{r-1}}{r}\sum\limits_{j=1}^{r}\omega_j^{2r-1} e^{\rho\omega_j(x-\xi)} & \ldots & \frac{1}{r}\sum\limits_{j=1}^{r}\omega_j^r e^{\rho\omega_j(x-\xi)} \end{pmatrix}$$

$$\times \begin{pmatrix} 0 \\ \ldots \\ 0 \\ -\sum\limits_{s=2}^{n} a_{s0}(\xi) y^{(n-s)}(\xi) d\xi \\ \sum\limits_{s=1}^{n} (db_{s1}(\xi) - a_{01}(\xi) a_{s0}(\xi) d\xi) y^{(n-s)}(\xi) \\ \sum\limits_{s=1}^{n} (db_{s2}(\xi) - a_{02}(\xi) a_{s0}(\xi) d\xi) y^{(n-s)}(\xi) + a_{02}(\xi) y^{[n]}(\xi) \\ \ldots \\ \sum\limits_{s=1}^{n} (db_{sm}(\xi) - a_{0m}(\xi) a_{s0}(\xi) d\xi) y^{(n-s)}(\xi) + a_{0m}(\xi) y^{[n]}(\xi) \end{pmatrix},$$

where the last column contains the null elements only in the first $n-1$ rows. Constants $\tilde{c}_j$, $j = \overline{1, r}$ can be choose such that the system of Volterra–Stieltjes integro-quasidifferential equations

$$
\begin{aligned}
y^{[\nu]}(x) = {} & \sum_{j=1}^{r} c_j \rho^{\nu} \omega_j^{\nu} e^{\rho\omega_j x} - \frac{\rho^{1-n+\nu}}{r} \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m+\nu+1} e^{\rho\omega_j(x-\xi)} a_{s0}(\xi) y^{(n-s)}(\xi) d\xi \\
& + \sum_{p=1}^{m} \frac{\rho^{1-n-p+\nu}}{r} \Bigg[\sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} y^{(n-s)}(\xi) db_{sp}(\xi) \\
& \qquad - \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) a_{s0}(\xi) y^{(n-s)}(\xi) d\xi \\
& \qquad + \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) y^{[n]}(\xi) d\xi \Bigg], \quad \nu = \overline{0, r-1},
\end{aligned}
\tag{11}
$$

holds. Indeed, from the equality

$$
\left(-\frac{\tilde{c}_1}{r}\omega_1^r - \ldots - \frac{\tilde{c}_r}{r\rho^{r-1}}\omega_1\right) e^{-\rho\omega_1 a} e^{\rho\omega_1 x} + \ldots + \left(-\frac{\tilde{c}_1}{r}\omega_r^r - \ldots - \frac{\tilde{c}_r}{r\rho^{r-1}}\omega_r\right) e^{-\rho\omega_r a} e^{\rho\omega_r x} = c_1 e^{\rho\omega_1 x} + \ldots + c_r e^{\rho\omega_r x}
$$

we obtain the system

$$
\begin{aligned}
- e^{-\rho\omega_1 a}(\tilde{c}_1 \omega_1^r \rho^{r-1} + \ldots + \tilde{c}_r \omega_1) &= c_1 r \rho^{r-1}, \\
&\ldots, \\
- e^{-\rho\omega_r a}(\tilde{c}_1 \omega_r^r \rho^{r-1} + \ldots + \tilde{c}_r \omega_r) &= c_r r \rho^{r-1}
\end{aligned}
$$

such that its determinant is nonzero for $|\rho| > 0$, because it is a Vandermonde determinant.

In the following theorem, asymptotic formulas for the solutions of equation (3) are derived on the basis of the analysis of the integro-quasidifferential equations (11).

Theorem 1. *Under the above-mentioned conditions imposed on the coefficients, in the entire domain S of the complex ρ-plane, the quasidifferential equation* (3) *has r linearly independent solutions $y_k(x,\rho)$, $k = \overline{1,r}$, which satisfy the relations*

$$
y_k^{[\nu]}(x,\rho) = \rho^{\nu} e^{\rho\omega_k x} z_{k\nu}(x,\rho) \tag{12}
$$

for $k = \overline{1,r}$, $\nu = \overline{0,r-1}$, $x \geq a \geq 0$, where the functions $z_{k\nu}(x,\rho)$ are bounded in the domain $a \leq x < \infty$, $\rho \in S$, $|\rho| \geq h > 0$.

The functions $y_k^{[\nu]}(x,\rho)$ are continuous with respect to the set of variables (x,ρ) for $x \in (0,\infty)$, $\rho \in S$, $|\rho| \geq h > 0$. These functions are regular (i.e., single-valued and analytic) with respect to $\rho \in S$, $|\rho| \geq h > 0$.

With $\rho \in S$ we have

$$
y_k^{[\nu]}(x,\rho) = \rho^{\nu} e^{\rho\omega_k x}\left[\omega_k^{\nu} + O\left(\frac{1}{\rho}\right)\right] \quad \text{as } \rho \to \infty \tag{13}
$$

uniformly with respect to $x \in [0,\infty)$.

Proof. Suppose

$$
y^{[\nu]}(x,\rho) = \rho^{\nu} e^{\rho\omega_k x} z_{\nu}(x,\rho), \quad \nu = \overline{0,r-1}, \tag{14}
$$

for some fixed k, $k = \overline{1, r}$.

Then we rewrite equations (11) in the form

$$\rho^{\nu} e^{\rho\omega_k x} z_{\nu}(x,\rho) = \sum_{j=1}^{r} c_j \rho^{\nu} \omega_j^{\nu} e^{\rho\omega_j x} - \frac{\rho^{1-n+\nu}}{r} \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m+\nu+1} e^{\rho\omega_j(x-\xi)} a_{s0}(\xi) \rho^{n-s} e^{\rho\omega_k\xi} z_{n-s}(\xi) d\xi$$
$$+ \sum_{p=1}^{m} \frac{\rho^{1-n-p+\nu}}{r} \Bigg[\sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} \rho^{n-s} e^{\rho\omega_k\xi} z_{n-s}(\xi) db_{sp}(\xi)$$
$$- \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) a_{s0}(\xi) \rho^{n-s} e^{\rho\omega_k\xi} z_{n-s}(\xi) d\xi$$
$$+ \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) \rho^{n} e^{\rho\omega_k\xi} z_{n}(\xi) d\xi \Bigg], \ \nu = \overline{0, r-1},$$

whence

$$z_{\nu}(x,\rho) = \sum_{j=1}^{r} c_j \omega_j^{\nu} e^{\rho(\omega_j-\omega_k)x} - \frac{1}{r} \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m+\nu+1} \rho^{1-s} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{s0}(\xi) z_{n-s}(\xi) d\xi$$
$$+ \sum_{p=1}^{m} \frac{\rho^{1-p}}{r} \Bigg[\sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} \rho^{-s} z_{n-s}(\xi) db_{sp}(\xi)$$
$$- \sum_{s=1}^{n} \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) a_{s0}(\xi) \rho^{-s} z_{n-s}(\xi) d\xi \tag{15}$$
$$+ \int_a^x \sum_{j=1}^{r} \omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) z_{n}(\xi) d\xi \Bigg], \quad \nu = \overline{0, r-1}.$$

Set

$$c_j' = c_j \ \text{ for } j = \overline{1, k}, \tag{16}$$

$$c_j' = c_j - \frac{1}{r} \sum_{s=1}^{n} \int_a^{\infty} \omega_j^{m+1} \rho^{1-s} e^{\rho(\omega_k-\omega_j)\xi} z_{n-s}(\xi) a_{s0}(\xi) d\xi$$
$$+ \sum_{p=1}^{m} \frac{\rho^{1-p}}{r} \Bigg[\sum_{s=1}^{n} \int_a^{\infty} \omega_j^{m-p+1} \rho^{-s} e^{\rho(\omega_k-\omega_j)\xi} z_{n-s}(\xi) db_{sp}(\xi)$$
$$- \sum_{s=1}^{n} \int_a^{\infty} \omega_j^{m-p+1} \rho^{-s} e^{\rho(\omega_k-\omega_j)\xi} z_{n-s}(\xi) a_{0p}(\xi) a_{s0}(\xi) d\xi \tag{17}$$
$$+ \int_a^{\infty} \omega_j^{m-p+1} e^{\rho(\omega_k-\omega_j)\xi} z_{n}(\xi) a_{0p}(\xi) d\xi \Bigg], \quad j = \overline{k+1, r},$$

for some fixed k, $k = \overline{1, r}$.

Each Riman-Stieltjes integral in formulas (17) exists and converges via continuity and boundedness of the functions $\rho^{1-s} e^{\rho(\omega_k-\omega_j)\xi} z_{n-s}(\xi)$, $\rho^{1-p-s} e^{\rho(\omega_k-\omega_j)\xi} z_{n-s}(\xi)$, $\rho^{1-p} e^{\rho(\omega_k-\omega_j)\xi} z_{n}(\xi)$, since $\mathrm{Re}(\rho\omega_k) \le \mathrm{Re}(\rho\omega_j)$, $j = \overline{k+1, r}$ (it follows from inequalities (4)).

Then system (15) can be written in the form

$$\begin{aligned} z_\nu(x,\rho) = {} & \sum_{j=1}^{r} c'_j \omega_j^\nu e^{\rho(\omega_j-\omega_k)x} - \frac{1}{r}\sum_{s=1}^{n}\int_a^x \sum_{j=1}^{k} \omega_j^{m+\nu+1}\rho^{1-s} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{s0}(\xi) z_{n-s}(\xi)d\xi \\ & + \sum_{p=1}^{m}\frac{\rho^{1-p}}{r}\Bigg[\sum_{s=1}^{n}\int_a^x \sum_{j=1}^{k}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)}\rho^{-s} z_{n-s}(\xi)db_{sp}(\xi) \\ & \qquad - \sum_{s=1}^{n}\int_a^x \sum_{j=1}^{k}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi)a_{s0}(\xi)\rho^{-s} z_{n-s}(\xi)d\xi \\ & \qquad + \int_a^x \sum_{j=1}^{k}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) z_n(\xi)d\xi\Bigg] \\ & + \frac{1}{r}\sum_{s=1}^{n}\int_x^\infty \sum_{j=k+1}^{r} \omega_j^{m+\nu+1}\rho^{1-s} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{s0}(\xi) z_{n-s}(\xi)d\xi \\ & - \sum_{p=1}^{m}\frac{\rho^{1-p}}{r}\Bigg[\sum_{s=1}^{n}\int_x^\infty \sum_{j=k+1}^{r}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)}\rho^{-s} z_{n-s}(\xi)db_{sp}(\xi) \\ & \qquad - \sum_{s=1}^{n}\int_x^\infty \sum_{j=k+1}^{r}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi)a_{s0}(\xi)\rho^{-s} z_{n-s}(\xi)d\xi \\ & \qquad + \int_x^\infty \sum_{j=k+1}^{r}\omega_j^{m-p+\nu+1} e^{\rho\omega_k(\xi-x)} e^{\rho\omega_j(x-\xi)} a_{0p}(\xi) z_n(\xi)d\xi\Bigg], \ \nu=\overline{0,r-1}. \end{aligned} \tag{18}$$

Suppose that equation (3) has a solution y_k such that $c'_\nu = 0$ for $\nu \neq k$, $c'_k = 1$. Let

$$y_k^{[\nu]} = \rho^\nu e^{\rho\omega_k x} z_{k\nu},$$

$$K_{kp\nu s}(x,\xi,\rho) = \begin{cases} \frac{1}{r} e^{\rho\omega_k(\xi-x)}\rho^{2-s-p-\nu}\sum\limits_{j=1}^{k}\omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)}, & \xi \le x, \\ -\frac{1}{r} e^{\rho\omega_k(\xi-x)}\rho^{2-s-p-\nu}\sum\limits_{j=k+1}^{n}\omega_j^{m-p+\nu+1} e^{\rho\omega_j(x-\xi)}, & \xi > x, \end{cases} \tag{19}$$

$$k=\overline{1,r},\quad \nu=\overline{0,r-1},\quad s=\overline{0,n},\quad p=\overline{0,m}.$$

Then for the functions $z_{k\nu}(x,\rho)$ we obtain the system of integral equations

$$\begin{aligned} z_{k\nu}(x,\rho) = {} & \omega_k^\nu - \frac{1}{\rho}\sum_{s=1}^{n}\int_a^\infty K_{k0\nu s}(x,\xi,\rho)a_{s0}(\xi) z_{n-s}(\xi,\rho)d\xi \\ & + \frac{1}{\rho}\sum_{p=1}^{m}\Bigg[\sum_{s=1}^{n}\int_a^\infty K_{kp\nu s}(x,\xi,\rho) z_{k,n-s}(\xi,\rho)db_{sp}(\xi) \\ & - \sum_{s=1}^{n}\int_a^\infty K_{kp\nu s}(x,\xi,\rho)a_{0p}(\xi)a_{s0}(\xi) z_{k,n-s}(\xi,\rho)d\xi + \int_a^\infty K_{kp\nu 0}(x,\xi,\rho)a_{0p}(\xi) z_{kn}(\xi,\rho)d\xi\Bigg]. \end{aligned} \tag{20}$$

We construct the functions $Q_{kp\nu s}(x,\xi,\rho)$ and $g_{sp}(x)$ ($k=\overline{1,r}$, $p=\overline{0,2m}$, $\nu=\overline{0,r-1}$, $s=\overline{0,n}$) as follows: $Q_{kp\nu s}(x,\xi,\rho) = K_{kp\nu s}(x,\xi,\rho)$ for $s=\overline{0,n}$, $p=\overline{1,m}$; $Q_{kp\nu s}(x,\xi,\rho) =$

$-K_{k,p-m,\nu s}(x,\xi,\rho)$ for $s=\overline{1,n}$, $p=\overline{m+1,2m}$; $Q_{k0\nu s}(x,\xi,\rho)=-K_{k0\nu s}(x,\xi,\rho)$ for $s=\overline{1,n}$; $Q_{kp\nu 0}(x,\xi,\rho)=0$ for $p=0,m+1,m+2,\ldots,2m$; $g_{sp}(x)=b_{sp}(x)$ for $s=\overline{1,n}$, $p=\overline{1,m}$; $g_{0p}(x)=\int_a^x a_{0p}(t)dt$ for $p=\overline{0,m}$; $g_{s0}(x)=\int_a^x a_{s0}(t)dt$ for $s=\overline{1,n}$; $g_{sp}(x)=\int_a^x a_{0,p-m}(t)a_{s0}(t)dt$ for $s=\overline{0,n}$, $p=\overline{m+1,2m}$. Obviously, all $g_{sp}(x)$ have bounded variation on any interval $[a,b]\subset[0,\infty)$. Then system (20) can be represented in the compact form

$$z_{k\nu}(x,\rho)=\omega_k^\nu+\frac{1}{\rho}\sum_{p=0}^{2m}\sum_{s=0}^{n}\int_a^\infty Q_{kp\nu s}(x,\xi,\rho)z_{k,n-s}(\xi,\rho)dg_{sp}(\xi). \tag{21}$$

Each function $Q_{kp\nu s}(x,\xi,\rho)$, $k=\overline{1,r}$, $\nu=\overline{0,r-1}$, $p=\overline{0,2m}$, $s=\overline{0,n}$, is left continuous for all $x,\xi\in[0,\infty)$ and it is regular for all $|\rho|\geq h>0$. There exists a constant $C_1>0$ such that

$$|Q_{kp\nu s}(x,\xi,\rho)|=\frac{1}{r}|\rho|^{2-s-p-\nu}\left|\sum_{j=1}^{k}\omega_j^{m-p+\nu+1}e^{(\rho\omega_j-\rho\omega_k)(x-\xi)}\right|\leq C_1, \tag{22}$$

$k=\overline{1,r}$, $\nu=\overline{0,r-1}$, $s=\overline{0,n}$, $p=\overline{0,2m}$, for $\xi\leq x$, because $\operatorname{Re}(\rho\omega_j)\leq\operatorname{Re}(\rho\omega_k)$, $j=\overline{1,k}$, since (4). Similarly, there exists a constant $C_2>0$ such that

$$|Q_{kp\nu s}(x,\xi,\rho)|=\frac{1}{r}|\rho|^{2-s-p-\nu}\left|\sum_{j=k+1}^{r}\omega_j^{m-p+\nu+1}e^{(\rho\omega_j-\rho\omega_k)(x-\xi)}\right|\leq C_2, \tag{23}$$

$k=\overline{1,r}$, $\nu=\overline{0,r-1}$, $s=\overline{0,n}$, $p=\overline{0,2m}$, for $\xi>x$, because $\operatorname{Re}(\rho\omega_j)\geq\operatorname{Re}(\rho\omega_k)$, $j=\overline{k+1,r}$, since (4).

Let $C=\max\{C_1,C_2\}$. Since bounded variation of the functions $g_{sp}(x)$, we can find a number $a\geq 0$ such that

$$c\frac{1}{h}\int_a^\infty|dg_{sp}(\xi)|=\frac{c}{h}\bigvee_a^\infty g_{sp}<\frac{1}{r},\quad s=\overline{0,n},\quad p=\overline{0,2m}.$$

Then all conditions of theorems 1 and 2 from [5] are satisfied and by these theorems system (21) has the bounded continuous solution $z_{k\nu}(x,\rho)$, $x\in[a,\infty)$, $\rho\in S$, $|\rho|\geq h>0$; moreover, there are asymptotic formulas (13) for $\rho\to\infty$.

Let us show that there exists solution (12) of equation (3) that satisfies system (21). To this end, it suffices to show that for all constants c'_ν there exists solution (14) of equation (3) satisfying system (18) for these values c'_ν.

Equalities (16), (17) are a linear transformation from c_j to c'_j. Obviously, it suffices to show that the determinant of the mapping (16), (17) is nonzero for sufficiently large $|\rho|$, $\rho\in S$. In this case for any c'_j equations (16), (17) can be solved for c_j.

If the determinant of the mapping (16), (17) is zero for arbitrarily large $|\rho|$, $\rho\in S$, then, for these ρ, equations (16), (17) have nontrivial solutions with respect to c_j for $c'_1=c'_2=\ldots=c'_r=0$. Then the corresponding function

$$z_\nu(x,\rho)=\rho^{-\nu}e^{-\rho\omega_k x}y^{[\nu]}(x,\rho) \tag{24}$$

is a nontrivial solution of the system

$$z_\nu(x,\rho)=\frac{1}{\rho}\sum_{p=0}^{2m}\sum_{s=0}^{n}\int_a^\infty Q_{kp\nu s}(x,\xi,\rho)z_{n-s}(\xi,\rho)dg_{sp}(\xi),$$

which can be obtained from (18) for $c_1' = c_2' = \ldots = c_r' = 0$ and designations (19). Let us show that this is impossible. Let $m(\rho) = \max |z_\nu(x, \rho)|$, $x \geq a$, $\nu = \overline{0, r-1}$. Using the inequalities (22), (23), we obtain

$$|z_\nu(x, \rho)| \leq \frac{C}{|\rho|} \int\limits_a^\infty |dg_{sp}(\xi)|\, m(\rho) \leq m(\rho) \frac{C_1}{|\rho|},$$

where C_1 is some constant. The last inequality should hold for all ρ. But for large $|\rho|$, this inequality is possible only if $m(\rho) = 0$; consequently, $z_\nu(x, \rho) = 0$. This, together with (24), implies that $y \equiv 0$ for $\nu = 0$.

It remains to prove a linear independence of the solutions $y_k(x, \rho)$. To do this, calculate the Wronskian of these functions for $\rho \to \infty$

$$W(x, \rho) = 1 \cdot \rho \ldots \rho^{r-1} e^{\rho(\omega_1 + \ldots + \omega_r)x} \begin{vmatrix} 1 & \cdots & 1 \\ \omega_1 & \cdots & \omega_r \\ \cdots & \cdots & \cdots \\ \omega_1^{r-1} & \cdots & \omega_r^{r-1} \end{vmatrix} = \rho^{\frac{r(r-1)}{2}} \begin{vmatrix} 1 & \cdots & 1 \\ \omega_1 & \cdots & \omega_r \\ \cdots & \cdots & \cdots \\ \omega_1^{r-1} & \cdots & \omega_r^{r-1} \end{vmatrix}.$$

Since the Vandermonde determinant of distinct numbers $\omega_1, \omega_2, \ldots, \omega_r$ is nonzero, we see that the Wronskian is nonzero for all $x \in [a, \infty)$, $\rho \in S$. □

Remark 1. *Each of obtained solutions $y_k(x, \rho)$, $k = \overline{1, r}$, can be extend to the interval $[0, a]$, constructing on it the solutions of equation* (3) *that satisfy the initial conditions $y^{[\nu]}(a) = y_k^{[\nu]}(a)$, $\nu = \overline{0, r-1}$, $k = \overline{1, r}$.*

Remark 2. *If h is so large that $\frac{c}{h} \int\limits_0^\infty |dg_{sp}(\xi)| < \frac{1}{r}$, $s = \overline{0, n}$, $p = \overline{0, 2m}$, in the designations of theorem 1 from this section, then in this theorem can be put $a = 0$.*

For a example we consider the quasidifferential equation

$$y^{\mathrm{IV}} + (a_{11} y')' + (a_{20} y)'' + a_{02} y'' + (a_{21} y)' + a_{12} y' + a_{22} y = \lambda y, \tag{25}$$

where $a_{20}(x), a_{02}(x) \in L_2[0, \infty)$, $a_{11}(x) = b_{11}'(x)$, $a_{12}(x) = b_{12}'(x)$, $a_{21}(x) = b_{21}'(x)$, $a_{22}(x) = b_{22}'(x)$, $b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x) \in BV^+[0, \infty)$. The quasiderivatives for this equation are defined by the formulas $y^{[1]} = y'$, $y^{[2]} = y'' + a_{20} y$, $y^{[3]} = (y'' + a_{20} y)' - a_{01} y'' - a_{11} y' - a_{21} y$. The conclusions of theorem 1 hold for equation (25).

The constructed asymptotic formulas for the linear independent system of the solutions of the quasidifferential equation with measures on the semiaxis allow to investigate an asymptotic behavior of eigenvalues and eigenfunctions of the corresponding boundary value problem. The presence of distributions in the coefficients of a quasidifferential equation does not affect these formulas.

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Махней О.В. *Асимптотика фундаментальної системи розв'язків квазідиференціального рівняння з мірами на півосі* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 113–122.

За допомогою концепції квазіпохідних побудовано асимптотичні формули для фундаментальної системи розв'язків квазідиференціального рівняння з мірами на півосі $[0,\infty)$. Отримані асимптотичні формули дозволяють досліджувати асимптотику власних значень і власних функцій відповідної крайової задачі.

Ключові слова і фрази: квазідиференціальне рівняння, міра, розподіл, квазіпохідна, піввісь, асимптотика розв'язків.

Махней А.В. *Асимптотика фундаментальной системы решений квазидифференциального уравнения с мерами на полуоси* // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 113–122.

С помощью концепции квазипроизводных построены асимптотические формулы для фундаментальной системы решений квазидифференциального уравнения с мерами на полуоси $[0,\infty)$. Полученные асимптотические формулы позволяют исследовать асимптотику собственных значений и собственных функций соответствующей краевой задачи.

Ключевые слова и фразы: квазидифференциальное уравнение, мера, распределение, квазипроизводная, полуось, асимптотика решений.