ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2016, **8** (2), 224–229
doi:10.15330/cmp.8.2.224-229

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2016, Т.8, №2, С.224–229

DILNYI V.M., HISHCHAK T.I.

ON THE INTERSECTION OF WEIGHTED HARDY SPACES

Let $H^p_\sigma(\mathbb{C}_+)$, $1 \le p < +\infty$, $0 \le \sigma < +\infty$, be the space of all functions f analytic in the half plane $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ and such that

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r |\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

We obtain some properties and description of zeros for functions from the space $\bigcap_{\sigma>0} H^p_\sigma(\mathbb{C}_+)$.

Key words and phrases: zeros of functions, weighted Hardy space, angular boundary values.

Ivan Franko State Pedagogical University, 24 Franka str., 82100, Drohobych, Ukraine
E-mail: dilnyi@ukr.net (Dilnyi V.M.), taniosered@mail.ru (Hishchak T.I.)

INTRODUCTION

Let $H^p(\mathbb{C}_+)$, $1 \leqslant p < +\infty$, be the Hardy space of holomorphic in $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ functions f such that

$$||f||^p = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\} < +\infty.$$

Let $H^p_\sigma(\mathbb{C}_+)$, $1 \le p < +\infty$, $0 \le \sigma < +\infty$, be the space of all functions f analytic in the half plane $\mathbb{C}_+$ and such that

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r |\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

We denote by $H^\infty_\sigma(\mathbb{C}_+)$, $0 \le \sigma < +\infty$, the space of all functions analytic in the right half-plane satisfying the condition

$$||f|| := \sup_{z \in \mathbb{C}_+} \left\{ |f(z)| e^{-\sigma |\mathrm{Im} z|} \right\} < +\infty.$$

The space $H^p_\sigma(\mathbb{C}_+)$, $1 \le p \le +\infty$, $0 \le \sigma < +\infty$, is a weighted Hardy space, as it follows from results of A. M. Sedletskii [9]. The theory of weighted Hardy space for the case if the weight is an exponential function considered in [2, 3, 10–13]. Functions $f \in H^p_\sigma(\mathbb{C}_+)$ have angular boundary values almost everywhere on $\partial\mathbb{C}_+$ (we denote the extension by the same

УДК 517.5
2010 *Mathematics Subject Classification:* 30D15, 30H10.

symbols f) and $f \in L^p(\partial\mathbb{C}_+)$. Thus, the space $H^p_\sigma(\mathbb{C}_+)$, $p \geq 1$, is a Banach space. For functions $f \in H^p_\sigma(\mathbb{C}_+)$ there exists [4,12] an integral boundary function defined by the equality

$$h(t_2) - h(t_1) = \lim_{x\to 0+} \int\limits_{t_1}^{t_2} \ln|f(x+it)|dt - \int\limits_{t_1}^{t_2} \ln|f(it)|dt, \quad t_1 < t_2$$

up to an additive constant and to values at continuity points. The integral boundary function h is nonincreasing on $\mathbb{R}$ and $h'(t) = 0$ almost everywhere on $\mathbb{R}$. The interest to the space $H^p_\sigma(\mathbb{C}_+)$ is generated by studies of completeness [3], by the theory of integral operators and the shift operator [1,8].

A number of papers have been devoted to the intersection of Hardy and related spaces (see [5,7]). The aim of our research is to describe some properties of the following space

$$H^p_\cap(\mathbb{C}_+) = \bigcap_{\sigma>0} H^p_\sigma(\mathbb{C}_+).$$

Obviously, $H^p_\cap(\mathbb{C}_+) \supset H^p(\mathbb{C}_+)$ and $H^p_\cap(\mathbb{C}_+) \subset H^p_\varepsilon(\mathbb{C}_+)$ for all ε.

1 THE MAIN RESULTS

Theorem 1. $H^p_\cap(\mathbb{C}_+) \neq H^p(\mathbb{C}_+)$.

Proof. Let $f(z) = e^{-z\sqrt{\ln(z+2)}}$. We choose the branch of the logarithm that $\ln 1 = 0$ and $\sqrt{1} = 1$. Let us prove that the function f belongs to $H^p_\sigma(\mathbb{C}_+)$ for all $\sigma > 0$. Indeed,

$$\begin{aligned}
\ln|f(re^{i\varphi})| &= -r\sqrt[4]{\ln^2\sqrt{4r\cos\varphi + r^2 + 4} + \operatorname{arctg}^2 \frac{r\sin\varphi}{r\cos\varphi+2}} \\
&\times \left(\cos\varphi \cos \frac{\operatorname{arctg} \frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2} - \sin\varphi \sin \frac{\operatorname{arctg} \frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2} \right) \\
&\leq r\sqrt[4]{\ln^2\sqrt{4r\cos\varphi + r^2 + 4} + \operatorname{arctg}^2 \frac{r\sin\varphi}{r\cos\varphi+2}} \sin\varphi \sin \frac{\operatorname{arctg} \frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2} \\
&\leq \frac{r}{2}\varphi \sin\varphi \frac{1}{\sqrt{\ln r}}, \quad r \to +\infty.
\end{aligned}$$

It follows easily that $f \in H^p_\sigma(\mathbb{C}_+)$. Consequently, $f \in H^p_\cap(\mathbb{C}_+)$.

Let us show that $f(z) = e^{-z\sqrt{\ln(z+2)}} \notin H^p(\mathbb{C}_+)$. Indeed,

$$\begin{aligned}
\ln|f(iy)| &= y\sqrt[4]{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2 \frac{y}{2}} \sin \frac{\operatorname{arctg} \frac{\operatorname{arctg}\frac{y}{2}}{\ln\sqrt{4+y^2}}}{2} \\
&= \frac{y}{\sqrt{2}}\sqrt[4]{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2 \frac{y}{2}} \sqrt{1 - \frac{\ln\sqrt{4+y^2}}{\sqrt{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2\frac{y}{2}}}} \\
&\geq \frac{y}{\sqrt{2\ln(4+y^2)}} \quad \text{for} \quad y \geq C > 0.
\end{aligned}$$

Therefore $f(iy) \notin L^p(0;+\infty)$. Hence, $f \notin H^p(\mathbb{C}_+)$. □

Proposition 1. *Suppose that $f \in H^p_\cap(\mathbb{C}_+)$, $1 \le p \le \infty$. Then the following conditions are fulfilled :*

a) angular boundary values exist almost everywhere on $i\mathbb{R}$;

b) $|f(it)|e^{-\varepsilon|t|} \in L^p(\mathbb{R})$ for any $\varepsilon > 0$;

c) $H^p_\cap(\mathbb{C}_+)$ is a Banach space for uniform convergence on compact sets.

Proof. Let $f \in H^p_\cap(\mathbb{C}_+)$, then $f \in H^p_\varepsilon(\mathbb{C}_+)$ for some $\varepsilon > 0$. In [11] B. V. Vinnitskii proved that a function $f \in H^p_\sigma(\mathbb{C}_+)$, $p \in (1;+\infty)$, has almost everywhere on $i\mathbb{R}$ angular boundary values $f(iy)$ and $f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R})$. Therefore $f(iy)e^{-\varepsilon|y|} \in L^p(\mathbb{R})$ for some positive ε.

In [10] B. V. Vinnitskii showed that a function $f \in H^\infty_\sigma(\mathbb{C}_+)$ has almost everywhere on $i\mathbb{R}$ angular boundary values $f(it)$ and $f(it)e^{-\varepsilon|t|} \in L^\infty(\mathbb{R})$ for all ε. In [11] inequality

$$|f(z)| \le \frac{c_2 \exp(c_2|z|)}{\operatorname{Re}(z)^{\frac{1}{p}}}$$

proved for each function f belonging to $H^p_\sigma(\mathbb{C}_+)$. Furthermore, $H^p_\cap(\mathbb{C}_+)$ is a Banach space with respect to uniform convergence on compact sets. □

Let B is a class of continuous, increasing functions $\eta : [0;+\infty) \to (0;+\infty)$ such that $\eta(r) = o(r)$ as $r \to +\infty$. We denote by $H^p_\ominus(\mathbb{C}_+)$ the space of functions analytic in $\mathbb{C}_+$ for which there exists $\eta \in B$

$$\sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr \right\}^{\frac{1}{p}} < +\infty,$$

where $\eta \in B$.

Theorem 2. *If $f \in H^p_\ominus(\mathbb{C}_+)$, then $f \in H^p_\cap(\mathbb{C}_+)$.*

Proof. Let $f \in H^p_\ominus(\mathbb{C}_+)$, then $f \in H^p_\sigma(\mathbb{C}_+)$ for all $\sigma > 0$. Furthermore,

$$\int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr = \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} e^{-pr\sigma|\sin\varphi| + \eta(r)|\sin\varphi|} dr.$$

Since $-pr\sigma|\sin\varphi| + \eta(r)|\sin\varphi| = |\sin\varphi|(-pr\sigma + \eta(r)) < 0$ as $r > r_0$, we have

$$\int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \le \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr < +\infty.$$

This implies that

$$\sup \left\{ \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \right\} \le \sup \left\{ \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr \right\}$$

and

$$\sup\left\{\int\limits_0^{r_0}|f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|}dr\right\}\le\sup\left\{\int\limits_0^{r_0}|f(re^{i\varphi})|^p\exp\{\min_{r\in[0;r_0]}\{-\eta(r)\}|\sin\varphi|\}dr\right\}$$
$$\le\sup\left\{\exp\{\min_{r\in[0;r_0]}\{-\eta(r)\}|\sin\varphi|\}\int\limits_0^{r_0}|f(re^{i\varphi})|^p dr\right\}\le c_1\int\limits_0^{r_0}|f(re^{i\varphi})|^p dr<+\infty.$$

In particular, choosing $c_2=\frac{2p\sigma r_0}{\eta(0)}$ we can achieve that

$$\int\limits_0^{+\infty}|f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}dr\le c_2<+\infty.$$

It follows that $f\in H^p_{\ominus}(\mathbb{C}_+)$. □

B. V. Vinnitskii described [11] zeros for functions $f\in H^p_\sigma(\mathbb{C}_+)$ in terms of the following function

$$S(r)=\sum_{1<|\lambda_n|\le r}\left(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\right)\frac{\mathrm{Re}\lambda_n}{|\lambda_n|},$$

where $\lambda_n\in\mathbb{C}_+$. We obtain the following statement.

Theorem 3. *If $f\in H^p_\cap(\mathbb{C}_+)$, then $S(r)=o(\ln r)$, $r\to+\infty$.*

Proof. Suppose $f\in H^p_\cap(\mathbb{C}_+)$, then $f\in H^p_\sigma(\mathbb{C}_+)$ for all $\sigma>0$. Use the following version of the Carleman formula [4,6,12]

$$S(r)=\frac{1}{\pi r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln|f(re^{i\varphi})|\cos\varphi d\varphi+\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt$$
$$-\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)|dh(t)|+O(1). \tag{1}$$

In [11] it is shown that for each function $f\in H^p_\sigma(\mathbb{C}_+),\sigma>0$, the first term on the right side of the last equality is bounded by an independent of r and σ constant. Hence, this term is bounded for each function of the space $H^p_\cap(\mathbb{C}_+)$. Consider the second addend

$$\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt=\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)(\ln|f(it)|e^{-\sigma|t|}+e^{\sigma|t|})dt$$
$$\le\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)(|f(it)|e^{-\sigma|t|}+\sigma|t|)dt.$$

Since $\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\sigma|t|dt=\frac{1}{\pi}\sigma\ln r$ and $f(iy)e^{-\sigma|y|}\in L^p(\mathbb{R})$, this yields

$$\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt\le c_3+\frac{1}{\pi}\sigma\ln r.$$

Therefore

$$S(r) = c_4 + \frac{1}{\pi}\sigma \ln r - \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) |dh(t)|.$$

Then, the last addend is negative, we deduce

$$S(r) \le c_4 + \frac{\sigma}{\pi} \ln r.$$

Since the result is true for on of an arbitrary σ, we obtain the statement of the theorem. □

Theorem 4. *If $f \in H^p_\cap(\mathbb{C}_+)$, then $P(r) = o(\ln r)$, $r \to +\infty$, where*

$$P(r) = \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) |dh(t)|.$$

Proof. Let $f \in H^p_\cap(\mathbb{C}_+)$, then $f \in H^p_\sigma(\mathbb{C}_+)$ for everyone $\sigma > 0$. Using (1), we get $P(r) = K(r) - S(r) + O(1), \quad r \to +\infty$, where

$$K(r) = \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln |f(it)| dt.$$

Since

$$\begin{aligned} K(r) &= \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln |f(it)| e^{-\sigma|t|} dt + \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \sigma |t| dt \\ &\le c_3 + \frac{1}{\pi}\sigma \ln r \quad \text{for all} \quad \sigma > 0, \end{aligned}$$

we deduce $K(r) = o(\ln r)$ as $r \to +\infty$. From Theorem 3 we get the following $S(r) = o(\ln r)$, $r \to +\infty$. Thus $P(r) = o(\ln r), r \to +\infty$. □

Theorem 5. *Let (λ_n) be an arbitrary sequence in $\mathbb{C}_+$. Then $S(r) = o(\ln r)$, $r \to +\infty$, if and only if $S_0(r) = o(\ln r), r \to +\infty$, where*

$$S_0(r) = \sum_{1<|\lambda_n|\le r} \frac{Re\lambda_n}{|\lambda_n|^2}.$$

Proof. It is clear that

$$S_0(r) - S(r) = \sum_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{r^2} \le \sum_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{|\lambda_n| r} = \frac{s(r)}{r},$$

where $s(r) = \sum\limits_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{|\lambda_n|}$.

In [10] B. V. Vinnitskii proved that

$$S(2r) \ge \frac{3s(r)}{4r}.$$

It follows that

$$S_0(r) - S(r) \le \frac{4rS(2r)}{3r} = \frac{4}{3} S(2r).$$

Since $S(r) = o(\ln r)$, we have $S(2r) = o(\ln r)$. Hence, $S_0(r) - S(r) = o(\ln r),\ r \to +\infty$.

The converse implication is trivial. □

REFERENCES

[1] Dilnyi V. M. *Splitting of some spaces of analytic functions*. Ufa Math. J. 2014, **6** (2), 25–34. dio:10.13108/2014-6-2-25

[2] Dil'nyi V. M. *On the equivalence of some conditions for weighted Hardy spaces*. Ukrainian Math. J. 2006, **58** (9), 1425–1432. doi:10.1007/s11253-006-0141-2 (translation of Ukrain. Mat. Zh. 2006, **58** (9), 1257–1263. (in Ukrainian))

[3] Dilnyi V. M. *On Cyclic Functions in Weighted Hardy Spaces*. J. Math. Phys., Anal., Geom. 2011, **7** (1), 19–33. (in Ukrainian)

[4] Fedorov M.A., Grishin A.F. *Some questions of the Nevanlinna theory for the Complex Half-Plane*. Interface Science 1998, **6** (3), 223–271. doi:10.1023/A:1008672404744

[5] Fedorov S.I. *On a recent result on the intersection of weighted Hardy spaces*. NZ J. Mathematics. 2001, **30** (2), 119–130.

[6] Govorov N.V. Riemann boundary value problem with infinite index. Nauka, Moscow, 1986. (in Russian)

[7] Jones P.W. *Recent advances in the theory of Hardy Spaces*. In: Proc. of the International Congress of Mathematicians, Warszawa, Poland, August 16–24, 1983. **1**, 829–839. http://www.mathunion.org/ICM/ICM1983.1/Main/icm1983.1.0829.0838.ocr.pdf

[8] Nikolskii N.K. Treatise on the shift operator. Spectral function theory. In: Grundlehren der mathematischen Wissenschaften, 273. Springer–Verlag, Berlin, 1986. doi:10.1007/978-3-642-70151-1

[9] Sedletskii A.M. *An equivalent definition of H^p spaces in the half-plane and some applications*. Mathematics of the USSR-Sbornik 1975, **25** (1), 69–76. doi:10.1070/SM1975v025n01ABEH002198 (translation of Math. Sb. (N.S.) 1975, **96** (1), 75–82. (in Russian))

[10] Vinnitskii B.V. *On zeros of functions analytic in a half plane and completeness of systems of exponents*. Ukrainian Math. J. 1994, **46** (5), 514–532. doi:10.1007/BF01058515 (translation of Ukrain. Mat. Zh. 1994, **46** (5), 484–500. (in Ukrainian))

[11] Vinnitskii B.V. *On zeros of functions analytic in a half plane*. Mat. Stud. 1996, **6** (1), 67–72. (in Ukrainian)

[12] Vynnytskyi B.V., Dil'nyi V.M. *On necessary conditions for existence of solutions of convolution type equation*. Mat. Stud. 2001, **16** (1), 61–70. (in Ukrainian)

[13] Vynnytsyi B.V., Sharan V.L. *On the factorization of one class of functions analytic in the half-plane*. Mat. Stud. 2000, **14** (1), 41–48. (in Ukrainian)

Received 30.09.2016

Revised 28.11.2016

Дільний В.М., Гіщак Т.І. *Про перетин вагових просторів Гарді* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 224–229.

Нехай $H^p_\sigma(\mathbb{C}_+)$, $1 \le p < +\infty$, $0 \le \sigma < +\infty$, – простір функцій, аналітичних у півплощині $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$, для яких

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

Отримано деякі властивості і опис нулів для функцій з простору $\bigcap_{\sigma>0} H^p_\sigma(\mathbb{C}_+)$.

Ключові слова і фрази: нулі функцій, ваговий простір Гарді, кутові граничні значення.