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# HYPERCYCLIC OPERATORS ON LIPSCHITZ SPACES

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We consider hypercyclic operators on free Banach spaces and little Lipschitz spaces which are some kind of generalizations of shift operators and composition operators respectively.

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Рассматриваются гиперциклические операторы на свободных банаховых пространствах и малых липшицевых пространствах, которые, в некотором смысле, обобщают операторы сдвига и операторы композиции соответственно.

**1. Introduction.** Let $X$ be a nonempty metric space and we fix a point $\theta_X \in X$. The pair $(X, \theta_X)$ is called a *pointed* space.

**Definition 1.** Let $X$ and $Y$ be metric spaces. A map $f\colon X \to Y$ is *Lipschitz* if there exists a constant $L_f \geq 0$ such that

$$\rho(f(p), f(q)) \leq L_f \rho(p, q)$$

for all $p, q \in X$.

The least such $L_f$ is called the *Lipschitz constant* of $f$. We denote by $\mathrm{Lip}_0(X, Y)$ the space of all Lipschitz maps between pointed metric spaces $(X, \theta_X)$ and $(Y, \theta_Y)$ which map $\theta_X$ into $\theta_Y$. In the case, when $Y$ is a linear space we suppose that $\theta_Y = 0$. It is known (see e.g. [4, 5]) that for an arbitrary metric pointed space $(X, \theta_X)$ there is a unique (up to isometrical isomorphism) Banach space $B(X)$ and a Lipschitz embedding $\nu\colon X \to B(X)$ such that for every normed space $E$ and any map $f(x) \in \mathrm{Lip}_0(X, E)$ there is a linear operator $\widetilde{f}(x)\colon B(X) \to E$ with $\widetilde{f}(\nu(x)) = f(x)$, $x \in X$ and $\|\widetilde{f}\| = L_f$. We denote by $\mathrm{span}X$ the linear span of $\nu(X)$ in $B(X)$ and elements by $\underline{x} = \nu(x)$. By the construction, elements $\sum_{k=1}^n \lambda_k \underline{x}_k$ are dense in $B(X)$. The space $B(X)$ is called a *free Banach space*.

A map $F$ from metric space $X$ to $X$ is called *topologically transitive* if there is a vector $x \in X$ such that the orbit $\mathrm{Orb}(F, x) = \{F^n(x) = \underbrace{F \circ \cdots \circ F}_{n}(x)\colon n \in \mathbb{N}\}$ is dense in $X$. In the case when $(X, \theta_X)$ is a metric pointed space we require that $\mathrm{Orb}(F, x)$ is dense in $X \setminus \theta$. Let $E$ be a Fréchet space. A linear continuous operator $T\colon E \to E$ is called *hypercyclic* if $T$ is topologically transitive. An element $x \in E$ is called a *hypercyclic vector* for $T$ if $\mathrm{Orb}(T, x)$

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is dense in $E$. A vector $x \in E$ is *cyclic* for $T$ provided the linear span of orbit $\mathrm{Orb}(T, x)$ is dense in $E$.

The study of hypercyclic operators started after Birkhoff's result ([3]) that the operator of composition with translation $x \mapsto x + a$, $a \neq 0$, $T_a\colon f(x) \mapsto f(x+a)$ is hypercyclic in the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$. R. Aron and J. Bès in [1] proved that the operator of composition with translation $T_a$ is hypercyclic in the space of weakly continuous analytic functions on all bounded subsets of a separable Banach space $X$ which are bounded on bounded subsets. A detailed survey of hypercyclic operators is given in [2].

## 2. Hypercyclic operators on $B(X)$.

**Theorem 1.** *Let $(X, \theta)$ be a complete metric pointed space and $F\colon X \to X$ be a topologically transitive map with $F(\theta) = \theta$. Then the linear operator $\widehat{F}\colon B(X) \to B(X)$ is cyclic.*

*Proof.* Let $x \in X$ be a hypercyclic vector of $F$. Then $\mathrm{Orb}(\widehat{F}, \underline{x})$ is dense in $\nu(X) = \underline{X}$. By the definition of $B(X)$, the space $\mathrm{span}X$ is dense in $B(X)$. Then the space $\mathrm{span}(\mathrm{Orb}(\widehat{F}, \underline{x}))$ is dense in $B(X)$ as well. □

We remark that in the general case, hypercyclicity of operator $\widehat{F}$ does not follow from the topological transitivity of $F$.

**Example 1.** Let $X = S^1 \bigcup \theta$ be the space with natural metric and $\theta = (0,0) \in \mathbb{R}^2$, where $S^1$ is the unit sphere in $\mathbb{R}^2$. We define a map of rotation $F$ on $X$ by an irrational angle $\alpha$. It is known that $F$ is topologically transitive and for every $x \in S^1$, $\mathrm{Orb}(F, x)$ is dense in $S^1$. So $\widehat{F}$ is a cyclic operator and for every $x \in S^1$, $\underline{x}$ is a cyclic vector. But the norm ofa hypercyclic operator must be strictly greater than 1 and we have $\|\widehat{F}\| = L_F = 1$.

We will use the Hypercyclicity Criterion (see [2]) to establish conditions of hypercyclicity of operators on a free Banach space.

**Theorem 2. (Hypercyclicity Criterion)** *Let $E$ be a separable Fréchet space. An operator $T\colon E \to E$ satisfies the Hypercyclicity Criterion provided there exist $X_0 \subset E$, $Y_0 \subset E$ dense subsets of $E$ and maps $S_n\colon Y \to E$, $n \in \mathbb{N}$, such that:*

*(i) $T_n x \to 0$, $n \to \infty$ for all $x \in X_0$,*

*(ii) $S_n y \to 0$, $n \to \infty$ for all $y \in Y_0$,*

*(iii) $(T_n \circ S_n)y \to y$, $n \to \infty$ for all $y \in Y$.*

**Theorem 3.** *Let $(X, \theta)$ be a separable complete metric space and $F\colon X \to X$, $F(\theta) = \theta$ be a 1–Lipschitz map. Suppose that $X$ can be represented as a countable union of nonempty, pairwise disjoint sets*

$$X = \bigcup_{n=0}^{\infty} A_n,$$

*where $A_0 = \theta$, $F(A_n) = A_{n-1}$ for any $n > 0$ and the restriction of $F$ to $A_n$ is injective for every $n > 1$. Then $T = \lambda \widehat{F}$ is a hypercyclic operator on $B(X)$ for any $\lambda$, $|\lambda| > 1$.*

*Proof.* We define $Y_0 = X_0 = \mathrm{span}(X \setminus \theta)$. It is a dense subset in $B(X)$ for every $z = \sum a_i \underline{x}_i \in X_0$, $S(z) = \sum a_i \frac{\widehat{F}^{-1}}{\lambda}(\underline{x}_i)$, $S_n(z) = S^n(z)$.

Since span$(X\backslash\theta)$ consists of formal finite sums, for every $z \in \text{span}(X \setminus \theta)$, $T^m(z) = 0$ starting with some number $m$. Therefore condition 1) is fulfilled. Thus if $|\lambda| > 1$ then

$$S^n(z) \leq \frac{1}{|\lambda|^n}\|z\| \to 0$$

as $n \to \infty$. Therefore condition 2) is fulfilled. Further $(T^n \circ S^n) = Id$ is the identical operator and therefore condition 3) is also fulfilled. □

**Example 2.** Let $X = \mathbb{N}\bigcup 0$ be the space with discrete metric and fixed point $\theta = 0$. It is known that $B(X) = \ell_1(\mathbb{N})$. We define $F\colon \mathbb{N} \to \mathbb{N}$, $F(n) = n - 1$ for $n \neq 0$ and $F(0) = 0$. Let $A_n = \{n\}$, then $F$ satisfies the conditions of Theorem 3. Observe, that $\lambda\widehat{F}(a_1, \dots, a_n, \dots) = \lambda(a_2, \dots, a_n, \dots)$ is the weighted left shift. It is well known that $\lambda\widehat{F}$ is hypercyclic for $|\lambda| > 0$.

**Theorem 4.** *Let $E$ be a separable Frechet space. If $T\colon E \to E$ is a hypercyclic operator satisfying the Hypercyclicity Criterion, then $\widehat{T}\colon B(E) \to B(E)$ is also a hypercyclic operator and satisfies the Hypercyclicity Criterion.*

*Proof.* Since $T$ satisfies the Hypercyclicity Criterion, there are apropriated spaces $X_0, Y_0$ and sequence of maps $S_n$. The spaces $X_0, Y_0$ are dense in $E$, then span$X_0$ and span$Y_0$ are dense sets in $B(E)$. We define $\widehat{S}_n(z) = \sum a_k S_n(x_k)$ for every $z = \sum a_k \underline{x}_k \in \text{span}Y_0$. It is easy to see that for $\widehat{T}$, $\widehat{S}_n$, span$X_0$ and span$Y_0$ the conditions of the criterion are fulfilled. Therefore $\widehat{T}$ satisfies the Hypercyclicity Criterion. □

**3. Hypercyclic operators on little Lipschitz space** lip$(X)$. We know that every $\text{Lip}_0(X)$ is a dual space. Like with a first predual, for a larger class of examples we actually have a nice, explicit description of a double predual. It is the subspace of $\text{Lip}_0(X)$ consisting of precisely those Lipschitz functions with a certain local flatness property ([5, p. 73]). This space is the "little" Lipschitz space $\text{lip}_0(X)$. The norm topology is the primary topology on $\text{lip}_0(X)$, while it plays the same role that the weak* topology in the case of $\text{Lip}_0(X)$. Also, the theory of $\text{lip}_0(X)$ breaks down when $X$ is not compact ([5]). We denote $\mathcal{M}_0^c$ the class of compact pointed metric spaces.

**Definition 2.** Let $X \in \mathcal{M}_0^c$ and $Y$ be a metric space, and let $f \in \text{Lip}(X, Y)$. Then $f$ is a *little Lipschitz function* if for every $\epsilon > 0$ the exists $\delta > 0$ such that

$$\rho(p, q) \leq \delta \quad \Rightarrow \quad \rho(f(p), f(q)) \leq \epsilon\rho(p, q).$$

The little Lipschitz space of real valued functions lip$(X)$ is the subset of Lip$(X)$. Similarly, $\text{lip}_0(X)$ is the subset of the space of Lipschitz functions $\text{Lip}_0(X)$. The space $\text{lip}_0(X)$ is a Banach space. It is known that $\text{lip}_0(X)^{**} \cong \text{Lip}_0(X)$ (see [5]).

Let $T \in \text{lip}_0(X, X)$ for a compact metric space $X$. Our purpose is to study the composition operator on $\text{lip}_0(X)$.

**Theorem 5.** *Let $X$ be a compact metric space and $T \in \text{lip}_0(X, X)$ be a surjective map. Suppose that $X$ can be represented as a countable union of nonempty, pairwise disjoint sets*

$$X = \bigcup_{n=0}^{\infty} A_n,$$

*where* $T(A_n) = A_{n+1}$ *for any* $n > 0$ *and* $A_0 = \theta$. *Then for every* $\lambda$, $|\lambda| > 1$, *the composition operator*

$$C_{\lambda T}\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X), \quad \lambda f \mapsto f \circ T$$

*is a hypercyclic operator on* $\mathrm{lip}_0(X)$.

*Proof.* Let us consider an operator $\lambda\widehat{T}\colon B(X) \to B(X)$, where $B(X)$ is a free Banach space. We know that $\mathrm{lip}_0^{**}(X) = B^*(X)$. We will assert that $\widehat{T}^*\colon B^*(X) \to B^*(X)$ satisfies the Hypercyclic Criterion. Let us define $Y_0 = X_0 = \mathrm{span}(X)$. It is a dense subset in $B^*(X)$. Since $T(A_n) = A_{n+1}$, $T^*(\mathrm{span}A_{n+1}) = \mathrm{span}A_n$. For every $z = \sum a_i \underline{x}_i \in X_0$, $S(z) = \sum a_i \frac{\widehat{T^*}^{-1}}{\lambda}(\underline{x}_i)$ for any $\lambda$, $|\lambda| > 1$, $S_n(z) = S^n(z)$. It is easy to see that $(\lambda\widehat{T}^*)^m = 0$ starting with some number $m$, $S^n(z) \to 0$ and $(\lambda T^n \circ S^n) = Id$. So $\lambda\widehat{T}^*$ satisfies the Hypercyclicity Criterion. Hence, the restriction of $\lambda\widehat{T}^*$ to $\mathrm{lip}_0(X) \subset B^*(X)$, $C_{\lambda T}$ satisfies the Hypercyclicity Criterion. Therefore the composition operator $C_{\lambda T}$ is hypercyclic on $\mathrm{lip}_0(X)$. □

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