УДК 517.95

M. M. Bokalo

UNBOUNDED, PERIODIC AND ALMOST PERIODIC SOLUTIONS OF ANISOTROPIC PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

M. M. Bokalo. *Unbounded, periodic and almost periodic solutions of anisotropic parabolic equations with variable exponents of nonlinearity*, Mat. Stud. **41** (2014), 81–91.

We prove the well-posedness of Fourier problems for anisotropic parabolic equations with variable exponents of nonlinearity without any assumptions on the solution behavior and growth of the initial data as time variable tends to minus infinity. We obtain estimates for generalized solutions of these problems as well as conditions for the existence of periodic and almost periodic solutions. Moreover, we prove some properties of the solutions of the problems under consideration.

М. М. Бокало. *Неограниченные, периодические и почти периодические решения анизотропных параболических уравнений с переменными показателями нелинейности* // Мат. Студії. – 2014. – Т.41, №1. – C.81–91.

Доказана корректность задач Фурье для анизотропных параболических уравнений с переменными показателями нелинейности без предположений о поведении решений и росте исходных данных при стремлении часовой переменной к минус бесконечности. Получены оценки обобщенных решений этих задач и условия существования периодических и почти периодических решений. Также установлены некоторые свойства решений рассматриваемой задачи.

Introduction. We examine a question of well-posedness of the Fourier problems (the problems without initial conditions) for anisotropic second order parabolic equations with variable exponents of nonlinearity. These equations are defined on unbounded cylindrical domains, which are the Cartesian products of bounded space domains and the whole time axis. Also the existence conditions of periodic and almost periodic solutions are investigated. Moreover, we examine the conditions on input data that guarantee specific behavior of the solutions at infinity.

Fourier problems for parabolic equations are examined in many papers ([1]–[11]). Fairly good survey of results regarding these problems can be found in [11]. It is worth to mention that Fourier problems for linear and a plenty of nonlinear parabolic equations are well posed only under some restrictions on the growth of solutions and input data as the time variable tends to $-\infty$, in addition to boundary conditions. However, there are nonlinear equations for which the Fourier problems are uniquely solvable with no conditions at infinity. This case for equations with variable exponents of nonlinearity is considered here. We look for solutions from the generalized Lebesgue and Sobolev spaces. More information on these

2010 *Mathematics Subject Classification:* 35K10, 35K55, 35K92.
Keywords: parabolic equation; Fourier problem; periodic solution; almost periodic solution.

spaces and about its applying can be find in [12]–[18]. The present paper can be viewed as a natural continuation of papers [4, 8, 9] for the case of equations with variable exponents of nonlinearity.

The paper consists of three parts: in the first part the formulation of problem and main results are presented, the second part includes auxiliary statements while the proofs of main results are in the third part.

1. Setting of the problem and main results. Let Ω be a bounded domain in $\mathbb{R}^n$ with the piecewise smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ is divided into two subsets Γ_0 and Γ_1, where Γ_0 is closed. The cases $\Gamma_0 = \varnothing$ and $\Gamma_0 = \partial\Omega$ are also possible. We denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit outward normal vector on $\partial\Omega$. Set $Q := \Omega \times \mathbb{R}$, $\Sigma_0 := \Gamma_0 \times \mathbb{R}$, $\Sigma_1 := \Gamma_1 \times \mathbb{R}$, and $Q_{t_1,t_2} := \Omega \times (t_1, t_2)$ for arbitrary real t_1 and t_2. Here and subsequently, we assume that $t_1 < t_2$.

Consider the problem of finding a function $u\colon \overline{Q} \to \mathbb{R}$ satisfying (in some sense) the equation

$$u_t - \sum_{i=1}^{n} \big(a_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i}\big)_{x_i} + a_0(x,t)|u|^{p_0(x)-2}u = f(x,t), \quad (x,t) \in Q, \tag{1}$$

and the boundary conditions

$$u\Big|_{\Sigma_0} = 0, \qquad \frac{\partial u}{\partial \nu_a}\Big|_{\Sigma_1} = 0, \tag{2}$$

where $\partial u(x,t)/\partial\nu_a := \sum_{i=1}^{n} a_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i}\,\nu_i(x)$ is the "conormal" derivative on Σ_1, and the functions $p_j\colon \Omega \to \mathbb{R}$, $a_j\colon Q \to \mathbb{R}$ $(j = 0, \dots, n)$, $f\colon Q \to \mathbb{R}$ are given.

First we introduce some function spaces. Suppose that either $G = \Omega$ or $G = \Omega \times S$, where S is an interval in $\mathbb{R}$. We also consider a function $r \in L_\infty(\Omega)$ such that $r(x) \ge 1$ for almost all $x \in \Omega$. We denote by $L_{r(\cdot)}(G)$ the generalized Lebesgue space consisting of functions $v \in L_1(G)$ such that $\rho_{G,r}(v) < \infty$, where $\rho_{G,r}(v) := \int_\Omega |v(x)|^{r(x)}dx$ for $G = \Omega$, and $\rho_{G,r}(v) := \int_G |v(x,t)|^{r(x)}dxdt$ for $G = \Omega \times S$. The space is equipped with the norm $\|v\|_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 \mid \rho_{G,r}(v/\lambda) \le 1\}$ ([12, p. 599]). If $\operatorname{ess\,inf}_{x\in\Omega} r(x) > 1$, then the dual space $[L_{r(\cdot)}(G)]'$ can be identified with $L_{r'(\cdot)}(G)$, where r' is the function defined by the equality $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for almost all $x \in \Omega$.

Let $G = \Omega \times S$, where S is an unbounded interval in $\mathbb{R}$ or $S = \mathbb{R}$. We denote by $L_{r(\cdot),\mathrm{loc}}(\overline{G})$ the space of measurable functions $g\colon G \to \mathbb{R}$ such that the restriction of g to Q_{t_1,t_2} belongs to $L_{r(\cdot)}(Q_{t_1,t_2})$ for each $t_1, t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\big\{\|\cdot\|_{L_{r(\cdot)}(Q_{t_1,t_2})} \,\big|\, t_1, t_2 \in S\big\}$. A sequence $\{g_m\}$ is said to be convergent strongly (resp., weakly) in $L_{r(\cdot),\mathrm{loc}}(\overline{G})$ provided the sequences of restrictions $\{g_m|_{Q_{t_1,t_2}}\}$ are convergent strongly (resp., weakly) in $L_{r(\cdot)}(Q_{t_1,t_2})$ for all $t_1, t_2 \in S$. Similarly we can define the space $L_{\infty,\mathrm{loc}}(\overline{G})$.

Let B be a Banach space with a norm $\|\cdot\|_B$. We also denote by $C(S;B)$ the space of functions $v\colon S \to B$ such that restriction of v to any interval $[t_1, t_2] \subset S$ belongs to $C([t_1, t_2]; B)$. The space $C(S;B)$ is complete locally convex with respect to the family of seminorms $\big\{\max_{t\in[t_1,t_2]} \|v(t)\|_B \,\big|\, t_1, t_2 \in S\big\}$. Therefore a sequence $\{g_m\}$ is convergent in $C(S; L_2(\Omega))$ provided the sequences of restrictions $\{g_m|_{[t_1,t_2]}\}$ are convergent in $C([t_1, t_2]; B)$ for each $t_1, t_2 \in S$.

Let $p = (p_0, \dots, p_n)\colon \Omega \to \mathbb{R}^{1+n}$ be a vector-function satisfying the following conditions:

($\mathcal{P}$) the functions $p_j\colon \Omega \to \mathbb{R}$ are measurable for all $j \in \{0, \dots, n\}$, $p_0^- := \operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 2$,
$p_i^- := \operatorname*{ess\,inf}_{x\in\Omega} p_i(x) \geq 2$ for $i \in \{1, \dots, n\}$,
$p_j^+ := \operatorname*{ess\,sup}_{x\in\Omega} p_i(x) < +\infty$ for $j \in \{0, \dots, n\}$.

We also denote by $p' := ({p_0}', \dots, {p_n}')$ the vector whose components are given by the equalities $1/p_j(x) + 1/{p_j}'(x) = 1$ for almost all $x \in \Omega$.

Let $W^1_{p(\cdot)}(\Omega)$ be the generalized Sobolev space consisting of functions $v \in L_{p_0(\cdot)}(\Omega)$ such that $v_{x_i} \in L_{p_i(\cdot)}(\Omega)$ for all $i \in \{1, \dots, n\}$. The space is equipped with the norm $\|v\|_{W^1_{p(\cdot)}(\Omega)} := \|v\|_{L_{p_0(\cdot)}(\Omega)} + \sum_{i=1}^n \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega)}$. We denote by $\widetilde{W}^1_{p(\cdot)}(\Omega)$ the closure of the set $\big\{v \in C^1(\overline{\Omega}) \ \big|\ v|_{\Gamma_0} = 0\big\}$ in the space $W^1_{p(\cdot)}(\Omega)$. Next, let $W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ denote the set of functions $w \in L_{p_0(\cdot)}(Q_{t_1,t_2})$ such that $w_{x_i} \in L_{p_i(\cdot)}(Q_{t_1,t_2})$ for all $i \in \{1, \dots, n\}$. We also define the norm $\|w\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} := \|w\|_{L_{p_0(\cdot)}(Q_{t_1,t_2})} + \sum_{i=1}^n \|w_{x_i}\|_{L_{p_i(\cdot)}(Q_{t_1,t_2})}$. We denote by $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ the subspace of $W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ consisting of functions v such that $v(\cdot,t) \in \widetilde{W}^1_{p(\cdot)}(\Omega)$ for a. e. $t \in [t_1, t_2]$.

Assume $G = \Omega \times S$, where S is a real interval or the real axis. Let $\widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{G})$ be the linear space of measurable functions such that its restrictions to Q_{t_1,t_2} belong to $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ for all $t_1, t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\big\{\|\cdot\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} \,\big|\, t_1, t_2 \in \mathbb{R}\big\}$.

We also introduce the space $\mathbb{U}_{p,\mathrm{loc}} = \widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q}) \cap C(\mathbb{R}; L_2(\Omega))$, which is a complete linear local convex space with respect to the family of seminorms

$$\Big\{\|w\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} + \max_{t\in[t_1,t_2]} \|w(\cdot,t)\|_{L_2(\Omega)} \,\Big|\, t_1, t_2 \in \mathbb{R}\Big\}.$$

For an interval I we consider the space $C^1_0(I)$ of $C^1(I)$-functions with compact support.

Let us denote by $\mathbb{A}$ the set of ordered arrays of functions $(a_0, a_1, \dots, a_n)$ satisfying the condition

($\mathcal{A}$): for each $j \in \{0, 1, \dots, n\}$ the function a_j belongs to the space $L_{\infty,\mathrm{loc}}(\overline{Q})$ and the following holds

$$a_j(x,t) \geq K_1 \quad \text{for almost all} \quad (x,t) \in Q \tag{3}$$

with some constant $K_1 > 0$ being dependent on $(a_0, a_1, \dots, a_n)$.

Definition 1. Suppose that p satisfies condition ($\mathcal{P}$), $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, and $f \in L_{{p_0}'(\cdot),\mathrm{loc}}(\overline{Q})$. A function u is called a *weak solution of* (1), (2) provided $u \in \mathbb{U}_{p,\mathrm{loc}}$ and the following integral identity holds

$$\iint_Q \Big\{\sum_{i=1}^n \big(a_i|u_{x_i}|^{p_i(x)-2}u_{x_i}\psi_{x_i} + a_0|u|^{p_0(x)-2}u\psi\big)\varphi - u\psi\varphi'\Big\}dxdt = \iint_Q f\psi\varphi dxdt \tag{4}$$

for all $\psi \in \widetilde{W}^1_{p(\cdot)}(\Omega)$, $\varphi \in C^1_0(\mathbb{R})$.

We say that a weak solution of (1), (2) *continuously depends on input data*, if for each sequence $\{f_k\}_{k=1}^\infty \subset L_{{p_0}'(\cdot),\mathrm{loc}}(\overline{Q})$ such that $f_k \underset{k\to\infty}{\longrightarrow} f$ in $L_{{p_0}'(\cdot),\mathrm{loc}}(\overline{Q})$ we have $u_k \underset{k\to\infty}{\longrightarrow} u$ in $\mathbb{U}_{p,\mathrm{loc}}$. Here u_k and u are weak solutions of (1), (2) with the right-hand sides f_k and f, respectively.

Theorem 1. *Suppose that p satisfies $(\mathcal{P})$, $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, and $f \in L_{p_0'(\cdot),loc}(\overline{Q})$. Then there exists a unique weak solution of (1), (2), and it continuously depends on the input data. Moreover, the estimate*

$$\max_{t\in[t_0-R_0,t_0]} \int_\Omega |u(x,t)|^2 dx + \int_{t_0-R_0}^{t_0} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)} \Big] dxdt \leq$$
$$\leq C_1 \Big\{ R^{-2/(p_0^+-2)} + \int_{t_0-R}^{t_0} \int_\Omega |f(x,t)|^{p_0'(x)} dxdt \Big\}, \tag{5}$$

holds for each R, R_0 such that $R \geq 1$, $0 < R_0 < R/2$, and $t_0 \in \mathbb{R}$. Here C_1 is a positive constant, which depends on K_1 and $p_j^\pm$ $(j \in \{0, \dots, n\})$ only.

Remark 1. Note that Theorem 1 has no conditions imposed on the behaviour of the solution and the growth of functions a_j $(j \in \{0, \dots, n\})$ as well as on the behaviour of f as $t \to -\infty$. But the theorem is not longer true for the case where $p_0(x) = p_1(x) = \dots = p_n(x) = 2$ for almost all $x \in \Omega$ (see, for example, [11]). Therefore condition $(\mathcal{P})$ is essential.

A solution u of (1), (2) is called *bounded* if $\sup\limits_{t\in\mathbb{R}} \int_\Omega |u(x,t)|^2 dx < \infty$.

Corollary 1. *Let $f \in L_{p_0'(\cdot)}(Q)$. Under the assumptions of Theorem 1, any weak solution of (1), (2) is bounded; it belongs to $\widetilde{W}^{1,0}_{p(\cdot)}(Q)$ and the following estimate holds*

$$\sup_{t\in\mathbb{R}} \int_\Omega |u(x,t)|^2 dx + \iint_Q \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)} \Big] dxdt \leq$$
$$\leq C_1 \iint_Q |f(x,t)|^{p_0'(x)} dxdt. \tag{6}$$

Corollary 2. *Under the assumptions of Theorem 1, if*

$$\sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t)|^{p_0'(x)} dxdt \leq C_2$$

for some positive constant C_2, then a weak solution u of (1), (2) is bounded. In addition,

$$\sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)} \Big] dxdt \leq C_3$$

with some positive constant C_3 being dependent on $K_1, p_j^\pm (j \in \{0, \dots, n\})$ and C_2 only.

Corollary 3. *Under the assumptions of Theorem 1, if moreover*

$$\lim_{\tau\to\pm\infty} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t)|^{p_0'(x)} dxdt = 0,$$

then for a weak solution u of problem (1), (2) the following relations hold

$$\lim_{t\to\pm\infty} \|u(\cdot,t)\|_{L_2(\Omega)} = 0, \quad \lim_{\tau\to\pm\infty} \int_{\tau-1}^{\tau} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)} \Big] dxdt = 0.$$

Theorem 2. *Under the assumptions of Theorem 1, if the functions f, $a_0,\dots,a_n$ are periodic in time with a period $\sigma > 0$, then a weak solution of (1), (2) is also σ-periodic in time.*

A set $X \subset \mathbb{R}$ is called *relatively dense*, if there exists a positive l such that the interval $[a, a+l]$ contains at least one element of the set X for any real a, i.e. $X \cap [a, a+l] \neq \varnothing$.

Let B be a Banach space with a norm $\|\cdot\|_B$. A function $v \in C(\mathbb{R}; B)$ is *Bohr almost periodic*, if for each $\varepsilon > 0$ the set $\{\sigma \mid \sup_{t\in\mathbb{R}} \|v(\cdot, t+\sigma) - v(\cdot, t)\|_B \le \varepsilon\}$ is relatively dense. A function $f \in L_{p_0(\cdot),\mathrm{loc}}(\overline{Q})$ is *Stepanov almost periodic* provided the set

$$\Big\{\sigma \mid \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau}\int_\Omega |f(x,t+\sigma) - f(x,t)|^{p_0(x)} dxdt \le \varepsilon\Big\}$$

is relatively dense for each positive ε. Viewing w as an element of $\widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q})$, we say that w is *almost periodic by Stepanov*, if for each $\varepsilon > 0$ the set $\{\sigma \mid \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau}\int_\Omega \big[\sum_{i=1}^n |w_{x_i}(x, t+\sigma) - w_{x_i}(x,t)|^{p_i(x)} + |w(x,t+\sigma) - w(x,t)|^{p_0(x)}\big] dxdt \le \varepsilon\}$ is relatively dense. We refer to [4, 19, 20] for the detailed information on the theory of almost periodic functions.

Theorem 3. *Let the hypotheses of Theorem 1 hold. In addition, suppose $a_0,\dots,a_n$ are Bohr almost periodic functions in $C(\mathbb{R}; L_\infty(\Omega))$. Assume also that f is Stepanov almost periodic in $L_{p_0(\cdot),loc}(\overline{Q})$. Moreover, the set*

$$F_\varepsilon := \{\sigma \mid \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau}\int_\Omega |f(x,t+\sigma) - f(x,t)|^{p_0(x)} dxdt \le \varepsilon,$$

$$\max_{j\in\{0,\dots,n\}} \sup_{t\in\mathbb{R}} \|a_j(\cdot,t+\sigma) - a_j(\cdot,t)\|_{L_\infty(\Omega)} \le \varepsilon\}$$

is relatively dense for each $\varepsilon > 0$.

Then the (unique) weak solution of (1), (2) is Bohr almost periodic in $C(\mathbb{R}; L_2(\Omega))$ and Stepanov almost periodic in $\widetilde{W}^{1,0}_{p(\cdot),loc}(\overline{Q})$.

2. Auxiliary statements. We start with some auxiliary results, which will be used below.

Lemma 1. *Given $t_1, t_2 \in \mathbb{R}$, we assume that a function $w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ satisfies the identity*

$$\int_{t_1}^{t_2}\int_\Omega \Big\{\Big(\sum_{i=1}^n g_i\psi_{x_i} + g_0\psi\Big)\varphi - w\psi\varphi'\Big\} dxdt = 0, \quad \psi \in \widetilde{W}^1_{p(\cdot)}(\Omega),\ \varphi \in C^1_0(t_1,t_2) \tag{7}$$

for some functions $g_j \in L_{p_j'(\cdot)}\big(Q_{t_1,t_2}\big)$ $(j \in \{0,\dots,n\})$. Then $w \in C([t_1,t_2]; L_2(\Omega))$ and the following equality

$$\theta(t)\int_\Omega |w(x,t)|^2 dx\Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2}\int_\Omega |w|^2\theta' dxdt + 2\int_{\tau_1}^{\tau_2}\int_\Omega \Big(\sum_{i=1}^n g_i w_{x_i} + g_0 w\Big)\theta dxdt = 0 \tag{8}$$

holds for all $\tau_1, \tau_2 \in [t_1,t_2]$ $(\tau_1 < \tau_2)$, $\theta \in C^1([t_1,t_2])$.

This statement can be proved similarly to Lemma 1 in [4].

Lemma 2. *Given $t_1, t_2 \in \mathbb{R}$ such that $t_2 - t_1 \geq 1$ and $a \in \mathbb{A}$, we suppose that functions u_1 and u_2 from $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2}) \cap C([t_1, t_2]; L_2(\Omega))$ fulfill the following identities*

$$\int_{t_1}^{t_2}\int_\Omega \Big\{\Big(\sum_{i=1}^n a_i|u_{l,x_i}|^{p_i(x)-2}u_{l,x_i}\psi_{x_i} + a_0|u_l|^{p_0(x)-2}u_l\psi\Big)\varphi - u_l\psi\varphi'\Big\}dxdt =$$
$$= \int_{t_1}^{t_2}\int_\Omega \Big(\sum_{i=1}^n f_{i,l}\psi_{x_i} + f_{0,l}\psi\Big)\varphi dxdt, \quad \psi \in \widetilde{W}^1_{p(\cdot)}(\Omega),\ \varphi \in C_0^1(t_1, t_2) \tag{9}$$

with the functions $f_{j,l} \in L_{p_j{}'(\cdot)}(Q_{t_1,t_2})$ $(j \in \{0, \dots, n\}; l \in \{1, 2\})$, respectively.

Then the following

$$\max_{t\in[t_0-R_0,t_0]}\int_\Omega |u_1(x,t) - u_2(x,t)|^2\,dx + \int_{t_0-R_0}^{t_0}\int_\Omega \Big(\sum_{i=1}^n |u_{1,x_i} - u_{2,x_i}|^{p_i(x)} + |u_1 - u_2|^{p_0(x)}\Big)\,dxdt \leq$$
$$\leq C_4\Big\{R^{-2/(p_0^+-2)} + \int_{t_0-R}^{t_0}\int_\Omega \sum_{j=0}^n |f_{j,1}(x,t) - f_{j,2}(x,t)|^{p_j{}'(x)}\,dxdt\Big\} \tag{10}$$

holds for each R, R_0 and t_0 such that $R \geq 1$, $0 < R_0 < R/2$, and $t_1 \leq t_0 - R < t_0 \leq t_2$. Here C_4 is a positive constant, which depends on K_1 and $p_j^\pm$ $(j \in \{0, \dots, n\})$ only.

Proof of Lemma 2. Let R, R_0, t_0 be as in the lemma statement, and $\eta(t) := t - t_0 + R$, $t \in \mathbb{R}$ (see [21]). For given $\psi \in \widetilde{W}^1_{p(\cdot)}(\Omega)$, $\varphi \in C_0^1(t_1, t_2)$ we subtract equality (9) with $l = 1$, and the same equality with $l = 2$. Then, putting

$$u_{12}(x,t) := u_1(x,t) - u_2(x,t),\ \ f_{j,12}(x,t) := f_{j,1}(x,t) - f_{j,2}(x,t),$$
$$a_{0,12}(x,t) := a_0(x,t)\big(|u_1(x,t)|^{p_0(x)-2}u_1(x,t) - |u_2(x,t)|^{p_0(x)-2}u_2(x,t)\big),$$
$$a_{i,12}(x,t) := a_i(x,t)\big(|u_{1,x_i}(x,t)|^{p_i(x)-2}u_{1,x_i}(x,t) - |u_{2,x_i}(x,t)|^{p_i(x)-2}u_{2,x_i}(x,t)\big)$$
$$(i \in \{1, \dots, n\};\ j \in \{0, \dots, n\};\ (x,t) \in Q),$$

we obtain an equality. From this equality using Lemma 1 with $w = u_{12}$, $g_j = a_{j,12} - f_{j,12}$ $(j \in \{0, \dots, n\})$, $\theta = \eta^s$, $s := p_0^-/(p_0^- - 2)$, $\tau_1 = t_0 - R$, $\tau_2 = \tau \in (t_0 - R, t_0]$, we get the following equality

$$\eta^s(\tau)\int_\Omega |u_{12}(x,\tau)|^2dx + 2\int_{t_0-R}^{\tau}\int_\Omega \Big\{\sum_{i=1}^n a_{i,12}(u_{12})_{x_i} + a_{0,12}u_{12}\Big\}\eta^s dxdt =$$
$$= s\int_{t_0-R}^{\tau}\int_\Omega |u_{12}|^2\eta^{s-1}dxdt + 2\int_{t_0-R}^{\tau}\int_\Omega \Big(\sum_{i=1}^n f_{i,12}(u_{12})_{x_i} + f_{0,12}u_{12}\Big)\eta^s dxdt. \tag{11}$$

We make corresponding estimates of the integrals in equality (11). First we note if $r \in L_\infty(\Omega)$ and $\operatorname{ess\,inf}_{x\in\Omega} r(x) \geq 2$, then in view of Lemma 1.2 of [5] we have the following inequality

$$(|s_1|^{r(x)-2}s_1 - |s_2|^{r(x)-2}s_2)(s_1 - s_2) \geq 2^{2-r^+}|s_1 - s_2|^{r(x)}$$

for each $s_1, s_2 \in \mathbb{R}$ and for almost all $x \in \Omega$ (here $r^+ := \operatorname{ess\,sup}_{x\in\Omega} r(x)$). Using this inequality we get

$$\int_{t_0-R}^{\tau}\int_\Omega \Big\{\sum_{i=1}^n a_{i,12}(u_{12})_{x_i} + a_{0,12}\,u_{12}\Big\}\eta^s dxdt \geq$$

$$\geq C_5 \int_{t_0-R}^{\tau} \int_{\Omega} \Big(\sum_{i=1}^{n} |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)} \Big) \eta^s dxdt, \tag{12}$$

where $C_5 > 0$ is a constant depending only on K_1 and p_j^+ $(j \in \{0, \dots, n\})$.

Further we need the following inequality

$$a\, b \leq \varepsilon |a|^q + \varepsilon^{-1/(q-1)} |b|^{q'}, \quad a, b \in \mathbb{R},\ q > 1,\ 1/q + 1/q' = 1,\ \varepsilon > 0, \tag{13}$$

which is a consequence of standard Young's inequality: $a\, b \leq |a|^q/q + |b|^{q'}/q'$.

Putting (for almost all $x \in \Omega$) $q = p_0(x)/2$, $q' = p_0(x)/(p_0(x) - 2)$, $a = |u_{12}|^2 \eta^{s/q}$, $b = \eta^{s/q'-1}$, $\varepsilon = \varepsilon_1 > 0$, under (13) we obtain

$$\int_{t_0-R}^{\tau} \int_{\Omega} |u_{12}|^2 \eta^{s-1} dxdt \leq \varepsilon_1 \int_{t_0-R}^{\tau} \int_{\Omega} |u_{12}|^{p_0(x)} \eta^s dxdt + \\ + \varepsilon_1^{-2/(p_0^- - 2)} \int_{t_0-R}^{\tau} \int_{\Omega} \eta^{s - p_0(x)/(p_0(x)-2)} dxdt, \tag{14}$$

where $\varepsilon_1 \in (0, 1)$ is an arbitrary number.

Again using inequality (13), we get

$$\int_{t_0-R}^{\tau} \int_{\Omega} \Big(\sum_{i=1}^{n} f_{i,12} (u_{12})_{x_i} + f_{0,12} u_{12} \Big) \eta^s dxdt \leq \varepsilon_2 \int_{t_0-R}^{\tau} \int_{\Omega} \Big(\sum_{i=1}^{n} |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)} \Big) \eta^s dxdt + \\ + \int_{t_0-R}^{\tau} \int_{\Omega} \Big(\sum_{j=0}^{n} \varepsilon_2^{-1/(p_j^- - 1)} |f_{j,12}|^{p_j{}'(x)} \Big) \eta^s dxdt, \tag{15}$$

where $\varepsilon_2 \in (0, 1)$ is an arbitrary number.

If ε_1 and ε_2 are sufficiently small positive, then from (11), (12), (14), (15) we get the following

$$\eta^s(\tau) \int_{\Omega_R} |u_{12}(x, \tau)|^2 dx + \int_{t_0-R}^{\tau} \int_{\Omega} \Big\{ \sum_{i=1}^{n} |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)} \Big\} \eta^s dxdt \leq \\ \leq C_6 \Big[\int_{t_0-R}^{\tau} \int_{\Omega} \eta^{s - p_0(x)/(p_0(x)-2)} dxdt + \int_{t_0-R}^{\tau} \int_{\Omega} \Big(\sum_{j=0}^{n} |f_{j,12}|^{p_j{}'(x)} \Big) \eta^s dxdt \Big], \tag{16}$$

where C_6 is a positive constant depending only on K_1 and $p_j^{\pm}$ $(j \in \{0, \dots, n\})$, and $\tau \in (t_0 - R, t_0]$ is an arbitrary number.

Note that $0 \leq \eta(t) \leq R$, if $t \in [t_0 - R, t_0]$, and $\eta(t) \geq R - R_0$, if $t \in [t_0 - R_0, t_0]$, where $R_0 \in (0, R)$ is an arbitrary number. Using this and that $R \geq \max\{1; 2R_0\}$ (then, in particular, we have $R/(R - R_0) = 1 + R_0/(R - R_0) \leq 2$), from (16) we get the needed statement. □

3. Proof of the main results.

Proof of Theorem 1. First we prove that there exists at most one weak solution of problem (1), (2). Assume the contrary. Let u_1, u_2 be (different) weak solutions of this problem. Using Lemma 2 we get

$$\max_{t \in [t_0 - R_0, t_0]} \int_{\Omega} |u_1(x, t) - u_2(x, t)|^2 dx \leq C_4 R^{-2/(p_0^+ - 2)}, \tag{17}$$

where R, R_0, t_0 are arbitrary numbers such that $R \geq 1$, $0 < R_0 < R/2$, $t_0 \in \mathbb{R}$.

We fix arbitrary numbers $R_0 > 0$, $t_0 \in \mathbb{R}$, and pass to the limit in (17) as $R \to +\infty$. As a result we obtain that $u_1 = u_2$ almost everywhere on $Q_{t_0-R_0,t_0}$. Since R_0, t_0 are arbitrary numbers, we get $u_1 = u_2$ almost everywhere on Q. The obtained contradiction proves our statement.

Now we are turning to a proof of the existence of a weak solution of problem (1), (2). For each $m \in \mathbb{N}$ we consider an initial-boundary value problem for equation (1) in the domain $Q_m = \Omega \times (-m, +\infty)$ with homogeneous initial condition and boundary conditions (2), namely: we are searching a function $u_m \in \widetilde{W}^{1,0}_{p(\cdot),loc}(\overline{Q_m}) \cap C([-m, +\infty); L_2(\Omega))$, which satisfies the initial condition: $u_m|_{t=-m} = 0$ and the integral equality

$$\iint_{Q_m} \Big\{ \Big(\sum_{i=1}^{n} a_i |u_{m,x_i}|^{p_i(x)-2} u_{m,x_i} \psi_{x_i} + a_0 |u_m|^{p_0(x)-2} u_m \psi \Big) \varphi - u_m \psi \varphi' \Big\} dxdt = \iint_{Q_m} f_m \psi \varphi dxdt \tag{18}$$

for each $\psi \in \widetilde{W}^1_{p(\cdot)}(\Omega)$, $\varphi \in C^1_0(-m, +\infty)$, where $f_m(x,t) := f(x,t)$ if $(x,t) \in Q_m$, and $f_m(x,t) := 0$ if $(x,t) \in Q \setminus Q_m$. The existence and uniqueness of the function u_m follows from the well-known fact (see, for example, [14]).

We extend u_m to Q by zero and this extension is denoted by u_m again. Further, we prove that the sequence $\{u_m\}$ converges in $\mathbb{U}_{p,loc}$ to a weak solution of problem (1), (2). Indeed, note that for each $m \in \mathbb{N}$ the fuction u_m is a weak solution of the problem, which differs from problem (1), (2) in f_m instead of f. Using Lemma 2 for each natural numbers m and k we have

$$\max_{t\in[t_0,t_0-R_0]} \int_\Omega |u_m(x,t) - u_k(x,t)|^2 dx + \int_{t_0-R_0}^{t_0} \int_\Omega \Big[\sum_{i=1}^{n} |u_{m,x_i} - u_{k,x_i}|^{p_i(x)} + |u_m - u_k|^{p_0(x)} \Big] dxdt \leq$$
$$\leq C_4 \Big\{ R^{-2/(p_0^+-2)} + \int_{t_0-R}^{t_0} \int_\Omega |f_m - f_k|^{p_0(x)} dxdt \Big\} \tag{19}$$

where R, R_0, t_0 are arbitrary numbers such that $t_0 \in R$, $R \geq 1$, $0 < R_0 < R/2$.

We show that for fixed t_0 and R_0 the left hand side of inequality (19) converges to zero as $m, k \to +\infty$. Actually, let $\varepsilon > 0$ be an arbitrary indefinitely small number. We choose $R \geq \max\{1, 2R_0\}$ to be large enough such that the following inequality holds

$$C_4 R^{-2/(p_0^+-2)} < \varepsilon. \tag{20}$$

It is possible as $p_0^+ - 2 > 0$. Under (20) for arbitrary $m, k \in \mathbb{N}$ such that $\max\{-m, -k\} \leq t_0 - R$ (then $f_m = f_k$ almost everywhere on $\Omega \times (t_0 - R, t_0)$) the right hand side of inequality (19) is less then ε. Hence the restriction of the terms of the sequence $\{u_m\}$ to $Q_{t_0-R_0,t_0}$ is a Cauchy sequence in $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_0-R_0,t_0}) \cap C([t_0 - R_0, t_0]; L_2(\Omega))$. Therefore, since t_0 and R_0 are arbitrary, it follows that there exists a function $u \in \mathbb{U}_{p,loc}$ such that $u_m \to u$ in $\mathbb{U}_{p,loc}$. Taking into account that in (18) the integration on Q_m can be replaced with the integration on Q, we pass to the limit in this equality as $m \to \infty$. As a result we get (4) for all $\psi \in \widetilde{W}^1_{p(\cdot)}(\Omega)$ and $\varphi \in C^1_0(\mathbb{R})$. It means that the function u is a weak solution of problem (1), (2). Estimate (5) directly follows from Lemma 2 when putting $u_1 = u$, $u_2 = 0$, $f_{0,1} = f$, $f_{i,1} = 0\,(i \in \{1, \ldots, n\})$, $f_{j,2} = 0\,(j \in \{0, \ldots, n\})$. Continuous dependence of a weak solution of problem (1), (2) on input data is easily proved using Lemma 2 with u_k and f_k instead of u_1 and $f_{0,1}$ respectively, and also u and f instead of u_2 and $f_{0,2}$ respectively, putting $f_{i,1} = f_{i,2} = 0\,(i \in \{1, \ldots, n\})$. $\square$

Proofs of Corollaries 1–3. These statements follow from estimate (5). □

Proof of Theorem 2. Let u denote a weak solution of problem (1), (2). Put $u^{(\mu)}(x,t) := u(x,t+\mu)$, $f^{(\mu)}(x,t) := f(x,t+\mu,)$ $a_j^{(\mu)}(x,t) := a_j(x,t+\mu)$, $(x,t)\in Q$, where $\mu\in\mathbb{R}$. Substitute t by $t+\mu$ ($\mu\in\mathbb{R}$ is arbitrary at present) in (4). As a result we get an identity, which we will write in the form

$$\iint_Q\Big\{\Big(\sum_{i=1}^n a_i^{(0)}|u_{x_i}^{(\mu)}|^{p_i(x)-2}u_{x_i}^{(\mu)}\psi_{x_i}+a_0^{(0)}|u^{(\mu)}|^{p_0(x)-2}u^{(\mu)}\psi\Big)\varphi-u\psi\varphi'\Big\}dxdt=$$
$$=\iint_Q\Big(\sum_{i=1}^n(a_i^{(0)}-a_i^{(\mu)})|u_{x_i}^{(\mu)}|^{p_i(x)-2}u_{x_i}^{(\mu)}\psi_{x_i}+(a_0^{(0)}-a_0^{(\mu)})|u^{(\mu)}|^{p_0(x)-2}u^{(\mu)}\psi\Big)\varphi dxdt \tag{21}$$

for all $\psi\in\widetilde{W}^1_{p(\cdot)}(\Omega)$, $\varphi\in C_0^1(\mathbb{R})$. From this, putting $\mu=\sigma$ and using periodicity of functions $a_j (j\in\{0,\dots,n\})$ and f, we get that the function $u^{(\sigma)}$ is a weak solution of problem (1), (2). Taking this into consideration and the fact of uniqueness of a weak solution of the problem (1), (2), we get $u^{(0)}=u^{(\sigma)}$ almost everywhere on Q. Therefore the statement of Theorem 2 is correct. □

Proof of Theorem 3. Similarly as in the proof of Theorem 2 we pass to equality (21). Let $\delta_*:=\min\{1;K_1/2\}$ and $\sigma\in F_{\delta_*}$, where F_ε is defined in the theorem statement. We consider identity (21) at first for $\mu=0$ and afterwards for $\mu=\sigma$. Then using Lemma 2 with $u_1=u^{(0)}$, $u_2=u^{(\sigma)}$, $a_j=a_j^{(0)}$ $(j\in\{0,\dots,n\})$, $f_{0,1}=f^{(0)}$, $f_{0,2}=(a_0^{(0)}-a_0^{(\sigma)})|u^{(\sigma)}|^{p_0(x)-2}u^{(\sigma)}+f^{(\sigma)}$, $f_{i,1}=0$, $f_{i,2}=(a_i^{(0)}-a_i^{(\sigma)})|u_{x_i}^{(\sigma)}|^{p_i(x)-2}u_{x_i}^{(\sigma)}$ $(i\in\{1,\dots,n\})$, $t_0=\tau\in\mathbb{R}$, $R_0=1$, $R=l\in\mathbb{N}$ $(l\geq 2)$, we get

$$\max_{t\in[\tau-1,\tau]}\int_\Omega|u^{(\sigma)}(x,t)-u^{(0)}(x,t)|^2dx+\int_{\tau-1}^\tau\int_\Omega\Big[\sum_{i=1}^n|u_{x_i}^{(\sigma)}-u_{x_i}^{(0)}|^{p_i(x)}+|u^{(\sigma)}-u^{(0)}|^{p_0(x)}\Big]dxdt\leq$$
$$\leq C_4\Big(l^{-2/(p_0^+-2)}+\int_{\tau-l}^\tau\int_\Omega\Big\{\Big(|f^{(\sigma)}-f^{(0)}|+|a_0^{(\sigma)}-a_0^{(0)}||u^{(\sigma)}|^{p_0(x)-1}\Big)^{p_0'(x)}+ \tag{22}$$
$$+\sum_{i=1}^n|a_i^{(\sigma)}-a_i^{(0)}|^{p_i'(x)}\cdot|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big\}dxdt\Big).$$

Under the inequality $(a+b)^q\leq 2^{q-1}(a^q+b^q)$, $a\geq0$, $b\geq0$, $q\geq1$, we have

$$\int_{\tau-l}^\tau\int_\Omega\Big(|f^{(\sigma)}-f^{(0)}|+|a_0^{(\sigma)}-a_0^{(0)}||u^{(\sigma)}|^{p_0(x)-1}\Big)^{p_0'(x)}dxdt\leq 2^{1/(p_0^--1)}\int_{\tau-l}^\tau\int_\Omega\Big(|f^{(\sigma)}-f^{(0)}|^{p_0'(x)}+$$
$$+|a_0^{(\sigma)}-a_0^{(0)}|^{p_0'(x)}|u^{(\sigma)}|^{p_0(x)}\Big)dxdt\leq 2^{1/(p_0^--1)}\int_{\tau-l}^\tau\int_\Omega\Big(|f^{(\sigma)}-f^{(0)}|^{p_0'(x)}dxdt+ \tag{23}$$
$$+\Big(\sup_{t\in\mathbb{R}}\|a_0^{(\sigma)}(\cdot,t)-a_0^{(0)}(\cdot,t)\|_{L_\infty(\Omega)}\Big)^{(p_0^+)'}\int_{\tau-l}^\tau\int_\Omega|u^{(\sigma)}|^{p_0(x)}dxdt,$$
$$\int_{\tau-l}^\tau\int_\Omega\Big(\sum_{i=1}^n|a_i^{(\sigma)}-a_i^{(0)}|^{p_i'(x)}\cdot|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big)dxdt\leq$$
$$\leq\max_{i\in\{1,\dots,n\}}\Big(\sup_{t\in\mathbb{R}}\|a_i^{(\sigma)}(\cdot,t)-a_i^{(0)}(\cdot,t)\|_{L_\infty(\Omega)}\Big)^{(p_i^+)'}\int_{\tau-l}^\tau\int_\Omega\sum_{i=n}^n|u_{x_i}^{(\sigma)}|^{p_i(x)}dxdt, \tag{24}$$

where $(p_j^+)' := p_j^+/(p_j^+-1)$ $(j \in \{0,\dots,n\})$.

Since σ belongs to F_{δ_*} and f is Stepanov almost periodic, then $a^{(\sigma)}(x,t) \geq K_1/2\,(j \in \{0,\dots,n\})$ for a. e. $(x,t) \in Q$ and $\sup_{s\in\mathbb{R}} \int_{s-1}^{s}\int_\Omega |f^{(\sigma)}(x,t)|^{p_0(x)}dxdt \leq C_6$, where $C_6 > 0$ is a number independent on σ. Hence, by Corollary 2 we have

$$\sup_{s\in\mathbb{R}} \int_{s-1}^{s}\int_\Omega \Big[|u^{(\sigma)}|^{p_0(x)} + \sum_{i=1}^{n} |u_{x_i}^{(\sigma)}|^{p_i(x)}\Big]dxdt \leq C_7, \tag{25}$$

where $C_7 > 0$ is a constant independent on σ.

Thus, from (22) using (23) and (24), we obtain

$$\int_\Omega |u^{(\sigma)}(x,\tau) - u^{(0)}(x,\tau)|^2dx + \int_{\tau-1}^{\tau}\int_\Omega \Big[\sum_{i=1}^{n} |u_{x_i}^{(\sigma)} - u_{x_i}^{(0)}|^{p_i(x)} + |u^{(\sigma)} - u^{(0)}|^{p_0(x)}\Big]dxdt \leq$$

$$\leq C_8\Big\{l^{-2/(p_0^+-2)} + \sum_{k=1}^{l}\int_{\tau-k}^{\tau-k+1}\int_\Omega |f^{(\sigma)} - f^{(0)}|^{p_0{}'(x)}dxdt + \tag{26}$$

$$+ \max_{j\in\{0,\dots,n\}}\Big(\sup_{t\in\mathbb{R}} ||a_j^{(\sigma)}(\cdot,t) - a_j^{(0)}(\cdot,t)||_{L_\infty(\Omega)}\Big)^{(p_j^+)'}\sum_{k=1}^{l}\int_{\tau-k}^{\tau-k+1}\int_\Omega\Big[|u^\sigma|^{p_0(x)} + \sum_{i=1}^{n}|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big]dxdt\Big\},$$

where C_8 is a constant independent on τ, σ and l.

Let $\varepsilon > 0$ be an arbitrary small fixed number. We show that the set

$$U_\varepsilon := \Big\{\sigma \in \mathbb{R}\Big| \sup_{t\in\mathbb{R}}\int_\Omega |u(x,t+\sigma) - u(x,t)|^2dx \leq \varepsilon,$$

$$\sup\int_{\tau-1}^{\tau}\int_\Omega\Big[\sum_{i=1}^{n}|u_{x_i}(x,t+\sigma) - u_{x_i}(x,t)|^{p_i(x)} + |u(x,t+\sigma) - u(x,t)|^{p_0(x)}|\Big]dxdt \leq \varepsilon\Big\}$$

contains a set F_δ for some $\delta \in (0,\delta_*]$ implying the relative density of the set. Indeed, choose large enough $l \in \mathbb{N}\,(l \geq 2)$ satisfying the following inequality $C_8 l^{-2/(p_0^+-2)} \leq \varepsilon/2$, and fix this value l. Then take $\delta \in (0,\delta_*]$ such that the following inequality remains true

$$C_8\Big(l\cdot\delta + \max_{j\in\{0,\dots,n\}} \delta^{(p_j^+)'}\cdot l \cdot C_7\Big) \leq \frac{\varepsilon}{2}. \tag{27}$$

Therefore, if $\delta \in F_\delta$, then the right hand side of inequality (26) is less than or equal to ε. This implies that $F_\delta \subset U_\varepsilon$, that is the fact we had to prove. □

REFERENCES

1. Tihonov A.N. *Uniqveness theorems for the heat equation*// Matem. sbornik. – 1935. – №2. – P. 199–216.
2. Lions J.-L. Quelques methodes de resolution des problems aux limities non lineaires. – Gauthier-Villars, Paris, 1969.
3. Olejnik O.A., Iosifjan G.A. *Analog of Sen-Venan principle and uniqueness of solutions of boundary problems for parabolic equations in unbounded domains*// Uspehi matem. nauk. – 1976. – V.31, №6. – P. 142–166. (in Russian)

4. Pankov A.A. Bounded and almost periodic solutions of nonlinear differential operator equations. – Kyiv: Naukova dumka, 1985. – 184 p.
5. Bokalo N.M. *On problem without initial conditions for some classes of nonlinear parabolic equations//* Trudy Seminara imeni I.G. Petrovskogo. – 1989. – V.14. – P. 3–44. (in Russian)
6. Pukach P.Ja. *On problem without initial conditions for some nonlinear degenarated parabolic system//* Ukr. Mat. Jour. – 1994. – V.46, №4. – P. 454–456. (in Ukrainian)
7. Showalter R.E., Monotone operators in Banach space and nonlinear partial differential equations. – Mathematical surveys and monographs, 49. – Amer. Math. Soc., Providence, 1997.
8. Bokalo M.M. *Well-posedness of problems without initial conditions for nonlinear parabolic variational inequalities//* Nelinejnye granichnye zadachi. – 1998. – V.8. – P. 58–63.
9. Bokalo M.M., Sikorskyy V.M. *About properties of solutions of problem without initial conditions for equations generalized politropic filtration equation//* Visnyk Lviv. Universytetu. Serija meh.-mat. – 1998. – V.51. – P. 85–98. (in Ukrainian)
10. Bokalo M.M., Pauchok I.B. *On the well-posedness of a Fourier problem for nonlinear parabolic equations of higher order with variable exponents of nonlinearity//* Mat. Stud. – 2006. – V.24, №1. – P. 25–48. (in Ukrainian)
11. Bokalo M., Lorenzi A. *Linear evolution first-order problems without initial conditions//* Milan Journal of Mathematics. – 2009. – V.77. – P. 437–494.
12. Kováčik O., Rákosnić J. *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$*//* Czechoslovak Mathematical Journal. – 1991. – V.41(116). – P. 592–618.
13. Růžička M., Electrorheological fluids: modeling and mathematical theory. – Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.
14. Fu Y., Pan N. *Existence of solutions for nonlinear parabolic problem with* $p(x)$*-growth//* Journal of Mathematical Analysis and Applications. – 2010. – V.362. – P. 313–326.
15. Alkhutov Y., Antontsev S., Zhikov V. *Parabolic equations with variable order of nonlinearity//* Collection of works of Institute of Mathematics NAS of Ukraine. – 2009. – V.6. – P. 23–50.
16. Buhrii O.M., Hlynyans'ka Kh.P. *Some parabolic variational inequalities with variable exponent of nonlinearity: unique solvability and comparison theorems//* Journal of Mathematical Sciences. – 2011. – V.174, №2. – P. 169–189 (Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya. – 2009. – V.52, №4. P. 42–57).
17. Mashiyev R.A., Buhrii O.M. *Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity//* Journal of Mathematical Analysis and Applications. – 2011. – V.377. – P. 450–463.
18. Mihailescu M., Radulescu V., Tersian S. *Homoclinic solutions of difference equations with variable exponents//* Topological Methods in Nonlinear Analysis. – 2011. – V.38. – P. 277–289.
19. Bohr G. Almost periodic functions. – M.: 1934.
20. Levitan B.M., Zhikov V.V. Almost periodic functions and differential equations. – M.: Izd-vo Mosk. un-ta., 1978. – 204 p. (in Russian)
21. Bernis F. *Elliptic and parabolic semilinear problems without conditions at infinity//* Arch. Rational Mech. Anal. – 1989. – V.106, №3. – P. 217–241.

Ivan Franko National University of Lviv
mm.bokalo@gmail.com

Received 27.08.2013