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ON THE STABILITY OF ENTIRE MULTIPLE DIRICHLET SERIES

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Let $D^p(\lambda)$ be the class of entire multiple Dirichlet series of the form $F(z) = \sum_{\|n\|=0}^{+\infty} a_n \mathrm{e}^{(z,\lambda_n)}$, $z \in \mathbb{C}^p$, $p \geq 1$, with exponents satisfying the conditions $\lambda_n = (\lambda_{n_1}^{(1)}, \ldots, \lambda_{n_p}^{(p)})$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$, $0 \leq \lambda_k^{(j)} < \lambda_{k+1}^{(j)} \to +\infty$ $(0 \leq k \to +\infty)$; $w\colon [0, +\infty) \to [0, +\infty)$ a nondecreasing function, and $\nu_1(t) = \sum_{\|\lambda_n\| \leq t} e^{w(\|\lambda_n\|)}$, $\|a\| = a_1 + \ldots + a_p$, $(a, b) = a_1b_1 + \ldots + a_pb_p$, for $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p) \in \mathbb{C}^p$. If $\int_1^{+\infty} t^{-1} d\ln \nu_1(t) < +\infty$ and $F_w \in D^p(\lambda)$, $F_w(z) = \sum_{\|n\|=0}^{+\infty} a_n \mathrm{e}^{w(\|\lambda_n\|)+(z,\lambda_n)}$. Then $\ln \max\{|a_n|e^{w(\|\lambda_n\|)+(\sigma,\lambda_n)}\colon n \in \mathbb{Z}_+^p\} \sim \ln \max\{|a_n|e^{(\sigma,\lambda_n)}\colon n \in \mathbb{Z}_+^p\}$ as $|\sigma| \to +\infty$ $(\sigma \in K \setminus E)$, for an arbitrary cone K in $\mathbb{R}_+^p$ with vertex at the point O such that $\overline{K}\setminus\{O\} \subset \mathbb{R}_+^p$, and a measurable set $E \subset \mathbb{R}_+^p$ such that $\tau_p(E \cap K) = \int_{E\cap K} \frac{d\sigma_1 \ldots d\sigma_p}{|\sigma|^{p-1}} < +\infty$.

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Пусть $D^p(\lambda)$ — класс целых кратных рядов Дирихле вида $F(z) = \sum_{\|n\|=0}^{+\infty} a_n \mathrm{e}^{(z,\lambda_n)}$, $z \in \mathbb{C}^p$, $p \geq 1$, с показателями, удовлетворяющими условиям $\lambda_n = (\lambda_{n_1}^{(1)}, \ldots, \lambda_{n_p}^{(p)})$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$, $0 \leq \lambda_k^{(j)} < \lambda_{k+1}^{(j)} \to +\infty$ $(0 \leq k \to +\infty)$; $w\colon [0, +\infty) \to [0, +\infty)$ неубывающая функция, а $\nu_1(t) = \sum_{\|\lambda_n\| \leq t} e^{w(\|\lambda_n\|)}$, $\|a\| = a_1 + \ldots + a_p$, $(a, b) = a_1b_1 + \ldots + a_pb_p$ для $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p) \in \mathbb{C}^p$. Если $\int_1^{+\infty} t^{-1} d\ln \nu_1(t) < +\infty$ и $F_w \in D^p(\lambda)$, $F_w(z) = \sum_{\|n\|=0}^{+\infty} a_n \mathrm{e}^{w(\|\lambda_n\|)+(z,\lambda_n)}$, то $\ln \max\{|a_n|e^{w(\|\lambda_n\|)+(\sigma,\lambda_n)}\colon n \in \mathbb{Z}_+^p\} \sim \ln \max\{|a_n|e^{(\sigma,\lambda_n)}\colon n \in \mathbb{Z}_+^p\}$ при $|\sigma| \to +\infty$ $(\sigma \in K \setminus E)$ для каждого конуса K в $\mathbb{R}_+^p$ с вершиной в точке O такого, что $\overline{K}\setminus\{O\} \subset \mathbb{R}_+^p$, а измеримое множество $E \subset \mathbb{R}_+^p$ такое, что $\tau_p(E \cap K) = \int_{E\cap K} \frac{d\sigma_1 \ldots d\sigma_p}{|\sigma|^{p-1}} < +\infty$.

1. Introduction. The stability of the Dirichlet series of one variable. Let $D(\lambda)$ be the class of Dirichlet series absolutely convergent in $\mathbb{C}$ of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \tag{1}$$

where $\lambda = (\lambda_n)$ is some sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \to +\infty)$. By $D_*(\lambda)$ we denote the class of formal series of form (1) such that $a_n e^{x\lambda_n} \to 0$ $(n \to +\infty)$ for every $x \in \mathbb{R}$, i.e., for every $x \in \mathbb{R}$ there exists the maximal term

$$\mu(x, F) = \max\{|a_n|e^{x\lambda_n}\colon n \geq 0\} < +\infty.$$

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Clearly, $D(\lambda) \subset D_*(\lambda)$. Besides such elementary statement holds.

Proposition 1. *If $F \in D_*(\lambda)$ and the condition $\ln n = o(\ln |a_n|)$ $(n \to +\infty)$ or the condition*

$$\ln n = o(\lambda_n) \quad (n \to +\infty) \tag{2}$$

holds then $F \in D(\lambda)$.

Indeed, the condition $F \in D_*(\lambda)$ implies $\varliminf\limits_{n\to+\infty} \frac{-\ln|a_n|}{\lambda_n} = +\infty$. By condition (2) (or $\ln n = o(\ln |a_n|)$ $(n \to +\infty)$, [3, p. 115], [4]), we can calculate the abscissa of absolute convergence of the series (1) by formulae $\sigma_a = \varliminf\limits_{n\to+\infty} \frac{-\ln|a_n|}{\lambda_n}$. Therefore, $F \in D(\lambda)$.

For a Dirichlet series $F \in D_*(\lambda)$ and any sequence (b_n), $b_n \in \mathbb{C}\setminus\{0\}$ $(n \geq 0)$ we consider

$$B^+(z) = \sum_{n=0}^{+\infty} a_n b_n e^{z\lambda_n}, \quad B^-(z) = \sum_{n=0}^{+\infty} a_n b_n^{-1} e^{z\lambda_n}.$$

Remark 1. If a sequence $\{b_n :\ n \geq 0\} \subset \mathbb{C}\setminus\{0\}$ satisfies the condition

$$b = \varlimsup_{n\to+\infty} \frac{1}{\lambda_n} \ln(|b_n| + |b_n|^{-1}) < +\infty, \tag{3}$$

then $F \in D(\lambda) \Longleftrightarrow B^+ \in D(\lambda) \Longleftrightarrow B^- \in D(\lambda)$.

Following A. M. Gaisin ([1]) we say that a Dirichlet series of form (1) is *stable (stable by Gaisin)* if the relations

$$\ln \mu(\sigma, F) = (1 + o(1)) \ln \mu(\sigma, B^+) = (1 + o(1)) \ln \mu(\sigma, B^-) \tag{4}$$

hold as $\sigma \to +\infty$ outside some set $E \subset [0, +\infty)$ of finite Lebesgue measure, i.e.

$$\operatorname{meas} E := \int_E d\sigma < +\infty.$$

Let L be the class of positive continuous on $[0, +\infty)$ functions $l(t)$ such that $l(t) \to +\infty$ $(t \to +\infty)$. By L_+ we denote the subclass of L of functions such that $l(t) \uparrow +\infty$ as $x \to +\infty$, and by $\mathcal{W}$ the class of functions $w \in L_+$ such that

$$\int_1^{+\infty} x^{-2} w(x) dx < +\infty.$$

The following theorem was announced and applied to investigation of the growth of entire Dirichlet series on curves in [1].

Theorem A ([1]). *Assume that conditions (3) and*

$$\varlimsup_{n\to+\infty} \frac{\ln n}{\ln \lambda_n} = a < +\infty \tag{5}$$

hold. For any $F \in D(\lambda)$ *asymptotic equations (4) hold as* $\sigma \to +\infty$ *outside some set* $E \subset [0; +\infty)$, $\text{meas}\, E < +\infty$, *if and only if there exists a function* $w \in \mathcal{W}$ *such that*

$$\ln(|b_n| + |b_n|^{-1}) \le w(\lambda_n) \quad (n \ge n_1). \tag{6}$$

Condition (5) in this statement is too "restrictive". In [2] one can find weaker sufficient conditions of the stability.

Theorem 1 (**Skaskiv, Trakalo [2]**). *Let* $\{F, B^+, B^-\} \subset D(\lambda)$, $w \in L$ *and condition (6) hold. If*

$$\int_0^{+\infty} t^{-2} \ln \nu(t) dt < +\infty, \tag{7}$$

where $\nu(t) = \int_0^t e^{w(x)} dn(x)$, $n(x) = \sum_{\lambda_n \le x} 1$, *then the Dirichlet series of form* (1) *is stable.*

Corollary 1 (**Skaskiv, Trakalo [2]**). *Let for a sequence* $\lambda = (\lambda_n)$ *we have*

$$\int_0^{+\infty} t^{-2} \ln n(t) dt < +\infty, \quad n(t) \stackrel{def}{=} \sum_{\lambda_n \le t} 1, \tag{8}$$

and for a sequence (b_n) *condition (6) hold. If* $\{F, B^+, B^-\} \subset D(\lambda)$ *and* $w \in \mathcal{W}$, *then the Dirichlet series* F *of form* (1) *is stable.*

It is also proved in [2, Theorem 3] that the statement of Corollary 1 cannot be improved in the following sense. For every sequence λ such that condition (8) holds and for each function $w \in L$ such that condition (7) does not hold there exists a function $F \in D(\lambda)$ such that for the function

$$B_w(z) = \sum_{n=0}^{+\infty} a_n e^{w(\lambda_n) + z\lambda_n}$$

we get

$$(\exists d > 0)(\forall x \ge x_0)\colon \quad \ln \mu(x, B_w) \ge (1 + d) \ln \mu(x, F),$$

i.e. the Dirichlet series F is not stable.

Remark 2. i) Since $\ln \nu(t) \le w(t) + \ln n(t)$ $(t \ge 0)$, conditions (8) and $w \in \mathcal{W}$ yield (7).
ii) Condition (8) implies relation (2).
iii) Using $\nu(t) \ge e^{w(0)}(n(t) - 1)$ $(t \ge 0)$, from condition (7) we get (8).

From Proposition 1 and Corollaries 1, 2 it follows that the condition $\{F, B^+, B^-\} \subset D(\lambda)$ in Theorem 1 and Corollary 1 one can replace with the condition $F \in D_*(\lambda)$, because (8) (as well as condition (7)) implies (2). So, the *conjecture from [2] that in Theorem 1 and Corollary 1, the condition* $\{F, B^+, B^-\} \subset D(\lambda)$ *can be replaced with* $\{F, B^+, B^-\} \subset D_*(\lambda)$, is true. Moreover, we can replace the condition $\{F, B^+, B^-\} \subset D(\lambda)$ with $B_w \in D_*(\lambda)$, $w \in L$ and reformulate Theorem 1 and Corollary 1 as follows.

Theorem 2. *Let* $w \in L$, $B_w \in D_*(\lambda)$ *and condition (7) be satisfied. Then the relation*

$$\ln \mu(\sigma, F) = (1 + o(1)) \ln \mu(\sigma, B_w) \tag{9}$$

holds as $\sigma \to +\infty$ *outside some set* $E \subset [0; +\infty)$, $\text{meas}\, E < +\infty$.

Corollary 2. *Suppose that for a sequence $\lambda = (\lambda_n)$ condition (8) holds. If $w \in \mathcal{W}$ and $B_w \in D_*(\lambda)$, then relation (9) is satisfied as $\sigma \to +\infty$ outside some set $E \subset [0;+\infty)$,* $\operatorname{meas} E < +\infty$.

Remark 3. Condition (7) does not imply that $w \in \mathcal{W}$.

Indeed, in order to prove this statement it is enough to consider the sequence $\lambda_n = \exp\{n^2\}$ and the function $w(t) = t/\ln(t+1)$. Obviously $w \in L \setminus \mathcal{W}$. Remark that $\nu(t) = \int_0^t e^{w(x)}dn(x) = \sum_{\lambda_n \le t} e^{w(\lambda_n)}$ and condition (7) holds if and only if $\int_0^{+\infty} t^{-1} d\ln\nu(t) < +\infty$. Then

$$\int\limits_0^t \frac{d\ln\nu(x)}{x} = \sum_{\lambda_n \le t} \frac{1}{\lambda_n} \ln\left(1 + \frac{e^{w(\lambda_n)}}{\sum\limits_{k\le n-1} e^{w(\lambda_k)}}\right). \tag{10}$$

The inequalities

$$\frac{1}{\lambda_n} \ln\left(1 + \frac{e^{w(\lambda_n)}}{\sum\limits_{k\le n-1} e^{w(\lambda_k)}}\right) \le \frac{(w(\lambda_n)+o(1))}{\lambda_n} \le \frac{2w(\lambda_n)}{\lambda_n} \le \frac{1}{n^2} \quad (n \to +\infty),$$

imply (7).

Suppose now that for the counting function $n(t)$ of the sequence λ the condition

$$(\exists\, \theta > 0)(\exists\, t_0 > 0)(\exists\, d > 0)(\forall\, t \ge t_0)\colon\ n((1+\theta)t) - n(t) \ge d \tag{11}$$

holds. Using

$$\ln\nu((1+\theta)t) \ge \ln \int\limits_t^{(1+\theta)t} e^{w(u)}dn(u) \ge w(t) + \ln d \quad (t \ge t_0),$$

one can show that (7) yields $w \in \mathcal{W}$.

Taking into account Remark 2, we obtain the following statement.

Proposition 2. *If conditions (7) and (11) hold, then we get (8) and $w \in \mathcal{W}$.*

For example, condition (11) is satisfied for the following sequences: $\lambda_n = e^n$, $\lambda_n = n^\alpha$ $(\alpha > 0)$, $\lambda_n = \ln^\alpha(n+1)$ $(\alpha > 0)$, $\lambda_n = \ln\ln(n+e), n \ge 0$. But from the example above we deduce that for the sequence $\lambda_n = e^{n^2}$, condition (11) does not hold because for this sequence we have $\lim\limits_{n\to+\infty} n/\ln\lambda_n = 0$, however, (11) implies that

$$\varliminf_{n\to+\infty} \frac{n}{\ln\lambda_n} \ge \frac{d}{\ln(1+\theta)} > 0.$$

In addition, examples of sequences λ indicate independence of conditions (11), (8).

2. The stability of the Dirichlet series of several variables. Let $p \in \mathbb{N}, p \ge 2$ and

$$\lambda = (\lambda_n),\ \lambda_n = (\lambda_{n_1}^{(1)}, \ldots, \lambda_{n_p}^{(p)}),\ n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p,\ \|n\| = \sum_{j=1}^p n_j,$$

$$(u,v) = \sum_{j=1}^p u_j v_j, \quad \text{for } u = (u_1, \ldots, u_p), v = (v_1, \ldots, v_p) \in \mathbb{C}^p.$$

Everywhere we continue to assume that the sequence (λ_n) satisfies the condition

$$(\forall\ j,\ 1 \le j \le p):\ \ 0 \le \lambda_k^{(j)} \uparrow +\infty \quad (0 \le k \uparrow +\infty).$$

By $D^p(\lambda)$ we denote the class of absolutely convergent in $\mathbb{C}^p$ Dirichlet series

$$F(z) = \sum_{\|n\|=0}^{+\infty} a_n e^{(z,\lambda_n)}, \quad z \in \mathbb{C}^p, \tag{12}$$

$D_*^p(\lambda)$ is the class of formal series of form (12) such that

$$(\forall\ z \in \mathbb{C}^p): \quad a_n e^{(z,\lambda_n)} \to 0 \quad (\|n\| \to +\infty).$$

For $F \in D_*^p(\lambda)$ and $\sigma \in \mathbb{R}^p$ we denote

$$\mu(\sigma, F) = \max\{|a_n| e^{(\sigma,\lambda_n)} :\ n \ge 0\}.$$

For $z \in \mathbb{C}^p$, $w \in L$ and a sequence of complex numbers $(b_n)_{n\in\mathbb{Z}_+^p}$, $b_n \ne 0$ $(n \in \mathbb{Z}_+^p)$, we put

$$B_\pm(z) = \sum_{\|n\|=0}^{+\infty} a_n (b_n)^{\pm 1} e^{(z,\lambda_n)}, \quad F_w(z) = \sum_{\|n\|=0}^{+\infty} |a_n| e^{w(\|\lambda_n\|)+(z,\lambda_n)}.$$

2.1. Sufficient conditions of stability.

Theorem 3. *Let $F \in D_*^p(\lambda)$. If there exists a function $w \in L$ such that $F_w \in D_*^p(\lambda)$, $\ln \nu_1 \in \mathcal{W}$ (here $\nu_1(t) = \sum_{\|\lambda_n\|\le t} e^{w(\|\lambda_n\|)}$) and*

$$e^{-w(\|\lambda_n\|)} \le |b_n| \le e^{w(\|\lambda_n\|)} \quad (\|n\| \ge k_1), \tag{13}$$

then there exists a Lebesgue measurable set $E \subset \mathbb{R}_+^p$ such that

$$\ln \mu(\sigma, F) = (1 + o(1)) \ln \mu(\sigma, B_+) = (1 + o(1)) \ln \mu(\sigma, B_-) \tag{14}$$

as $|\sigma| \to +\infty$ $(\sigma \in K \setminus E)$ and

$$\tau_p(E \cap K) \overset{def}{=} \int\limits_{E\cap K} \frac{d\sigma_1 \dots d\sigma_p}{|\sigma|^{p-1}} < +\infty$$

for an arbitrary cone K in $\mathbb{R}_+^p$ with vertex at the origin O such that $\overline{K}\setminus\{O\} \subset \mathbb{R}_+^p$.

Corollary 3. *Suppose that for $\lambda = (\lambda_n)_{n\in\mathbb{Z}_+^p}$, $\lambda_n = (\lambda_{n_1}^{(1)}, \dots, \lambda_{n_p}^{(p)})$ $(n = (n_1, \dots, n_p))$, we have*

$$(\forall\ j,\ 1 \le j \le p): \quad \sum_{k=1}^{+\infty} \frac{1}{k\lambda_k^{(j)}} < +\infty, \tag{15}$$

$w \in \mathcal{W}$, and for $(b_n), F_w, F$ the assumptions of Theorem 3 hold. Then the statement of Theorem 3 is true.

The assertion of Theorem 3 confirms the conjecture expressed by the second author at the conference dedicated to the 125th anniversary of H. Hans ([5]) (Chernivtsi, June 2004).

In the proof of Theorem 3 we use asymptotic estimates of the form

$$I(\sigma)=\int\limits_{\mathbb{R}^p_+} a(x)e^{(\sigma,x)}\nu(dx),\quad \sigma\in\mathbb{R}^p_+,\tag{16}$$

where ν is a nonnegative locally finite measure on $\mathbb{R}^p$, and $a\colon \mathbb{R}^p\to\mathbb{R}_+$ is a ν-measurable function ([8, Theorem 1, p. 130], [9, Theorem 1, p. 134]).

By $\mathcal{I}^p(\nu)$ we denote the class of the functions $I\colon\mathbb{R}^p\to\mathbb{R}_+$ of form (16). For $\sigma\in\mathbb{R}^p$ we put

$$\mu_*(\sigma)\overset{def}{=}\sup\{a(x)e^{\langle\sigma,x\rangle}\colon x\in\operatorname{supp}\nu\},$$

where supp ν is the support of the measure ν. For the function F, without loss of generality we may assume that $a(x)>0$ for every $x\in\operatorname{supp}\nu$.

In [9] the following statement is proved, which completes the corresponding statement from [8] in the description of the exceptional set.

Lemma 1 (**[8, 9]**). *Let $I\in\mathcal{I}^p(\nu)$. If*

$$\int\limits_0^{+\infty} t^{-2}\ln\nu_0(t)dt<+\infty,\quad \nu_0(0,t]\overset{def}{=}\nu(\{x\in\mathbb{R}^p\colon 0<\|x\|\le t\}),\tag{17}$$

then there exists a set $E\subset\mathbb{R}^p_+$ such that for an arbitrary cone $K\subset\mathbb{R}^p_+$ with vertex at the origin O such that $\overline{K}\backslash\{O\}\subset\mathbb{R}^p_+$ the relation

$$\ln I(\sigma)\le(1+o(1))\ln\mu_*(\sigma)\tag{18}$$

holds for $|\sigma|\to+\infty$ ($\sigma\in K\backslash E$), and $\tau_p(E\cap K)<+\infty$.

Proof of Theorem 3. We prove first that

$$\ln\mu(\sigma,F)=(1+o(1))\ln\mu(\sigma,F_w)\tag{19}$$

as $|\sigma|\to+\infty$ ($\sigma\in K\backslash E$), where the set E and the cone K are the same as in Lemma 1. Let $a(t),b(t)$ be measurable nonnegative functions on $\mathbb{R}^p_+$ such that $a(\lambda_n)=|a_n|$, $b(\lambda_n)=e^{w(\|\lambda_n\|)}$ and

$$\mu(\sigma,F)=\sup\{a(t)e^{(t,\sigma)}:\ t\in\mathbb{R}^p_+\},\quad \mu(\sigma,B_w)=\sup\{a(t)b(t)e^{(t,\sigma)}:\ t\in\mathbb{R}^p_+\}.$$

It is enough to put $a(t)=0$ for $t\notin\{\lambda_n\colon n\in\mathbb{Z}^p_+\}$.

Then for all $\sigma\in\mathbb{R}^p$ we get

$$\mu(\sigma,F)\le\mu(\sigma,F_w)\le F_w(\sigma)=\sum_{\|n\|=0}^{+\infty}|a_n|b(\lambda_n)e^{(\sigma,\lambda_n)}=\int\limits_{\mathbb{R}^p_+}a(x)e^{(x,\sigma)}\nu(dx),\tag{20}$$

where the measure ν is such that $\nu(G)=\sum_{\|n\|=0}^{+\infty}b(\lambda_n)\delta_{\lambda_n}(G)$ for each bounded set $G\subset\mathbb{R}^p_+$ and $\delta_\lambda(G)=1$ for $\lambda\in G$ and $\delta_\lambda(G)=0$ for $\lambda\notin G$. Clearly,

$$\nu_0(0,t]=\nu(\{x\in\mathbb{R}^p_+\colon 0<\|x\|\le t\})\le\sum_{\|\lambda_n\|\le t}b(\lambda_n)=\sum_{\|\lambda_n\|\le t}e^{w(\|\lambda_n\|)}=\nu_1(t).$$

Observe that the condition $\ln\nu_1 \in \mathcal{W}$ immediately implies assumption (17) of Lemma 1.

Using Lemma 1 to the integral in (20), as $|\sigma| \to +\infty$ ($\sigma \in K\backslash E$) (here the set E and the cone K are the same as in Lemma 1) we obtain

$$\ln\mu(\sigma, F) \le \ln\mu(\sigma, F_w) \le (1 + o(1))\ln\mu_*(\sigma),$$

where $\mu_*(\sigma) = \max\{a(x)e^{(x,\sigma)} \colon x \in \mathbb{R}^p_+\}$. By the choice of the function $a(t)$ we get $\mu_*(\sigma) = \mu(\sigma, F)$ and obtain relation (19).

Set

$$B_{-w}(x) := \sum_{\|n\|=0}^{+\infty} |a_n| e^{-w(\|\lambda_n\|)} e^{(x,\lambda_n)}.$$

Using inequalities (13), we get

$$\mu(\sigma, B_{-w}) \le \mu(\sigma, B_+) \le \mu(\sigma, F_w), \quad \mu(\sigma, B_{-w}) \le \mu(\sigma, B_-) \le \mu(\sigma, F_w). \tag{21}$$

Therefore, applying just proved statement to the function B_{-w}, as $|\sigma| \to +\infty$ ($\sigma \in K\backslash E$), (here the set E and the cone K are the same as in Lemma 1) we have

$$\ln\mu(\sigma, B_{-w}) = (1 + o(1))\ln\mu(\sigma, (B_{-w})_w) = (1 + o(1))\ln\mu(\sigma, F),$$

because $\mu(\sigma, (B_{-w})_w) = \mu(\sigma, F)$. From the previous relation and inequalities (21) we deduce (14). □

Proof of Corollary 3. Note that for each $t > 0$

$$\ln\nu_1(t) \le w(t) + \ln\Big(\sum_{\|\lambda_n\|\le t} 1\Big) \le w(t) + \sum_{j=1}^{p} \ln n_j(t) := w_0(t),$$

where $n_j(t) = \sum_{\lambda_k^{(j)} \le t} 1$ is the counting function of the sequence of j-th coordinates of the vector sequence (λ_n). It remains to note that the conditions $0 < \lambda_k^{(j)} \uparrow +\infty$ $(1 \le k \uparrow +\infty)$ and (15) imply $\ln n_j \in \mathcal{W}$ ([10]). So, $w_0 \in \mathcal{W}$, and we conclude that $\ln\nu_1 \in \mathcal{W}$. □

2.2. Necessity of the condition $w \in \mathcal{W}$. Analysis of the proof of Theorem 3 from [9] and Theorem 3 from [2] gives that the statement of Theorem 3 ([9]) holds in a stronger form.

Lemma 2 ([9]). *Let ν be a nonnegative countably additive measure on $\mathbb{R}^p$, which is a direct product of countably-additive measures ν_j on $\mathbb{R}_+$, $\nu = \nu_1 \times \nu_2 \times \ldots \times \nu_j$. If condition (17) does not hold and $\ln\nu_0(0, t] = O(t)$ $(t \to +\infty)$ then there exist a function $I \in \mathcal{I}^p(\nu)$ and constants $d > 0,\ t_0 > 0$ such that for all*

$$\sigma \in E_0 := \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p_+ \colon\ x_1 \ge t_0,\ t_0 \le x_j \le x_1,\ j \in \{2, \ldots, p\}\},$$

we get

$$\ln I(\sigma) \ge (1 + d)\ln\mu(\sigma), \quad \mu(\sigma) := \max\{a(x)e^{(x,\sigma)} \colon\ x \in \mathbb{R}^p_+\}, \tag{22}$$

and $\tau_p(E_0) = +\infty$.

Remark that $K_0 := \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p_+ \colon x_j \le x_1, j \in \{2, \ldots, p\}\}$ is a cone in $\mathbb{R}^p_+$ with vertex at the origin and

$$E_0 = K_0 \setminus \{x \in \mathbb{R}^p_+ \colon x_1 < t_0 \vee x_2 < \min\{t_0, x_1\} \vee \ldots \vee x_p < \min\{t_0, x_1\}\}.$$

Suppose that condition (15) holds for a sequence $\lambda = (\lambda_n)$ but $\ln \nu_1 \notin \mathcal{W}$, $\nu_1(t) = \sum_{\|\lambda_n\| \le t} e^{w(\|\lambda_n\|)}$. Using

$$\nu_1(t) = \sum_{\|\lambda_n\| \le t} e^{w(\|\lambda_n\|)} = \nu(\{x \in \mathbb{R}^p_+ \colon 0 < \|x\| \le t\}) +$$
$$+ \nu(\{x \in \mathbb{R}^p \colon \|x\| = 0\}) = \nu_0(0, t] + \nu(\{x \in \mathbb{R}^p \colon \|x\| = 0\}),$$

we conclude that (17) does not hold. By Lemma 2, there exists a positive function $I \in \mathcal{I}^p(\nu)$ such that $\ln I(\sigma) > (1 + d) \ln \mu(\sigma)\ \ (\sigma \in E)$, where

$$I(\sigma) = \int\limits_{\mathbb{R}^p_+} a(x) e^{(\sigma, x)} \nu(dx) = \int\limits_{\mathbb{R}^p_+} a(x) e^{w(\|x\|)} e^{(\sigma, x)} \nu_2(dx) := B(\sigma), \tag{23}$$

and the measure ν_2 is such that $\nu_2(G) = \sum_{\|n\|=0}^{+\infty} \delta_{\lambda_n}(G)$ for each bounded set $G \subset \mathbb{R}^p_+$. Note now that the measure ν_2 satisfies (17). So, if we choose $a_n = a(\lambda_n)$ and apply the statement of Lemma 1 to the second integral in (23), then for the Dirichlet series $F \in D^p(\lambda)$ of form (12) we obtain as $|\sigma| \to +\infty$ $(\sigma \in K_0 \setminus E,\ \tau_p(E) < +\infty)$

$$(1 + d) \ln \mu(\sigma, F) \le (1 + d) \ln \mu_*(\sigma, I) \le \ln I(\sigma) = \ln B(\sigma) \le$$
$$\le (1 + o(1)) \ln \sup\{a(x) e^{w(\|x\|)} e^{(\sigma, x)} : x \in \text{supp } \nu_2\} =$$
$$= (1 + o(1)) \ln \sup\{a_n e^{w(\|\lambda_n\|)} e^{(\sigma, \lambda_n)} :\ n \in \mathbb{Z}^p_+\} = (1 + o(1)) \ln \mu(\sigma, F_w).$$

It remains to remark that K_0 is a cone in $\mathbb{R}^p_+$ with vertex at the origin and $\tau_p(K_0 \setminus E) = +\infty$.

So, from Corollary 3 we deduce the following statement.

Theorem 4. *Let $\lambda = (\lambda_n)$ be a vector sequence such that condition* (15) *holds. Then the following assertions are equivalent.*

1. *For every function $F \in D^p_*(\lambda)$ there exists a Lebesgue measurable set $E \subset \mathbb{R}^p_+$ such that $\tau_p(E \cap K) < +\infty$ and asymptotic relations* (14) *hold as $|\sigma| \to +\infty$ $(\sigma \in K \setminus E)$ for an arbitrary cone K in $\mathbb{R}^p_+$ with vertex at the origin O such that $\overline{K} \setminus \{O\} \subset \mathbb{R}^p_+$*
2. *There exists a function $w \in L$ such that $\ln \nu_1 \in \mathcal{W}$, $F_w \in D^p_*(\lambda)$ and inequalities* (13) *hold.*

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