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EXISTENCE OF OPTIMAL CONTROL IN THE COEFFICIENTS FOR PROBLEM WITHOUT INITIAL CONDITION FOR STRONGLY NONLINEAR PARABOLIC EQUATIONS

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An optimal control problem for systems described by Fourier problem for nonlinear parabolic equations is studied. Control function occur in the coefficients of the state equations. Different types of observation is considered. The existence of the optimal control is proved.

Introduction. Optimal control of determined systems governed by partial differential equations (PDEs) is currently of much interest. Many ideas and methods of solving different optimal control problems for systems governed by evolutionary equations and variational inequalities were considered in monograph [25]. Numerous generalizations of problems considered there were investigated in many papers. In particular, papers [1], [4], [5], [6], [16]-[18], [19], [21], [26], [27], [32], [33] are devoted to this topic. In all these papers the state of controlled system is described by the initial-boundary value problems for parabolic equations.

Optimal control problems for PDEs are most completely studied for the case in which the control functions occur either on the right-hand sides of the state equations, or the boundary or initial conditions (see for example, [13], [30], [34]). So far, problems in which control functions occur in the coefficients of the state equations are less studied (see for example, [1], [27], [32], [33]). A simple model of such type problem is the following.

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ, $T > 0$, $Q := \Omega\times(0,T)$, $\Sigma := \Gamma\times(0,T)$. A state of controlled system for given control $v \in U := L^\infty(Q)$ is defined by a weak solution $y = y(v) = y(x,t;v)$, $(x,t)\in Q$, from the space $L^2(0,T;H^1_0(\Omega))\cap C([0,T];L^2(\Omega))$, of the following problem

$$y_t - \Delta y + vy = f, \quad y\Big|_\Sigma = 0, \quad y\Big|_{t=0} = y_0,$$

where $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$.

The cost functional is $J(v) := \|y(\cdot,T;v) - z_0(\cdot)\|^2_{L^2(\Omega)} + \mu\|v\|^2_{L^\infty(Q)}$ $\forall v\in U$, where $\mu > 0$, $z_0 \in L^2(\Omega)$ are given. An optimal control problem is to find a function $u \in U_\partial := \{v \in U : v \geq 0 \text{ a. e. on } Q\}$ such that

$$J(u) = \inf_{v\in U_\partial} J(v).$$

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In [1] and [27] control functions appears as coefficients at lower derivatives, and in [32], [33] the control functions are coefficients at higher derivatives. In [27] the existence and uniqueness of optimal control in the case of final observation was shown and a necessary optimality condition in the form of the generalized rule of Lagrange multipliers was obtained. In paper [1] authors proved the existence of at least one optimal control for system governed by a system of general parabolic equations with degenerate discontinuous parabolicity coefficient. In papers [32], [33] the authors consider cost function in general form, and as special case it includes different kinds of specific practical optimization problems. The well-posedness of the problem statement is investigated and a necessary optimality condition in the form of the generalized principle of Lagrange multiplies is established in this papers.

In papers [4], [16]–[19], [21], [26] authors investigate optimal control of systems governed by nonlinear PDEs. In particular, in [4] the problem of allocating resources to maximize the net benefit in the conservation of a single species is studied. In [17] the optimal control problem is converted to an optimization problem which is solved using a penalty function technique. Paper [21] presents analytical and numerical solutions of an optimal control problem for quasilinear parabolic equations. In [23] the authors consider the optimal control of a degenerate parabolic equation governing a diffusive population with logistic growth terms. In paper [26] optimal control for semilinear parabolic equations without Cesari-type conditions is investigated.

In this paper, we study an optimal control problem for systems whose states are described by problems without initial conditions or, other words, Fourier problems for nonlinear parabolic equations.

The problem without initial conditions for evolution equations describes processes that started a long time ago and initial conditions do not affect on them in the actual time moment. Such problem were investigated in the works of many mathematicians (see [7, 12, 31] and bibliography there). Fourier problem for linear and a plenty of nonlinear evolution equations are correct only under some restrictions on the growth of solutions and input data as the time variable leads to $-\infty$ ([7], [24], [28], [29], [31]). However, there are some nonlinear parabolic equations for which the Fourier problem are uniquely solvable without any conditions at infinity ([8]–[10]). In our paper the state of control system is governed by Fourier problem for a nonlinear parabolic equation of such type. The model example of considered optimal control problem is a problem which differs from the previous one (see beginning of this section) by the following facts: the initial moment is $-\infty$ and, correspondingly, the state equation and control functions are considered in the domain $Q = \Omega \times (-\infty, T)$, a boundary condition is given on the surface $\Sigma = \partial\Omega \times (-\infty, T)$. A state of controlled system for given control $v \in U := L^\infty(Q)$ is defined by a weak solution y from the space $L^2_{\rm loc}(-\infty, T; H^1_0(\Omega)) \cap L^p_{\rm loc}(-\infty, T; L^p(\Omega)) \cap C((-\infty, T]; L^2(\Omega))$, of the following problem

$$y_t - \Delta y + |y|^{p-2}y + vy = f, \quad y\Big|_\Sigma = 0,$$

where $p > 2$ is constant and $f \in L^{p'}_{\rm loc}(-\infty, T; L^{p'}(\Omega))$, $1/p + 1/p' = 1$.

As we know among numerous works devoted to the optimal control problems for PDEs, only in papers [5], [6] the state of controlled system is described by the solution of Fourier problem for parabolic equations. In the current paper, unlike the above two, we consider optimal control problem in case when the control functions occur in the coefficients of the state equation and cost functional unites observations of different types (final, distributed, etc.). The main result of this paper is existence of the solution of this problem.

The outline of this paper is as follows. In Section 1, we give notations, definitions of function spaces and auxiliary results. In Section 3, we formulate the optimal control problem. In Section 2, we prove existence and uniqueness of the solutions for the state equations. Furthermore, we obtain a priori estimates for the weak solutions of the state equations. Finally, the existence of the optimal control is presented in Section 4.

1. Preliminaries. Let n be a natural number, $\mathbb{R}^n$ be the linear space of ordered collections $x=(x_1,\ldots,x_n)$ of real numbers with the norm $|x|:=(|x_1|^2+\ldots+|x_n|^2)^{1/2}$. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ. Set $S:=(-\infty,0]$, $Q:=\Omega\times S$, $Q_{t_1,t_2}:=\Omega\times(t_1,t_2)$ for all $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, $\Sigma:=\Gamma\times S$.

For every $q\in[1,\infty]$ denote by $L^q_{\rm loc}(\overline{Q})$ the linear space of measurable functions on Q such that their restrictions to any bounded measurable set $Q'\subset Q$ belong to the space $L^q(Q')$. The sequence $\{z_k\}$ is said to be strongly (resp., weakly) convergent to z in $L^q_{\rm loc}(\overline{Q})$ $(q\in[1,\infty))$ if it is strongly (resp., weakly) convergent to z in $L^q(Q')$ for every $Q'\subset Q$.

Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$. Denote by $L^q_{\rm loc}(S;X)$ $(q\in[1,\infty])$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[a,b]\subset S$ belong to the space $L^q(a,b;X)$. We say that $z_m \underset{m\to\infty}{\longrightarrow} z$ strongly (resp., weakly) in $L^q_{\rm loc}(S;X)$ if for each $t_1,t_2\in S$ $(t_1<t_2)$ we have $z_m \underset{m\to\infty}{\longrightarrow} z$ strongly (resp., weakly) in $L^q(t_1,t_2;X)$.

Denote by $C^1_c(a,b)$, where $-\infty\le a<b\le+\infty$, the linear space of continuously differentiable functions on (a,b) with compact supports. By $C(S;X)$ we denote the space of continuous functions determined on S with values in X. We say that $z_m \underset{m\to\infty}{\longrightarrow} z$ in $C(S;X)$ if for each $t_1,t_2\in S$ $(t_1<t_2)$ we have $\max_{\tau\in[t_1,t_2]}\|z(\tau)-z_k(\tau)\|_X \underset{k\to\infty}{\longrightarrow} 0$.

Let $H^1(\Omega):=\{v\in L_2(\Omega)\mid v_{x_i}\in L_2(\Omega)\ (i=\overline{1,n})\}$ be a Sobolev space, which is a Hilbert space with respect to the scalar product $(v,w)_{H^1(\Omega)}:=\int_\Omega\big\{\sum_{i=1}^n v_{x_i}w_{x_i}+vw\big\}\,dx$ and the corresponding norm $\|v\|_{H^1(\Omega)}:=(\int_\Omega\big\{\sum_{i=1}^n|v_{x_i}|^2+|v|^2\big\}dx)^{1/2}$. Under $H^1_0(\Omega)$ we mean the closure in $H^1(\Omega)$ of the space $C^\infty_c(\Omega)$ consisting of infinitely differentiable functions on Ω with compact supports.

Also define $\partial_0 z:=z$, $\partial_j z:=z_{x_j}$ if $j\in\{1,\ldots,n\}$.

We denote $V^q(\Omega):=H^1_0(\Omega)\cap L^q(\Omega)$, where $q>1$ is arbitrary.

It is well known that

$$\big(V^q(\Omega)\big)':=H^{-1}(\Omega)+L^{q'}(\Omega),\quad q'=\frac{q}{q-1}.$$

Also we define $Y^q_{\rm loc}(Q):=L^2_{\rm loc}(S;H^1_0(\Omega))\cap L^q_{\rm loc}(\overline{Q})\cap C(S;L^2(\Omega))$. We say that $z_m \underset{m\to\infty}{\longrightarrow} z$ strongly in $Y^q_{\rm loc}(Q)$ if for each $t_1,t_2\in S(t_1<t_2)$ we have $z_m \underset{m\to\infty}{\longrightarrow} z$ strongly in $L^2(t_1,t_2;H^1_0(\Omega))\cap L^q(Q_{t_1,t_2})\cap C([t_1,t_2];L^2(\Omega))$.

2. Formulation of the optimal control problem and the main result. Let U be a closed linear subspace of $L^\infty(Q)$, and one be a space of controls, for example, $U:=L^\infty(Q)$ or $U:=\{u\in L^\infty(Q)\mid v(x,t)=0$ for a.e. $(x,t)\in Q\setminus Q_{t^*,0}\}$, where $t^*<0$ is arbitrary fixed. Assume that $U_\partial:=\{v\in U|v\ge 0$ a. e. in $Q\}$ be the set of admissible controls.

We assume that the state of the investigated evolutionary system for a given control $v\in U_\partial$ is described by a weak solution of the problem

$$y_t-\sum_{i=1}^n\frac{d}{dx_i}a_i(x,t,y,\nabla y)+\widehat{a}_0(x,t,y,\nabla y)+v(x,t)g(x,t,y)=f(x,t),\quad (x,t)\in Q,\qquad (1)$$

$$y\big|_{\Sigma} = 0, \tag{2}$$

where functions $a_0, a_1, \dots, a_n$, f and g satisfy the following conditions:

($\mathcal{A}_1$) functions

$$Q \times \mathbb{R} \times \mathbb{R}^n \ni (x,t,s,\xi) \mapsto a_i(x,t,s,\xi) \in \mathbb{R} \ (i = \overline{1,n}),$$
$$Q \times \mathbb{R} \times \mathbb{R}^n \ni (x,t,s,\xi) \mapsto \widehat{a}_0(x,t,s,\xi) \in \mathbb{R}$$

are Caratheodory functions, i.e., $\widehat{a}_0(x,t,\cdot,\cdot)$, $a_i(x,t,\cdot,\cdot)\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ are continuous functions for a.e. $(x,t) \in Q$, and $\widehat{a}_0(\cdot,\cdot,s,\xi)$, $a_i(\cdot,\cdot,s,\xi)\colon Q \to \mathbb{R}$ is the measurable function for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$; moreover, $\widehat{a}_0(x,t,0,0) = 0$, $a_i(x,t,0,0) = 0$ $(i = \overline{1,n})$ for a. e. $(x,t) \in Q$;

($\mathcal{A}_2$) there exists $p > 2$ such that for every $i \in \{1,\dots,n\}$, for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$, and for a.e. $(x,t) \in Q$ the following estimates are valid

$$|\widehat{a}_0(x,t,s,\xi)| \le C_1\big(|s|^{p-1} + |\xi|^{2(p-1)/p}\big) + h_0(x,t),$$
$$|a_i(x,t,s,\xi)| \le C_2\big(|s|^{p/2} + |\xi|\big) + h_i(x,t), \quad i = \overline{1,n},$$

where $C_1, C_2 = \mathrm{const} > 0$, $h_0 \in L^{p'}_{\mathrm{loc}}(\overline{Q})$, $h_i \in L^2_{\mathrm{loc}}(\overline{Q})$ $(i = \overline{1,n})$;

($\mathcal{A}_3$) for every (s_1,ξ^1), $(s_2,\xi^2) \in \mathbb{R}\times\mathbb{R}^n$ and for a.e. $(x,t) \in Q$ the following inequality holds

$$\sum_{i=1}^{n} \big(a_i(x,t,s_1,\xi^1) - a_i(x,t,s_2,\xi^2)\big)(\xi_i^1 - \xi_i^2)$$
$$+ \big(\widehat{a}_0(x,t,s_1,\xi^1) - \widehat{a}_0(x,t,s_2,\xi^2)\big)(s_1 - s_2) \ge K\big[|s_1-s_2|^p + |\xi^1-\xi^2|^2\big],$$

where $K = \mathrm{const} > 0$;

($\mathcal{F}$) $f \in L^{p'}_{\mathrm{loc}}(\overline{Q})$;

($\mathcal{G}_1$) the function $Q\times\mathbb{R} \ni (x,t,s) \mapsto g(x,t,s) \in \mathbb{R}$ is the Caratheodory function, i.e., $g(x,t,\cdot)\colon \mathbb{R}\to\mathbb{R}$ is the continuous function for a.e. $(x,t)\in Q$, $g(\cdot,\cdot,s)\colon Q\to\mathbb{R}$ is the measurable function for every $s\in\mathbb{R}$; moreover, $g(\cdot,\cdot,0) \in L^2_{\mathrm{loc}}(\overline{Q})$;

($\mathcal{G}_2$) for every $s_1, s_2 \in \mathbb{R}$ and for a.e. $(x,t)\in Q$ the following inequalities hold

$$0 \le \big(g(x,t,s_1) - g(x,t,s_2)\big)\big(s_1 - s_2\big) \le M|s_1-s_2|^2,$$

where $M > 0$ is a constant.

Hereafter $p' = \frac{p}{p-1}$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$; $\nabla y = (y_{x_1},\dots,y_{x_n})$, $|\nabla y|^2 = \sum_{i=1}^n |y_{x_i}|^2$.

Remark 1. Example of g: $g(x,t,s) = g_0(x,t)g_1(s)$, where $g_0 \in L^\infty(Q)$, $g_0 \ge 0$ for a.e. $(x,t)\in Q$, and $g_1\colon\mathbb{R}\to\mathbb{R}$ and $|g_1(s_1) - g_1(s_2)| \le M|s_1-s_2|$ for all $s_1,s_2\in\mathbb{R}$.

Definition 1. The function y is called a *weak solution of problem* (1), (2) if it belongs to $Y^p_{\mathrm{loc}}(Q)$ and the following integral equality holds

$$\iint\limits_Q \Big\{ -y\psi\varphi' + \sum_{i=1}^n a_i(x,t,y,\nabla y)\partial_i\psi\varphi + \widehat{a}_0(x,t,y,\nabla y)\psi\varphi + vg(x,t,y)\psi\varphi\Big\}dxdt$$
$$= \iint\limits_Q f\psi\varphi dxdt, \quad \psi \in V^p(\Omega),\ \varphi \in C^1_c(-\infty,0). \tag{3}$$

Remark 1. Research methodology of problems similar to problem (1), (2) is quite well developed, in particular, in papers of one of the authors ([8]–[10], [12]). But exactly the same problem as considered here, more precisely, Fourier problem for semilinear parabolic equation in bounded spatial variables domains, is not investigated in literature. Beside this local estimates of the weak solution are important for us. So, for a complete presentation of the material, in Section 3 we give full proof of existence and uniqueness of the weak solution and its local estimates.

A weak solution y of the specified problem will be called a weak solution of problem (1), (2) for control v, and will be denoted by y, or $y(v)$, or $y(x,t)$, $(x,t) \in Q$, or $y(x,t;v)$, $(x,t) \in Q$. The existence and uniqueness of a weak solution of problem (1), (2) (for a given $v \in U_\partial$) is shown in Section 3 (see Theorem 2).

We assume that the cost functional has the form

$$J(v) := G(y(\cdot,\cdot;v)) + \mu\|v\|_{L^\infty(Q)}, \tag{4}$$

where $\mu > 0$ is arbitrary fix and functional G satisfies following condition:

($\mathcal{J}$) $G\colon Y^p_{\mathrm{loc}}(Q) \to [0,+\infty)$ is lower semicontinuous in $L^2_{\mathrm{loc}}(S;L^2(\Omega))$, or $C(S;L^2(\Omega))$, or $L^2_{\mathrm{loc}}(S;L^2(\Omega)) \cap C(S;L^2(\Omega))$.

Remark 2. *For example we may choose functional G as following*

$$G(y(\cdot,\cdot;v)) := \mu_1 \int_S \gamma(t)\|y(\cdot,t;v) - y_{d,1}(\cdot,t)\|^2_{L^2(\Omega)}dt$$
$$+\mu_2 \max_{t\in S}\left[\rho(t)\|y(\cdot,t;v) - y_{d,2}(\cdot,t)\|^2_{L^2(\Omega)}\right] + \sum_{i=1}^N \mu_{3,i}\|y(\cdot,t_i;v) - z_{d,i}(\cdot)\|^2_{L^2(\Omega)}, \quad v \in U,$$

where $N \in \mathbb{N}$, $y_{d,1} \in L^2_{\mathrm{loc}}(S;L^2(\Omega))$, $y_{d,2} \in C(S;L^2(\Omega))$, $z_{d,i} \in L^2(\Omega)$ $(i = \overline{1,N})$, $\gamma \in L^\infty(S)$ and $\rho \in C(S)$ are nonnegative functions, which vanish outside some bounded interval, $\mu_1 \geq 0, \mu_2 \geq 0, \mu_{3,i} \geq 0$ $(i = \overline{1,N})$ are given and $\mu_1 + \mu_2 + \sum_{i=1}^N \mu_{3,i} > 0$, and $t_i \in S$ $(i = \overline{1,L})$ are any fixed points for some $L \in \mathbb{N}$.

We consider the following **optimal control problem**: find a control $u \in U_\partial$ such that

$$J(u) = \inf_{v\in U_\partial} J(v). \tag{5}$$

We briefly call this problem (5), and its solutions will be called *optimal controls*.

The main result of this paper is the following statement.

Theorem 1. *Suppose that conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, $(\mathcal{F})$, $(\mathcal{G}_1), (\mathcal{G}_2)$ and $(\mathcal{J})$ hold. Then problem (5) has a solution.*

3. Well-posedness of the problem without initial conditions for nonlinear parabolic equations.

3.1. Formulation of the problem and corresponding results.

Theorem 2. *Suppose that conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, $(\mathcal{G}_1)$, $(\mathcal{G}_2)$ *and* $(\mathcal{F})$ *hold. Then there exists a unique weak solution of* (1), (2). *In addition, the estimate*

$$\max_{t\in[t_0-R_0,t_0]}\int_\Omega |y(x,t)|^2dx+\int_{t_0-R_0}^{t_0}\int_\Omega \big[|\nabla y|^2+|y|^p\big]dxdt\le$$
$$\le C\Big\{R^{-2/(p-2)}+\int_{t_0-R}^{t_0}\int_\Omega |f|^{p'}dxdt\Big\} \tag{6}$$

holds for each t_0, R_0 *and* R *such that* $t_0\in S$, $R_0>0$ *and* $R>\max\{1;2R_0\}$. *Here* C *is a positive constant which depends on* K, p *and* $\mathrm{mes}_n\Omega$ *only.*

Hereafter $\mathrm{mes}_n\Omega$ means the Lebesgue measure of Ω.

Remark 3. Note that Theorem 2 has no conditions imposed on the behaviour of the solution and the growth of the functions a_j $(j=0,\dots,n)$ as well as on the behaviour of f as $t\to-\infty$. However, the theorem is not true for the case when $p=2$ (see, for example, [12]). Therefore the condition $p>2$ is essential.

3.2. Auxiliary statements.

Lemma 1. *Suppose that a function* $z\in L^2(t_1,t_2;H_0^1(\Omega))\cap L^p(Q_{t_1,t_2})$, *where* $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, *satisfies the identity*

$$\int_{t_1}^{t_2}\int_\Omega\Big\{-z\psi\varphi'+\sum_{i=0}^n g_i\partial_i\psi\varphi\Big\}dxdt=0,\quad \psi\in V^p(\Omega),\ \varphi\in C_c^1(t_1,t_2), \tag{7}$$

for some $g_i\in L^2(Q_{t_1,t_2})$ $(i=\overline{1,n})$, $g_0\in L^{p'}(Q_{t_1,t_2})$.
Then

(i) *the function* z *belongs to the space* $C([t_1,t_2];L^2(\Omega))$ *and for every* $\theta\in C^1([t_1,t_2])$ *and for all* $\tau_1,\tau_2\in[t_1,t_2]$ $(\tau_1<\tau_2)$ *we have*

$$\frac12\theta(t)\int_\Omega |z(x,t)|^2dx\Big|_{t=\tau_1}^{t=\tau_2}-\frac12\int_{\tau_1}^{\tau_2}\int_\Omega |z|^2\theta'dxdt+\int_{\tau_1}^{\tau_2}\int_\Omega\sum_{i=0}^n g_i\partial_i z\theta dxdt=0; \tag{8}$$

(ii) *the derivative* z_t *of the function* z *in the sense* $D'\big(t_1,t_2;(V^p(\Omega))'\big)$ *(the distributions space) belongs to* $L^{p'}\big(t_1,t_2;(V^p(\Omega))'\big)$, *furthermore*

$$\int_{t_1}^{t_2}\|z_t(\cdot,t)\|^{p'}_{(V^p(\Omega))'}dt\le C_3\Big[\sum_{i=1}^n\|g_i\|^{p'}_{L^{p'}(t_1,t_2;L^2(\Omega))}+\|g_0\|^{p'}_{L^{p'}(Q_{t_1,t_2})}\Big], \tag{9}$$

where $C_3>0$ *is a constant depending on* t_1,t_2,p *and* n *only.*

Proof. The first statement follows directly from Lemma 2 of [11]. Let us prove the second statement. Firstly note that the following continuous and dense embeddings hold

$$V^p(\Omega)\subset L^2(\Omega)\subset\big(V^p(\Omega)\big)'. \tag{10}$$

Since the spaces $L^2(t_1,t_2;V^p(\Omega))$, $L^{p'}(t_1,t_2;(V^p(\Omega))')$ can be identified with subspaces of the space of distributions $D'(t_1,t_2;(V^p(\Omega))')$, then it allows us to speak about derivatives of functions from $L^2(t_1,t_2;V^p(\Omega))$ in the sense of $D'(t_1,t_2;(V^p(\Omega))')$ and their belonging to the space $L^{p'}(t_1,t_2;(V^p(\Omega))')$.

Let us rewrite equality (7) in the form

$$-\int_{t_1}^{t_2}\int_\Omega z\psi\varphi' dxdt = -\int_{t_1}^{t_2}\int_\Omega \sum_{i=0}^n g_i\partial_i\psi\varphi dxdt, \quad \psi\in V^p(\Omega),\ \varphi\in C_c^1(t_1,t_2). \tag{11}$$

According to the definition of the derivative of distributions from $D'(t_1,t_2;\big(V^p(\Omega)\big)')$, (11) implies that z_t belongs to the space $L^{p'}(t_1,t_2;(V^p(\Omega))')$, and for almost all $t\in(t_1,t_2)$

$$\langle z_t(\cdot,t),\psi(\cdot)\rangle_{V^p(\Omega)} = -\int_\Omega \sum_{i=0}^n g_i(x,t)\partial_i\psi(x)dx,$$

where $\langle\cdot\,,\,\cdot\rangle_{V^p(\Omega)}$ denotes the canonical scalar product between $(V^p(\Omega))'$ and $V^p(\Omega)$.

From this, using the Cauchy-Schwarz inequality, for almost all $t\in(t_1,t_2)$ we obtain

$$|\langle z_t(\cdot,t),\psi(\cdot)\rangle_{V^p(\Omega)}| \le \sum_{i=1}^n \|g_i(\cdot,t)\|_{L^2(\Omega)}\|\partial_i\psi(\cdot)\|_{L^2(\Omega)} + \|g_0(\cdot,t)\|_{L^{p'}(\Omega)}\|\psi(\cdot)\|_{L^p(\Omega)}$$

$$\le \Big(\sum_{i=1}^n \|g_i(\cdot,t)\|^2_{L^2(\Omega)}\Big)^{1/2}\|\psi(\cdot)\|_{H^1(\Omega)} + \|g_0(\cdot,t)\|_{L^{p'}(\Omega)}\|\psi(\cdot)\|_{L^p(\Omega)}. \tag{12}$$

From (12) it follows that for almost all $t\in(t_1,t_2)$ the following estimate is valid

$$\|z_t(\cdot,t)\|_{(V^p(\Omega))'} \le \Big(\sum_{i=1}^n \|g_i(\cdot,t)\|^2_{L^2(\Omega)}\Big)^{1/2} + \|g_0(\cdot,t)\|_{L^{p'}(\Omega)} \le \tag{13}$$

$$\le \sum_{i=1}^n \|g_i(\cdot,t)\|_{L^2(\Omega)} + \|g_0(\cdot,t)\|_{L^{p'}(\Omega)}.$$

Hölder's inequality implies

$$\big(\sum_{i=0}^n a_i\big)^{p'} \le (n+1)^{p'/p}\sum_{i=0}^n a_i^{p'} \quad \text{if } a_i\ge 0, \quad i=\overline{0,n}. \tag{14}$$

From (13), using (14), we obtain

$$\|z_t(\cdot,t)\|^{p'}_{(V^p(\Omega))'} \le C_4\Big(\sum_{i=1}^n \|g_i(\cdot,t)\|^{p'}_{L^2(\Omega)} + \|g_0(\cdot,t)\|^{p'}_{L^{p'}(\Omega)}\Big), \tag{15}$$

where $C_4 := (n+1)^{p'/p}$.

Integrating (15) we get (9). □

Lemma 2. *Suppose that conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ and $(\mathcal{G}_1)$, $(\mathcal{G}_2)$ hold. Given $t_1,t_2\in\mathbb{R}$ such that $t_2-t_1\ge 1$, we suppose that functions y_l $(l=1,2)$ from $L^2(t_1,t_2;H_0^1(\Omega))\cap L^p(Q_{t_1,t_2})\cap C([t_1,t_2];L^2(\Omega))$ satisfy the following identities*

$$\int_{t_1}^{t_2}\int_\Omega\Big(-y_l\psi\varphi' + \sum_{i=1}^n a_i(x,t,y_l,\nabla y_l)\partial_i\psi\varphi + \widehat{a}_0(x,t,y,\nabla y)\psi\varphi + vg(x,t,y)\psi\varphi\Big)dxdt =$$

$$= \int_{t_1}^{t_2}\int_\Omega f_l\psi\varphi dxdt, \quad \psi\in V^p(\Omega),\ \varphi\in C_c^1(t_1,t_2), \tag{16}$$

with the functions $f_l \in L^{p'}(Q_{t_1,t_2})$ $(l = 1, 2)$.

Then the inequality

$$\max_{t\in[t_0-R_0,t_0]} \int_\Omega |y_1(x,t) - y_2(x,t)|^2 dx + \int_{t_0-R_0}^{t_0}\int_\Omega \big(|\nabla(y_1 - y_2)|^2 + |y_1 - y_2|^p\big) dxdt$$
$$\le C\Big\{R^{-2/(p-2)} + \int_{t_0-R}^{t_0}\int_\Omega |f_1 - f_2|^{p'} dxdt\Big\} \qquad (17)$$

holds for each t_0*,* R_0 *and* R *such that,* $R_0 > 0$*,* $R \ge \max\{1; 2R_0\}$ *and* $t_1 \le t_0 - R < t_0 \le t_2$*. Here* C *is such as in* (6)*.*

Proof. Let t_0, R_0, R be such as in the formulation of the lemma, and $\eta(t) := t - t_0 + R, t \in \mathbb{R}$. For given $\psi \in V^p(\Omega)$, $\varphi \in C_c^1(t_1, t_2)$ we subtract equality (16) when $l = 1$, and the same equality when $l = 2$. Then, putting

$$y_{12}(x,t) := y_1(x,t) - y_2(x,t), \quad f_{12}(x,t) := f_1(x,t) - f_2(x,t),$$
$$g_{12}(x,t,y_1) := g(x,t,y_1) - g(x,t,y_2),$$
$$\widehat{a}_{0,12}(x,t) := \widehat{a}_0(x,t,y_1(x,t),\nabla y_1(x,t)) - \widehat{a}_0(x,t,y_2(x,t),\nabla y_2(x,t)),$$
$$a_{i,12}(x,t) := a_i(x,t,y_1(x,t),\nabla y_1(x,t)) - a_i(x,t,y_2(x,t),\nabla y_2(x,t))$$
$$(i = 1,\dots,n;\ (x,t) \in Q),$$

we receive an equality. From this equality using Lemma 1 with $w = y_{12}$, $g_0 = \widehat{a}_{0,12} + vg_{12} - f_{12}$, $g_j = a_{j,12}$ $(j = 1,\dots,n)$, $\theta = \eta^s$, $s := 2p/(p-2)$, $\tau_1 = t_0 - R$, $\tau_2 = \tau \in (t_0 - R, t_0]$, we get the equality

$$\eta^s(\tau)\int_\Omega |y_{12}(x,\tau)|^2 dx + 2\int_{t_0-R}^{\tau}\int_\Omega \Big[\sum_{i=1}^n a_{i,12}(y)(\partial_i y_{12}) + \widehat{a}_{0,12}y_{12} + vg_{12}y_{12}\Big]\eta^s dxdt \qquad (18)$$
$$= s\int_{t_0-R}^{\tau}\int_\Omega |y_{12}|^2\eta^{s-1} dxdt + 2\int_{t_0-R}^{\tau}\int_\Omega f_{12}y_{12}\eta^s dxdt.$$

We make the corresponding estimates of the integrals of equality (18).

From conditions $(\mathcal{A}_3)$ and $(\mathcal{G}_2)$ we obtain

$$\int_{t_0-R}^{\tau}\int_\Omega \Big[\sum_{i=1}^n a_{i,12}(y)(\partial_i y_{12}) + \widehat{a}_{0,12}y_{12} + vg_{12}y_{12}\Big]\eta^s dxdt \ge K\int_{t_0-R}^{\tau}\int_\Omega \big(|\nabla y_{12}|^2 + |y_{12}|^p\big)\eta^s dxdt. \qquad (19)$$

Further we need the following inequality:

$$ab \le \varepsilon|a|^q + \varepsilon^{-1/(q-1)}|b|^{q'}, \quad a,b \in \mathbb{R},\ q > 1,\ 1/q + 1/q' = 1,\ \varepsilon > 0, \qquad (20)$$

which is a corollary from standard Young's inequality: $ab \le |a|^q/q + |b|^{q'}/q'$.

Putting $q = p/2$, $q' = p/(p-2)$, $a = |y_{12}|^2\eta^{s/q}$, $b = \eta^{s/q'-1}$, $\varepsilon = \varepsilon_1 > 0$, under (20) we obtain

$$\int_{t_0-R}^{\tau}\int_\Omega |y_{12}|^2\eta^{s-1} dxdt \le \varepsilon_1\int_{t_0-R}^{\tau}\int_\Omega |y_{12}|^p\eta^s dxdt + \varepsilon_1^{-2/(p-2)}\int_{t_0-R}^{\tau}\int_\Omega \eta^{s-p/(p-2)} dxdt, \qquad (21)$$

where $\varepsilon_1 > 0$ is an arbitrary number.

Again using inequality (20), we obtain

$$\int_{t_0-R}^{\tau}\int_{\Omega} f_{12}y_{12}\eta^s dxdt \le \varepsilon_2 \int_{t_0-R}^{\tau}\int_{\Omega} |y_{12}|^p\eta^s dxdt + \varepsilon_2^{-1/(p-1)}\int_{t_0-R}^{\tau}\int_{\Omega} |f_{12}|^{p'}\eta^s dxdt, \tag{22}$$

where $\varepsilon_2 > 0$ is an arbitrary number.

From (18) using (19), (21), (22) and $(\mathcal{G}_2)$, if $\varepsilon_1 = K/(2s)$ and $\varepsilon_2 = K/4$, we obtain the following

$$\eta^s(\tau)\int_{\Omega_R} |y_{12}(x,\tau)|^2 dx + K\int_{t_0-R}^{\tau}\int_{\Omega}\Big\{|\nabla y_{12}|^2 + |y_{12}|^p\Big\}\eta^s dxdt \le$$
$$\le C_5\Big[\int_{t_0-R}^{\tau}\int_{\Omega}\eta^{s-p/(p-2)}dxdt + \int_{t_0-R}^{\tau}\int_{\Omega}|f_{12}|^{p'}\eta^s dxdt\Big], \tag{23}$$

where $C_5 > 0$ is a constant depending on K and p only.

Note that $0 \le \eta(t) \le R$, if $t \in [t_0 - R, t_0]$, and $\eta(t) \ge R - R_0$, if $t \in [t_0 - R_0, t_0]$. Using this and that $R \ge \max\{1; 2R_0\}$ (then, in particular, we have $R/(R-R_0) = 1 + R_0/(R-R_0) \le 2$), from (23) we obtain the required statement. □

3.3. Proof of Theorem 2.

Proof. First we prove that there exists at most one weak solution of problem (1), (2). Assume the contrary. Let y_1, y_2 be (distinct) weak solutions of this problem. Using Lemma 2 we get

$$\max_{t\in[t_0-R_0,t_0]}\int_{\Omega}|y_1(x,t) - y_2(x,t)|^2 dx \le CR^{-2/(p-2)}, \tag{24}$$

where t_0, R_0, R are arbitrary numbers such that such that $t_0 \in S$, $R_0 > 0$, $R > \max\{1; 2R_0\}$.

We fix numbers $R_0 > 0$, $t_0 \in S$, and take the limit when $R \to +\infty$ in (24). As a result we receive that $y_1 = y_2$ almost everywhere on $Q_{t_0-R_0,t_0}$. Since R_0 and t_0 are arbitrary numbers, we obtain $y_1 = y_2$ almost everywhere on Q. The obtained contradiction proves our statement.

Now we are turning to the proof of the existence of a weak solution of problem (1), (2). For each $m \in \mathbb{N}$ we consider an initial-boundary value problem for equation (1) in the domain $Q_m = \Omega \times (-m, 0)$ with a homogeneous initial condition and boundary conditions (2), namely: we are searching a function $y_m \in L^2(-m,0;H_0^1(\Omega)) \cap L^p(Q_m) \cap C([-m,0];L^2(\Omega))$ which satisfies the initial condition $y_m|_{t=-m} = 0$ and the integral equality

$$\iint_{Q_m}\Big\{-y_m\psi\varphi' + \sum_{i=1}^{n} a_i(y_m)\partial_i\psi\varphi + \widehat{a}_0(y_m)\psi\varphi + vg(y_m)\psi\varphi\Big\}dxdt =$$
$$= \iint_{Q_m} f_m\psi\varphi dxdt, \quad \psi \in V^p(\Omega), \varphi \in C_c^1(-m,0), \tag{25}$$

where $f_m(x,t) := f(x,t)$ if $(x,t) \in Q_m$, and $f_m(x,t) := 0$ if $(x,t) \in Q \setminus Q_m$.

The existence and uniqueness of the function y_m follows from a well-known fact (see, for example, [22, p. 539]).

We extend y_m on Q by zero and for this extension we keep the same notation y_m. Further we prove that the sequence $\{y_m\}$ converges in $Y^p_{\mathrm{loc}}(Q)$ to a weak solution of problem (1), (2). Indeed, note that for each $m \in \mathbb{N}$ the fuction y_m is a weak solution of the problem which differs from problem (1), (2) in f_m instead of f. Using Lemma 2 for each natural numbers m and k we have

$$\max_{t\in[t_0,t_0-R_0]}\int_\Omega |y_m(x,t)-y_k(x,t)|^2dx+\int_{t_0-R_0}^{t_0}\int_\Omega \big[|\nabla(y_m-y_k)|^2+|y_m-y_k|^p\big]dxdt\le$$
$$\le C\Big\{R^{-2/(p-2)}+\int_{t_0-R}^{t_0}\int_\Omega |f_m-f_k|^{p'}dxdt\Big\}, \tag{26}$$

where t_0, R_0, R are arbitrary numbers such that $t_0\in S$, $R_0>0$, $R>\max\{1;2R_0\}$.

Now we show that for fixed t_0 and R_0 the left side of inequality (26) converges to zero when $m,k\to+\infty$. Actually, let $\varepsilon>0$ be an arbitrary small number. We choose R to be big enough such that the following inequality holds

$$CR^{-2/(p-2)}<\varepsilon. \tag{27}$$

This is possible as $p>2$. Under (27) for arbitrary $m,k\in\mathbb{N}$ such that $\max\{-m,-k\}\le t_0-R$ (then $f_m=f_k$ almost everywhere on $\Omega\times(t_0-R,t_0)$) the right side of inequality (26) is less than ε. From this it follows that the restriction of the terms of the sequence $\{y_m\}$ on $Q_{t_0-R_0,t_0}$ is a Cauchy sequence in $L^2(t_0-R_0,t_0;H_0^1(\Omega))\cap L^p(Q_{t_0-R,t_0})\cap C([t_0-R_0,t_0];L^2(\Omega))$. Therefore, since t_0 and R_0 are arbitrary, it follows that there exists a function $y\in Y^p_{\rm loc}(Q)$ such that $y_m\to y$ strongly in $Y^p_{\rm loc}(Q)$. From this and [20, Lemma 2.2] it easily follows

$$\iint_Q\Big(\sum_{i=1}^n a_i(y_m)\partial_i\psi\varphi+\widehat{a}_0(y_m)\psi\varphi+vg(y_m)\psi\varphi\Big)dxdt\underset{m\to\infty}{\longrightarrow}$$
$$\underset{m\to\infty}{\longrightarrow}\iint_Q\Big(\sum_{i=1}^n a_i(y)\partial_i\psi\varphi+\widehat{a}_0(y)\psi\varphi+vg(y)\psi\varphi\Big)dxdt.$$

Taking into account that in (25) integration on Q_m can be replaced by integration on Q, we pass to the limit in equality (25) as $m\to\infty$. So, we abtain (52). It means that the function y is a weak solution of problem (1), (2). Estimate (6) directly follows from Lemma 2 putting $y_1=y,\ y_2=0,\ f_1=f,\ f_2=0$. □

4. Proof of the main result.

Proof of Theorem 1. Since the cost functional J is bounded below, there exists a minimizing sequence $\{v_k\}$ for J in U_∂, i.e., $J(v_k)\underset{k\to\infty}{\longrightarrow}\inf_{v\in U_\partial}J(v)$. This and (4) imply that the sequence $\{v_k\}$ is bounded in the space $L^\infty(Q)$, that is

$$\operatorname*{ess\,sup}_{(x,t)\in Q}|v_k(x,t)|\le C_6\quad \forall k\in\mathbb{N}, \tag{28}$$

where $C_6>0$ is a constant, which does not depend on k.

Since for each $k\in\mathbb{N}$ the function $y_k:=y(v_k)$ $(k\in\mathbb{N})$ is a weak solution of problem (1), (2) for $v=v_k$, the following identity holds

$$\iint_Q\Big\{-y_k\psi\varphi'+\sum_{i=1}^n a_i(y_k)\partial_i\psi\varphi+\widehat{a}_0(y_k)\psi\varphi+v_kg(y_k)\psi\varphi\Big\}dxdt=$$
$$=\iint_Q f\psi\varphi dxdt,\ \psi\in V^p(\Omega),\ \varphi\in C_c^1(-\infty,0). \tag{29}$$

According to Theorem 2 for each $k \in \mathbb{N}$ we have the estimate

$$\max_{t\in[t_0-R_0,t_0]} \int_\Omega |y_k(x,t)|^2 dx + \int_{t_0-R_0}^{t_0}\int_\Omega \big[|\nabla y_k|^2 + |y_k|^p\big] dxdt \le$$
$$\le C\Big\{R^{-2/(p-2)} + \int_{t_0-R}^{t_0}\int_\Omega |f|^{p'} dxdt\Big\}, \tag{30}$$

where t_0, R_0, R are arbitrary such that $t_0 \in S$, $R_0 > 0, R \ge \max\{1, 2R_0\}$ and constant C is independent on $k \in \mathbb{N}$.

Let $\tau_1, \tau_2 \in S(\tau_1 < \tau_2)$ be arbitrary. From (30) and condition $(\mathcal{F})$ we obtain

$$\|\nabla y_k\|_{L^2(Q_{\tau_1,\tau_2})} \le C_7, \quad \|y_k\|_{L^p(Q_{\tau_1,\tau_2})} \le C_7, \quad k \in \mathbb{N}, \tag{31}$$

where $C_7 > 0$ is a constant independent on k.

From $(\mathcal{A}_2)$ and (31) it follows

$$\int_{\tau_1}^{\tau_2}\int_\Omega |\widehat{a}_0(y_k)|^{p'} dxdt \le C_8 \int_{\tau_1}^{\tau_2}\int_\Omega \big(|y_k|^p + |\nabla y_k|^2 + |h_0|^{p'}\big) dxdt \le C_9, \tag{32}$$

$$\int_{\tau_1}^{\tau_2}\int_\Omega |a_i(y_k)|^2 dxdt \le C_{10} \int_{\tau_1}^{\tau_2}\int_\Omega \big(|y_k|^p + |\nabla y_k|^2 + \sum_{i=1}^n |h_i|^2\big) dxdt \le C_{11}, \quad i = \overline{1,n}, \tag{33}$$

where $C_8, \ldots, C_{11}$ are positive constants independent on k.

Since $p > 2$, then $1 < p' < 2$. Hence we have continuous embeddings

$$L^p(Q_{\tau_1,\tau_2}) \subset L^2(Q_{\tau_1,\tau_2}) \subset L^{p'}(Q_{\tau_1,\tau_2}). \tag{34}$$

According to (33) and (34), we have

$$\Big(\int_{\tau_1}^{\tau_2} \|a_i(y_k)\|^{p'}_{L^2(\Omega)} dt\Big)^{1/p'} \le \Big(\int_{\tau_1}^{\tau_2} \|a_i(y_k)\|^2_{L^2(\Omega)} dt\Big)^{1/2} \le \sqrt{C_{11}}, \tag{35}$$

$$\|y_k\|_{L^{p'}(Q_{\tau_1,\tau_2})} \le C_{12} \|y_k\|_{L^p(Q_{\tau_1,\tau_2})} \le C_{13}, \tag{36}$$

where C_{12}, C_{13} are positive constants independent on k.

From $(\mathcal{G}_1), (\mathcal{G}_2)$ we easily get

$$|g(x,t,y_k)| \le M|y_k| + |g(x,t,0)| \quad \text{для м. в.} \quad (x,t) \in Q.$$

Thus, using previous inequality, (14), (28), (31) and (36) we obtain

$$\int_{\tau_1}^{\tau_2}\int_\Omega |v_k g(y_k)|^{p'} dxdt \le (C_6)^{p'} \int_{\tau_1}^{\tau_2}\int_\Omega |g(y_k)|^{p'} dxdt \tag{37}$$
$$\le C_{14} \int_{\tau_1}^{\tau_2}\int_\Omega (|y_k|^{p'} + |g(x,t,0)|^{p'}) dxdt \le C_{15},$$

where C_{14}, C_{15} are positive constants independent on k.

Taking into account statement *(ii)* of Lemma 1 and (14), according to condition $(\mathcal{F})$, (32), (35), (37) from (29) we obtain

$$\int_{\tau_1}^{\tau_2} \|y_{k,t}\|^{p'}_{(V^p(\Omega))'} dt \le C_3 \Big[\sum_{i=1}^n \|a_i(y_k)\|^{p'}_{L^{p'}(t_1,t_2;L^2(\Omega))} + \|a_0(y_k) + v_k g(y_k) - f\|^{p'}_{L^{p'}(Q_{t_1,t_2})}\Big] \le C_{16}, \tag{38}$$

where $C_{16} > 0$ is a constant independent on k.

Further, we will need the following statement.

Proposition 1. *(Aubin theorem, see [2] and [3, p. 393]). If $q > 1, r > 1$ are any real numbers, $t_1, t_2 \in \mathbb{R}$ $(t_1 < t_2)$, $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are any Banach spaces such that $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

$$\{u \in L^q(t_1, t_2; \mathcal{W}) \mid u' \in L^r(t_1, t_2; \mathcal{B})\} \overset{K}{\subset} \Big(L^q(t_1, t_2; \mathcal{L}) \cap C([t_1, t_2]; \mathcal{B})\Big),$$

that is, if $\{u'_m\}_{m\in\mathbb{N}}$ is bounded sequence in the space $L^q(t_1, t_2; \mathcal{W})$ and $\{u'_m\}_{m\in\mathbb{N}}$ is bounded sequence in the space $L^r(t_1, t_2; \mathcal{B})$, then there exists a subsequence $\{u_{m_j}\}_{j\in\mathbb{N}} \subset \{u_m\}_{m\in\mathbb{N}}$ and function $u \in L^q(t_1, t_2; \mathcal{L}) \cap C([t_1, t_2]; \mathcal{B})$ such that $u_{m_j} \underset{j\to\infty}{\longrightarrow} u$ strongly in $L^q(t_1, t_2; \mathcal{L})$ and in $C([t_1, t_2]; \mathcal{B})$.

Since $V^p(\Omega) \circlearrowleft H^1_0(\Omega) \overset{K}{\subset} L^2(\Omega)$ (see [25], p. 245), then $V^p(\Omega) \overset{K}{\subset} L^2(\Omega)$. According to Proposition 1 with $\mathcal{W} = V^p(\Omega)$, $\mathcal{L} = L^2(\Omega)$, $\mathcal{B} = (V^p(\Omega))'$, $q = 2$, $r = p'$, estimates (28), (31), (32), (33), (38) yield that there exists a subsequence of the sequence $\{v_k, y_k\}$ (still denoted by $\{v_k, y_k\}$) and functions $u \in U_\partial$, $y \in L^2_{\rm loc}(S; H^1_0(\Omega)) \cap L^p_{\rm loc}(\overline{Q})$ and $\chi_0 \in L^{p'}_{\rm loc}(\overline{Q})$, $\chi_i \in L^2_{\rm loc}(\overline{Q})$ $(i = \overline{1,n})$ such that

$$v_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^\infty(Q), \tag{39}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in} \quad L^2_{\rm loc}(S; H^1_0(\Omega)), \tag{40}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in} \quad L^p_{\rm loc}(\overline{Q}), \tag{41}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{strongly in} \quad L^2_{\rm loc}(S; L^2(\Omega)), \tag{42}$$

$$\widehat{a}_0(y_k) \underset{k\to\infty}{\longrightarrow} \chi_0 \quad \text{weakly in} \quad L^{p'}_{\rm loc}(\overline{Q}), \tag{43}$$

$$a_i(y_k) \underset{k\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L^2_{\rm loc}(\overline{Q}), \quad i = \overline{1,n}. \tag{44}$$

Note that (40) implies the following

$$\partial_i y_k \underset{k\to\infty}{\longrightarrow} \partial_i y \quad \text{weakly in} \quad L^2_{\rm loc}(\overline{Q}), \quad i = \overline{0,n}. \tag{45}$$

Let us show that (39) and (42) yield

$$\iint_Q v_k g(y_k)\psi\varphi dxdt \underset{k\to\infty}{\longrightarrow} \iint_Q u g(y)\psi\varphi dxdt \quad \forall\psi \in V^p(\Omega), \forall\varphi \in C^1_c(-\infty, 0). \tag{46}$$

Indeed, let φ be an arbitrary, and $t_1, t_2 \in S$ be such that ${\rm supp}\varphi \subset [t_1, t_2]$. Then we have

$$\iint_Q v_k g(y_k)\psi\varphi dxdt = \int_{t_1}^{t_2}\int_\Omega (v_k g(y_k) - v_k g(y) + v_k g(y))\psi\varphi dxdt =$$
$$= \int_{t_1}^{t_2}\int_\Omega v_k g(y)\psi\varphi dxdt + \int_{t_1}^{t_2}\int_\Omega v_k (g(y_k) - g(y))\psi\varphi dxdt. \tag{47}$$

Using condition $(\mathcal{G}_2)$ we easily obtain $|g(y_k) - g(y)| \le M|y_k - y|$. Hence, using Cauchy-Schwarz inequality, (28) and (42), we obtain

$$\Big|\int_{t_1}^{t_2}\int_\Omega v_k(g(y_k) - g(y))\psi\varphi dxdt\Big| \le M\int_{t_1}^{t_2}\int_\Omega v_k|y_k - y||\psi\varphi|dxdt$$
$$\le M\Big(\int_{t_1}^{t_2}\int_\Omega |v_k\psi\varphi|^2 dxdt\Big)^{1/2}\Big(\int_{t_1}^{t_2}\int_\Omega |y_k - y|^2 dxdt\Big)^{1/2} \underset{k\to\infty}{\longrightarrow} 0. \tag{48}$$

Thus, using (39) and (48), (47) implies (46).

Similarly to (46) it can be easily shown that (39) and (42) yield

$$\iint_Q v_k g(y_k) y_k \varphi dxdt \underset{k\to\infty}{\longrightarrow} \iint_Q ug(y)y\varphi dxdt \quad \forall\, \varphi \in C_c^1(-\infty,0). \tag{49}$$

Indeed,

$$\iint_Q v_k g(y_k)y_k\varphi dxdt = \iint_Q \big(v_k g(y_k)y_k - v_k g(y)y_k + v_k g(y)y_k\big)\varphi dxdt =$$

$$= \iint_Q v_k y_k (g(y_k)-g(y))\varphi dxdt + \iint_Q v_k g(y) y_k \varphi dxdt.$$

Similarly to (46), from (39) and (42), we easily get

$$\iint_Q v_k g(y) y_k \varphi dxdt \underset{k\to\infty}{\longrightarrow} \iint_Q ug(y)y\varphi dxdt. \tag{50}$$

Condition $(\mathcal{G}_2)$, Cauchy-Schwarz inequality, (28), (31) and (42) imply

$$\Big|\iint_Q v_k y_k (g(y_k)-g(y))\varphi dxdt\Big| \le M \iint_Q v_k |y_k||y_k - y||\varphi| dxdt \underset{k\to\infty}{\longrightarrow} 0. \tag{51}$$

From (50) and (51) we obtain (49).

Letting $k\to\infty$ in (29), using (43)–(46) we obtain

$$\iint_Q \Big\{-y\psi\varphi' + \sum_{i=0}^n \chi_i \partial_i \psi\varphi + ug(y)\psi\varphi\Big\} dxdt =$$

$$= \iint_Q f\psi\varphi dxdt, \quad \psi \in V^p(\Omega), \varphi \in C_c^1(-\infty,0). \tag{52}$$

According to Lemma 1, identity (52) implies that $y \in C(S; L^2(\Omega))$. This and the fact that $y \in L^2_{\mathrm{loc}}(S; H_0^1(\Omega)) \cap L^p_{\mathrm{loc}}(\overline{Q})$ imply $y \in Y^p_{\mathrm{loc}}(Q)$.

Now let us show that the equality

$$\int_\Omega \Big\{\sum_{i=0}^n \chi_i \partial_i \psi\Big\} dx = \int_\Omega \Big\{\sum_{i=1}^n a_i(y)\partial_i\psi + \widehat{a}_0(y)\psi\Big\} dx \tag{53}$$

is valid for every $\psi \in V^p(\Omega)$ and for a. e. $t \in S$. For this we use the monotonicity method (see [24, Section 2]).

Let us take an arbitrary functions $w \in L^2_{\mathrm{loc}}(S; H^1(\Omega))$ and $\theta \in C_c^1(-\infty,0)$, $\theta(t) \ge 0$ for all $t \in (-\infty,0)$. Using condition $(\mathcal{A}_3)$ for every $k \in \mathbb{N}$ we have

$$W_k := \iint_Q \Big\{\sum_{i=1}^n \big[(a_i(y_k) - a_i(w))(\partial_i y_k - \partial_i w) + (\widehat{a}_0(y_k) - \widehat{a}_0(w))(y_k - w)\big]\Big\}\theta\, dxdt \ge 0.$$

From this we obtain

$$W_k = \iint_Q \Big(\sum_{i=1}^n a_i(y_k)\partial_i y_k + \widehat{a}_0(y_k)y_k\Big)\theta dxdt - \tag{54}$$

$$- \iint_Q \Big(\sum_{i=1}^n \big[a_i(y_k)\partial_i w + a_i(w)(\partial_i y_k - \partial_i w)\big] + \widehat{a}_0(y_k)w + \widehat{a}_0(w)(y_k - w)\Big)\theta \ge 0, \quad k \in \mathbb{N}.$$

According to Lemma 1, (29) implies

$$-\frac{1}{2}\iint_Q |y_k|^2\theta' dxdt + \iint_Q \Big\{\sum_{i=1}^n a_i(y_k)\partial_i y_k + \widehat{a}_0(y_k)y_k + v_k g(y_k)y_k\Big\}\theta dxdt = \iint_Q f y_k \theta dxdt. \tag{55}$$

From (54), using (55), we obtain

$$W_k = \iint_Q \Big\{\frac{1}{2}|y_k|^2\theta' + \big(fy_k - v_k g(y_k)y_k\big)\theta\Big\} dxdt- \tag{56}$$

$$-\iint_Q \Big(\sum_{i=1}^n \big[a_i(y_k)\partial_i w + a_i(w)(\partial_i y_k - \partial_i w)\big] + \widehat{a}_0(y_k)w + \widehat{a}_0(w)(y_k - w)\Big)\theta dxdt \geq 0,\ k \in \mathbb{N}.$$

Taking into account (42) and (49) we have

$$\lim_{k\to\infty}\iint_Q\Big\{\frac{1}{2}|y_k|^2\theta' + \big(fy_k - v_k g(y_k)y_k\big)\theta\Big\}dxdt = \iint_Q\Big\{\frac{1}{2}|y|^2\theta' + \big(fy - ug(y)y\big)\theta\Big\}dxdt. \tag{57}$$

By (43)–(45) and (57) from (56) we get

$$0 \leq \lim_{k\to\infty} W_k = \iint_Q \Big\{\frac{1}{2}|y|^2\theta' + \big(fy - ug(y)y\big)\theta\Big\}dxdt-$$

$$-\iint_Q\Big(\sum_{i=1}^n \big[\chi_i\partial_i w + a_i(w)(\partial_i y - \partial_i w)\big] + \chi_0 w + \widehat{a}_0(w)(y-w)\Big)\theta dxdt. \tag{58}$$

From (52), using Lemma 1, we obtain

$$\iint_Q \sum_{i=0}^n \chi_i \partial_i y\theta\ dxdt = \iint_Q \Big\{\frac{1}{2}|y|^2\theta' + \big(fy - ug(y)y\big)\theta\Big\}dxdt. \tag{59}$$

Thus, (58) and (59) imply that

$$\iint_Q \Big\{\sum_{i=1}^n (\chi_i - a_i(w))(\partial_i y - \partial_i w) + (\chi_0 - \widehat{a}_0(w))(y-w)\Big\}\theta dxdt \geq 0. \tag{60}$$

Substituting $w = y - \lambda\psi$ in the above inequality, where $\psi \in H_0^1(\Omega)$, $\lambda > 0$ are arbitrary, and dividing the obtained inequality by λ we get

$$\iint_Q \Big\{\sum_{i=1}^n (\chi_i - a_i(u - \lambda\psi))\partial_i\psi + (\chi_0 - \widehat{a}_0(u-\lambda\psi))\psi\Big\}\theta\ dxdt \geq 0. \tag{61}$$

Letting $\lambda \to 0+$ in (61), using condition $(\mathcal{A}_2)$ and the Dominated Convergence Theorem (see [15, p. 648]), we have

$$\iint_Q \Big\{\sum_{i=1}^n (\chi_i - a_i(y))\partial_i\psi + (\chi_0 - \widehat{a}_0(y))\psi\Big\}\theta\ dxdt = 0. \tag{62}$$

Since $\psi \in H_0^1(\Omega)$, $\theta \in C_c^1(-\infty, 0)$ are arbitrary functions, then (62) impliest (53). Identity (52) and (53) imply (3) with $v = u$.

Therefore y is a weak solution of problem (1), (2) with $v = u$. Hence, we have shown that $y = y(u) = y(x, t; u)$, $(x, t) \in Q$, is the state of the controlled system for the control u. Now we are going to show that u is an optimal control. First we prove that

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in} \quad C(S; L^2(\Omega)), \tag{63}$$

i.e., for every closed interval $[\alpha, \beta] \subset S$,

$$\max_{\tau\in[\alpha,\beta]} \int_\Omega |y(x,\tau) - y_k(x,\tau)|^2 dx \underset{k\to\infty}{\longrightarrow} 0. \tag{64}$$

For this purpose, we subtract identity (29) from identity (3) with $v = u$

$$\iint_Q \Big\{ -(y - y_k)\psi\varphi' + \sum_{i=1}^n \big(a_i(y) - a_i(y_k)\big)\partial_i\psi\varphi + \big(\widehat{a}_0(y) - \widehat{a}_0(y_k)\big)\psi\varphi + \\ +(ug(y) - v_k g(y_k))\psi\varphi\Big\} dxdt = 0, \psi \in V^p(\Omega),\ \varphi \in C_c^1(-\infty, 0). \tag{65}$$

To the resulting identity (65), we apply Lemma 1 with $\theta(t) = 2(t - \tau + 1)$, $\tau_1 = \tau - 1$, $\tau_2 = \tau$, where $\tau \in S$ is any fixed. Consequently, we get

$$\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2 dx - \int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2 dxdt + \int_{\tau-1}^{\tau}\int_\Omega \Big[\sum_{i=1}^n \big(a_i(y) - a_i(y_k)\big)(\partial_i y - \partial_i y_k) + \\ +\big(\widehat{a}_0(y) - \widehat{a}_0(y_k)\big)(y - y_k) + (ug(y) - v_k g(y_k))(y - y_k)\Big]\theta dxdt = 0. \tag{66}$$

From (66), taking into account conditions $(\mathcal{A}_3)$ and $(\mathcal{G}_2)$, we obtain

$$\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2 dx \le \tag{67}$$
$$\le \int_{\tau-1}^{\tau}\int_\Omega \big[|y - y_k|^2 - (ug(y) - v_k g(y) + v_k g(y) - v_k g(y_k))(y - y_k)\theta\big] dxdt \le$$
$$\le \int_{\tau-1}^{\tau}\int_\Omega \big[|y - y_k|^2 - (u - v_k)g(y)(y - y_k)\theta - v_k(g(y) - g(y_k))(y - y_k)\theta\big] dxdt \le$$
$$\le 2\int_{\tau-1}^{\tau}\int_\Omega \big[|y - y_k|^2 + |u - v_k||g(y)||y - y_k|\big] dxdt.$$

Using (28), $(\mathcal{G}_1)$, $(\mathcal{G}_2)$ and Cauchy-Schwarz inequality, from (67) we obtain

$$\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2 dx \le C_{17}\Big(\Big[\int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2 dxdt\Big]^{1/2} + \int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2 dxdt\Big), \tag{68}$$

where $C_{17} > 0$ is a constant which does not depend on k.

For every $\tau \in [\alpha, \beta]$, we obviously have $[\tau - 1, \tau] \subset [\alpha - 1, \beta]$. Hence, from estimate (68) we easily get for $\forall \tau \in [\alpha, \beta]$

$$\max_{\tau\in[\alpha,\beta]} \int_\Omega |y(x,\tau) - y_k(x,\tau)|^2 dx \le C_{17}\Big(\Big[\int_{\alpha-1}^{\beta}\int_\Omega |y - y_k|^2 dxdt\Big]^{1/2} + \int_{\alpha-1}^{\beta}\int_\Omega |y - y_k|^2 dxdt\Big). \tag{69}$$

Thus, according to (42), estimate (69) implies (64).

It remains to prove that u is a minimizing element of the functional J. Indeed, since functional $G\colon Y^p_{\rm loc}(Q)\to[0,+\infty)$ is lower semicontinuous in either $L^2_{\rm loc}(S;L^2(\Omega))$, or $C(S;L^2(\Omega))$, or in both and we have convergence (42), (64), then

$$\lim_{k\to\infty}\inf G(y_k)\ge G(y). \tag{70}$$

Also, (39) and properties of $*$-weakly convergent sequences (see [14, Proposition 3.13, p. 63]) yield $\lim_{k\to\infty}\inf\|v_k\|_{L^\infty(Q)}\ge\|u\|_{L^\infty(Q)}$.

From (4), (70) and previous inequality it easily follows that

$$\lim_{k\to\infty}J(v_k)\ge\lim_{k\to\infty}\inf G(y_k)+\mu\lim_{k\to\infty}\inf\|v_k\|_{L^\infty}(Q)\ge J(u).$$

Thus, we have shown that u is a solution of problem (5), i.e., an optimal control. □

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