УДК 512.536.7+512.568.2

O. Gutik, O. Sobol

ON FEEBLY COMPACT TOPOLOGIES ON THE SEMILATTICE $\mathrm{EXP}_n\lambda$

O. Gutik, O. Sobol. *On feebly compact topologies on the semilattice* $\exp_n\lambda$, Mat. Stud. **46** (2016), 29–43.

We study feebly compact topologies τ on the semilattice $(\exp_n\lambda,\cap)$ such that $(\exp_n\lambda,\tau)$ is a semitopological semilattice. All compact semilattice T_1-topologies on $\exp_n\lambda$ are described. Also we prove that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a T_1-topology τ on $\exp_n\lambda$ the following conditions are equivalent: (i) $(\exp_n\lambda,\tau)$ is a compact topological semilattice; (ii) $(\exp_n\lambda,\tau)$ is a countably compact topological semilattice; (iii) $(\exp_n\lambda,\tau)$ is a feebly compact topological semilattice; (iv) $(\exp_n\lambda,\tau)$ is a compact semitopological semilattice; (v) $(\exp_n\lambda,\tau)$ is a countably compact semitopological semilattice. We construct a countably pracompact H-closed quasiregular non-semiregular topology τ_{fc}^2 such that $\left(\exp_2\lambda,\tau_{\mathsf{fc}}^2\right)$ is a semitopological semilattice with discontinuous semilattice operation and prove that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every T_1-semiregular feebly compact semitopological semilattice $\exp_n\lambda$ is a compact topological semilattice.

We shall follow the terminology of [2, 3, 4, 5, 15]. If X is a topological space and $A\subseteq X$, then by $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ we denote the topological closure and interior of A in X, respectively. By ω we denote the first infinite cardinal.

A semigroup S is called an *inverse semigroup* if every a in S possesses an unique inverse, i.e. if there exists an unique element a^{-1} in S such that

$$aa^{-1}a=a \quad\text{and}\quad a^{-1}aa^{-1}=a^{-1}.$$

A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *topological* (*inverse*) *semigroup* is a topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ a *semigroup* (*inverse*) *topology* on S. A *semitopological semigroup* is a topological space together with a separately continuous semigroup operation.

If S is a semigroup, then by $E(S)$ we denote the subset of all idempotents of S. On the set of idempotents $E(S)$ there exists a natural partial order: $e\leqslant f$ *if and only if* $ef=fe=e$. A *semilattice* is a commutative semigroup of idempotents. A *topological* (*semitopological*) *semilattice* is a topological space together with a continuous (separately continuous) semilattice

2010 *Mathematics Subject Classification:* 22A26, 22A15, 54D10, 54D30, 54H12.
Keywords: topological semilattice; semitopological semilattice; compact; countably compact; feebly compact; H-closed; semiregular space; regular space.
doi:10.15330/ms.46.1.29-43

operation. If S is a semilattice and τ is a topology on S such that (S,τ) is a topological semilattice, then we shall call τ a *semilattice topology* on S.

Let λ be an arbitrary non-zero cardinal. A map α from a subset D of λ into λ is called a *partial transformation* of X. In this case the set D is called the *domain* of α and it is denoted by $\operatorname{dom}\alpha$. The image of an element $x\in\operatorname{dom}\alpha$ under α we shall denote by $x\alpha$ Also, the set $\{x\in X\colon y\alpha=x \text{ for some } y\in Y\}$ is called the *range* of α and is denoted by $\operatorname{ran}\alpha$. The cardinality of $\operatorname{ran}\alpha$ is called the *rank* of α and denoted by $\operatorname{rank}\alpha$. For convenience we denote by $\varnothing$ the empty transformation, that is a partial mapping with $\operatorname{dom}\varnothing=\operatorname{ran}\varnothing=\varnothing$.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta)=(x\alpha)\beta \quad \text{if} \quad x\in\operatorname{dom}(\alpha\beta)=\{y\in\operatorname{dom}\alpha\colon y\alpha\in\operatorname{dom}\beta\}, \quad \text{for} \quad \alpha,\beta\in\mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [3]). The symmetric inverse semigroup was introduced by V. V. Wagner [17] and it plays a major role in the theory of semigroups.

Put $\mathscr{I}_\lambda^n=\{\alpha\in\mathscr{I}_\lambda\colon \operatorname{rank}\alpha\leqslant n\}$, for $n=1,2,3,\ldots$. Obviously, $\mathscr{I}_\lambda^n$ $(n=1,2,3,\ldots)$ are inverse semigroups, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$, for each $n=1,2,3,\ldots$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$ [7, 12]. The empty partial map $\varnothing\colon\lambda\rightharpoonup\lambda$ we denote by 0. It is obvious that 0 is zero of the semigroup $\mathscr{I}_\lambda^n$.

Let λ be a non-zero cardinal. On the set $B_\lambda=(\lambda\times\lambda)\cup\{0\}$, where $0\notin\lambda\times\lambda$, we define the semigroup operation "$\cdot$" as follows

$$(a,b)\cdot(c,d)=\begin{cases}(a,d), & \text{if } b=c;\\ 0, & \text{if } b\neq c,\end{cases}$$

and $(a,b)\cdot 0=0\cdot(a,b)=0\cdot 0=0$ for $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units* (see [3]). Obviously, for any cardinal $\lambda>0$, the semigroup of $\lambda\times\lambda$-matrix units B_λ is isomorphic to $\mathscr{I}_\lambda^1$.

A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A))=A$.

We recall that a topological space X is said to be

- *functionally Hausdorff* if for every pair of distinct points $x_1,x_2\in X$ there exists a continuous function $f\colon X\to[0,1]$ such that $f(x_1)=0$ and $f(x_2)=1$;
- *semiregular* if X has a base consisting of regular open subsets;
- *quasiregular* if for any non-empty open set $U\subset X$ there exists a non-empty open set $V\subset U$ such that $\operatorname{cl}_X(V)\subseteq U$;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_i\}_{i\in\mathbb{N}}$ of X has a convergent subsequence in X;
- *countably compact* if each open countable cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it contained;
- *countably compact at a subset* $A\subseteq X$ if every infinite subset $B\subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;

- *feebly compact* if each locally finite open cover of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded.

According to Theorem 3.10.22 of [4], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [1]), and every H-closed space is feebly compact too (see [11]).

Topological properties of an infinite (semi)topological semigroup $\lambda\times\lambda$-matrix units studied in [8, 9, 10]. In [9] it is shown that on the infinite semitopological semigroup $\lambda\times\lambda$-matrix units B_λ there exists a unique Hausdorff topology τ_c such that (B_λ,τ_c) is a compact semitopological semigroup and there proved that every pseudocompact Hausdorff topology τ on B_λ such that (B_λ,τ_c) is a semitopological semigroup, is compact. Also, in [9] proved that every non-zero element of a Hausdorff semitopological semigroup $\lambda\times\lambda$-matrix units B_λ is an isolated point in the topological space B_λ. In [8] proved that infinite semigroup $\lambda\times\lambda$-matrix units B_λ does not embed into a compact Hausdorff topological semigroup, every Hausdorff topological inverse semigroup S such that contains B_λ as a subsemigroup, contains B_λ as a closed subsemigroup, i.e., B_λ is *algebraically complete* in the class of Hausdorff topological inverse semigroups. This result in [7] is extended onto so called inverse semigroups with *tight ideal series* and as a corollary onto the semigroup $\mathscr{I}_\lambda^n$. Also, in [12] there proved that for every positive integer n the semigroup $\mathscr{I}_\lambda^n$ is *algebraically h-complete* in the class of Hausdorff topological inverse semigroups, i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological inverse semigroups. In the paper [13] this result is extended onto the class of Hausdorff semitopological inverse semigroups and there it is shown that for an infinite cardinal λ the semigroup $\mathscr{I}_\lambda^n$ admits a unique Hausdorff topology τ_c such that $(\mathscr{I}_\lambda^n,\tau_c)$ is a compact semitopological semigroup. Also there proved the every countably compact Hausdorff topology τ on B_λ such that $(\mathscr{I}_\lambda^n,\tau_c)$ is a semitopological semigroup is compact. In [10] it is shown that a topological semigroup of finite partial bijections $\mathscr{I}_\lambda^n$ with a compact subsemigroup of idempotents is absolutely H-closed (i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological semigroups) and any countably compact topological semigroup does not contain $\mathscr{I}_\lambda^n$ as a subsemigroup for infinite λ. In [10] it was given sufficient conditions onto a topological semigroup $\mathscr{I}_\lambda^1$ to be non-H-closed. Also in [6] it is proved that an infinite semitopological semigroup of $\lambda\times\lambda$-matrix units B_λ is H-closed in the class of semitopological semigroups if and only if the space B_λ is compact.

For an arbitrary positive integer n and an arbitrary non-zero cardinal λ we put

$$\exp_n\lambda=\{A\subseteq\lambda\colon |A|\leqslant n\}.$$

It is obvious that for any positive integer n and any non-zero cardinal λ the set $\exp_n\lambda$ with the binary operation $\cap$ is a semilattice. Later in this paper by $\exp_n\lambda$ we shall denote the semilattice $(\exp_n\lambda,\cap)$. It is easy to see that $\exp_n\lambda$ is isomorphic to the subsemigroup of idempotents (the band) of the semigroup $\mathscr{I}_\lambda^n$ for any positive integer n.

In this paper we study feebly compact topologies τ on the semilattice $\exp_n\lambda$ such that $(\exp_n\lambda,\tau)$ is a semitopological semilattice. All compact semilattice topologies on $\exp_n\lambda$ are described. We prove that for an arbitrary positive integer n and λ every T_1-semitopological countably compact semilattice $(\exp_n \lambda, \tau)$ is a compact topological semilattice. Also, we construct a countably pracompact H-closed quasiregular non-semiregular topology τ^2_{fc} such that $(\exp_2 \lambda, \tau^2_{\mathsf{fc}})$ is a semitopological semilattice with the discontinuous semilattice operation and show that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every T_1-semiregular feebly compact semitopological semilattice $\exp_n \lambda$ is a compact topological semilattice.

We recall that a topological space X is said to be

- *scattered* if X does not contain a non-empty dense-in-itself subspace;
- *hereditarily disconnected* (or *totally disconnected*) if X does not contain any connected subsets of cardinality larger than one.

Proposition 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every T_1-topology τ on $\exp_n \lambda$ such that $(\exp_n \lambda, \tau)$ is a semitopological semilattice the following assertions hold:*

(i) *$(\exp_n \lambda, \tau)$ is a closed subset of any T_1-semitopological semilattice S which contains $\exp_n \lambda$ as a subsemilattice;*

(ii) *for every $x \in \exp_n \lambda$ there exists an open neighbourhood $U(x)$ of the point x in the space $(\exp_n \lambda, \tau)$ such that $U(x) \subseteq {\uparrow}x$;*

(iii) *${\uparrow}x$ is a closed-and-open subset of the space $(\exp_n \lambda, \tau)$ for every $x \in \exp_n \lambda$;*

(iv) *the topological space $(\exp_n \lambda, \tau)$ is functionally Hausdorff and quasiregular, and hence is Hausdorff;*

(v) *$(\exp_n \lambda, \tau)$ is a scattered hereditarily disconnected space.*

Proof. (i) We shall prove our assertion by induction.

Let $n = 1$ and let S be an arbitrary T_1-semitopological semilattice which contains $\exp_1 \lambda$ as a proper subsemilattice. We fix an arbitrary element $x \in S \setminus \exp_1 \lambda$. Suppose to the contrary that every open neighbourhood $U(x)$ of the point x in the topological space S intersects the semilattice $\exp_1 \lambda$. First we shall show that $ex = 0$ for any $e \in \exp_1 \lambda$, where 0 is zero of the semilattice $\exp_1 \lambda$. Suppose to the contrary that there exists $e \in \exp_1 \lambda$ such that $ex = y \neq 0$. Since S is a T_1-space there exists an open neighbourhood $U(y)$ of the point y in S such that $0 \notin U(y)$. Then by the definition the semilattice operation of $\exp_1 \lambda$ and by separate continuity of the semilattice operation of S we have that $0 \in e \cdot V(x) \subseteq U(y)$ for every open neighbourhood $V(x)$ of the point x in S, because the neighbourhood $V(x)$ contains infinitely many points from the semilattice $\exp_1 \lambda$. This contradicts the choice of the neighbourhood $U(y)$. The obtained contradiction implies that $ex = 0$ for any $e \in \exp_1 \lambda$. Fix an arbitrary open neighbourhood $U(x)$ of x in S such that $0 \notin U(x)$. Then by the separate continuity of the semilattice operation of S we get that there exists an open neighbourhood $V(x)$ of x in S such that $x \cdot V(x) \subseteq U(x)$. Since $V(x)$ intersects the semilattice $\exp_1 \lambda$, the above arguments imply that $0 \in x \cdot V(x)$, a contradiction. Therefore, $\exp_1 \lambda$ is a closed subsemilattice of S.

Suppose that for every $j < k$ the semilattice $\exp_j \lambda$ is a closed subsemilattice of any T_1-semitopological semilattice which contains $\exp_j \lambda$ as a proper subsemilattice, where $k \leqslant n$. We proved that this implies that $\exp_k \lambda$ is a closed subsemilattice of any T_1-semitopological semilattice which contains $\exp_k \lambda$ as a proper subsemilattice. Suppose to the contrary that there exists a T_1-semitopological semilattice S which contains $\exp_k \lambda$ as a non-closed subsemilattice. Then there exists an element $x \in S \setminus \exp_k \lambda$ such that every open neighbourhood $U(x)$

of the point x in the topological space S intersects the semilattice $\exp_k\lambda$. The assumption of induction implies that there exists an open neighbourhood $U(x)$ of the point x in S such that $U(x)\cap\exp_k\lambda\subseteq\exp_k\lambda\setminus\exp_{k-1}\lambda$. Now, as in the case of the semilattice $\exp_1\lambda$ the separate continuity of the semilattice operation of S implies that $e\cdot x\in\exp_{k-1}\lambda$ for any $e\in\exp_k\lambda\setminus\exp_{k-1}\lambda$. Indeed, suppose to the contrary that there exists $e\in\exp_k\lambda\setminus\exp_{k-1}\lambda$ such that $e\cdot x=z\notin\exp_{k-1}\lambda$. Then the assumption of induction implies that $\exp_{k-1}\lambda$ is a closed subsemilattice of S and hence there exists an open neighbourhood $U(y)$ of the point y in S such that $U(y)\cap\exp_{k-1}\lambda=\varnothing$. Now, by the separate continuity of the semilattice operation of S there exists an open neighbourhood $U(x)$ of the point x in S such that $e\cdot U(x)\subseteq U(y)$. Then the semilattice operation of $\exp_k\lambda$ implies that $(e\cdot U(x))\cap\exp_{k-1}\lambda\neq\varnothing$, which contradicts the choice of the neighbourhood $U(y)$.

Fix an arbitrary open neighbourhood $U(x)$ of x in S such that $U(x)\cap\exp_k\lambda\subseteq\exp_k\lambda\setminus\exp_{k-1}\lambda$. Then by the separate continuity of the semilattice operation of S we get that there exists an open neighbourhood $V(x)\subseteq U(x)$ of x in S such that $x\cdot V(x)\subseteq U(x)$. By our assumption we have that the set $V(x)\cap\exp_k\lambda\setminus\exp_{k-1}\lambda$ is infinite and hence the above part of our proof implies that $(x\cdot V(x))\cap\exp_{k-1}\lambda\neq\varnothing$, which contradicts the choice of the neighbourhood $U(x)$. The obtained contradiction implies that $\exp_k\lambda$ is a closed subset of S, which completes the proof of our assertion.

(ii) In the case when $x=0$ the statement is trivial, and hence we assume that $x\neq 0$. Then the definition of the semilattice $\exp_n\lambda$ implies that there exists the minimum positive integer k such that $x\in\exp_k\lambda$ and $x\notin\exp_{k-1}\lambda$. By item *(i)* there exists an open neighbourhood $U(x)$ of the point x in the space $(\exp_n\lambda,\tau)$ such that $U(x)\subseteq\exp_n\lambda\setminus\exp_{k-1}\lambda$. Then the separate continuity of the semilattice operation in $(\exp_n\lambda,\tau)$ implies that there exists an open neighbourhood $V(x)\subseteq U(x)$ such that $x\cdot V(x)\subseteq U(x)$. If $V(x)\nsubseteq{\uparrow}x$ then by the definition of the semilattice operation on $\exp_n\lambda$ we have that there exists $y\in V(x)$ such that $xy\in\exp_{k-1}\lambda$, a contradiction. Hence we get that $V(x)\subseteq{\uparrow}x$.

(iii) Since a topological space is T_1-space if and only if every its point is a closed subset of itself, the separate continuity of the semilattice operation implies that ${\uparrow}x$ is a closed subset of $(\exp_n\lambda,\tau)$ for any $x\in\exp_n\lambda$. Also, item *(ii)* implies that

$${\uparrow}x=\bigcup\{V(y)\colon y\in{\uparrow}x \text{ and } V(y) \text{ is an open neighbourhood of } y \text{ such that } V(y)\subseteq{\uparrow}y\}$$

is an open subset of $(\exp_n\lambda,\tau)$ for any $x\in\exp_n\lambda$.

(iv) Fix arbitrary distinct elements x_1 and x_2 of the semitopological semilattice $(\exp_n\lambda,\tau)$. Then we have either $x_1\notin{\uparrow}x_2$ or $x_2\notin{\uparrow}x_1$. In the case when $x_1\notin{\uparrow}x_2$ we define the map $f\colon(\exp_n\lambda,\tau)\to[0,1]$ by the formula

$$f(x)=\begin{cases}1, & \text{if } x\in{\uparrow}x_2;\\ 0, & \text{if } x\notin{\uparrow}x_2.\end{cases}$$

Then we have that $f(x_1)=0$ and $f(x_2)=1$ and by item *(iii)* ${\uparrow}x_2$ is an open-and-closed subset of the space $(\exp_n\lambda,\tau)$, and hence so defined map $f\colon(\exp_n\lambda,\tau)\to[0,1]$ is continuous.

The definition of the semilattice $\exp_n\lambda$ implies that every non-empty open subset of $(\exp_n\lambda,\tau)$ has a maximal element x with the respect to the natural partial order on $\exp_n\lambda$. Then by item *(iii)*, ${\uparrow}x$ is an open-and-closed subset of $(\exp_n\lambda,\tau)$, and hence x is an isolated point of $(\exp_n\lambda,\tau)$. Since τ is a T_1-topology, $\operatorname{cl}_{\exp_n\lambda}(\{x\})=\{x\}\subseteq U$, which implies that $(\exp_n\lambda,\tau)$ is a quasiregular space.

(v) We shall prove that every non-empty subset A of $(\exp_n \lambda, \tau)$ has an isolated point in itself. Fix an arbitrary non-empty subset A of $(\exp_n \lambda, \tau)$. If $A \cap \exp_n \lambda \setminus \exp_{n-1} \lambda \neq \varnothing$ then by item (ii) every point $x \in A \cap \exp_n \lambda \setminus \exp_{n-1} \lambda$ is isolated in $(\exp_n \lambda, \tau)$ and hence x is an isolated point of A. In the other case there exists a positive integer $k < n$ such that $A \subseteq \exp_k \lambda$ and $A \nsubseteq \exp_{k-1} \lambda$. Then by item (ii) every point $x \in A \cap \exp_k \lambda \setminus \exp_{k-1} \lambda$ is isolated in A.

The hereditary disconnectedness of the space $(\exp_n \lambda, \tau)$ follows from item (iii). Indeed, if $x \nleqslant y$ in $\exp_n \lambda$ then by item (iii), ${\uparrow}x$ is an open-and-closed neighbourhood of x in $(\exp_n \lambda, \tau)$ such that $y \notin {\uparrow}x$. This implies that the space $(\exp_n \lambda, \tau)$ does not contain any connected subsets of cardinality larger than one. □

Recall [6] an algebraic semilattice S is called *algebraically complete* in the class $\mathfrak{STSL}$ of semitopological semilattices if S is a closed subsemilattice of every semitopological semilattice $L \in \mathfrak{STSL}$ which contains S as a subsemilattice.

Proposition 1(i) implies the following corollary.

Corollary 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then the semilattice $\exp_n \lambda$ is algebraically complete in the class of T_1-semitopological semilattices.*

The following example shows that the statement (iv) of Proposition 1 does not hold in the case when $(\exp_n \lambda, \tau)$ is a T_0-space.

Example 1. For an arbitrary positive integer n and an arbitrary infinite cardinal λ we define a topology τ_0 on $\exp_n \lambda$ in the following way:

(i) all non-zero elements of the semilattice $\exp_n \lambda$ are isolated points in $(\exp_n \lambda, \tau_0)$; and

(ii) $\exp_n \lambda$ is the unique open neighbourhood of zero in $(\exp_n \lambda, \tau_0)$.

Simple verifications show that the semilattice operation on $(\exp_n \lambda, \tau_0)$ is continuous.

Example 2. For an arbitrary positive integer n and an arbitrary infinite cardinal λ we define a topology τ_{c}^n on $\exp_n \lambda$ in the following way: the family $\{\mathscr{B}_{\mathsf{c}}^n(x) \colon x \in \exp_n \lambda\}$, where

$$\mathscr{B}_{\mathsf{c}}^n(x) = \{U_x(x_1, \ldots, x_j) = {\uparrow}x \setminus ({\uparrow}x_1 \cup \cdots \cup {\uparrow}x_j) : x_1, \ldots, x_j \in {\uparrow}x \setminus \{x\}\},$$

forms a neighbourhood system for the topological space $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$. Simple verifications show that the family $\{\mathscr{B}_{\mathsf{c}}^n(x) \colon x \in \exp_n \lambda\}$ satisfies the properties **(BP1)–(BP3)** of [4]. Also, it is obvious that the family $\{\mathscr{B}_{\mathsf{c}}^n(x) \colon x \in \exp_n \lambda\}$ satisfies the property **(BP4)** of [4, Proposition 1.5.2], and hence the topological space $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$ is Hausdorff.

Recall [4] a topological space X is called 0-*dimensional* if X has a base which consists of open-and-closed subsets of X.

Proposition 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$ is a compact 0-dimensional topological semilattice.*

Proof. The definition of the family $\{\mathscr{B}_{\mathsf{c}}^n(x) \colon x \in \exp_n \lambda\}$ implies that for arbitrary $x \in \exp_n \lambda$ the set ${\uparrow}x$ is open-and-closed in $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$, and hence $\{\mathscr{B}_{\mathsf{c}}^n(x) \colon x \in \exp_n \lambda\}$ is the base of the topological space $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$ which consists of open-and-closed subsets.

Now, by induction we shall show that the space $(\exp_n \lambda, \tau_{\mathsf{c}}^n)$ is compact. In the case when $n = 1$ the compactness of $(\exp_1 \lambda, \tau_{\mathsf{c}}^1)$ follows from the definition of the family

$\{\mathscr{B}^1_{\mathsf{c}}(x)\colon x\in\exp_1\lambda\}$. Next, we shall prove that the statement the space $(\exp_i\lambda,\tau^i_{\mathsf{c}})$ is compact for all positive integers $i<k\leqslant n$ implies that the space $\left(\exp_k\lambda,\tau^k_{\mathsf{c}}\right)$ is compact too. Fix an arbitrary open cover $\mathscr{U}$ of the topological space $\left(\exp_k\lambda,\tau^k_{\mathsf{c}}\right)$. The definition of the topology τ^k_{c} implies that there exists an element $U_0\in\mathscr{U}$ such that $0\in U_0$. Then there exists $U_0(x_1,\ldots,x_j)\in\mathscr{B}^k_{\mathsf{c}}(0)$ such that $U_0(x_1,\ldots,x_j)\subseteq U_0$. The definition of the semilattice $\exp_n\lambda$ implies that for any $x_1,\ldots,x_j\in\exp_k\lambda$ the subsemilattices ${\uparrow}x_1,\ldots,{\uparrow}x_j$ of $\exp_k\lambda$ are isomorphic to the semilattices $\exp_{i_1}\lambda,\ldots,\exp_{i_j}\lambda$, respectively, for some non-negative integers $i_1,\ldots,i_j<k$. This, the definition of the topology τ^k_{c} and the assumption of induction imply that ${\uparrow}x_1,\ldots,{\uparrow}x_j$ are compact subsets of $\left(\exp_k\lambda,\tau^k_{\mathsf{c}}\right)$. Then there exist finitely many $U_1,\ldots,U_m\in\mathscr{U}$ such that ${\uparrow}x_1\cup\cdots\cup{\uparrow}x_j\subseteq U_1\cup\cdots\cup U_m$, and hence $\{U_0,U_1,\ldots,U_m\}\subseteq\mathscr{U}$ is a finite subcover of the topological space $\left(\exp_k\lambda,\tau^k_{\mathsf{c}}\right)$.

Since in a Hausdorff compact semitopological semilattice the semilattice operation is continuous (see [5, Proposition VI-1.13] or [14, p. 242, Theorem 6.6]), it is sufficient to show that the semilattice operation in $(\exp_n\lambda,\tau^n_{\mathsf{c}})$ is separately continuous.

Let a and b are arbitrary elements of the semilattice $\exp_n\lambda$. We consider the following three cases:

$$\textbf{(I)}\ a=b;\quad \textbf{(II)}\ a<b;\quad \text{and}\quad \textbf{(III)}\ a \text{ and } b \text{ are incomparable.}$$

In case **(I)** we have that $a\cdot U_a(x_1,\ldots,x_k)=\{a\}\subseteq U_a(x_1,\ldots,x_k)$ for any $U_a(x_1,\ldots,x_k)\in\mathscr{B}^n_{\mathsf{c}}(a)$.

In case **(II)** we get that $a\cdot U_b(b_1,\ldots,b_l)=\{a\}\subseteq U_a(x_1,\ldots,x_k)$ and $U_a(x_1,\ldots,x_k)\cdot b\subseteq U_a(x_1,\ldots,x_k)$ for any $U_a(x_1,\ldots,x_k)\in\mathscr{B}^n_{\mathsf{c}}(a)$ and $U_b(b_1,\ldots,b_l)\in\mathscr{B}^n_{\mathsf{c}}(b)$, because if $a\subseteq x\subseteq y$ and $a\subseteq b$ in $\exp_n\lambda$, then $a\subseteq x\cap b\subseteq y$.

In case **(III)** we consider two possible subcases: ${\uparrow}a\cap{\uparrow}b=\varnothing$ and ${\uparrow}a\cap{\uparrow}b\neq\varnothing$. Put $d=ab=a\cap b$. If ${\uparrow}a\cap{\uparrow}b=\varnothing$ then $a\cdot U_b(b_1,\ldots,b_l)=\{d\}\subseteq U_d(z_1,\ldots,z_k)$ and $U_a(x_1,\ldots,x_k)\cdot b\subseteq U_d(z_1,\ldots,z_k)$ for any $U_a(x_1,\ldots,x_k)\in\mathscr{B}^n_{\mathsf{c}}(a)$, $U_b(b_1,\ldots,b_l)\in\mathscr{B}^n_{\mathsf{c}}(b)$ and $U_d(z_1,\ldots,z_k)\in\mathscr{B}^n_{\mathsf{c}}(d)$, because in this subcase we have that ${\uparrow}a\cdot{\uparrow}b=d$. If ${\uparrow}a\cap{\uparrow}b\neq\varnothing$ then similar arguments as in the above case imply that

$$a\cdot U_b(b_1,\ldots,b_l,u)=\{d\}\subseteq U_d(z_1,\ldots,z_k),\quad U_a(x_1,\ldots,x_k,u)\cdot b\subseteq U_d(z_1,\ldots,z_k)$$

for any $U_a(x_1,\ldots,x_k,u)\in\mathscr{B}^n_{\mathsf{c}}(a)$, $U_b(b_1,\ldots,b_l,u)\in\mathscr{B}^n_{\mathsf{c}}(b)$ and $U_d(z_1,\ldots,z_k)\in\mathscr{B}^n_{\mathsf{c}}(d)$, where $u=a\cup b$ in $\exp_n\lambda$.

This completes the proof of our proposition. □

Remark 1. By Proposition 1(v) the topological space $(\exp_n\lambda,\tau^n_{c})$ is scattered. Since every countably compact scattered T_3-space is sequentially compact (see [16, Theorem 5.7]), $(\exp_n\lambda,\tau^n_{c})$ is a sequentially compact space.

Theorem 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for any T_1-topology τ on $\exp_n\lambda$ the following conditions are equivalent:*

(i) $(\exp_n\lambda,\tau)$ *is a compact topological semilattice;*

(ii) $\tau=\tau^n_{\mathsf{c}}$;

(iii) $(\exp_n\lambda,\tau)$ *is a countably compact topological semilattice;*

(iv) $(\exp_n\lambda,\tau)$ *is a feebly compact topological semilattice;*

(v) $(\exp_n\lambda,\tau)$ *is a compact semitopological semilattice;*

(vi) $(\exp_n\lambda,\tau)$ *is a countably compact semitopological semilattice.*

Proof. By Proposition 1 without loss of generality we may assume that τ is a Hausdorff topology on $\exp_n\lambda$. It is obvious that the following implications $(i)\Rightarrow(iii)$, $(iii)\Rightarrow(iv)$, $(iii)\Rightarrow(vi)$, $(i)\Rightarrow(v)$ and $(v)\Rightarrow(vi)$ are trivial, and implication $(ii)\Rightarrow(i)$ follows from Proposition 2.

$(i)\Rightarrow(ii)$. Suppose that τ is a compact topology on $\exp_n\lambda$ such that $(\exp_n\lambda,\tau)$ is a topological semilattice. Then by Proposition 1(iii) the identity map

$$\mathsf{id}_{\exp_n\lambda}\colon (\exp_n\lambda,\tau)\to(\exp_n\lambda,\tau_{\mathsf{c}}^n)$$

is a continuous, and hence by Theorem 2.1.13 of [4] is a homeomorphism. Thus, we get that $\tau=\tau_{\mathsf{c}}^n$.

Implication $(v)\Rightarrow(i)$ follows from Proposition VI-1.13 of [5] (also from Theorem 6.6 of [14, p. 242]).

$(vi)\Rightarrow(v)$. We shall prove this implication by induction.

Assume that $n=1$. Suppose to the contrary that there exists a non-compact topology τ on $\exp_1\lambda$ such that $(\exp_1\lambda,\tau)$ is a countably compact semitopological semilattice. Then there exists an open cover $\mathscr{U}$ of the space $(\exp_1\lambda,\tau)$ which contans no a finite subcover. This implies that there exists $U\in\mathscr{U}$ such that $0\in U$ and $\exp_1\lambda\setminus U$ is infinite subset of $\exp_1\lambda$. Then by Proposition 1(iii) the space $(\exp_1\lambda,\tau)$ contains an open-and-closed discrete subspace, which contradicts Theorem 3.10.3 from [4]. Thus, $(\exp_1\lambda,\tau)$ is a compact semitopological semilattice.

Next, we shall prove that the statement the countably compact semitopological semilattice $(\exp_i\lambda,\tau)$ is compact for all positive integers $i<k\leqslant n$ implies that the countably compact semitopological semilattice $(\exp_k\lambda,\tau)$ is compact too. Then there exists an open cover $\mathscr{U}$ of the topological space $(\exp_k\lambda,\tau)$ which contains no a finite subcover. Then by Proposition 1(i), $\exp_{k-1}\lambda$ is a closed subset of $(\exp_k\lambda,\tau)$, and hence by Theorem 3.10.4 of [4] $\exp_{k-1}\lambda$ is countably compact. The assumption of induction implies that $\exp_{k-1}\lambda$ is a compact subspace of $(\exp_k\lambda,\tau)$, and hence the open cover $\mathscr{U}$ of the topological space $(\exp_k\lambda,\tau)$ contains a finite subcover $\mathscr{U}_0$ of $\exp_{k-1}\lambda$. If the open cover $\mathscr{U}$ of the topological space $(\exp_k\lambda,\tau)$ contains no a finite subcover of $(\exp_k\lambda,\tau)$ then by Proposition 1(iii) we have that $\exp_k\lambda\setminus\bigcup\mathscr{U}_0$ is an open-and-closed discrete subspace, which contradicts Theorem 3.10.3 from [4]. Thus, $(\exp_k\lambda,\tau)$ is a compact semitopological semilattice. This completes the proof of our implication.

$(iv)\Rightarrow(iii)$. We shall prove this implication by induction.

Assume that $n=1$. Suppose to the contrary that there exists a feebly compact topological semilattice τ on $\exp_1\lambda$ such that $(\exp_1\lambda,\tau)$ is not a countably compact space. Then there exists a countable open cover $\mathscr{U}$ of the space $(\exp_1\lambda,\tau)$ which contains no a finite subcover. This implies that there exists $U\in\mathscr{U}$ such that $0\in U$ and $\exp_1\lambda\setminus U$ is infinite subset of $\exp_1\lambda$. Then by Proposition 1(iii) the space $(\exp_1\lambda,\tau)$ contains an open-and-closed discrete subspace of $(\exp_k\lambda,\tau)$, which contradicts the feeble compactness of $(\exp_1\lambda,\tau)$, a contradiction. Hence $(\exp_1\lambda,\tau)$ is a countably compact space.

Next, we shall prove that the statement that every feebly compact topological semilattice $(\exp_i\lambda,\tau)$ is countably compact for all positive integers $i<k\leqslant n$ implies that the feebly compact topological semilattice $(\exp_k\lambda,\tau)$ is countably compact too.

Suppose to the contrary that every feebly compact topological semilattice $(\exp_i\lambda,\tau)$ is countably compact for all positive integers $i<k\leqslant n$ but there exists a feebly compact

topological semilattice $(\exp_k \lambda, \tau)$ which is not countably compact. Then by Theorem 3.10.3 from [4] the topological semilattice $(\exp_k \lambda, \tau)$ contains an infinite closed discrete subspace A. Since by Proposition 1(ii), $\exp_k \lambda \setminus \exp_{k-1} \lambda$ is an open discrete subspace of $(\exp_k \lambda, \tau)$, the feeble compactness of $(\exp_k \lambda, \tau)$ implies that $A \subseteq \exp_{k-1} \lambda$. Also, by Proposition 1$(iii)$ since ${\uparrow}x$ is an open-and-closed subset of the space $(\exp_k \lambda, \tau)$ for every $x \in \exp_k \lambda$ we have that ${\uparrow}x$ is a feebly compact subspace of $(\exp_k \lambda, \tau)$. It is obvious that for any non-zero element $x \in \exp_k \lambda$ the subsemilattice ${\uparrow}x$ of $\exp_k \lambda$ is isomorphic to semilattice $\exp_m \lambda$ for some non-negative integer $m < k$. This and the assumption of induction imply that ${\uparrow}x$ is a countably compact subspace of $(\exp_k \lambda, \tau)$ for any non-zero element x of the semilattice $\exp_k \lambda$. Hence we get that the set $A \cap {\uparrow}x$ is finite for any non-zero element x of the semilattice $\exp_k \lambda$.

Next, by induction we shall show that if for some positive integer i with $2 \leqslant i < n$ in a feebly compact topological semilattice $(\exp_n \lambda, \tau)$ there exists an open neighbourhood $U(0)$ of zero 0 in $(\exp_n \lambda, \tau)$ such that $U(0)$ does not contain an infinite subset A of $\exp_i \lambda \setminus \exp_{i-1} \lambda$ such that $A \cap {\uparrow}x$ is finite for any non-zero element $x \in \exp_i \lambda$ and ${\uparrow}x$ is countably compact, then there exists an open neighbourhood $V(0) \subseteq U(0)$ of zero 0 in $(\exp_n \lambda, \tau)$ such that $V(0)$ does not contain an infinite subset A_+ of $\exp_{i+1} \lambda \setminus \exp_i \lambda$ such that $A_+ \cap {\uparrow}x$ is finite for any non-zero element $x \in \exp_i \lambda$.

Suppose that in a feebly compact topological semilattice $(\exp_n \lambda, \tau)$ there exists an open neighbourhood $U(0)$ of zero 0 such that $U(0) \cap A = \varnothing$ for some infinite subset

$$A = \{x_i \colon i \in \mathbb{N}\} \subseteq \exp_1 \lambda \setminus \{0\}.$$

Then the continuity of the semilattice operation in $(\exp_n \lambda, \tau)$ implies that there exists an open neighbourhood $V(0) \subseteq U(0)$ of zero in $(\exp_n \lambda, \tau)$ such that $V(0) \cdot V(0) \subseteq U(0)$. Suppose that there exist some distinct $x_{i_0}, x_{i_1} \in A$ such that $\{x_{i_0}, x_{i_1}\} \in V(0)$. Then by the inclusion $V(0) \cdot V(0) \subseteq U(0)$ we have that

$$\{\{x_{i_0}, x_i\} : i \in \mathbb{N} \setminus \{i_0, i_1\}\} \cap V(0) = \varnothing.$$

This implies that the subspace ${\uparrow}\{x_{i_0}\}$ of $(\exp_n \lambda, \tau)$ contains a closed discrete subspace, which contradicts the countable compactness of ${\uparrow}\{x_{i_0}\}$. Hence we get that $V(0) \cap A_+ = \varnothing$, where

$$A_+ = \{\{x_k, x_l\} : x_k, x_l \text{ are distinct elements of } A\}.$$

Suppose that in a feebly compact topological semilattice $(\exp_n \lambda, \tau)$ there exist an open neighbourhood $U(0)$ of zero 0 and infinite subset $A \subseteq \exp_n \lambda$ such that $U(0) \cap A = \varnothing$ and $|x| = j > 1$ for any $x \in A$. Since for any non-zero element $a \in \exp_n \lambda$ the subspace ${\uparrow}a$ is countably compact, without loss of generality we may assume that there exists a countable set $A_1 = \{x_i \colon i \in \mathbb{N}\}$ which consists of singletons from $\exp_n \lambda$ such that $A \cap {\uparrow}x_i$ is a singleton for any positive integer i. Then the continuity of the semilattice operation in $(\exp_n \lambda, \tau)$ implies that there exists an open neighbourhood $V(0) \subseteq U(0)$ of zero in $(\exp_n \lambda, \tau)$ such that $V(0) \cdot V(0) \subseteq U(0)$. We claim that for any distinct elements $x_p, x_s \in A_1$, $s, p \in \mathbb{N}$ there exists no $x \in {\uparrow}x_p$ such that $y = \{\{x_s\} \cup x\} \notin V(0)$. Indeed, in the other case the neighbourhood $V(0)$ does not contain the set $\{\{x_q\} \cup x \colon x_q \in A \setminus \{x_s\}\}$. This implies that the subspace ${\uparrow}x_p$ of $(\exp_n \lambda, \tau)$ contains an infinite closed discrete subspace, which contradicts the assumption that ${\uparrow}x_p$ is a countably compact subspace of $(\exp_n \lambda, \tau)$. Hence we get that $V(0) \cap A_+ = \varnothing$, where

$$A_+ = \{\{x_i\} \cup x \colon x_i \in A_1 \text{ and } x \in A\}.$$

The above presented arguments imply that the topological semilattice $(\exp_n \lambda, \tau)$ contains an infinite open-and-closed discrete subspace, which contradicts the feeble compactness of the space $(\exp_n \lambda, \tau)$. The obtained contradiction implies the requested implication. □

Proposition 1(iii) implies the following corollary.

Corollary 2. *Let λ be an arbitrary infinite cardinal. Then every feebly compact T_1-topology τ on the semilattice $\exp_1 \lambda$ such that $(\exp_1 \lambda, \tau)$ is a semitopological semilattice, is compact, and hence $(\exp_1 \lambda, \tau)$ is a topological semilattice.*

But, the following example shows that for any infinite cardinal λ and any positive integer $n \geqslant 2$ there exists a Hausdorff feebly compact topology τ on the semilattice $\exp_n \lambda$ such that $(\exp_n \lambda, \tau)$ is a non-countably compact semitopological semilattice.

Example 3. Let λ be any infinite cardinal and τ_{c}^2 be the topology on the semilattice $\exp_2 \lambda$ defined in Example 2. We construct more stronger topology τ_{fc}^2 on $\exp_2 \lambda$ them τ_{c}^2 in the following way. By $\pi\colon \lambda \to \exp_2 \lambda\colon a \mapsto \{a\}$ we denote the natural embedding of λ into $\exp_2 \lambda$. Fix an arbitrary infinite subset $A \subseteq \lambda$ of cardinality $\leqslant \lambda$. For every non-zero element $x \in \exp_2 \lambda$ we put the base $\mathscr{B}_{\mathsf{fc}}^2(x)$ of the topology τ_{fc}^2 at the point x coincides with the base of the topology τ_{c}^2 at x, and

$$\mathscr{B}_{\mathsf{fc}}^2(0) = \left\{U_B(0) = U(0) \setminus \pi(B)\colon U(0) \in \mathscr{B}_{\mathsf{c}}^2(0), B \subseteq \lambda \text{ and the set } A \setminus B \cup B \setminus A \text{ is finite}\right\}$$

form a base of the topology τ_{fc}^2 at zero 0 of the semilattice $\exp_2 \lambda$. Simple verifications show that the family $\{\mathscr{B}_{\mathsf{fc}}^2(x)\colon x \in \exp_2 \lambda\}$ satisfies the conditions **(BP1)–(BP4)** of [4], and hence τ_{fc}^2 is a Hausdorff topology on $\exp_2 \lambda$.

Proposition 3. *Let λ be an arbitrary infinite cardinal. Then $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ is a countably pracompact semitopological semilattice such that $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ is an H-closed non-semiregular space.*

Proof. The definition of the topology τ_{fc}^2 implies that it is sufficient to show that the semilattice operation is separately continuous in the case $x \cdot 0$. Fix an arbitrary basic neighbourhood $U_B(0)$ of zero in $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$. If x is a singleton of λ, i.e., $x = \{x_0\}$ in $\exp_2 \lambda$, then we have that $x \cdot V_B(0) = \{0\} \subseteq U_B(0)$, where $V(0) = U(0) \setminus {\uparrow}x$. In the case when x is a two-elements subset of λ, where $x = \{x_1, x_2\}$ for some $x_1, x_2 \in \lambda$, the we get that $x \cdot W_B(0) = \{0\} \subseteq U_b(0)$, where $W(0) = U(0) \setminus ({\uparrow}\{x_1\} \cup {\uparrow}\{x_2\})$.

Also, the definition of the topology τ_{fc}^2 on $\exp_2 \lambda$ implies that the set $\exp_2 \lambda \setminus \exp_1 \lambda$ is dense in $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ and every infinite subset of $\exp_2 \lambda \setminus \exp_1 \lambda$ has an accumulation point in $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$, and hence the space $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ is countably pracompact.

Suppose to the contrary that $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ is not an H-closed topological space. Then there exists a Hausdorff topological space X which contains $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ as a dense proper subspace. Fix an arbitrary $x \in X \setminus \exp_2 \lambda$. Since X is Hausdorff there exist disjunctive open neighbourhoods $U(x)$ and $U(0)$ of x and zero 0 of the semilattice $\exp_2 \lambda$ in X, respectively. Then there exists a basic neighbourhood $V_B(0)$ of zero in $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ such that $V_B(0) \subseteq \exp_2 \lambda \cap U(0)$. Also, the definition of the base $\mathscr{B}_{\mathsf{fc}}^2(0)$ of the topology τ_{fc}^2 at zero 0 of the semilattice $\exp_2 \lambda$ implies that there exist finitely many non-zero elements $x_1, \ldots, x_m$ of the semilattice $\exp_2 \lambda$ such that

$$\exp_2 \lambda \setminus ({\uparrow}x_1 \cup \ldots \cup {\uparrow}x_m \cup V_B(0)) \subseteq B,$$

and since by Proposition 1(iii) the subsets ${\uparrow}x_1,\ldots,{\uparrow}x_m$ are open-and-closed in $(\exp_2\lambda,\tau^2_{\mathsf{fc}})$ without loss of generality we may assume that $U(x)\cap\exp_2\lambda\subseteq B$. If the set $U(x)\cap\exp_2\lambda\subseteq B$ is infinite then the space $(\exp_2\lambda,\tau^2_{\mathsf{fc}})$ contains a discrete infinite open-and-close subspace, which contradicts the feeble compactness of $(\exp_2\lambda,\tau^2_{\mathsf{fc}})$. The obtained contradiction implies that the space $(\exp_2\lambda,\tau^2_{\mathsf{fc}})$ is H-closed. □

Remark 2. If n is an arbitrary positive integer $\geqslant 3$, λ is any infinite cardinal and τ^n_{c} is the topology on the semilattice $\exp_n\lambda$ defined in Example 2, then we construct more stronger topology τ^n_{fc} on $\exp_n\lambda$ them τ^2_{c} in the following way. Fix an arbitrary element $x\in\exp_n\lambda$ such that $|x|=n-2$. It is easy to see that the subsemilattice ${\uparrow}x$ of $\exp_n\lambda$ is isomorphic to $\exp_2\lambda$, and by $h\colon \exp_2\lambda\to{\uparrow}x$ we denote this isomorphism.

Fix an arbitrary subset $A\subseteq\lambda$ of cardinality $\leqslant\lambda$. For every zero element $y\in\exp_n\lambda\setminus{\uparrow}x$ we put the base $\mathscr{B}^n_{\mathsf{fc}}(y)$ of the topology τ^n_{fc} at the point y coincides with the base of the topology τ^n_{c} at y, and put ${\uparrow}x$ is an open-and-closed subset and the topology on ${\uparrow}x$ is generated by map $h\colon(\exp_2\lambda,\tau^2_{\mathsf{fc}})\to{\uparrow}x$. Simple verifications as in the proof of Proposition 3 show that $(\exp_n\lambda,\tau^n_{\mathsf{fc}})$ is a countably pracompact semitopological semilattice such that $(\exp_n\lambda,\tau^n_{fc})$ is an H-closed quasiregular non-semiregular space.

Remark 3. Simple verifications show that $(\exp_2\lambda,\tau^2_{fc})$ is not a 0-dimensional space. This implies that the term "hereditarily disconnected" in item (v) of Proposition 1 can not be replaced by "0-dimensional".

A T_1-space X is called *collectionwise normal* if for every discrete family $\{F_s\}_{s\in\mathscr{A}}$ of closed subsets of X there exists a discrete family $\{U_s\}_{s\in\mathscr{A}}$ of open subsets of X such that $F_s\subseteq U_s$ for every $s\in\mathscr{A}$ [4].

Proposition 4. *Let λ be an arbitrary infinite cardinal and τ be a T_1-topology on $\exp_1\lambda$ such that $(\exp_1\lambda,\tau)$ is a semitopological semilattice. Then the space $(\exp_1\lambda,\tau)$ is collectionwise normal.*

Proof. Suppose that $\{F_s\}_{s\in\mathscr{A}}$ is a discrete family of closed subsets of $(\exp_1\lambda,\tau)$. By Proposition 1(iii) all non-zero elements of the semilattice $\exp_1\lambda$ are isolated points in the space $(\exp_1\lambda,\tau)$. Hence, if there exists an open neighbourhood $U(0)$ of zero in $(\exp_1\lambda,\tau)$ such that $U(0)\cap F_s=\varnothing$ for all $s\in\mathscr{A}$ then we put $U_s=F_s$ for all $s\in\mathscr{A}$. In other case there exists an open neighbourhood $U(0)$ of zero in $(\exp_1\lambda,\tau)$ such that $U(0)\cap F_{s_0}\neq\varnothing$ for some $s_0\in\mathscr{A}$ and $U(0)\cap F_s=\varnothing$ for all $s\in\mathscr{A}\setminus\{s_0\}$. We put

$$U_s=\begin{cases} F_s, & \text{if } s\in\mathscr{A}\setminus\{s_0\}\,;\\ F_{s_0}\cup U(0), & \text{if } s=s_0.\end{cases}$$

Then Proposition 1(ii) implies that $\{U_s\}_{s\in\mathscr{A}}$ is a discrete family $\{U_s\}_{s\in\mathscr{A}}$ of open subsets of $(\exp_1\lambda,\tau)$ such that $F_s\subseteq U_s$ for every $s\in\mathscr{A}$, and hence the space $(\exp_1\lambda,\tau)$ is collectionwise normal. □

Remark 4. A topological space X is called *perfectly normal* if X is normal and every closed subset of X is a G_δ-set. It is obvious that if λ is any uncountable cardinal then $(\exp_1\lambda,\tau^1_{\mathsf{c}})$ is a compact space which is not perfectly normal (see: [4, Section 1.5]).

Theorem 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every semilattice T_1-topology on $\exp_n\lambda$ is regular.*

Proof. Suppose that τ is a T_1-topology on $\exp_n\lambda$ such that $(\exp_n\lambda,\tau)$ is a topological semilattice. In the case when $n=1$ the statement of the theorem follows from Proposition 4. Hence, later we assume that $n\geqslant 2$.

By Proposition 1(iii), ${\uparrow}x$ is an open-and-closed subsemilattice of $(\exp_n\lambda,\tau)$ for any $x\in\exp_n\lambda$, and hence it will be sufficient to show that for every open neighbourhood $U(0)$ of zero in $(\exp_n\lambda,\tau)$ there exists an open neighbourhood $V(0)$ of zero in $(\exp_n\lambda,\tau)$ such that $\operatorname{cl}_{\exp_n\lambda}(V(0))\subseteq U(0)$.

Fix an arbitrary open neighbourhood $U(0)$ of zero in $(\exp_n\lambda,\tau)$. Then the continuity of the semilattice operation in $(\exp_n\lambda,\tau)$ implies that there exists an open neighbourhood $V(0)\subseteq U(0)$ of zero in $(\exp_n\lambda,\tau)$ such that $V(0)\cdot V(0)\subseteq U(0)$. Suppose that there exists

$$x\in\operatorname{cl}_{\exp_n\lambda}(V(0))\setminus V(0).$$

By Proposition 1(ii) we have that $x\in{\downarrow}V(0)$. We assume that $x=\{a_1,\ldots,a_i\}$ as a finite subset of the cardinal λ, where $i<n$, i.e., $x\in\exp_i\lambda\setminus\exp_{i-1}\lambda$. Then $V(x)\cap V(0)\neq\varnothing$ for every open neighbourhood $V(x)$ of the point x in $(\exp_n\lambda,\tau)$. Proposition 1(iii) implies that without loss of generality we may assume that $V(x)\subseteq{\uparrow}x$. Fix an arbitrary $y\in(V(0)\cap V(x))\setminus\{x\}$. Then we may assume that $y=\{a_1,\ldots,a_i,a_{i+1},\ldots,a_j\}$ as a finite subset of the cardinal λ, where $i<j\leqslant n$. We put

$$x_1=\{a_1,\ldots,a_i,a_{i+1}\}\,,\ldots,x_{j-i}=\{a_1,\ldots,a_i,a_j\}\,,$$

as finite subsets of the cardinal λ. Then the semilattice operation of $\exp_n\lambda$ implies that

$$y\in{\uparrow}x_1\cup\cdots\cup{\uparrow}x_{j-i}\subseteq{\uparrow}x$$

and $y\cdot z=x$ for every

$$z\in{\uparrow}x\setminus({\uparrow}x_1\cup\cdots\cup{\uparrow}x_{j-i})\,.$$

Since $x\in\operatorname{cl}_{\exp_n\lambda}(V(0))\setminus V(0)$, Proposition 1$(iii)$ implies that

$$W(x)=V(x)\setminus({\uparrow}x_1\cup\cdots\cup{\uparrow}x_{j-i})$$

is an open neighbourhood of the point x in $(\exp_n\lambda,\tau)$. Then the above arguments imply that $x=y\cdot W(x)\subseteq V(0)\cdot V(0)\subseteq U(0)$ and hence $\operatorname{cl}_{\exp_n\lambda}(V(0))\subseteq U(0)$. □

Since in any countable T_1-space X every open subset of X is a F_σ-set, Theorem 1.5.17 from [4] and Theorem 2 imply the following corollary.

Corollary 3. *Let n be an arbitrary positive integer. Then every semilattice T_1-topology on $\exp_n\omega$ is perfectly normal.*

Later we need the following lemma:

Lemma 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Let $(\exp_n\lambda,\tau)$ be a Hausdorff feebly compact semitopological semilattice. Then for every open neighbourhood $U(0)$ of zero in $(\exp_n\lambda,\tau)$ there exist finitely many non-zero elements $x_1,\ldots,x_i\in\exp_n\lambda$ such that*

$$\exp_n\lambda\setminus\exp_{n-1}\lambda\subseteq U(0)\cup{\uparrow}x_1\cup\ldots\cup{\uparrow}x_i.$$

Proof. By Proposition 1(*ii*) every point $x\in\exp_n\lambda\setminus\exp_{n-1}\lambda$ is isolated in $(\exp_n\lambda,\tau)$. Next, we apply the feeble compactness of $(\exp_n\lambda,\tau)$ and Proposition 1(*iii*). □

Theorem 3. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every semiregular feebly compact T_1-topology τ on $\exp_n\lambda$ such that $(\exp_n\lambda,\tau)$ is a semitopological semilattice, is compact, and hence the semilattice operation in $(\exp_n\lambda,\tau)$ is continuous.*

Proof. We shall prove the statement of the theorem by induction. In the case when $n=1$ the statement of the theorem follows from Corollary 2. First we consider the initial step: $n=2$. Suppose to the contrary that there exists a semiregular feebly compact non-compact T_1-semitopological semilattice $(\exp_2\lambda,\tau)$. By Theorem 1 the topological space $(\exp_2\lambda,\tau)$ is not countably compact, and hence Theorem 3.10.3 of [4] implies that $(\exp_2\lambda,\tau)$ contains an infinite closed discrete subspace X. Now, Proposition 1(*iii*) implies that $\exp_2\lambda\setminus\exp_1\lambda$ is an open discrete subspace of $(\exp_2\lambda,\tau)$, and since $(\exp_2\lambda,\tau)$ is feebly compact, without loss of generality we may assume that $X\subseteq\exp_1\lambda\setminus\{0\}$. Fix an arbitrary regular open neighbourhood $U(0)$ of zero in $(\exp_2\lambda,\tau)$ such that $U(0)\cap X=\varnothing$.

For every $x\in\exp_1\lambda\setminus\{0\}$ the subset ${\uparrow}x$ is open-and-closed in $(\exp_2\lambda,\tau)$ and hence ${\uparrow}x$ is feebly compact. Since ${\uparrow}x$ is algebraically isomorphic to $\exp_1\lambda$, Corollary 2 implies that ${\uparrow}x$ is compact. By Lemma 1 there exist finitely many non-zero elements $x_1,\ldots,x_i\in\exp_2\lambda$ such that

$$\exp_2\lambda\setminus\exp_1\lambda\subseteq U(0)\cup{\uparrow}x_1\cup\ldots\cup{\uparrow}x_i.$$

The semilattice operation of $(\exp_2\lambda,\tau)$ implies that without loss of generality we may assume that $x_1,\ldots,x_i$ are singleton subsets of the cardinal λ. This and above presented arguments imply that $\mathrm{cl}_{\exp_2\lambda}(U(0))\cap X\neq\varnothing$. Moreover, we have that the narrow $\mathrm{cl}_{\exp_2\lambda}(U(0))\setminus U(0)$ consists of singleton subsets of the cardinal λ. Then for every $x\in\mathrm{cl}_{\exp_2\lambda}(U(0))\setminus U(0)$ by Corollary 2, ${\uparrow}x$ is a compact topological subsemilattice of $(\exp_2\lambda,\tau)$. Now, Theorem 1 and Lemma 1 imply that

$$x\in\mathrm{int}_{\exp_2\lambda}\left(\mathrm{cl}_{\exp_2\lambda}(U(0))\right)=U(0),$$

which contradicts the assumption $U(0)\cap X=\varnothing$. The obtained contradiction implies that $(\exp_2\lambda,\tau)$ is a compact semitopological semilattice.

Next we shall show the step of induction, i.e., that the statement *if for every positive integer $l<n$ a semiregular feebly compact T_1-semitopological semilattice* $(\exp_l\lambda,\tau)$ *is compact* implies that a semiregular feebly compact T_1-semitopological semilattice $(\exp_n\lambda,\tau)$ is compact too. Suppose to the contrary that there exists a semiregular feebly compact non-compact T_1-semitopological semilattice $(\exp_n\lambda,\tau)$ which is not compact. By Theorem 1 the topological space $(\exp_n\lambda,\tau)$ is not countably compact, and hence Theorem 3.10.3 of [4] implies that $(\exp_n\lambda,\tau)$ contains an infinite closed discrete subspace X. Now, by Proposition 1(*iii*), $\exp_n\lambda\setminus\exp_{n-1}\lambda$ is an open discrete subspace of $(\exp_n\lambda,\tau)$, and since $(\exp_n\lambda,\tau)$ is feebly compact, without loss of generality we may assume that $X\subseteq\exp_{n-1}\lambda\setminus\{0\}$.

Put $k<n$ is the maximum positive integer such that the set $\exp_k\lambda\setminus\exp_{k-1}\lambda\cap X$ is infinite. We observe that for any non-zero element $x\in\exp_n\lambda$ the subsemilattice ${\uparrow}x$ of $\exp_n\lambda$ is algebraically isomorphic to the semilattice $\exp_j\lambda$ for some positive integer $j<n$, and since by Proposition 1(*iii*), ${\uparrow}x$ is an open-and-closed subset of a feebly compact semitopological semilattice $(\exp_n\lambda,\tau)$, the assumption of induction implies that ${\uparrow}x$ is a compact subsemilattice of $(\exp_n\lambda,\tau)$. This implies that there do not exist finitely many non-zero

elements $y_1, \ldots, y_s$ of the semitopological semilattice $(\exp_n \lambda, \tau)$ such that

$$X \subseteq {\uparrow}y_1 \cup \ldots \cup {\uparrow}y_s.$$

Fix an arbitrary regular open neighbourhood $U(0)$ of zero in $(\exp_n \lambda, \tau)$ such that $U(0) \cap X = \varnothing$. Then the above arguments imply that

$$\mathrm{cl}_{\exp_n \lambda}(V(0)) \cap (\exp_k \lambda \cap X) \neq \varnothing.$$

Moreover, we have that the narrow $\mathrm{cl}_{\exp_n \lambda}(U(0)) \setminus U(0)$ contains infinitely many k-element subsets of the cardinal λ which belongs to the set X. Then for every such element $x \in \mathrm{cl}_{\exp_n \lambda}(U(0)) \setminus U(0)$ the assumtion of induction implies that ${\uparrow}x$ is a compact topological subsemilattice of $(\exp_n \lambda, \tau)$. Now, Theorem 1 and Lemma 1 imply that every such element x belongs to the set

$$\mathrm{int}_{\exp_n \lambda}\left(\mathrm{cl}_{\exp_n \lambda}(U(0))\right) = U(0),$$

which contradicts the assumption $U(0) \cap X = \varnothing$. The obtained contradiction implies that $(\exp_n \lambda, \tau)$ is a compact semitopological semilattice.

The last assertion of the theorem follows from Theorem 1. □

Acknowledgements. We acknowledge Taras Banakh and the referee for they useful comments and suggestions.

REFERENCES

1. A. V. Arkhangel'skii, Topological Function Spaces, Kluwer Publ., Dordrecht, 1992.
2. J.H. Carruth, J.A. Hildebrant, R.J. Koch, The Theory of Topological Semigroups, V.I, Marcel Dekker, Inc., New York and Basel, 1983; V.II, Marcel Dekker, Inc., New York and Basel, 1986.
3. A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups*, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.
4. R. Engelking, General Topology, 2nd ed., Heldermann, Berlin, 1989.
5. G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott, Continuous Lattices and Domains, Cambridge Univ. Press, Cambridge, 2003.
6. O. Gutik, *On closures in semitopological inverse semigroups with continuous inversion*, Algebra Discr. Math. **18** (2014), №1, 59–85.
7. O. Gutik, J. Lawson, D. Repovš, *Semigroup closures of finite rank symmetric inverse semigroups*, Semigroup Forum, **78** (2009), №2, 326–336.
8. O.V. Gutik, K.P. Pavlyk, *On topological semigroups of matrix units*, Semigroup Forum, **71** (2005), №3, 389–400.
9. O.V. Gutik, K.P. Pavlyk, *Topological semigroups of matrix units*, Algebra Discrete Math., №3 (2005), 1–17.
10. O. Gutik, K. Pavlyk, A. Reiter, *Topological semigroups of matrix units and countably compact Brandt λ^0-extensions*, Mat. Stud., **32** (2009), №2, 115–131.
11. O. Gutik, O. Ravsky, *Pseudocompactness, products and topological Brandt λ^0–extensions of semitopological monoids*, Math. Methods and Phys.-Mech. Fields, **58** (2015), №2, 20–37.

12. O.V. Gutik, A.R. Reiter, *Symmetric inverse topological semigroups of finite rank* $\leqslant n$, Math. Methods and Phys.-Mech. Fields, **52** (2009), №3, 7–14; reprinted version: J. Math. Sc. **171** (2010), №4, 425–432.

13. O. Gutik, A. Reiter, *On semitopological symmetric inverse semigroups of a bounded finite rank*, Visnyk Lviv Univ. Ser. Mech. Math., **72** (2010), 94–106. (in Ukrainian)

14. J.D. Lawson, *Intrinsic lattices and lattice topologies*, S. Fajtlowicz and K. Kaiser (eds.), Proceedings of the University of Houston Lattice Theory Conference, Houston, Texas, 1973. University of Houston, 206–230.

15. W. Ruppert, Compact Semitopological Semigroups: An Intrinsic Theory, Lect. Notes Math., V.1079, Springer, Berlin, 1984.

16. J.E. Vaughan, Countably compact and sequentially compact spaces, Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan (eds.), Amsterdam, North-Holland, 1984, 569–602.

17. V.V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR, **84** (1952), 1119–1122. (in Russian)

Department of Mechanics and Mathematics
Ivan Franko National University of Lviv
o_ gutik@franko.lviv.ua
ovgutik@yahoo.com

olesyasobol@mail.ru

Received 4.07.2016