THE CARTAN-MONGE GEOMETRIC APPROACH TO THE CHARACTERISTIC METHOD FOR HAMILTON-JACOBI TYPE EQUATIONS AND ITS GENERALIZATION FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

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The Cartan-Monge geometric approach to the characteristic method for Hamilton-Jacobi type equations and nonlinear partial differential equations of higher orders is analyzed. The Hamiltonian structure of characteristic vector fields related with nonlinear partial differential equations of first order is analyzed, the tensor fields of special structure are constructed for defining characteristic vector fields naturally related with nonlinear partial differential equations of higher orders. The generalized characteristic method is developed in the framework of the symplectic theory within geometric Monge and Cartan pictures. Based on their inherited geometric properties, the related functional-analytic Hopf-Lax type solutions to a wide class of boundary and Cauchy problems for nonlinear partial differential equations of Hamilton-Jacobi type are studied. For the non-canonical Hamilton-Jacobi equations there is stated a relationship between their solutions and a good posed functional-analytic fixed point problem, related with Hopf-Lax type solutions to specially constructed dual canonical Hamilton-Jacobi

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equations. Functional-analytic solutions to a Hamilton-Jacobi equation of Riccati type are obtained and investigated reducing them to the classical Brouwer-Banach type fixed point theory.

1. INTRODUCTION: GEOMETRIC BACKGROUNDS OF THE CLASSICAL CHARACTERISTIC METHOD

Solutions to linear partial differential equations, as is well known [12,15,17], can be studied enough effectively using many classical approaches, such as Fourier method, spectral theory and Green function method. Nevertheless, all of them, regrettably, can not be applied for analyzing solution manifolds of general nonlinear partial differential equations even of the first and second orders. Since the classical Cauchy works on the problem by now there exist [5,12,16,17] only a few approaches to treating such equations, among which the famous characteristics method proved to be the most effective and fruitful. During the last century this method was further developed by many mathematicians, amongst them such as P.Lax, H.Hopf, O.Oleinik, S.Kruzhkov, V.Maslov, P.Lions, L.Evans, D.Blackmore [6, 12, 13, 16–19, 21, 30] and others. Still long ago it was observed the deep connection of the characteristics method with Hamiltonian analysis, reducing the problem to studying some systems of ordinary differential equations. This aspect was prevailing in works of H.Hopf, P.Lax and O.Oleinik (see [6, 11, 12]), who described doing this way a wide class of so called generalized solutions to first order nonlinear partial differential equations. The most known result within this field is attributed to H.Hopf and P.Lax, who have found for the first time a very interesting variational representation for solutions of first order nonlinear partial differential equations called a Hopf–Lax type representation. As these results were strongly based on some geometric notions, it was natural to analyze the Cauchy characteristics method from the differential-geometric point of view, initiated still in the classical works of G.Monge and E.Cartan [10]. Within the framework of the Monge geometric approach to studying solutions of partial differential equations we proposed in [25] a generalization of the classical Cauchy characteristic method for equations of first and higher orders, making use of the special tensor fields, intimately related with them. These tensor fields appear very naturally within a developed Monge approach as some geometric objects, generalizing the classical Hamilton type equations for characteristic vector fields. Moreover, this geometric approach jointly with some Cartan's compatibility considerations [3, 4, 10] is also naturally extended to a wide class of nonlinear partial differential equations of first and higher orders. And even more, if an introduced tensor field is chosen in such a way that it carries an associated symplectic structure, the corresponding solutions to generalized Hamilton-Jacobi equations can be found, in general, effectively in the implicit functional-analytic Hopf-Lax type form, which is equivalent [26] to some good posed fixed point problem.

The characteristic method [4, 12, 17, 30] proposed in XIX century by A. Cauchy was very nontrivially developed by G. Monge, having introduced the geometric notion of characteristic surface, related with partial differential equations of first order. The latter, being augmented with a very important notion of characteristic vector fields, appeared to be fundamental [16,23,24, 30] for the characteristic method, whose main essence consists in bringing about the problem of studying solutions to our partial differential equation to an equivalent one of studying some set of ordinary differential equations. This way of reasoning succeeded later in development of the Hamilton-Jacobi theory, making it possible to describe a wide class of solutions to partial differential equations of first order of the form

$$H(x;u,u_x)=0, \tag{1.1}$$

where $H\in C^2(\mathbb{R}^{n+1}\times\mathbb{R}^n;\mathbb{R})$, $||H_x||\neq 0$, is called a Hamiltonian function and $u\in C^2(\mathbb{R}^n;\mathbb{R})$ is unknown function under search. The equation (1.1) is endowed still with a boundary value condition

$$u|_{\Gamma_\varphi}=u_0, \tag{1.2}$$

with $u_0\in C^1(\Gamma_\varphi;\mathbb{R})$, defined on some smooth almost everywhere hypersurface

$$\Gamma_\varphi:=\{x\in\mathbb{R}^n:\varphi(x)=0,\quad ||\varphi_x||\neq 0\}, \tag{1.3}$$

where $\varphi\in C^1(\mathbb{R}^n;\mathbb{R})$ is some smooth function. Following to the Monge's ideas, let us introduce the characteristic surface $S_H\subset\mathbb{R}^{n+1}\times\mathbb{R}^n$ as

$$S_H:=\{(x;u,p)\in\mathbb{R}^{n+1}\times\mathbb{R}^n:H(x;u,p)=0\}, \tag{1.4}$$

where we put, by definition, $p:=u_x\in\mathbb{R}^n$ for all $x\in\mathbb{R}^n$. The characteristic surface (1.4) was effectively described by Monge within his geometric approach by means of the so called Monge cones $K\subset T(\mathbb{R}^{n+1})$ and their duals $K^*\subset T^*(\mathbb{R}^{n+1})$ [24,30]. The corresponding differential-geometric analysis of this Monge scenario was later done by E. Cartan, who reformulated [10,30] the geometric picture, drown by Monge, by means of the related compatibility conditions for dual Monge cones and the notion of integral submanifold

$\Sigma_H \subset S_H$, naturally assigned to special vector fields on the characteristic surface S_H. In particular, Cartan had introduced on S_H the differential 1-form

$$\alpha^{(1)} := du - < p, dx >, \tag{1.5}$$

where $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$, and demanded its vanishing along the dual Monge cones $K^* \subset T^*(\mathbb{R}^{n+1})$, concerning the corresponding integral submanifold imbedding mapping

$$\pi : \Sigma_H :\rightarrow S_H. \tag{1.6}$$

This means that the 1-form

$$\pi^* \alpha_1^{(1)} := du - < p, dx > |_{\Sigma_H} \Rightarrow 0 \tag{1.7}$$

for all points $(x; u, p) \in \Sigma_H$ of a solution surface Σ_H, defined in such a way that $K^* = T^*(\Sigma_H)$. The obvious corollary from the condition (1.7) is the second Cartan condition

$$d\pi^* \alpha_1^{(1)} = \pi^* d\alpha_1^{(1)} =< dp, \wedge dx > |_{\Sigma_H} \Rightarrow 0. \tag{1.8}$$

These two Cartan's conditions (1.7) and (1.8) should be still augmented with the characteristic surface S_H invariance condition for the differential 1-form $\alpha_2^{(1)} \in \Lambda^1(S_H)$ as

$$\alpha_2^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{1.9}$$

The conditions (1.7), (1.8) and (1.9), when imposed on the characteristic surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$, make it possible to construct the proper characteristic vector fields on S_H, whose suitable characteristic strips [24, 30] generate the searched solution surface Σ_H. Thereby, having solved the corresponding Cauchy problem related with the boundary value conditions (1.2) and (1.3) for these characteristic vector fields, considered as ordinary differential equations on S_H, one can construct a solution to our partial differential equation (1.1). And what is interesting, this solution in many cases can be represented [12, 26] in exact functional-analytic Hopf-Lax type form. The latter is a natural consequence from the related Hamilton-Jacobi theory, whose main ingredient consists in proving the fact that the solution to our equation (1.1) is exactly the extremal value of some Lagrangian functional, naturally associated [3, 4, 23] with a given Hamiltonian function.

Below we will construct the proper characteristic vector fields for partial differential equations of first order (1.1) on the characteristic surface S_H, generating the solution surface Σ_H as suitable characteristic strips related with the boundary conditions (1.2) and (1.3), and next generalize the Cartan-Monge geometric approach for partial differential equations of second and higher orders.

2. THE CHARACTERISTIC VECTOR FIELDS METHOD: FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Consider on the surface $S_H \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ a characteristic vector field $K_H : S_H \to T(S_H)$ in the form

$$\left.\begin{aligned} dx/d\tau &= a_H(x;u,p), \\ dp/d\tau &= b_H(x;u,p), \\ du/d\tau &= c_H(x;u,p), \end{aligned}\right\} := K_H(x;u,p), \tag{2.1}$$

where $\tau \in \mathbb{R}$ is a suitable evolution parameter and $(x;u,p) \in S_H$. Since, owing to the Cartan-Monge geometric approach, there hold conditions (1.7), (1.8) and (1.9) along the solution surface Σ_H, we can satisfy them, applying the interior anti-differentiation operation $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ of the Grassmann algebra $\Lambda(S_H)$ of differential forms [3, 14, 28] on S_H to the corresponding differential forms $\alpha_1^{(1)}$ and $d\alpha_1^{(1)} \in \Lambda(S_H)$:

$$i_{K_H}\alpha_1^{(1)} = 0, \quad i_{K_H}d\alpha_1^{(1)} = 0. \tag{2.2}$$

As a result of simple calculations one finds that

$$c_H = \langle p, a_H \rangle, \quad \beta^{(1)} := \langle b_H, dx \rangle - \langle a_H, dp \rangle |_{S_H} = 0 \tag{2.3}$$

for all points $(x;u,p) \in S_H$. The obtained 1-form $\beta^{(1)} \in \Lambda^1(S_H)$ must be, evidently, compatible with the defining invariance condition (1.9) on S_H. This means that there exists a scalar function $\mu \in C^1(S_H;\mathbb{R})$, such that the condition

$$\mu\alpha_2^{(1)} = \beta^{(1)} \tag{2.4}$$

holds on S_H. This gives rise to such final relationships:

$$a_H = \mu\partial H/\partial p, \quad b_H = -\mu(\partial H/\partial x + p\partial H/\partial u), \tag{2.5}$$

which together with the first equality of (2.3) complete the search for the structure of the characteristic vector fields $K_H : S_H \to T(S_H)$:

$$K_H = (\mu\partial H/\partial p; \langle p, \mu\partial H/\partial p \rangle, -\mu(\partial H/\partial x + p\partial H/\partial u))^{\intercal}. \tag{2.6}$$

Now we can pose a suitable Cauchy problem for the equivalent set of ordinary differential equations (2.1) on S_H as follows:

$$\begin{aligned} &dx/d\tau = \mu\partial H/\partial p, \quad x|_{\tau=0} \stackrel{?}{=} x_0(x) \in \Gamma_\varphi, \quad x|_{\tau=t(x)} = x \in \mathbb{R}^n\backslash\Gamma_\varphi; \\ &du/d\tau = \langle p, \mu\partial H/\partial p \rangle, \quad u|_{\tau=0} = u_0(x_0(x)), \quad u|_{\tau=t(x)} \stackrel{?}{=} u(x), \\ &dp/d\tau = -\mu(\partial H/\partial x + p\partial H/\partial u), \quad p|_{\tau=0} = \partial u_0(x_0(x))/\partial x_0, \end{aligned} \tag{2.7}$$

where $x_0(x) \in \Gamma_\varphi$ is the intersection point of the corresponding vector field orbit, starting at a fixed point $x \in \mathbb{R}^n \backslash \Gamma_\varphi$, with the boundary hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ at the moment of „time" $\tau = t(x) \in \mathbb{R}$. As a result of solving the corresponding „inverse" Cauchy problem (2.7) one finds the following exact functional-analytic expression for a solution $u \in C^2(\mathbb{R}^n; \mathbb{R})$ to the boundary value problem (1.2) and (1.3):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \bar{\mathcal{L}}(x; u, p) d\tau, \tag{2.8}$$

where, by definition,

$$\bar{\mathcal{L}}(x; u, p) := < p, \mu \partial H / \partial p > \tag{2.9}$$

for all $(x; u, p) \in S_H$. If the Hamiltonian function $H : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$ is nondegenerate, that is $HessH := \det(\partial^2 H / \partial p \partial p\} \neq 0$ for all $(x; u, p) \in S_H$, then the first equation of (2.7) can be solved with respect to the variable $p \in \mathbb{R}^n$ as

$$p = \psi(x, \dot{x}; u) \tag{2.10}$$

for $(x, \dot{x}) \in T(\mathbb{R}^n)$, where $\psi : T(\mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is some smooth mapping. This gives rise to the followingcanonical Lagrangian function expression:

$$\mathcal{L}(x, \dot{x}; u) := \bar{\mathcal{L}}(x; u, p)|_{p=\psi(x,\dot{x};u)} \tag{2.11}$$

and to the resulting solution (2.8):

$$u(x) = u_0(x_0(x)) + \int_0^{t(x)} \mathcal{L}(x, \dot{x}; u) d\tau. \tag{2.12}$$

The functional-analytic form (2.12) is already proper for constructing its equivalent Hopf-Lax type form, being very important for finding so called generalized solutions [11,12,16] to the partial differential equation (1.1). This aspect of the Cartan-Monge geometric approach we suppose to analyze in detail elsewhere.

3. THE CHARACTERISTIC VECTOR FIELDS METHOD: SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Assume we are given a second order partial differential equation

$$H(x; u, u_x, u_{xx}) = 0, \tag{3.1}$$

where solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$ and the generalized „Hamiltonian" function $H \in C^2(\mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n);\mathbb{R})$. Putting $p^{(1)} := u_x$, $p^{(2)} := u_{xx}$, $x \in \mathbb{R}^n$, one can construct within the Cartan-Monge generalized geometric approach the characteristic surface

$$S_H := \{(x; u, p^{(1)}, p^{(2)}) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times (\mathbb{R}^n \otimes \mathbb{R}^n) : H(x; u, p^{(1)}, p^{(2)}) = 0\} \tag{3.2}$$

and a suitable Cartan's set of differential one- and two-forms:

$$\begin{aligned} \alpha_1^{(1)} &:= du- < p^{(1)}, dx > |_{\Sigma_H} \Rightarrow 0, d\alpha_1^{(1)} :=< dx, \wedge dp^{(1)} > |_{\Sigma_H} \Rightarrow 0, \\ \alpha_2^{(1)} &:= dp^{(1)}- < p^{(2)}, dx > |_{\Sigma_H} \Rightarrow 0, d\alpha_2^{(1)} :=< dx, \wedge dp^{(2)} > |_{\Sigma_H} \Rightarrow 0, \end{aligned} \tag{3.3}$$

vanishing upon the corresponding solution submanifold $\Sigma_H \subset S_H$. The set of differential forms (3.3) should be augmented with the characteristic surface S_H invariance differential 1-form

$$\alpha_3^{(1)} := dH|_{S_H} \Rightarrow 0, \tag{3.4}$$

vanishing, respectively, upon the characteristic surface S_H. Let the characteristic vector field $K_H : S_H \to T(S_H)$ on S_H is given by expressions

$$\left. \begin{aligned} dx/d\tau &= a_H(x; u, p^{(1)}, p^{(2)}), \\ du/d\tau &= c_H(x; u, p^{(1)}, p^{(2)}), \\ dp^{(1)}/d\tau &= b_H^{(1)}(x; u, p^{(1)}, p^{(2)}), \\ dp^{(2)}/d\tau &= b_H^{(2)}(x; u, p^{(1)}, p^{(2)}), \end{aligned} \right\} := K_H(x; u, p^{(1)}, p^{(2)}), \tag{3.5}$$

for all $(x; u, p^{(1)}, p^{(2)}) \in S_H$. To find the vector field (3.5) it is necessary to satisfy the Cartan compatibility conditions in the following geometric form:

$$\begin{aligned} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \quad i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \end{aligned} \tag{3.6}$$

where, as above, $i_{K_H} : \Lambda(S_H) \to \Lambda(S_H)$ is the internal differentiation of differential forms along the vector field $K_H : S_H \to T(S_H)$. As a result of conditions (3.6) one finds that

$$\begin{aligned} &c_H =< p^{(1)}, a_H >, \quad b_H^{(1)} =< p^{(2)}, a_H >, \\ &\beta_1^{(1)} :=< a_H, dp^{(1)} > - < b_H^{(1)}, dx > |_{S_H} \Rightarrow 0, \\ &\beta_2^{(1)} :=< a_H, dp^{(2)} > - < b_H^{(2)}, dx > |_{S_H} \Rightarrow 0, \end{aligned} \tag{3.7}$$

being satisfied upon S_H identically. The conditions (3.7) must be augmented still with the characteristic surface invariance condition (3.4). Notice now that 1-form $\beta_1^{(1)} = 0$ owing to the second condition of (3.7) and the third condition of (3.3). Thus, we need now to make compatible the basic scalar 1-form (3.4) with the vector-valued 1-form $\beta_2^{(1)} \in \Lambda(S_H) \otimes \mathbb{R}^n$. To do this let us construct, making use of the $\beta_2^{(1)}$, the following parametrized set of, respectively, scalar 1-forms:

$$\beta_2^{(1)}[\mu] := < \bar{\mu}^{(1|0)} \otimes a_H, dp^{(2)} > - < b_H^{(2)}, \bar{\mu}^{(1|0)} \otimes dx > |_{S_H} \Rightarrow 0, \qquad (3.8)$$

where $\bar{\mu}^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ is any smooth vector-valued function on S_H. The compatibility condition for (3.8) and (3.4) gives rise to the next relationships:

$$\begin{aligned} &\bar{\mu}^{(1|0)} \otimes a_H = \partial H/\partial p^{(2)}, \\ &< \bar{\mu}^{(1|0)}, b_H^{(2)} > = -\partial H/\partial x + p^{(1)} \partial H/\partial u + < \partial H/\partial p^{(1)}, p^{(2)} >, \end{aligned} \qquad (3.9)$$

holding on S_H. Take now such a dual vector function $\mu^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ that $< \mu^{(1|0)}, \bar{\mu}^{(1|0)} > = 1$ for all points of S_H. Then from (3.9) one finds easily that

$$\begin{aligned} a_H &= < \mu^{(1|0)}, \partial H/\partial p^{(2)} >, \\ b_H^{(2)} &= -\mu^{(1|0),*} \otimes (\partial H/\partial x + p^{(1)} \partial H/\partial u + < \partial H/\partial p^{(1)}, p^{(2)} >). \end{aligned} \qquad (3.10)$$

Combining now the first two relationships of (3.7) with the found above relationships (3.10) we get the final form for the vector field (3.5):

$$\begin{aligned} K_H = (a_H; < p^{(1)}, a_H >, < p^{(2)}, a_H >, -\mu^{(1|0),*} \otimes \\ \otimes (\partial H/\partial x + p^{(1)} \partial H/\partial u + < \partial H/\partial p^{(1)}, p^{(2)} >))^{\intercal}, \end{aligned} \qquad (3.11)$$

where $a_H = < \mu^{(1|0)}, \partial H/\partial p^{(2)} >$ and $\mu^{(1|0)} \in C^1(S_H;\mathbb{R}^n)$ is some smooth vector-valued function on S_H. Thereby, we can construct as before solutions to our partial differential equation of second order (3.1) by means of solving the equivalent Cauchy problem for the set of ordinary differential equations (3.5) on the characteristic surface S_H.

4. THE CHARACTERISTIC VECTOR FIELDS METHOD: PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

Consider a general nonlinear partial differential equation of higher order $m \in \mathbb{Z}_+$ as

$$H(x; u, u_x, u_{xx}, \ldots, u_{mx}) = 0, \qquad (4.1)$$

where there is assumed that $H \in C^2(\mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2}; \mathbb{R})$. Within the generalized Cartan-Monge geometric characteristic method we need to construct the related characteristic surface S_H as

$$S_H := \{(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^n)^{\otimes m(m+1)/2} : \\ H(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}) = 0\}, \tag{4.2}$$

where we put $p^{(1)} := u_x \in \mathbb{R}^n$, $p^{(2)} := u_{xx} \in \mathbb{R}^n \otimes \mathbb{R}^n, \ldots, p^{(m)} \in (\mathbb{R}^n)^{\otimes m}$ for $x \in \mathbb{R}^n$. The corresponding solution manifold $\Sigma_H \subset S_H$ is defined naturally as the integral submanifold of the following set of one- and two-forms on S_H :

$$\alpha_1^{(1)} := du - < p^{(1)}, dx > |_{\Sigma_H} \Rightarrow 0, d\alpha_1^{(1)} :=< dx, \wedge dp^{(1)} > |_{\Sigma_H} \Rightarrow 0, \tag{4.3}$$

$$\alpha_2^{(1)} := dp^{(1)} - < p^{(2)}, dx > |_{\Sigma_H} \Rightarrow 0, d\alpha_2^{(1)} :=< dx, \wedge dp^{(2)} > |_{\Sigma_H} \Rightarrow 0, \tag{4.4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\alpha_m^{(1)} := dp^{(m-1)} - < p^{(m)}, dx > |_{\Sigma_H} \Rightarrow 0, d\alpha_m^{(1)} :=< dx, \wedge dp^{(m)} > |_{\Sigma_H} \Rightarrow 0, \tag{4.5}$$

vanishing upon Σ_H. The set of differential forms (4.4) is augmented with the determining characteristic surface S_H invariance condition

$$\alpha_{m+1}^{(1)} := dH|_{S_H} \Rightarrow 0. \tag{4.6}$$

Proceed now to constructing the characteristic vector field $K_H : S_H \to T(S_H)$ on the hypersurface S_H within the developed above generalized characteristic method. Take the expressions

$$\left.\begin{array}{r} dx/d\tau = a_H(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}), \\ du/d\tau = c_H(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}), \\ dp^{(1)}/d\tau = b_H^{(1)}(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}), \\ dp^{(2)}/d\tau = b_H^{(2)}(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}), \\ \cdots\cdots\cdots\cdots\cdots, \\ dp^{(m)}/d\tau = b_H^{(m)}(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}), \end{array}\right\} := K_H(x; u, p^{(1)}, p^{(2)}), \tag{4.7}$$

for $(x; u, p^{(1)}, p^{(2)}, \ldots, p^{(m)}) \in S_H$ and satisfy the corresponding Cartan compatibility conditions in the following geometric form:

$$\begin{array}{ll} i_{K_H}\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, & i_{K_H}d\alpha_1^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ i_{K_H}\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, & i_{K_H}d\alpha_2^{(1)}|_{\Sigma_H} \Rightarrow 0, \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots, \\ i_{K_H}\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0, & i_{K_H}d\alpha_m^{(1)}|_{\Sigma_H} \Rightarrow 0. \end{array} \tag{4.8}$$

As a result of suitable calculations in (4.8) one gets the following expressions:

$$\begin{aligned}
&c_H =< p^{(1)}, a_H >, \quad b_H^{(1)} =< p^{(2)}, a_H >,\\
&\beta_1^{(1)} :=< a_H, dp^{(1)} > - < b_H^{(1)}, dx > |_{S_H} \Rightarrow 0,\\
&\beta_2^{(1)} :=< a_H, dp^{(2)} > - < b_H^{(2)}, dx > |_{S_H} \Rightarrow 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\beta_m^{(1)} :=< a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0,
\end{aligned} \tag{4.9}$$

being satisfied upon S_H identically. It is now easy to see that all of 1-forms $\beta_j^{(1)} \in \Lambda^1(S_H) \otimes (\mathbb{R}^n)^{\otimes j}$, $j = \overline{1, m-1}$ are vanishing identically upon S_H owing to the relationships (4.4). Thus, as a result we obtain the only relationship

$$\beta_m^{(1)} :=< a_H, dp^{(m)} > - < b_H^{(m)}, dx > |_{S_H} \Rightarrow 0, \tag{4.10}$$

which should be compatibly combined with that of (4.6). To do this suitably with the tensor structure of the 1-forms (4.10), we take a smooth tensor function $\bar{\mu}^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H and construct the parameterized set of scalar 1-forms

$$\beta_m^{(1)}[\mu] :=< \bar{\mu}^{(m-1|0)} \otimes a_H, dp^{(m)} > - < b_H^{(m)}, \bar{\mu}^{(m-1|0)} \otimes dx > |_{S_H} \Rightarrow 0, \tag{4.11}$$

which can be now identified with the 1-form (4.6). This gives rise right away to the relationships

$$\begin{aligned}
&\bar{\mu}^{(m-1|0)} \otimes a_H = \partial H/\partial p^{(m)},\\
&< \bar{\mu}^{(m-1|0)}, b_H^{(m)} >= -(\partial H/\partial x + p^{(1)}\partial H/\partial u +\\
&+ < \partial H/\partial p^{(1)}, p^{(2)} > + \ldots + < \partial H/\partial p^{(m-1)}, p^{(m)} >),
\end{aligned} \tag{4.12}$$

holding on S_H. Now we can take such a dual tensor-valued function $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ on S_H that $< \mu^{(m-1|0)}, \bar{\mu}^{(m-1|0)} >= 1$ for all points of S_H. Then from (4.12) we easily get the searched unknown expressions

$$\begin{aligned}
&a_H =< \mu^{(m-1|0)}, \partial H/\partial p^{(m)} >,\\
&b_H^{(m)} = -\mu^{(1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u +\\
&+ < \partial H/\partial p^{(1)}, p^{(2)} > + \ldots + < \partial H/\partial p^{(m-1)}, p^{(m)} >).
\end{aligned} \tag{4.13}$$

The obtained above result (4.13) combined with suitable expressions from (4.9) give rise to the following final form for the vector field (4.7):

$$\begin{aligned}
&K_H = (a_H; < p^{(1)}, a_H >, < p^{(2)}, a_H >, \ldots, < p^{(m)}, a_H >,\\
&-\mu^{(m-1|0),*} \otimes (\partial H/\partial x + p^{(1)}\partial H/\partial u +\\
&+ < \partial H/\partial p^{(1)}, p^{(2)} > + \ldots + < \partial H/\partial p^{(m-1)}, p^{(m)} >))^{\intercal},
\end{aligned} \tag{4.14}$$

where $a_H =< \mu^{(m-1|0)}, \partial H/\partial p^{(m)} >$ and $\mu^{(m-1|0)} \in C^1(S_H; (\mathbb{R}^n)^{\otimes(m-1)})$ is some smooth tensor-valued function on S_H. The resulting set (4.7) of ordinary differential equations on S_H makes it possible to construct exact solutions to our partial differential equation (4.1) in a suitable functional-analytic form, being often very useful for analyzing its properties important for applications. On these and related questions we plan to stop in detail elsewhere later.

Namely, if for instance a first order differential equation is given as

$$H(x; u, u_x) = 0, \tag{4.15}$$

where $x \in \mathbb{R}^n$, $H \in C^1(\mathbb{R}^{2n+1}; \mathbb{R})$, $||H_{u_x}|| \neq 0$, the characteristics vector fields on the related Monge hypersurface

$$S_H := \{(x; u, p) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : \bar{H}(x; u, p) := H(x; u, \pi)|_{\pi=\psi(x;u,p)} = 0\} \tag{4.16}$$

are represented [25] as follows:

$$\begin{gathered} dx/d\tau = \mu^{(1|1)} \partial \bar{H}/\partial p, \quad dp/d\tau = -\mu^{(1|1),*} \left(\partial \bar{H}/\partial x + \psi \partial \bar{H}/\partial u\right), \\ du/d\tau =< \psi, \mu^{(1|1)} \partial \bar{H}/\partial p > . \end{gathered} \tag{4.17}$$

Here $\mu^{(1|1)} := (\partial \psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1}; \mathbb{R}^n \otimes \mathbb{R}^n)$ is a nondegenerate smooth tensor field on the hipersurface S_H, related with its parametrization $\pi := \psi(x; u, p) \in \mathbb{R}^n$, and $\tau \in \mathbb{R}$ is an evolution parameter. Vector field (4.17) ensures [25] the tangency to the hyper-surface $S_H \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$ and the projection compatibility condition with the dual Monge cone K^* upon the corresponding solution hypersurface $\bar{S}_H \subset \mathbb{R}^{n+1}$, (see Fig. 1) generated by the characteristic strips $\Sigma_H \subset S_H$ through smoothly embedded sets $\Sigma \subset S_H$, consisting of points carrying the solutions to our problem (4.15). Similar results were obtained in [25] also for both partial differential equations of higher orders and systems.

In general, the problem (4.15) is endowed with some boundary condition on a smooth hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ as

$$u|_{\Gamma_\varphi} = u_0, \tag{4.18}$$

where $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is a given function. The hypersurface $\Gamma_\varphi \subset \mathbb{R}^n$ can be, for simplicity, defined as

$$\Gamma_\varphi := \{x \in \mathbb{R}^n : \varphi(x) = 0\}, \tag{4.19}$$

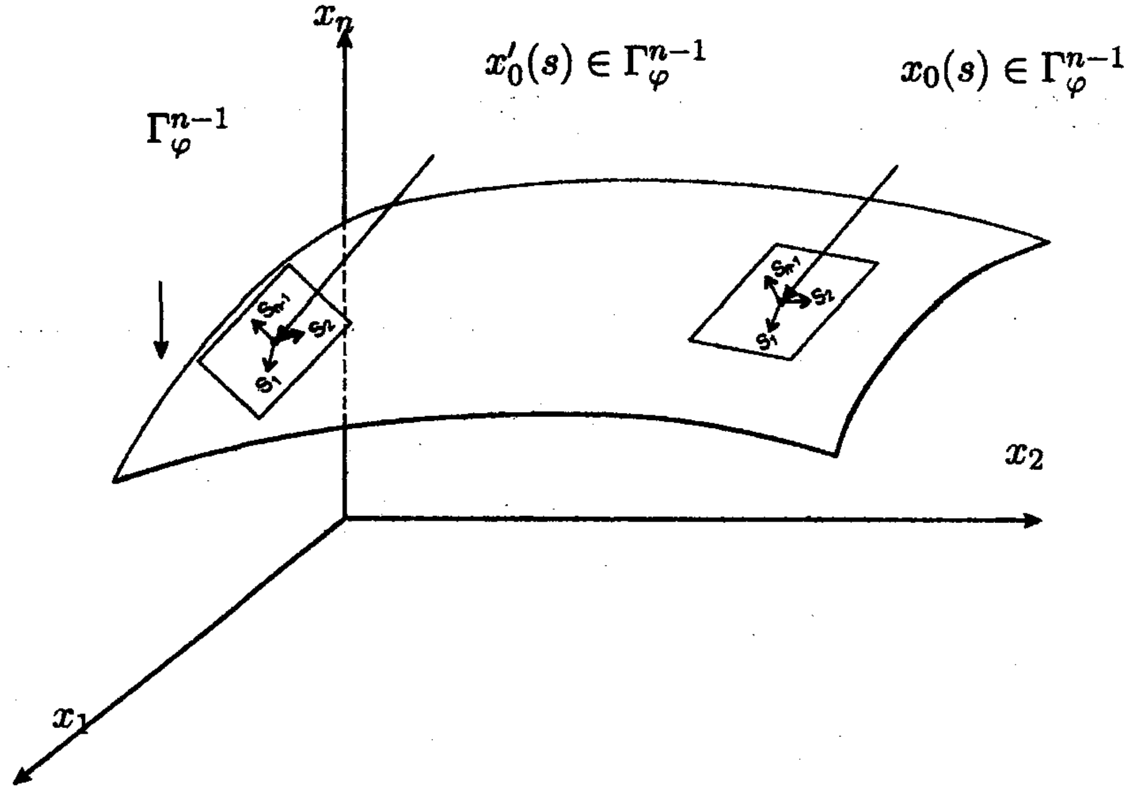

Fig. 1. The boundary $\Gamma_\varphi^{n-1} = \{x_0 \in \mathbb{R}^n : \varphi(x_0) = 0\}$,
$x_0(s) \in \Gamma_\varphi^{n-1}$, $s \in \mathbb{R}^{n-1}$ – local coordinates.

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth mapping endowed with some local coordinates $s(x) \in \mathbb{R}^{n-1}$ in the corresponding open neighborhoods $O_\varepsilon(x) \subset \Gamma_\varphi$ of all points $x \in \Gamma_\varphi$ at some $\varepsilon > 0$. Thus, we are interested in constructing analytical solutions to the boundary problem (4.15), (4.18) and (4.19) and studying their properties. This and related aspects of this problem will be discussed in detail below.

5. BOUNDARY PROBLEM ANALYSIS

Consider the set of characteristic equations (4.17) on the hypersurface $S_H \subset \mathbb{R} \times \mathbb{R}^{n+1}$, which start at points $(x_0; u_0, p_0) \in \Sigma$ under the additional condition that the corresponding projection $\Sigma \to \bar{\Sigma}$ upon the subspace $\mathbb{R}^{n+1}$ (see Fig. 2) coincides with the boundary set $(\Gamma_\varphi; u_0) \subset \mathbb{R}^{n+1}$, that is

$$\bar{\Sigma} := (\Gamma_\varphi; u_0), \tag{5.1}$$

where $u_0 \in C^1(\Gamma_\varphi; \mathbb{R})$ is our boundary condition. The condition (5.1) assumes evidently that the set $\Sigma \subset S_H$ can be defined as follows:

$$\Sigma = (\bar{\Sigma}; p_0) \tag{5.2}$$

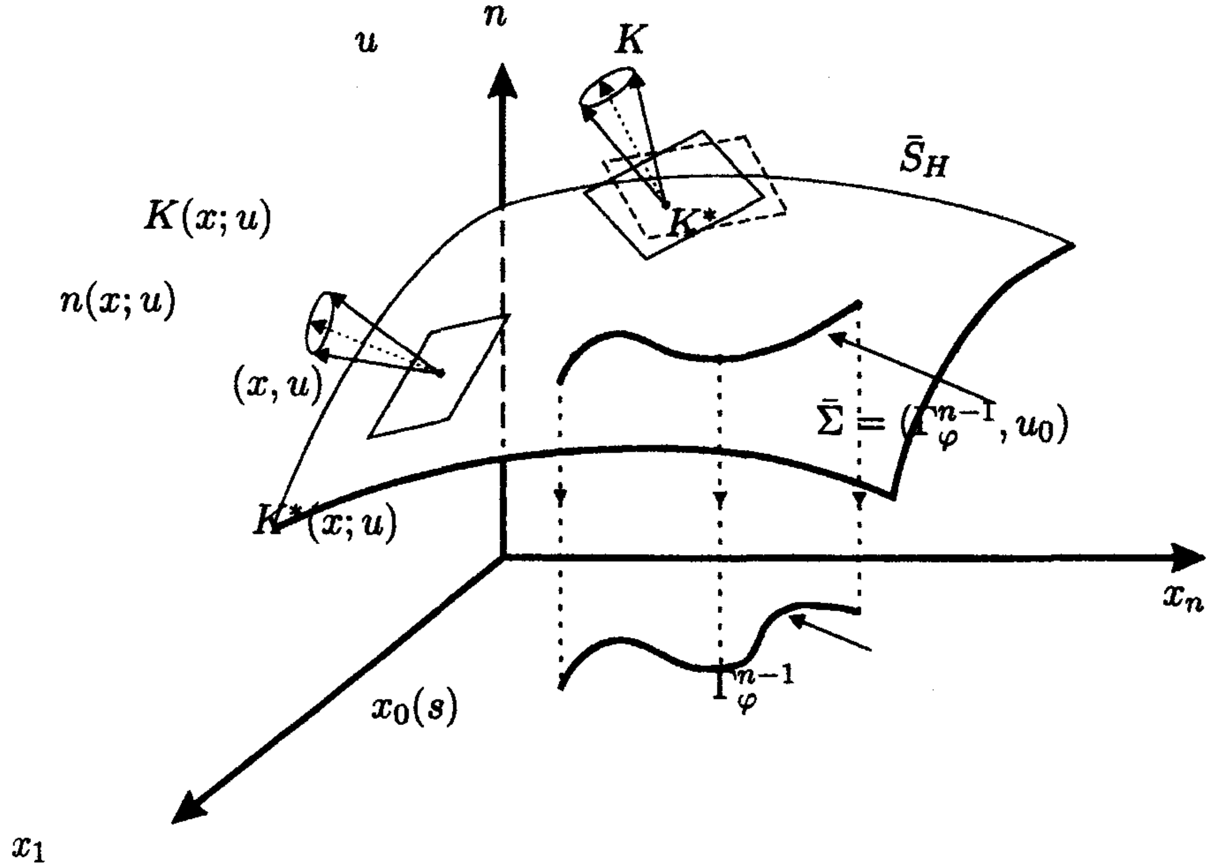

Fig. 2. Geometric Monge method. The boundary conditions:
$\bar{\Sigma} = (\Gamma_\varphi^{n-1}, u_0) \subset \bar{S}_H$, $u_0 \in C^1(\Gamma_\varphi^{n-1}; \mathbb{R})$,
the boundary problem solution hypersurface: $\bar{S}_H := \{(x,u) \in \mathbb{R}^{n+1} : u = \psi(x)\}$

withsome $p_0 \in C^1(\Gamma_\varphi; \mathbb{R}^n)$ being yet unknown smooth mappings. For them to be determined we need to ensure for all points $\Sigma \subset S_H$ the mentioned above Cartan compatibility conditions, that is the conditions

$$du|_\Sigma = < p, dx > |_\Sigma, \qquad < d\psi, \wedge dx > |_\Sigma = 0, \tag{5.3}$$

where $\Sigma \subset S_H$ is given by (5.2). As a result of (5.3) one finds easily that

$$\begin{cases} \partial u_0(s)/\partial s - < \psi(x_0(s); u_0(x_0(s)), p_0(s)), \partial x_0(s)/\partial s >= 0, \\ \bar{H}(x_0(s); u_0(x_0(s)), p_0(s)) = 0 \end{cases} \tag{5.4}$$

for all points $x_0 := x_0(s) \in \Gamma_\varphi$, $s \in \mathbb{R}^{n-1}$. Here we took into account that any point $x \in \Gamma_\varphi$ is parametrized by means of the corresponding local coordinates $s = s(x_0) \in \mathbb{R}^{n-1}$, defined in the corresponding $\varepsilon-$vicinities $O_\varepsilon(x) \subset \Gamma_\varphi$, $\varepsilon > 0$.

The system of relationships (5.4) must be solvable for a mapping $p_0 : \Gamma_\varphi \to \mathbb{R}^n$ at all points $x_0 \in \Gamma_\varphi$, what gives rise to the determinant condition

$$\det \left[\left(\frac{\partial \psi}{\partial p} \right)^* \frac{\partial x_0}{\partial s}; \left(\frac{\partial \bar{H}}{\partial p} \right)^\intercal \right] \bigg|_{(x_0; u_0, p_0)} \neq 0 \tag{5.5}$$

owing to the implicit function theorem [29]. If the condition (5.5) is satisfied at points $(x_0; u_0, p_0^{(j)}) \in S_H$, where $j = \overline{1,N}$ for some $N \in \mathbb{Z}_+$ and all points $(x_0; u_0) \in \bar{\Sigma}$, the system of equation (5.4) possesses exactly $N \in \mathbb{Z}_+$ different smooth solution $p_0^{(j)} \in C^1(\Gamma_\varphi; \mathbb{R}^n)$, $j = \overline{1,N}$, thereby determining the corresponding Cauchy data (5.2) for the characteristic vector fields (4.17). It is clear enough that our boundary problem (4.15), (4.18) and (4.19) possesses, in general, many solutions of different functional classes, depending on the kind of chosen boundary conditions. For instance, as it was studied and analyzed in [12,17,26] this boundary problem can possess also so called generalized solutions, which allow at some additional conditions the so called Hopf-Lax inf-type extremality form, being often very useful for studying their asymptotic and other properties.

Concerning the important problem of constructing functional-analytic solutions to our equation (4.15) under the boundary conditions (4.18) and (4.19) we will ponder it in detail below.

6. THE HOPF-LAX TYPE INF-TYPE FUNCTIONAL-ANALYTIC REPRESENTATION

Assume now that $p_0 \in C^1(\Gamma_\varphi; \mathbb{R}^n)$ is a smooth solution to the system (5.4), thereby defining completely the sought Cauchy data $\Sigma \subset S_H$ for the characteristic vector fields (4.17). Thus, making use of suitable classical methods for solving these ordinary differential equations, one can find, in particular, that the function $u \in C^2(\mathbb{R}^n; \mathbb{R})$ for each reachable point $x = x(t) \in \mathbb{R}^n$ can be represented in the analytical form

$$u(x(t)) = u(x(0)) + \int_0^t < \psi(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau) > d\tau \tag{6.1}$$

at any moment of „time“ $t \in \mathbb{R}$. Since, by definition, $x(0) := x_0(s) \in \Gamma_\varphi$ and $u(x(0)) := u_0(x_0(s))$, $s \in \mathbb{R}^{n-1}$, the solution (6.1) is rewriten as

$$u(x(t)) = u_0(x_0(s)) + \int_0^t < \psi(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau) > d\tau \tag{6.2}$$

for any $t \in \mathbb{R}$, where the integrand function in (6.2) is assumed to be found analytically.

Pose now for the vector field equations (4.17) the following „inverse“ Cauchy problem

$$x|_{\tau=t(x)} = x \in \mathbb{R}^n, \ x|_{\tau=0} = x_0(s[x_0; x]) \in \Gamma_\varphi \tag{6.3}$$

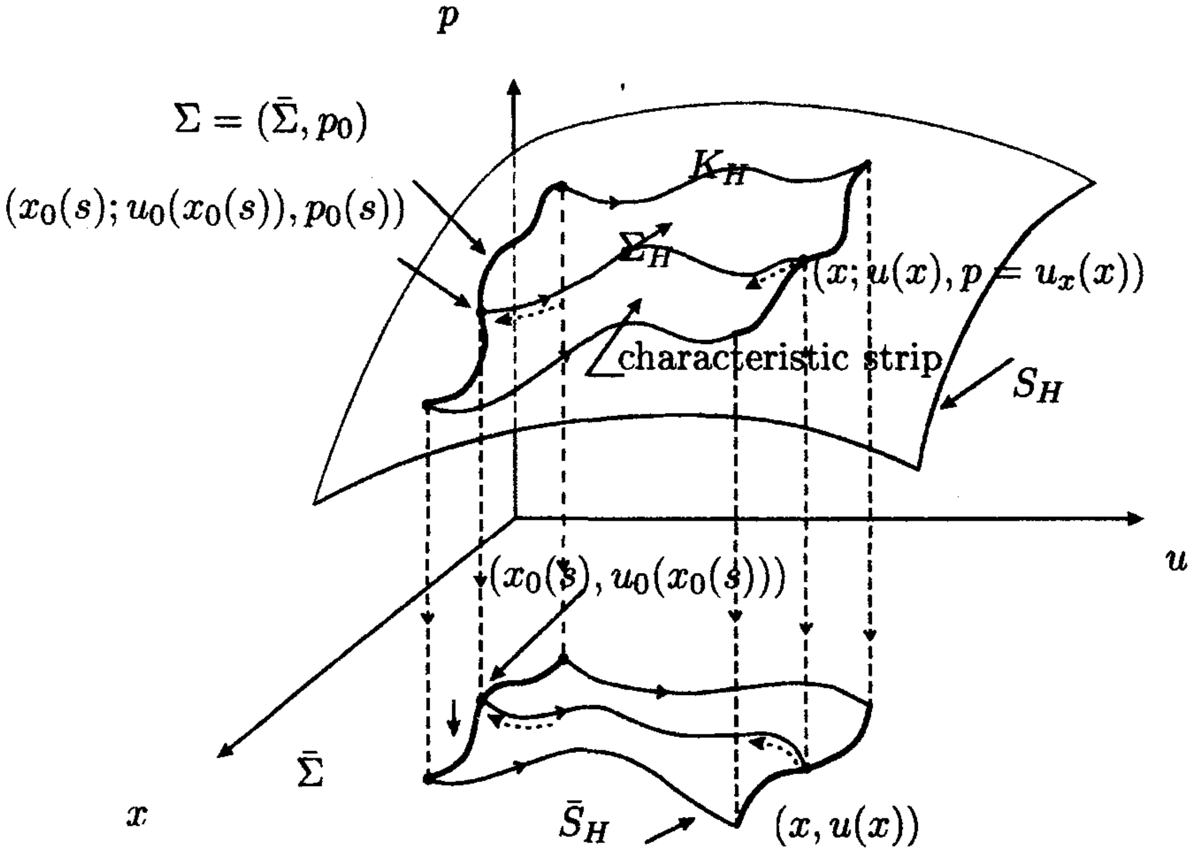

Fig. 3. Geometric Monge method. The characteristic surface: $S_H = \{(x; u, p) \in \mathbb{R}^{2n+1} : \ H(x; u, p) = 0\}$ and intial conditions for the vector field $K_H : S_H \to T(S_H)$, satisfying the Cartan's comatibility conditions: $du - < p, dx >_{|K_H, \Gamma_\varphi^{n-1}} = 0$ iff $\bar{S}_H || K^*$ and there exist data $\Sigma = (\bar{\Sigma}, p_0)$ defining the characteristic strip Σ_H.

for some local parameter $s[x_0; x] \in \mathbb{R}^{n-1}$ at the moment of „time“ $t(x) \in \mathbb{R}$ corresponding to an arbitrary reachable point $x \in \mathbb{R}^n$ as it is shown on Fig. 3.

Here we assumed that the evolution mapping $(\Gamma_\varphi, \mathbb{R}) \ni (x_0, \tau) \to x(\tau; x_0) := x \in \mathbb{R}^n$ is invertible for almost all reachable points $x \in \mathbb{R}^n$ and, respectively, for each found above point $x_0(s[x_0; x]) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$ one can suitably determine the unique point $p_0(s[x_0; x]) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. As a result, one can write down, owing to the conditions (6.3), the following expression:

$$u(x) = u_0(x_0([x_0; x])) + \int_{\tau=0}^{\tau=t(x)} \mathcal{L}(\tau | x_0(s[x_0; x]); x) d\tau, \qquad (6.4)$$

where $\mathcal{L} : \mathbb{R} \times (\Gamma_\varphi \times \mathbb{R}^n) \to \mathbb{R}$ is the so called „quasi-Lagrangian“ function:

$$\mathcal{L}(\tau | x_0([x_0; x]); x) := < p(\tau), \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p}(\tau) >, \qquad (6.5)$$

which is defined by solutions to the characteristic vector field equations (4.17) under conditions (6.3). The expression (6.4) on integrating it with respect the parameter $\tau \in [0, t(x)] \subset \mathbb{R}$ reduces to the analytical form

$$u(x) = u_0(x_0(s[x_0; x])) + \mathcal{P}(x_0([x_0; x]); x), \qquad (6.6)$$

where points $x_0(s(x)) \in \Gamma_\varphi$, $x \in \mathbb{R}^n$, and, by definition, the „kernel" function

$$\mathcal{P}(x_0(s[x_0;x]);x) := \int_{\tau=0}^{\tau=t(x)} \mathcal{L}(\tau|x_0([x_0;x]);x)d\tau. \tag{6.7}$$

The obtained expression (6.6) does solve the equation (4.15) under the boundary conditions (4.18) and can be effective enough for applications, if the kernel-function (6.7) is constructed analytically. But, in general, if $\partial\bar{H}/\partial u \neq 0$ identically on S_H, the quasi-Lagrangian function (6.5) depends effectively on the unknown still solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$, that makes the expressions (6.7) and (6.6) senseless. Since the latter expressions depend, obviously, strongly on a choice of the parametrisation $\pi := \psi(x;u,p) \in \mathbb{R}^n$, $(x;u,p) \in S_H$, at which the tensor field $\mu^{(1|1)} = (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1};\mathbb{R}^n \otimes \mathbb{R}^n)$ is under our disposition, one can propose a partial remedy to this problem.

Namely, to make use of this possibility as much as one can, let us assume additionally that our tensor field $\mu^{(1|1)} = (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+1};\mathbb{R}^n \otimes \mathbb{R}^n)$ carries the associated symplectic structure. This means, speaking more generally, the existence of such a „symplectic" element $\Psi := (\psi_1,\psi_2)^\intercal \in C^2(\mathbb{R}^n \times \mathbb{R}^{n+1};T^*(T^*(\mathbb{R}^n)))$ that for all $(x,p) \in T^*(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathbb{R}^n$ the following equality

$$\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dp}{d\tau} \end{pmatrix} = -\vartheta \begin{pmatrix} \frac{\partial\bar{H}}{\partial x} + \psi\frac{\partial\bar{H}}{\partial u} \\ \frac{\partial\bar{H}}{\partial p} \end{pmatrix}, \tag{6.8}$$

holds, where the co-symplectic operator $\vartheta : T^*(T^*(\mathbb{R}^n)) \to T(T^*(\mathbb{R}^n))$ in the form

$$\vartheta := \begin{pmatrix} 0 & -\mu^{(1|1)} \\ \mu^{(1|1),*} & 0 \end{pmatrix} \tag{6.9}$$

is defined as $\vartheta = \Omega^{-1}$ under the condition that the symplectic matrix

$$\Omega := \Psi' - \Psi'^{,*} = \begin{pmatrix} \frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1^*}{\partial x} & \frac{\partial\psi_1}{\partial p} - \frac{\partial\psi_2^*}{\partial x} \\ \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1^*}{\partial p} & \frac{\partial\psi_2}{\partial p} - \frac{\partial\psi_2^*}{\partial p} \end{pmatrix} \tag{6.10}$$

is nondegenerate. This gives rise, in particular, to the next important corollary: the charactersitic vector field system (4.17) is Hamiltonian, allowing the natural Lagrangian extremality interpretation:

$$\frac{\delta}{\delta x}\int_{\tau=0}^{\tau=t(x)} \tilde{\mathcal{L}}(x,\dot{x};u)d\tau = 0, \tag{6.11}$$

holding over the set of all smooth curves $x \in C^2([0,t(x)];\mathbb{R}^n)$, $x(0)=x_0 \in \Gamma_\varphi$, $x(t(x)) = x \in \mathbb{R}^n \backslash \Gamma_\varphi$. Here, by definition, we put $\dot{x} := dx/d\tau$ for $\tau \in [0,t(x)]$,

$$\tilde{\mathcal{L}}(x,\dot{x};u) := <\psi_1,\dot{x}> + <\psi_2,\dot{p}> - \bar{H}(x;u,p)\big|_{p=\alpha(x,\dot{x};u)}, \tag{6.12}$$

$\dot{p} := dp/d\tau$, where the vector $p := \alpha(x,\dot{x};u) \in \mathbb{R}^n$ solves the equivalent to (6.8) system of equations

$$\begin{gathered}\left(\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1^*}{\partial x}\right)\dot{x} + \left(\frac{\partial\psi_1}{\partial p} - \frac{\partial\psi_2^*}{\partial x}\right)\dot{p} = \frac{\partial\bar{H}}{\partial x} + \psi\frac{\partial\bar{H}}{\partial u},\\ \left(\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1^*}{\partial p}\right)\dot{x} + \left(\frac{\partial\psi_2}{\partial p} - \frac{\partial\psi_2^*}{\partial p}\right)\dot{p} = \frac{\partial\bar{H}}{\partial p}\end{gathered} \tag{6.13}$$

at points $(x,\dot{x};u) \in \mathbb{R}^{2n} \times \mathbb{R}$.

The Lagrangian extremality condition (6.11) makes it possible to introduce a new „momentum“ variable $\tilde{p} \in \mathbb{R}^n$, canonically conjugated with the variable $x \in \mathbb{R}^n$ as follows:

$$\tilde{p} := \partial\tilde{\mathcal{L}}/\partial\dot{x}. \tag{6.14}$$

This gives rise to a new canonical Hamiltonian system for conjugated variables $(x,\tilde{p}) \in \mathbb{R}^n \times \mathbb{R}^n$ and a new Hamiltonian function $\tilde{H} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$, completely equivalent to the system (6.8)

$$\frac{dx}{d\tau} = \frac{\partial\tilde{H}}{\partial\tilde{p}}, \quad \frac{d\tilde{p}}{d\tau} = -\left(\frac{\partial\tilde{H}}{\partial x} + \psi\frac{\partial\tilde{H}}{\partial u}\right)\Big|_{p=\tilde{\alpha}(x,;u,\tilde{p})} \tag{6.15}$$

together with the compatibility equation

$$du/d\tau = <\tilde{\alpha}(x;u,\tilde{p}),\tilde{\beta}(x;u,\tilde{p})>, \tag{6.16}$$

where, by definition, we put

$$\begin{gathered}\tilde{\alpha}(x;u,\tilde{p}) := \alpha(x,\dot{x};u)|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})}, p = \tilde{\alpha}(x;u,\tilde{p}) := \alpha(x,\dot{x};u)|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})},\\ \tilde{H}(x;\tilde{p}|u) := H(x;u,p) + <\tilde{p}-\psi_1,\tilde{\beta}> - <\psi_2,d\tilde{\alpha}/d\tau>|_{p:=\tilde{\alpha}(x;u,\tilde{p})},\end{gathered} \tag{6.17}$$

based on the following relationships

$$\tilde{p} = \partial\tilde{\mathcal{L}}(x,\dot{x};u)/\partial\dot{x}\Big|_{\dot{x}=\tilde{\beta}(x;u,\tilde{p})}, \tag{6.18}$$

owing to the implicit function theorem, applied to (6.14) with respect to the variable $\dot{x} \in \mathbb{R}^n$.

Now we are in a position to write down the corresponding to expressions (6.15) and (6.17) Hamilton-Jacobi equation on the canonical transformations „generating" function $\tilde{u} \in C^2(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$:

$$\frac{\partial \tilde{u}}{\partial \tau} + \tilde{H}\left(x; \frac{\partial \tilde{u}}{\partial x}|u\right) = 0, \tag{6.19}$$

where the sought function $u \in C^2(\mathbb{R}^n; \mathbb{R})$ satisfies equation (6.16).

Assume now for a moment that the function $u \in C^2(\mathbb{R}^n; \mathbb{R})$ is constant along the vector field (6.8), that is

$$du/d\tau =< \tilde{\alpha}(x; u, \tilde{p}), \tilde{\beta}(x; u, \tilde{p}) >= 0 \tag{6.20}$$

for all $(x; u, \tilde{p}) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$. The condition (6.20) involves some constraints on the „symplectic" vector $\Psi = (\psi_1, \psi_2)^\intercal \in C^2(\mathbb{R}^n \times \mathbb{R}^{n+1}; \mathbb{R}^n \times \mathbb{R}^n)$, which can be satisfied by means of choosing a suitable parametrization $\pi := \psi(x; u, p) \in \mathbb{R}^n$, $(x; u, p) \in S_H$, of the characterstic hypersurface S_H. Proceed now to solving the canonical Hamilton-Jacobi equation (6.19) under some Cauchy data $\tilde{u}|_{t=0} = \tilde{u}_0 \in C^2(\Gamma_\varphi; \mathbb{R})$. This task can be solved easily enough via the standard Hopf-Lax type [12,26] scheme. Namely, consider the inverse Cauchy problem (6.3) for the canonical Hamilton equations (6.15) in the form

$$\frac{dx}{d\tau} = \frac{\partial \tilde{H}}{\partial \tilde{p}}, \quad \frac{d\tilde{p}}{d\tau} = -\left(\frac{\partial \tilde{H}}{\partial \tilde{p}} + \tilde{\alpha}\frac{\partial \tilde{H}}{\partial u}\right), \tag{6.21}$$

where the parameter $\tau \in [0, t(x)] \in \mathbb{R}$. The corresponding solution to Hamilton-Jacobi equation (6.19) possesses then the functional-analytical Hopf-Lax type form

$$\tilde{u}(x, t|u) = \inf_{y \in \Gamma_\varphi} \{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t, x; y|u)\}, \tag{6.22}$$

following right away from the expression analogous to (6.6), where, by definition, the „kernel"

$$\tilde{\mathcal{P}}(t, x; y|u) := \int_0^t \tilde{\mathcal{L}}(\tau; y, x|u) d\tau \tag{6.23}$$

is obtained from the Lagrangian function

$$\begin{aligned} \tilde{\mathcal{L}}(\tau; y, x|u) &= \tilde{\mathcal{L}}(x, \dot{x}; u)\Big|_{x=\tilde{x}(\tau; y, x|u)}, \\ \tilde{\mathcal{L}}(x, \dot{x}; u) &:=< \tilde{p}, \tfrac{\partial \tilde{H}}{\partial \tilde{p}} > -\tilde{H}(x; u, \tilde{p}), \end{aligned} \tag{6.24}$$

calculated on solutions to the equations (6.21) under conditions (6.3). Then, owing to conditions (6.24), (6.17) and (6.16), the equality

$$\tilde{H}(x;\tilde{p}|u) = H(x;u,p) + <\tilde{p} - \psi_1, \tilde{\beta}> - <\psi_2, d\tilde{\alpha}/d\tau>|_{\tau=t(x)}, \quad (6.25)$$

holds for suitable $x_0 = x_0(x) \in \Gamma_\varphi$ and $\tau = t(x) \in \mathbb{R}$. Moreover, as $H(x;u,p) = 0$ for all points $(x;p|u) \in T^*(\mathbb{R}^n) \times \mathbb{R}$, the equality (6.25) reduces to

$$\tilde{H}(x;u,\tilde{p}) = <\tilde{u}_x - \psi_1, \tilde{\beta}> - <\psi_2, d\tilde{\alpha}/d\tau>|_{\tau=t(x)}, \quad (6.26)$$

which will be used further for determining the sought solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$ in an implicit form. To do this much more effectively, we consider the expression (6.22) at $t = t(x) \in \mathbb{R}$ obtained in the following functional analytic form:

$$\tilde{u}(x,t(x)|u) = \inf_{y\in\Gamma_\varphi} \{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t(x),x;y|u)\}, \quad (6.27)$$

taking into account the boundary condition (4.18) for the corresponding solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$ of the Hamilton-Jacobi equation (4.15) at $x_0 = x_0(x|u) \in \Gamma_\varphi$ for all reachable points $x \in \mathbb{R}\backslash\Gamma_\varphi$. The Cauchy data $\tilde{u}_0 \in C^2(\Gamma_\varphi;\mathbb{R})$ can be taken, in general, arbitrary, but such that the infimum (6.27) exists and the conditions

$$\tilde{p}_0(x_0) = \partial\tilde{u}(x_0,\tau|u)/\partial x|_{\tau=0} = \partial\tilde{\mathcal{L}}(x,\dot{x};u)/\partial\dot{x}\Big|_{\tau=0} \quad (6.28)$$

hold, if equations (6.15) and (6.20) are sataisfied. Therefore, if the point $\bar{y} := y(x|u) \in \Gamma_\varphi$ is such that

$$\inf_{y\in\Gamma_\varphi} \{\tilde{u}_0(y) + \tilde{\mathcal{P}}(t(x),x;y|u)\} = \tilde{u}_0(\bar{y}) + \tilde{\mathcal{P}}(t(x),x;\bar{y}|u), \quad (6.29)$$

then the trajectory of the point $y(x|u) = x_0(x|u) \in \Gamma_\varphi$ along the vector field (6.21) will necessary satisfy the condition (6.20), which makes it possible to write down the following implicit expression for the sought solution $u \in C^2(\mathbb{R}^n;\mathbb{R})$:

$$u(x) = u_0(y(x|u(x))), \quad (6.30)$$

where $\bar{y} := y(x|u) \in \Gamma_\varphi$ satisfies the following determining relationship

$$\tilde{p}_0(\bar{y}) + \partial\tilde{\mathcal{P}}(t(x),x;\bar{y}|u)/\partial y = 0, \quad (6.31)$$

stemming from the condition (6.28). Thereby, we can formulate the obtained result as the following theorem.

Theorem 1. *The implicit expression* (6.30) *gives rise to a functional-analytic solution to the boundary problem* (4.15) *and* (4.18), *depending on the given boundary data* $u_0 \in C^2(\Gamma_\varphi;\mathbb{R})$.

Based on the derivation of the result above, we can deduce a corollary that the statement of the above theorem holds, in general, for any nontrivial smooth Hamiltonian function $H \in C^2(\mathbb{R}^{2n+1};\mathbb{R})$, for which the following two conditions

$$rank\,\Omega = 2n, \quad rank\left(\frac{\partial}{\partial p}\left[\vartheta_{11}\left(\frac{\partial \bar{H}}{\partial x}+\psi\frac{\partial \bar{H}}{\partial u}\right)+\vartheta_{12}\frac{\partial \bar{H}}{\partial p}\right]\right)=n \qquad (6.32)$$

are satisfied almost everywhere on $T^*(\mathbb{R}^n)\times\mathbb{R}$. The conditions (6.32) should hold simultaneously with that of (6.20), bringing about the implicit solution (6.29) to the Hamilton-Jacobi equation (4.15) under the boundary condition (4.16).

It is easy to see now that expression (6.30) is quivalent to some fixed point problem $P(u)=u$, $u\in C^2(\mathbb{R}^n;\mathbb{R})$, for an associated nonlinear mapping $P: C^2(\mathbb{R}^n;\mathbb{R}) \to C^2(\mathbb{R}^n;\mathbb{R})$, where, by definition,

$$P(u)(x) := u_0(y(x|u(x))) \qquad (6.33)$$

for all reachable points $x\in\mathbb{R}^n$. This observation can be formulated as the following important theorem.

Theorem 2. *A solution to the functional-analytic fixed point problem* (6.33) *solves simultaneously the boundary problem* (4.18) *to our generalized Hamilton-Jacobi equation* (4.15).

7. THE STRUCTURE OF HOPF-LAX TYPE FUNCTIONAL-ANALYTIC SOLUTIONS TO GENERALIZED HAMILTON-JACOBI EQUATIONS

Consider the following generalized nonlinear Hamilton-Jacobi equation

$$\partial u/\partial t + H(x,t;u,u_x) = 0 \qquad (7.1)$$

with a Hamiltonian function $H\in C^2(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1};\mathbb{R})$ and pose the Cauchy problem

$$u|_{t=0} = u_0, \qquad (7.2)$$

where $u_0\in C^1(\mathbb{R}^n;\mathbb{R})$ and $t\in\mathbb{R}$ is an evolution parameter. For investigating functional-analytic solutions to Hamiton-Jacobi equation (7.1) we will apply the generalized characteristic method, proposed above. Namely, consider the following non-canonical Hamiltonian vector field on the cotangent space $T^*(\mathbb{R}^{n+1}) \ni (x,t;p,\sigma)$, generated by a nondegenerate Hamiltonian function $H \in C^2(\mathbb{R}^{2n+2};\mathbb{R})$, where the function $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$ is *a priori* assumed to solve the equation (7.1) under condition (7.2), that is

$$\begin{gathered}\begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dp}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 & \mu^{(1|1)} \\ -\mu^{(1|1),*} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}}{\partial x} + \psi \frac{\partial \bar{H}}{\partial u} \\ \frac{\partial \bar{H}}{\partial p} \end{pmatrix}, \\ \frac{d\sigma}{d\tau} = -\left(\frac{\partial \bar{H}}{\partial t} + \sigma \frac{\partial \bar{H}}{\partial u} \right), \quad \frac{dt}{d\tau} = 1, \end{gathered} \tag{7.3}$$

where the tensor field $\mu^{(1|1)} := (\partial\psi/\partial p)^{*,-1} \in C^1(\mathbb{R}^{2n+2};\mathbb{R}^n \otimes \mathbb{R}^n)$ is chosen with respect to a suitable parametrization $\pi := \psi(x,t;u,p)$, $(x,t;u,p) \in S_H$, of the charactersitic surface

$$S_H := \{(x,t;u,p,\sigma) \in \mathbb{R}^{2n+2} : \sigma + \bar{H}(x,t;u,p) = 0\},$$

$$\bar{H}(x,t;u,p) := H(x,t;u,\pi)|_{\pi=\psi(x,t;u,p)},$$

compatible with the Cartan condition

$$du/d\tau =< \psi, \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p} > - \bar{H}(x,t;u,p). \tag{7.4}$$

Since the flow (7.3) is Hamiltonian, it can be represented [22, 28] dually in the related Lagrangian variatonal form:

$$\frac{\delta}{\delta x} \int_{\tau=0}^{\tau=t} \mathcal{L}(x,t;\dot{x}|u)d\tau \Bigg|_{\substack{x(0)=x_0\in\mathbb{R}^n \\ x(t)=x\in\mathbb{R}^n}} = 0 \tag{7.5}$$

for any $t \in \mathbb{R}$ and fixed points $x(0) = x_0 \in \mathbb{R}^n$ and $x(t) = x \in \mathbb{R}^n$. Here, as before, we denoted by $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, and by

$$\mathcal{L}(x,t;\dot{x}|u) := < \psi, \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p} > - \bar{H}(x,t;u,p) \Bigg|_{p:=\alpha(x,t;\dot{x}|u)} \tag{7.6}$$

the corresponding quasi-Lagrangian function, and put, by definition, $p := \alpha(x,t;\dot{x}|u)$, and $\dot{p} = dp/d\tau$, solving implicitly the system of equations

$$\dot{x} - \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p} = 0, \quad \dot{p} + \mu^{(1|1),*} \left(\frac{\partial \bar{H}}{\partial x} + \psi \frac{\partial \bar{H}}{\partial u} \right) = 0 \tag{7.7}$$

under the inverse Cauchy data

$$x|_{\tau=t} = x \in \mathbb{R}^n, \; x|_{\tau=0} = x_0(x,t) \in \mathbb{R}^n, \;\; p|_{\tau=0} = p_0(x,t) \in \mathbb{R}^n \qquad (7.8)$$

for any fixed point $(x,t) \in \mathbb{R}^{n+1}$. Note also, that the first equation of (7.7) is always uniquely solvable with respect to the variable $p \in \mathbb{R}^n$, owing to the nondegeneracy condition

$$rank(\frac{\partial}{\partial p}[< \psi, \mu^{(1|1)} \frac{\partial \bar{H}}{\partial p} >]) = n, \qquad (7.9)$$

assusmed before. Based now on the Lagrangian variational form (7.5), one can construct the following functional-analytical Hopf-Lax type representation for the solution of Hamilton-Jacobi equation under condition (7.14):

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \{u_0(y) + \mathcal{P}(x,t;y|u)\}, \qquad (7.10)$$

where, by definition, the „kernel“ function

$$\mathcal{P}(x,t;y|u) := \int_{\tau=0}^{\tau=t} \mathcal{L}(x,t;\dot{x}|u)d\tau \qquad (7.11)$$

is calculated on solutions to the Hamiltonian equations (7.7) under conditions (7.8). In the case, when $\partial H/\partial u \neq 0$ identically on S_H, we need to make the next step to skirt this problem, as the expression (7.10) becomes senseless, depending on the unknown solution $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$. Assume now that the parametrization $\pi := \psi(x,t;u,p) \in \mathbb{R}^n$, $(x,t;u,p) \in S_H$, of the charactersitic surface S_H is taken in such a way that the condition (7.4) is transformed into identical zero:

$$du/d\tau = < \psi, \mu^{(1|1)} \frac{\partial \tilde{H}}{\partial p} > - \bar{H}(x,t;u,p)\Big|_{S_H} = 0. \qquad (7.12)$$

Since the infimum (7.10) is then reachable at some point $x_0 = \bar{y} := y(x,t|u) \in \mathbb{R}^n$ for taken arbitrary but fixed point $(x,t|u) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and constant value $u = u_0(\bar{y}) \in \mathbb{R}$, we can write down two important relationships:

$$\partial u_0(\bar{y})/\partial y := \psi(x(\tau),\tau;u(\tau),p(\tau))|_{\tau=0} = \psi(\bar{y},0,u_0(\bar{y}),p_0(\bar{y}|u)), \quad (7.13)$$

where the initial vector $p_0(x,t) := p_0(\bar{y}|u) \in \mathbb{R}^n$ depends on the chosen constant value $u = u_0(\bar{y}) \in \mathbb{R}$, and

$$u(x,t) = u_0(y(x,t|u(x,t)), \qquad (7.14)$$

holding for all $(x,t) \in \mathbb{R}^n$. Thereby, having solved the equation(7.13) with respect to the critical point $\bar{y} := y(x,t|u) \in \mathbb{R}^n$, one can write down right away the solution to our Hamilton-Jacobi equation (7.1) with Cauchy data (7.2) for all $(x,t) \in \mathbb{R}^{n+1}$ in the implicit functional-analytic form (7.14). The obtained expression (7.14) is, evidently, equivalent to a fixed point problem $P(u) = u$, $u \in C^2(\mathbb{R}^{n+1};\mathbb{R})$, for an associated nonlinear mapping $P : C^2(\mathbb{R}^n \times \mathbb{R};\mathbb{R}) \to C^2(\mathbb{R}^n \times \mathbb{R};\mathbb{R})$, where, by definition,

$$P(u)(x,t) := u_0(y(x,t|u(x,t))) \tag{7.15}$$

for all reachable points $(x,t) \in \mathbb{R}^{n+1}$. The result, obtained above, one can formulate as the following theorem.

Theorem 3. *A solution to the functional-analytic fixed point problem* (7.15) *solves simultaneously the Cauchy problem* (7.1)*) to our generalized Hamilton-Jacobi equation* (7.2).

The fixed point problem (7.15), in general, is solved [29] under some weak enough conditions on the operator (7.15), but its solution is not, as is well known [12, 17, 29], often unique, thereby one needs more of its additional properties to be studied. We hope to investigate in detail such and related problems elsewhere.

Consider a canonical Hamilton-Jacobi equation

$$u_t + ||u_x||^2/2 = 0, \tag{7.16}$$

where $||\cdot||$ is the standard norm in the Euclidean space $\mathbb{E}^n := (\mathbb{R}^n, <\cdot,\cdot>)$ and try to construct its exact functional-analytic [12, 26, 27] generalized solutions $u : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{R}$, satisfying the Cauchy condition

$$u|_{t=+0} = u_0 \tag{7.17}$$

for a given function $u_0 : \mathbb{E}^n \to \mathbb{R}$. One can easily enough to state, making use of the characteristic method [12,24,26,30], that equation (7.16) possesses for smooth Cauchy data $u_0 \in C^1(\mathbb{E}^n;\mathbb{R})$ an exact functional-analytic generalized solution in the form

$$u(x,t) = u_0(y) + ||x-y||^2/(2t), \tag{7.18}$$

where a vector $y := y(x,t) \in \mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ satisfies the following determining equation

$$\partial u_0(y)/\partial y - (x-y)/t = 0. \tag{7.19}$$

It was proved in [11, 12, 26] that in a more general case of convex and below semicontinuos Cauchy data $u_0 \in BSC_{(c)}(\mathbb{R}^n;\mathbb{R})$ the expression (7.19) allows the completely equivalent to(7.18) so called Hopf-Lax type representation

$$u(x,t) = \inf_{y\in\mathbb{E}^n}\{u_0(y) + ||x-y||^2/(2t)\}, \tag{7.20}$$

being a generalized [12] solution to the Hamilton-Jacobi equation (7.16). The solution (7.18) satisfies the following natural asymptotic viscosity property: $\lim_{t\to\infty} u(x,t) = \inf_{y\in\mathbb{E}^n}\{u_0(y)\}$ for almost all $x \in \mathbb{E}^n$. The Cauchy problem (7.16), (7.17) for functions $u_0 \in BSC(\mathbb{E}^n;\mathbb{R})\cap C^1(\mathbb{E}^n;\mathbb{R})$ possesses a unique functional-analytic representation for its generalized solutions and satisfying the standard viscosity property. Below we consider a generalized geometric Monge characteristic method of solving noncanonical Hamilton-Jacobi type equations in the general form $u_t+H(x,t;u,u_x) = 0$ with Cauchy data (7.17), and give two examples, where $H := H_1 =< u_x, u_x > /2$ and $H := H_2 = (< u_x, u_x > +u^2)/2$, an evolution Riccati type equation.

8. A GENERALIZED MONGE CHARACTERISTIC METHOD: SHORT BACKGROUNDS

A noncanonical Hamilton-Jacobi equation

$$u_t + H(x,t;u,u_x) = 0$$

within the geometric Monge approach [4, 24, 25, 27, 30] can be considered as a characteristic surface $S_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2$ in the following form:

$$S_H := \{(x,t;u,p,\sigma) \in \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2 : \sigma + \bar{H}(x,t;u,p) = 0, \quad \bar{H}(x,t;u,p) := H(x,t;u,\pi)|_{\pi=\psi(x;u,p)}\}, \tag{8.1}$$

where a related Monge cones parametrization is taken as $\pi = \psi(x;u,p) \in \mathbb{E}^n$, $(x;u,p) \in S_H$, for some nondegenerate mapping $\psi \in C^1(\mathbb{R}^{2n+1};\mathbb{E}^n)$, that is $det(\partial\psi/\partial p) \neq 0$. We denoted here $u_t := \sigma \in \mathbb{R}$, $u_x := \pi \in \mathbb{E}^n$ for $(x,t) \in \mathbb{E}^n\times\mathbb{R}_+$ and $< \cdot,\cdot >$ is the standard scalar product in the Euclidean vector space $\mathbb{E}^n := (\mathbb{E}^n, < \cdot,\cdot >)$. As an example of equation (7.16), we will put below

$$H_1(x,t;u,\pi) :=< \pi,\pi > /2, \quad H_2(x,t;u,\pi) := (< \pi,\pi > +u^2)/2. \tag{8.2}$$

Now one can construct [24, 25, 27] a solution surface $\bar{S}_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{R}$, which is compatible with the characteristic surface $S_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2$, and satisfying the following Cartan's compatibility relationships:

$$du = \sigma dt + < \psi, dx >, \quad < d\sigma, \wedge dt > + < d\psi, \wedge dx > = 0, \tag{8.3}$$

holding upon $\bar{S}_H$ along any solution $u : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{R}$.

Consider now a related characteristic vector field on the surface S_H in the form

$$\begin{aligned} \tfrac{dx}{d\tau} &= \mu^{(1|1)} \partial \bar{H}/\partial p, \\ \tfrac{dp}{d\tau} &= -\mu^{(1|1),*}(\partial \bar{H}/\partial x + \psi \partial \bar{H}/\partial u - \bar{H} \partial \psi/\partial u), \\ \tfrac{du}{d\tau} &= < p, \mu^{(1|1)} \partial \bar{H}/\partial p > - \bar{H}, \quad d\sigma/d\tau = \sigma \partial \bar{H}/\partial u, \quad dt/d\tau = 1, \end{aligned} \tag{8.4}$$

under the following mixed Cauchy data:

$$\begin{aligned} &x|_{\tau=0} = y := y(x,t), \quad x|_{\tau=t} = x, \\ &u|_{\tau=0} = u_0(y), \quad p|_{\tau=0} = p_0(y), \quad \psi(x_0; u_0, p_0) := \partial u_0(x_0)/\partial x, \end{aligned} \tag{8.5}$$

where, by definition, the tensor $\mu^{(1|1)} := (\partial \psi/\partial p)^{*,-1}$ and the condition $u|_{\tau=t} = u(x,t)$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ is assumed, owing to the relationships (8.3), to be satisfied. The equations (8.4) fully ensure [10, 24, 25, 27, 30] the invariance of the characteristic S_H and fulfillment of the related Cartan's compatibility conditions (8.3). The problem (8.4) and (8.5) is, actually, an inverse one subject to the corresponding initial data at $\tau = 0 \in \mathbb{R}_+$, if the corresponding data at $\tau = t \in \mathbb{R}_+$ are *a priori* given. The general solution to this inverse problem gives rise [24, 25] to the following exact functional-analytic expression:

$$u(x,t) = (u_0(y) + \mathcal{P}(x,t;y))|_{y=x_0(x,t)}, \tag{8.6}$$

where a vector $y := x_0(x,t) \in \mathbb{E}^n$, defined by (8.5), must belong to the set of points $U(x) \subset \mathbb{E}^n$, reachable at $\tau = t \in \mathbb{R}_+$ by the vector field (8.4) starting at $x \in \mathbb{E}^n$, and the kernel

$$\mathcal{P}(x,t;y) := \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u) d\tau, \tag{8.7}$$

$\dot{x} := dx/d\tau$, is defined by the Lagrangian function

$$\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u) := \Big(< \psi, \mu^{(1|1)} \partial \bar{H}/\partial p > - \bar{H}(x, \tau;, p) \Big)\Big|_{\dot{x} = \mu^{(1|1)} \partial \bar{H}/\partial p} \tag{8.8}$$

for all reachable points $(x(\tau), \tau) \in U(x) \times \mathbb{R}_+$. The functional analytic expression (8.6) for the Cauchy problem (7.16) and (7.17) gives rise [27] right away to its generalized solution in the Hopf-Lax type form, since the tensor field $\mu^{(1|1)} \in C^1(\mathbb{E}^n \times \mathbb{R}_+; \mathbb{E}^n \otimes \mathbb{E}^n)$ on the corresponding characteristic surface S_H is symplectic [3–5, 28]. This means, in particular, that the differential 2-form

$$\omega^{(2)} :=< d\psi, \wedge dx > |_{\Sigma_H} \tag{8.9}$$

is nondegenerate on the characteristic strip $\Sigma_H \subset S_H$. We can now easily obtain from (8.4), (8.5) and (8.6) that the Hamilton-Jacobi equation (7.16) with Cauchy data $u_0 \in C^1(\mathbb{E}^n; \mathbb{R})$ possesses solution (7.18) for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, defined by the vector $y = x_0(x,t) \in \mathbb{E}^n$, which solves the determining equation (7.19).

9. EXAMPLES

Example 1. *The canonical Hamiltonian function*

$$H_1(x,t;u,\pi) := \frac{1}{2} < \pi, \pi > .$$

Actually, the Lagrangian function (8.8) corresponding to this case, owing to the (8.2), equals the expression

$$\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u) = ||\dot{x}||^2/2, \tag{9.1}$$

where, owing to equations (8.4) and conditions (8.5), the following relationships

$$x(\tau) - y = \psi(x_0; u_0, p_0)\tau, \ \ \psi(x_0; u_0, p_0) := \partial u_0(y)/\partial y, \tag{9.2}$$

hold for all $\tau \in \mathbb{R}_+$. Therefore, the expressions (8.6) and (8.7) give rise to such an exact solution to the equation (7.16):

$$u(x,t) = u_0(y) + \frac{1}{2t}||x-y||^2 \Big|_{y=x_0(x,t)}, \tag{9.3}$$

where a vector $y := x_0(x,t) \in \mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ satisfies the determining equation (7.19), easily following from (9.2), that is

$$\partial u_0(y)/\partial y - (x-y)/t = 0. \tag{9.4}$$

The result obtained one can interestedly interpret making use of the Lagrangian variational principle: the system (8.4) of characteristic Hamiltonian vector fields is completely equivalent to the variational equation

$$\delta\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)/\delta x = 0,$$

solving the extremum problem

$$\tilde{u}(x,t) = \inf_{\substack{x\in C^2([0,t];\mathbb{R}) \\ \{x(0)=y\in\mathbb{E}^n, x(t)=x\in\mathbb{E}^n\}}} (u_0(y) + \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)d\tau) \tag{9.5}$$

in the space of smooth functions $x \in C^2([0,t];\mathbb{R})$, $t \in \mathbb{R}_+$.

A very important fact concerning the constructed function (9.5) consists in that it satisfies [8] the Hamilton-Jacobi equation for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ exactly the same as (7.16), that is

$$\partial\tilde{u}/\partial t + ||\tilde{u}_x||^2/2 = 0 \tag{9.6}$$

under the evident initial condition

$$\tilde{u}|_{t=0} = u_0. \tag{9.7}$$

Thereby, we can identify the obtained function (8.4) with our solution to the Hamilton-Jacobi equation (7.16) with Cauchy data (7.17), that is $u = \tilde{u}$. Since the infimum problem (9.5) is equivalent to that

$$u(x,t) = \inf_{y\in U(x)} \{u_0(y) + \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)d\tau\} \tag{9.8}$$

in the space of solutions to the equations (8.4) with Cauchy data (8.5), we obtain right away its well known [12, 26] the Hopf-Lax type representation

$$u(x,t) = \inf_{y\in\mathbb{R}^n} \left\{u_0(y) + ||x-y||^2/(2t)\right\}, \tag{9.9}$$

where we took into account that the neighborhood $U(x) \subseteq \mathbb{R}^n$ and the kernel (8.7) equals

$$\mathcal{P}(x,t;y) = ||x-y||^2/(2t) \tag{9.10}$$

for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. If the Cauchy data $u_0 \in BSC_{(c)}(\mathbb{E}^n;\mathbb{R})\cap C^1(\mathbb{E}^n;\mathbb{R})$, a vector $y = y(x,t) \in \mathbb{E}^n$, solving the problem (9.9), satisfies evidently the equation (9.4), thereby confirming the result, obtained previously.

Example 2. *The evolutionary Hamilton-Jacobi equation of the Riccati type: the Hamiltonian function* $H := H_2 = \frac{1}{2}(<\pi,\pi> + u^2)$.

The corresponding characteristic vector fields equations on the surface S_H, parametrized as $\pi := \psi(x;u,p) \in \mathbb{E}^n$, $(x;u,p) \in S_H$, are given as

$$\begin{aligned} &dx/d\tau = \psi(x;u,p), \\ &(\partial\psi/\partial p)dp/d\tau = -u\psi - (\partial\psi/\partial x)\psi - \partial\psi/\partial u\tfrac{1}{2}(<\psi,\psi> - u^2), \\ &du/d\tau = \tfrac{1}{2}(<\psi,\psi> - u^2), \end{aligned} \tag{9.11}$$

where parameter $\tau \in \mathbb{R}$. It is easy to see that a suitable pramaetrization of S_H can be given by the trivial mapping, for which $\psi(x;u,p) := p$ for all $(x;u,p) \in S_H$. Then, the system of equations (9.11) transforms into

$$dx/d\tau = p, \quad dp/d\tau = -up, \quad du/d\tau = \tfrac{1}{2}(<p,p> - u^2). \tag{9.12}$$

Solve now the following inverse Cauchy problem related with system (9.12):

$$\begin{aligned} x|_{\tau=0} &= x_0 := y(x,t), \qquad x|_{\tau=t} = x, \\ p|_{\tau=0} &= p_0(y) = \partial u_0(y)/\partial y \end{aligned} \tag{9.13}$$

for any fixed $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. The inverse problem (9.12) and (9.13) can be easily enough solved, giving rise to the following expression for the solution

$$u(x,t) = \frac{s_0(y)(\frac{t}{4}s_0(y) + c_\pm(y))}{2[1 + (\frac{t}{4}s_0(y) + c_\pm(y))^2]}, \tag{9.14}$$

where

$$\begin{aligned} &y = x + p_0(y)\operatorname{arctg}(\tfrac{t}{4}s_0(y) + c_\pm(y)), \\ &s_0(y) := (||p_0(y)||^2 + u_0^2(y))/||p_0(y)||, \quad p_0(y) = \partial u_0(y)/\partial y, \\ &c_\pm(y) := \tfrac{1}{4u_0(y)}[\, s_0(y) \pm \sqrt{s_0^2(y) - 16u_0^2(y)}\,]. \end{aligned} \tag{9.15}$$

The unknown function $y : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{E}^n$ should be found from the exact functional-analytic expression (9.15), being equivalent, evidently, to the following four finite dimensional fixed point problems:

$$\mathcal{P}_\pm(y) = y \tag{9.16}$$

in $\mathbb{E}^n$ at any fixed parameters $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$, where the smooth mappings $\mathcal{P}_\pm : \mathbb{E}^n \to \mathbb{E}^n$ are defined as

$$\mathcal{P}_\pm(y) := x + p_0(y)\operatorname{arctg}\left(\frac{t}{4}s_0(y) + c_\pm(y)\right) \tag{9.17}$$

for any $y \in \mathbb{E}^n$. As $s_0^2(y) - 16u_0^2(y) \geq 0$ for any $y \in \mathbb{E}^n$, the functions $c_\pm : \mathbb{E}^n \to \mathbb{R}$ are positive definite and the fixed point problem (9.16) is well posed in $\mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$.

Denote now by $Ricc(\mathbb{E}^n;\mathbb{R}_+)$ the functional subspace of bounded mappings $u_0 \in C^2(\mathbb{E}^n;\mathbb{R}_+)$, for which the fixed point problem (9.16) is solvable owing to the Brouwer-Banach type theorems [12] for all parameters $(x,t) \in$

$\mathbb{E}^n \times \mathbb{R}_+$. Then, taking into account the exact expression (9.14) and considerations above, we can formulate the following theorem.

Theorem 4. *Exact functional-analytic solutions to the canonical Riccati type equation*

$$\partial u/\partial t + (||u_x||^2 + u^2)/2 = 0 \tag{9.18}$$

under the Cauchy condition $u_0 \in Ricc(\mathbb{E}^n;\mathbb{R}_+)$ *are given by expression* (9.14), *where the function* $y : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{E}^n$ *is a fixed point of the mapping* (9.17). *If the fixed point problem* (9.16) *possesses a unique solution under the condition* $u_0 \in Ricc(\mathbb{E}^n;\mathbb{R}_+)$, *the corresponding solution to the Riccati type equation* (9.18) *is given by the following two different branches*

$$u_\pm(x,t) = \frac{s_0((y_\pm))(c_\pm)(y_\pm) + \frac{t}{4}s_0((y_\pm)))}{2[1 + (c_\pm((y_\pm)) + \frac{t}{4}s_0((y_\pm)))^2]}, \tag{9.19}$$

which owing to expressions (9.15) *are regular for all* $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. *Moreover, the viscosity limiting properties*

$$\lim_{t\to\infty} u_\pm(x,t) = 0, \qquad \lim_{t\to\infty} y_\pm(x,t) = \bar{y} \tag{9.20}$$

hold, where the vector $\bar{y} \in \mathbb{E}^n$ *satisfies the stationary equation*

$$2(x - \bar{y}) = \pi p_0(\bar{y}) \tag{9.21}$$

for all $x \in \mathbb{E}^n$.

Owing to Theorem 4 the obtained fixed point of the mapping (9.16) solves, obviously, our Cauchy problem (7.17) for any given function $u_0 \in Ricc(\mathbb{E}^n;\mathbb{R}_+)$, finishing our calculations. In particular, we showed that solutions to the canonical Hamilton-Jacobi equation (7.16) with Cauchy data (7.17) can be constructed effectively in the functional-analytic form, using the generalized characteristics method. Within those exact functional-analytic solutions can exist, in particular, such ones, whose asymptotic properties possess nontrivial asymptotic viscosity behavior, being important for applications.

10. CONCLUSION

The results of the previous [25, 26] and this work convince us firmly that the geometric Monge-Cartan approach to studying solution of a wide class of nonlinear partial differential equations of Hamilton-Jacobi type, based on our generalized characteristic method and complemented with the modern symplectic theory, is very perspective for many possible applications. The constructed Hopf-Lax type functional-analytic representation of the corresponding solutions to both boundary and Cauchy problems makes it possible to find many new, and in some sense, generalized solutions for a wide class of boundary and Cauchy data. Another still weakly investigated but important aspect of this approach is related with its applications to analyzing the suitable multi-dimensional symplectic reductions of boundary and Cauchy problems, giving rise [7, 9, 20, 22] to new types of associated Hamiltonian nonlinear dynamical systems on functional manifolds of smaller spatial dimension. We plan to discuss this topic in another place.

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ГЕОМЕТРИЧНИЙ ПІДХІД КАРТАНА-МОНЖА ДО МЕТОДУ ХАРАКТЕРИСТИК ДЛЯ РІВНЯНЬ ТИПУ ГАМІЛЬТОНА-ЯКОБІ ТА ЙОГО УЗАГАЛЬНЕННЯ ДЛЯ НЕЛІНІЙНИХ РІВНЯНЬ ІЗ ЧАСТИННИМИ ПОХІДНИМИ ВИЩИХ ПОРЯДКІВ

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Дано аналіз геометричного підходу Картана-Монжа до методу характеристик для рівнянь Гамільтона-Якобі та нелінійних рівнянь із частинними похідними вищих порядків. Вивчено гамільтонову структуру характеристичних векторних полів, асоційованих з нелінійними рівняннями із частинними похідними першого порядку, побудовано тензорні поля спеціальної структури для визначення характеристичних векторних полів, природнім чином асоційованих з нелінійними рівняннями із частинними похідними вищих порядків. Розвинуто узагальнений метод характеристик у рамках симплектичної теорії на основі геометричних картин Монжа та Картана. На підставі відповідних геометричних властивостей вивчено функціонально-аналітичнні розв'язки типу Хопфа-Лакса для широкого класу крайових задач та задач Коші для нелінійних рівнянь із частинними похідними Гамільтона-Якобі. У випадку неканонічних рівнянь Гамільтона-Якобі встановлено співвідношення між їх розв'язками та добре визначеною функціонально-аналітичною проблемою про нерухому точку, асоційованою з розв'язками типу Хопфа-Лакса для спеціально сконструйованих дуальних рівнянь Гамільтона-Якобі. На основі класичної теорії про нерухому точку типу Брауера-Банаха отримано функціонально-аналітичні розв'язки для рівняння Гамільтона-Якобі типу Ріккаті та досліджено їх властивості.