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ADDITIONAL REDUCTIONS IN THE K-CONSTRAINED MODIFIED KP HIERARCHY

ДОДАТКОВІ РЕДУКЦІЇ В K-РЕДУКОВАНІЙ МОДИФІКОВАНІЙ КП-ІЄРАРХІЇ

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Additional reductions in the modified k-constrained KP hierarchy are proposed. As a result we obtain generalizations of Kaup – Broer system, Korteweg – de Vries equation and a modification of Korteweg – de Vries equation that belongs to modified k-constrained KP hierarchy. We also propose solution generating technique based on binary Darboux transformations for the obtained equations.

Запропоновано додаткові редукції в k-редукованій модифікованій КП-ієрархії. Як наслідок, отримано узагальнення системи Каупа – Броера, рівняння Кортевега – де Фріза та модифікованого рівняння Кортевега – де Фріза, які належать до модифікованої k-редукованої ієрархії. Також запропоновано метод побудови розв'язків для отриманих рівнянь, який базується на бінарних перетвореннях Дарбу.

1. Introduction. The algebraic constructions of the well-known Kyoto group [1], which are called the Sato theory, play an important role in the contemporary theory of nonlinear integrable systems of mathematical and theoretical physics. The leading place in these investigations is occupied by the theory of equations of Kadomtsev – Petviashvili type (KP hierarchy) and their generalizations and applications [1 – 3].

One of known generalizations of the KP hierarchy arises as a result of k-symmetry constraints (so-called k-cKP hierarchy) that were investigated in [4 – 8]. k-cKP hierarchy are closely connected with so-called KP equation with self-consistent sources (KPSCS) [9 – 12]. Multicomponent k-constraints of the KP hierarchy were introduced in [13] and investigated in [14 – 18]. This extension of k-cKP hierarchy contains vector (multicomponent) generalizations of physically relevant systems like the nonlinear Schrödinger equation, the Yajima – Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system.

The modified k-constrained KP (k-cmKP) hierarchy was proposed in [19, 20]. It contains, for example, the vector Chen – Lee – Liu, the modified Korteweg – de Vries (mKdV) equation and their multicomponent extensions. The k-cmKP hierarchy and dressing methods for it via integral transformations were investigated in [21 – 23].

In [24, 25] (2+1)-dimensional extensions of the k-cKP hierarchy ((2+1)-dimensional k-cKP hierarchy) were introduced and dressing methods via differential transformations were investigated. Some systems of this hierarchy were investigated via binary Darboux transformations in [22, 23]. This hierarchy was also rediscovered recently in [26, 27]. Matrix generalizations of $(2+1)$-dimensional k-cKP hierarchy were considered in [28, 29].

In this paper our aim is to consider additional reductions of the k-cmKP hierarchy that lead to new generalizations of well-known integrable systems. We also investigated dressing methods for the obtained systems via integral transformations that arise from Binary Darboux Transformations (BDT).

This work is organized as follows. In Section 2 we present a short survey of results on constraints for the KP hierarchies including the k-cmKP hierarchy. In Section 3 we investigate Lax representations obtained as a result of additional reductions in the k-cmKP hierarchy and corresponding nonlinear systems. Section 4 presents results on dressing methods for Lax pairs obtained in Section 3. In the final section, we discuss the obtained results and mention problems for further investigations.

2. Symmetry constraints of the KP hierarchy. Let us recall some basic objects and notations concerning KP hierarchy, modified KP hierarchy, their multicomponent k-constraints and their $(2+1)$-extensions. A Lax representation of the KP hierarchy is given by

$$L_{t_n} = [B_n, L], \quad n \geq 1, \tag{1}$$

where $L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots$ is a scalar pseudodifferential operator, $t_1 := x$, $D := \dfrac{\partial}{\partial x}$, and $B_n := (L^n)_+ := (L^n)_{\geq 0} = D^n + \sum_{i=0}^{n-2} u_i D^i$ is the differential operator part of L^n. The consistency condition (zero-curvature equations), arising from the commutativity of flows (1), is

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{2}$$

Let B_n^τ denote the formal transpose of B_n, i. e., $B_n^\tau := (-1)^n D^n + \sum_{i=0}^{n-2} (-1)^i D^i u_i^\top$, where $^\top$ denotes the matrix transpose. We will use curly brackets to denote the action of an operator on a function whereas, for example, $B_n q$ means the composition of the operator B_n and the operator of multiplication by the function q. The following formula holds for $B_n q$ and $B_n\{q\} :=$ $:= (B_n q)_{=0} = B_n q - (B_n q)_{>0}$. In the case $k = 2$, $n = 3$ formula (2) presents a Lax pair for the Kadomtsev – Petviashvili equation. Its Lax pair was obtained in [33] (see also [34]).

The multicomponent k-constraints of the KP hierarchy is given by [13]

$$L_{t_n} = [B_n, L], \tag{3}$$

with the k-symmetry reduction

$$L_k := L^k = B_k + \sum_{i=1}^{m} \sum_{j=1}^{m} q_i m_{ij} D^{-1} r_j = B_k + \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \tag{4}$$

where $\mathbf{q} = (q_1, \ldots, q_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ are vector functions, $\mathcal{M}_0 = (m_{ij})_{i,j=1}^m$ is a constant $(m \times m)$-matrix. In the scalar case $(m = 1)$ we obtain a k-constrained KP hierarchy [4 – 8]. The hierarchy given by (3), (4) admits the Lax representation (here $k \in \mathbf{N}$ is fixed):

$$[L_k, M_n] = 0, \quad L_k = B_k + \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \quad M_n = \partial_{t_n} - B_n. \tag{5}$$

Lax equation (5) is equivalent to the following system:

$$[L_k, M_n]_{\geq 0} = 0, \quad M_n\{\mathbf{q}\} = 0, \quad M_n^\tau\{\mathbf{r}\} = 0. \tag{6}$$

Below we will also use the formal adjoint $B_n^* := \bar{B}_n^\tau = (-1)^n D^n + \sum_{i=0}^{n-2}(-1)^i D^i u_i^*$ of B_n, where * denotes the Hermitian conjugation (complex conjugation and transpose).

For $k = 1$, the hierarchy given by (6) is a multicomponent generalization of the AKNS hierarchy. For $k = 2$ and $k = 3$, one obtains vector generalizations of the Yajima – Oikawa and Melnikov [9 – 11] hierarchies, respectively. An essential extension of the k-cKP hierarchy is its $(2+1)$-dimensional generalization introduced in [24, 25] and rediscovered in [26, 27].

In [19, 20], a k-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. Dressing methods for k-cmKP hierarchy under additional D-Hermitian reductions were also investigated in [21, 22]. At first we recall the definition of the modified KP hierarchy.

A Lax representation of this hierarchy is given by

$$L_{t_n} = [B_n, L], \quad n \geq 1, \tag{7}$$

where $L = D + U_0 + U_1 D^{-1} + U_2 D^{-2} + \ldots$ and $B_n := (L^n)_{>0} := D^n + \sum_{i=1}^{n-1} u_i D^i$ is the purely differential operator part of L^n. The consistency condition arising from the commutativity of flows (7) is

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0.$$

The multicomponent k-constraints of the modified KP hierarchy are given by the operator equation

$$L_{t_n} = [B_n, L], \tag{8}$$

with the k-symmetry reduction

$$L_k := L^k = B_k - \sum_{i=1}^{m}\sum_{j=1}^{m} q_i m_{ij} D^{-1} r_j D = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \tag{9}$$

where $\mathbf{q} = (q_1, \ldots, q_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ are vector functions, $\mathcal{M}_0 = (m_{ij})_{i,j=1}^m$ is a constant $(m \times m)$-matrix. The hierarchy (8), (9) admits the Lax representation (here $k \in \mathbf{N}$ is fixed)

$$\begin{gathered} [L_k, M_n] = 0, \quad L_k = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \\ M_n = \alpha_n \partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i. \end{gathered} \tag{10}$$

We can rewrite the Lax pair (10) in the following way:

$$\begin{gathered} [L_k, M_n] = 0, \quad L_k = B_k - \mathbf{q}\mathcal{M}_0 \mathbf{r}^\top + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_x^\top, \\ M_n = \alpha_n \partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i. \end{gathered} \tag{11}$$

From Lax representation for k-cKP hierarchy (5), (6) and representation (11) we come to conclusion that the equation $[L_k, M_n] = 0$ in (10) is equivalent to the following system:

$$[L_k, M_n]_{>0} = 0, \quad M_n\{\mathbf{q}\} = 0, \quad (M_n^\tau)\{\mathbf{r}_x\} = 0$$

($[L_k, M_n]_{=0} = 0$ since $[L_k, M_n]\{1\} = 0$). We can rewrite the last equation in the following form: $(D^{-1}M_n^\tau D)\{\mathbf{r}\} = 0$ to keep the order of differentiation equal to n. As a result we obtain

$$[L_k, M_n]_{>0} = 0, \quad M_n\{\mathbf{q}\} = 0, \quad (D^{-1}M_n^\tau D)\{\mathbf{r}\} = 0. \tag{12}$$

The hierarchy (10) contains vector generalizations of the Chen – Lee – Liu ($k = 1$), the modified multicomponent Yajima – Oikawa ($k = 2$) and Melnikov ($k = 3$) hierarchies. Consider some equations that can be obtained from (10) under certain choice of k and n (see [23]).

(1) $k = 1, n = 2$. Then (10) becomes

$$L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \quad M_2 = \alpha_2\partial_{t_2} - D^2 + 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top D. \tag{13}$$

In this case equation (12) becomes the following system:

$$\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} + 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top\mathbf{q}_x = 0, \quad \alpha_2\mathbf{r}^\top_{t_2} + \mathbf{r}^\top_{xx} + 2\mathbf{r}^\top_x\mathbf{q}\mathcal{M}_0\mathbf{r}^\top = 0. \tag{14}$$

Under the additional Hermitian conjugation reduction $\alpha_2 = i$, $\mathcal{M}_0 = -\mathcal{M}_0^*$, $\mathbf{r}^\top = \mathbf{q}^*$ ($L_1^* = -D^{-1}L_1D$, $M_2^* = D^{-1}M_2D$) in (14), we obtain the Chen – Lee – Liu equation

$$i\mathbf{q}_{t_2} - \mathbf{q}_{xx} + 2\mathbf{q}\mathcal{M}_0\mathbf{q}^*\mathbf{q}_x = 0. \tag{15}$$

(2) $k = 1, n = 3$. In this case (10) takes the form

$$L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D,$$
$$M_3 = \alpha_3\partial_{t_3} - D^3 + 3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top D^2 + 3[\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top - (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2]D, \tag{16}$$

and equations (12) read

$$\alpha_3\mathbf{q}_{t_3} = \mathbf{q}_{xxx} - 3(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx} - 3(\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top - (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2)\mathbf{q}_x,$$
$$\alpha_3\mathbf{r}^\top_{t_3} = \mathbf{r}^\top_{xxx} + 3\mathbf{r}^\top_{xx}(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top) + 3\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top_x + (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2). \tag{17}$$

After reduction of Hermitian conjugation: $\alpha_3 = 1$, $\mathbf{r}^\top = \mathbf{q}^*$, $\mathcal{M}_0 = -\mathcal{M}_0^*$ ($L_1^* = -D^{-1}L_1D$, $M_3^* = -D^{-1}M_3D$), (17) becomes:

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} - 3(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_{xx} - 3(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^* - (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)^2)\mathbf{q}_x. \tag{18}$$

(3) $k = 2, n = 2$. After additional reduction in (10): $\alpha_2 = i$, $u_1 := iu$, $u = u(x, t_2) \in \mathbb{R}$, $\mathcal{M}_0 = \mathcal{M}_0^*$, the Lax pair in (12) reads

$$[L_2, M_2] = 0, \quad L_2 = D^2 + iuD - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^* D, \quad M_2 = i\partial_{t_2} - D^2 - iuD, \tag{19}$$

and equation (12) becomes the modified Yajima – Oikawa equation

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + iu\mathbf{q}_x, \quad u_{t_2} = 2(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x.$$

In the next section we will introduce additional reductions in Chen – Lee – Liu hierarchy. As a result we will obtain generalizations of the Kaup – Broer system, KdV equation, modified KdV equation and their scalar coupled versions.

3. Additional reductions in the modified k-constrained KP hierarchy. For further convenience let us make a change in formulae (10),

$$\mathbf{q} \to \tilde{\mathbf{q}}, \quad \mathbf{r} \to \tilde{\mathbf{r}}, \quad \mathcal{M}_0 \to \tilde{\mathcal{M}_0}. \tag{20}$$

After the change (20) the hierarchy (10) reads

$$\begin{gathered}[L_k, M_n] = 0, \quad L_k = B_k - \tilde{\mathbf{q}}\tilde{\mathcal{M}_0}D^{-1}\tilde{\mathbf{r}}^\top D, \\ M_n = \alpha_n\partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i.\end{gathered} \tag{21}$$

Let us make the additional reduction in (21),

$$\begin{gathered}\tilde{\mathbf{q}} := (q_1, \ldots, q_m, -v - \beta D^{-1}\{u\}, 1) = (\mathbf{q}, -v - \beta D^{-1}\{u\}, 1), \\ \tilde{\mathcal{M}_0} = \begin{pmatrix} \mathcal{M}_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\mathbf{r}} := (r_1, \ldots, r_m, 1, \beta D^{-1}\{u\}) = (\mathbf{r}, 1, \beta D^{-1}\{u\}),\end{gathered} \tag{22}$$

where $\mathcal{M}_0$ is an $(m \times m)$-constant matrix, $\mathbf{q}$ and $\mathbf{r}$ are m-component vectors, u and v are scalar functions, $\beta \in \mathbb{R}$, $D^{-1}\{u\}$ denotes indefinite integral of the function u with respect to x. After reduction (22) k-cmKP hierarchy (21) takes the form

$$\begin{gathered}[L_k, M_n] = 0, \quad L_k = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + v + \beta D^{-1}u, \\ M_n = \alpha_n\partial_{t_n} - B_n, \quad B_n = D^n + \sum_{i=1}^{n-1} u_i D^i.\end{gathered} \tag{23}$$

In the following subsections we will investigate hierarchy (23) in case $k = 1$.

3.1. Reductions of the Chen – Lee – Liu system. Let us put $k = 1$, $n = 2$. Then Lax pair (23) becomes

$$\begin{gathered}[L_1, M_2] = 0, \quad L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + \beta D^{-1}u + v, \\ M_2 = \alpha_2\partial_{t_2} - D^2 + 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)D.\end{gathered} \tag{24}$$

A system that corresponds to equation (24) has the form

$$\alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\mathbf{q}_x, \quad \alpha_2 \mathbf{r}^\top_{t_2} = -\mathbf{r}^\top_{xx} - 2\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v),$$
$$\alpha_2 u_{t_2} + u_{xx} + 2\left(u(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\right)_x = 0, \tag{25}$$
$$-\alpha_2 v_{t_2} + 2\beta u_x + v_{xx} - 2\left(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v\right)v_x = 0.$$

Consider additional reductions of Lax pair (24) and system (25).

(1) Assume that $\mathcal{M}_0 = -\mathcal{M}_0^*$, $\mathbf{r}^\top = \mathbf{q}^*$, $v = -2i\mathrm{Im}\,(\beta D^{-1}\{u\})$ ($L_1^* = -DL_1D^{-1}$, $M_2^* = = DM_2D^{-1}$). Then equation (25) takes the form

$$\alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(2i\mathrm{Im}\,(\beta D^{-1}\{u\}) + \mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_x,$$
$$\alpha_2 u_{t_2} + u_{xx} + 2\left(u(2i\mathrm{Im}\,(\beta D^{-1}\{u\}) + \mathbf{q}\mathcal{M}_0\mathbf{q}^*)\right)_x = 0.$$

(2) Let us put $\mathcal{M}_0 = 0$ in the operators L_1 and M_2, $L_1 = D + \beta D^{-1}u + v$, $M_2 = \alpha_2\partial_{t_2} - -D^2 - 2vD$. Then equation (25) becomes the Kaup – Broer system

$$\alpha_2 u_{t_2} + u_{xx} - 2(uv)_x = 0, \quad -\alpha_2 v_{t_2} + 2\beta u_x + v_{xx} + 2vv_x = 0. \tag{26}$$

In case $u = 0$ in (26) we obtain the Burgers equation $-\alpha_2 v_{t_2} + v_{xx} - vv_x = 0$.

(3) Consider the case $u = 0$ for the operators L_1 and M_2 (25), $L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + v$, $M_2 = \alpha_2\partial_{t_2} - D^2 + 2(v + \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D$. Then (25) reads

$$\alpha_2 \mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\mathbf{q}_x, \quad \alpha_2 \mathbf{r}^\top_{t_2} = -\mathbf{r}^\top_{xx} - 2\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v),$$
$$-\alpha_2 v_{t_2} + v_{xx} - \left(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v\right)v_x = 0.$$

3.2. Reductions of the modification of KdV system (18). Now let us consider the hierarchy (23) in case $k = 1$, $n = 3$. Then its Lax pair L_1, M_3 in (23) reads

$$[L_1, M_3] = 0, \quad L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + \beta D^{-1}u + v,$$
$$M_3 = \alpha_3\partial_{t_3} - D^3 - 3(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D^2 - 3\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + \beta u + v_x\right)D. \tag{27}$$

Commutator equation in (27) is equivalent to the system

$$-\alpha_3 v_{t_3} + v_{xxx} + 3vv_{xx} + 3v^2v_x + 3v_x^2 + 6\beta(uv)_x +$$
$$+ 3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top\right\}v_x - 3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top v_{xx} -$$
$$- 6\mathbf{q}\mathcal{M}_0\mathbf{r}^\top vv_x - 3\beta(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top u)_x - 3\beta\mathbf{q}\mathcal{M}_0\mathbf{r}^\top u_x = 0,$$

$$\begin{aligned}\alpha_3\mathbf{q}_{t_3} &= \mathbf{q}_{xxx}+3(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx}+3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top+v_x+\beta u\right\}\mathbf{q}_x,\\ \alpha_3\mathbf{r}^\top_{t_3} &= \mathbf{r}^\top_{xxx}-3\left(\mathbf{r}^\top_x\left(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top\right)\right)_x+3\mathbf{r}^\top_x\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top+v_x+\beta u\right\},\\ \alpha_3 u_{t_3} &= u_{xxx}-3\left(u(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\right)_{xx}+3\left(u\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top+v_x+\beta u\right)\right)_x.\end{aligned} \tag{28}$$

Consider additional reductions in Lax pair (27) and corresponding system (28).

(1) Assume that $v = -2i\mathrm{Im}\,(\beta D^{-1}\{u\})$, $\mathbf{q}^* = \mathbf{r}^\top$, $u \in \mathbb{R}$, $\mathcal{M}_0 = -\mathcal{M}_0^*$ ($L_1^* = -DL_1D^{-1}$, $M_3^* = -DM_3D^{-1}$). Then system (28) takes the form

$$\begin{aligned}\alpha_3\mathbf{q}_{t_3} &= \mathbf{q}_{xxx}-3(2i\mathrm{Im}\,(\beta u)+\mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_{xx}+\\ &+3\left((\mathbf{q}\mathcal{M}_0\mathbf{q}^*+2i\mathrm{Im}\,(\beta u))^2-\mathbf{q}_x\mathcal{M}_0\mathbf{q}^*+\beta u-2i\mathrm{Im}\,(\beta u)\right)\mathbf{q}_x,\\ \alpha_3 u_{t_3} &= u_{xxx}+3\left\{u\left(2i\mathrm{Im}\,(\beta u)+\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\right\}_{xx}+\\ &+3\left(u\left\{(\mathbf{q}\mathcal{M}_0\mathbf{q}^*+2i\mathrm{Im}\,(\beta u))^2-\mathbf{q}_x\mathcal{M}_0\mathbf{q}^*+\beta u-2i\mathrm{Im}\,(\beta u)\right\}\right)_x.\end{aligned} \tag{29}$$

(a) Let us assume that in addition to the reductions described in item 1, the functions $\mathbf{q}$ and u with the matrix $\mathcal{M}_0$ are real-valued (i.e., the matrix $\mathcal{M}_0$ is skew-symmetric, $\mathcal{M}_0^\top = -\mathcal{M}_0$) and $v = 0$. Then the scalar $\mathbf{q}\mathcal{M}_0\mathbf{q}^\top = 0$ since $\mathbf{q}\mathcal{M}_0\mathbf{q}^\top = -(\mathbf{q}\mathcal{M}_0\mathbf{q}^\top)^\top$, and equation (29) reads

$$\begin{aligned}\alpha_3\mathbf{q}_{t_3} &= \mathbf{q}_{xxx}-3\mathbf{q}_x\mathcal{M}_0\mathbf{q}^\top\mathbf{q}_x+3\beta u\mathbf{q}_x,\\ \alpha_3 u_{t_3} &= u_{xxx}-3(u\mathbf{q}_x\mathcal{M}_0\mathbf{q}^\top)_x+6\beta uu_x.\end{aligned} \tag{30}$$

(2) Let us put $\mathcal{M}_0 = 0$ in the operators L_1, M_3 (27),

$$L_1 = D+\beta D^{-1}u+v,\quad M_3 = \alpha_3\partial_{t_3}-D^3-3vD^2-3(v^2+v_x+\beta u)D.$$

Then equation (28) takes the form

$$\begin{aligned}-\alpha_3 v_{t_3}+v_{xxx}+3vv_{xx}+3v^2v_x+3v_x^2+6\beta(uv)_x &= 0,\\ \alpha_3 u_{t_3} &= u_{xxx}-3(uv)_{xx}+3(u(v^2+v_x+\beta u))_x.\end{aligned} \tag{31}$$

(a) Under the additional restrictions $v = -2i\mathrm{Im}\,(D^{-1}\{\beta u\})$ ($L_1^* = -DL_1D^{-1}$, $M_3^* = -DM_3\,D^{-1}$) in item 2 we obtain a complex generalization of the modified KdV equation,

$$\alpha_3 u_{t_3} = u_{xxx}+6i(u\mathrm{Im}\,(D^{-1}\{\beta u\}))_{xx}+3(u(-4Im(D^{-1}\{\beta u\})^2-2i\mathrm{Im}\,(\alpha u)+\beta u))_x. \tag{32}$$

In the real case ($\beta \in \mathbb{R}$, u is a real-valued function, $v = 0$) the operators L_1 and M_3 take the form $L_1 = D+\beta D^{-1}u$, $M_3 = \beta\partial_t - D^3 - 3\beta uD$, and we obtain the KdV equation in (32),

$$\alpha_3 u_{t_3} = u_{xxx}+6\beta uu_x. \tag{33}$$

(3) Let us put $u = 0$ in Lax pair (27), $L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + v$, $M_3 = \alpha_3\partial_{t_3} - D^3 - 3(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D^2 - 3\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top + v_x\right)D$. Equation (28) becomes

$$-\alpha_3 v_{t_3} + v_{xxx} + 3vv_{xx} + 3v^2 v_x + 3v_x^2 + 3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top\right\}v_x -$$

$$- 3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top v_{xx} - 6\mathbf{q}\mathcal{M}_0\mathbf{r}^\top v v_x = 0,$$

$$\alpha_3\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx} + 3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + v_x\right\}\mathbf{q}_x, \tag{34}$$

$$\alpha_3\mathbf{r}_{t_3}^\top = \mathbf{r}_{xxx}^\top - 3\left(\mathbf{r}_x^\top\left(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top\right)\right)_x + 3\mathbf{r}_x^\top\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + v_x\right\}.$$

4. Dressing methods for k-cmKP hierarchy. In this section our aim is to elaborate dressing methods for the k-cmKP hierarchy (10). At first we recall a main result from paper [35]. Let $(1 \times K)$-matrix functions φ and ψ be solutions of linear problems with (2+1)-dimensional generalization of the operator L_k (4) with a more general differential part B_k,

$$L_k\{\varphi\} = \varphi\Lambda, \quad L_k^\tau\{\psi\} = \psi\tilde{\Lambda}, \quad \Lambda, \tilde{\Lambda} \in Mat_{K\times K}(\mathbb{C}),$$
$$L_k = \beta_k\partial_{\tau_k} + B_k + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top, \quad B_k = \sum_{j=0}^{k} u_j D^j. \tag{35}$$

Introduce a binary Darboux transformation (BDT) in the following way:

$$W = I - \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1} D^{-1}\psi^\top := I - \varphi\Delta^{-1}D^{-1}\psi^\top, \tag{36}$$

where C is a $(K \times K)$-constant nondegenerate matrix. The inverse operator W^{-1} has the form

$$W^{-1} = I + \varphi D^{-1}\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top = I + \varphi D^{-1}\Delta^{-1}\psi^\top. \tag{37}$$

The following theorem is proved in [35].

Theorem 1 [35]. *The operator $\hat{L}_k := WL_kW^{-1}$ obtained from L_k in (35) via BDT (36) has the form*

$$\hat{L}_k := WL_kW^{-1} = \beta_k\partial_{\tau_k} + \hat{B}_k + \hat{\mathbf{q}}\mathcal{M}_0 D^{-1}\hat{\mathbf{r}}^\top + \Phi\mathcal{M}D^{-1}\Psi^\top,$$
$$\hat{B}_k = \sum_{j=0}^{k}\hat{u}_j D^j, \tag{38}$$

where

$$\mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C, \quad \Phi = \varphi\Delta^{-1}, \quad \Psi = \psi\Delta^{-1,\top},$$
$$\Delta = C + D^{-1}\{\psi^\top\varphi\}, \quad \hat{\mathbf{q}} = W\{\mathbf{q}\}, \quad \hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\}. \tag{39}$$

The coefficients $\hat{u}_k$ and $\hat{u}_{k-1}$ of the operator $\hat{L}_k$ remain the same, i.e., $\hat{u}_k = u_k$, $\hat{u}_{k-1} = u_{k-1}$. All other coefficients $\hat{u}_j$, $j = \overline{0, k-2}$, depend on the functions φ, ψ and u_i, $i = \overline{0, j}$. Exact forms of all coefficients $\hat{u}_j$ can be found in [35].

Using the previous theorem we obtain the following result for the $(2+1)$-generalization of operator L_k from the k-cmKP hierarchy (10):

Theorem 2. *Let $(1 \times K)$-vector functions φ and ψ satisfy linear problems:*

$$L_k\{\varphi\} = \varphi\Lambda, \quad L_k^{\tau}\{\psi\} = \psi\tilde{\Lambda}, \quad \Lambda, \tilde{\Lambda} \in \mathrm{Mat}_{K\times K}(\mathbb{C}),$$
$$L_k = \beta_k\partial_{\tau_k} + B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^{\top}D, \quad B_k = \sum_{i=1}^{k} u_i D^i. \tag{40}$$

Then the operator $\hat{L}_k$ transformed via the operator

$$W_m := w_0^{-1}W = w_0^{-1}\left(I - \varphi\Delta^{-1}D^{-1}\psi^{\top}\right) = I - \varphi\tilde{\Delta}^{-1}D^{-1}(D^{-1}\{\psi\})^{\top}D, \tag{41}$$

where

$$w_0 = I - \varphi\Delta^{-1}D^{-1}\{\psi^{\top}\}, \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^{\top}\}\varphi_x\}, \quad \Delta = C + D^{-1}\{\psi^{\top}\varphi\},$$

has the form

$$\tilde{L}_k := W_m L_k W_m^{-1} = \beta_k\partial_{\tau_k} + \tilde{B}_k - \tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^{\top}D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^{\top}D,$$
$$\tilde{B}_k = \sum_{j=1}^{k}\tilde{u}_j D^j, \quad \tilde{u}_k = u_k, \quad \tilde{u}_{k-1} = u_{k-1} + k u_k w_0^{-1} w_{0,x}, \ldots, \tag{42}$$

where

$$\mathcal{M} = C\Lambda - \tilde{\Lambda}^{\top}C, \quad \tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1},$$
$$\tilde{\Psi} = D^{-1}\{W_m^{\tau,-1}\{\psi\}\}C^{-1,\top} = D^{-1}\{\psi\}\Delta^{-1,\top}, \quad \tilde{\mathbf{q}} = W_m\{\mathbf{q}\}, \tag{43}$$
$$\tilde{\mathbf{r}} = D^{-1}W_m^{-1,\tau}D\{\mathbf{r}\}, \quad \tilde{\Delta} = C + D^{-1}\{D^{-1}\{\psi^{\top}\}\varphi_x\}.$$

Proof. Let us check that

$$w_0^{-1} = I - \varphi\tilde{\Delta}D^{-1}\{\psi^{\top}\}, \quad \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^{\top}\}\varphi_x\}.$$

In order to do that we have to verify the equality $w_0 w_0^{-1} = I$,

$$w_0 w_0^{-1} = I - \varphi\Delta^{-1}D^{-1}\{\psi^{\top}\} - \varphi\tilde{\Delta}^{-1}D^{-1}\{\psi^{\top}\} +$$
$$+ \varphi\tilde{\Delta}^{-1}\left(C + D^{-1}\{\psi^{\top}\varphi\} - C + D^{-1}\{D^{-1}\{\psi^{\top}\}\varphi_x\}\right)\varphi\Delta^{-1}D^{-1}\{\psi^{\top}\} = I.$$

Analogously it can be verified that $w_0^{-1}w_0 = I$. By Theorem 1 we obtain

$$W_m L_k W_m^{-1} = w_0^{-1} W \left(\beta_k \partial_{\tau_k} + B_k - \mathbf{q}\mathcal{M}_0 \mathbf{r}^\top + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_x^\top\right) W^{-1} w_0 =$$

$$= \beta_k \partial_{\tau_k} + (W_m L_k W_m^{-1})_{\geq 0} - w_0^{-1} W\{\mathbf{q}\}\mathcal{M}_0 D^{-1}\left(W^{-1,\tau}\{\mathbf{r}_x\}\right)^\top w_0 +$$

$$+ w_0^{-1}\Phi\mathcal{M}D^{-1}\Psi^\top w_0. \tag{44}$$

We shall point out that $\Psi^\top w_0 = \Delta^{-1}\psi^\top(I - \varphi\Delta^{-1}D^{-1}\{\psi^\top\}) = \left(\Delta^{-1}D^{-1}\{\psi^\top\}\right)_x = \tilde{\Psi}_x^\top$. We shall also observe that

$$(W^{-1,\tau}\{\mathbf{r}_x\})^\top w_0 = \left(\mathbf{r}_x^\top - D^{-1}\{\mathbf{r}_x^\top\varphi\}\Delta^{-1}\psi^\top\right)\left(I - \varphi\Delta^{-1}D^{-1}\{\psi^\top\}\right) =$$

$$= \left(\mathbf{r}^\top - D^{-1}\{\mathbf{r}_x^\top\varphi\}\Delta^{-1}D^{-1}\{\psi^\top\}\right)_x = (D^{-1}W_m^{-1,\tau}D\{\mathbf{r}\})_x^\top = \tilde{\mathbf{r}}_x^\top.$$

Thus (44) can be rewritten as

$$\tilde{L}_k = W_m L_k W_m^{-1} = w_0^{-1} W \left(\beta_k \partial_{\tau_k} + B_k - \mathbf{q}\mathcal{M}_0 \mathbf{r}^\top + \mathbf{q}\mathcal{M} D^{-1}\mathbf{r}_x^\top\right) W^{-1} w_0 =$$

$$= \beta_k \partial_{\tau_k} + (W_m L_k W_m^{-1})_{\geq 0} + \tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}_x^\top - \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}_x^\top =$$

$$= \beta_k \partial_{\tau_k} + (W_m L_k W_m^{-1})_{\geq 0} + \tilde{\mathbf{q}}\mathcal{M}_0\tilde{\mathbf{r}}^\top - \tilde{\Phi}\mathcal{M}\tilde{\Psi}^\top - \tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^\top D. \tag{45}$$

Using that $\tilde{L}_k\{1\} = \tilde{u}_0 = 0$ we obtain the form of $\tilde{B}_k$, i.e., $\tilde{B}_k := (W_m L_k W_m^{-1})_{\geq 0} + \tilde{\mathbf{q}}\mathcal{M}_0\tilde{\mathbf{r}}^\top - -\tilde{\Phi}\mathcal{M}\tilde{\Psi}^\top = \sum_{j=1}^{k}\tilde{u}_j D^j$.

Theorem 2 is proved.

Theorem 2 provides us with a dressing method for k-cmKP hierarchy (10), i.e., the following corollary directly follows from the previous theorem:

Corollary 1. *Assume that the operators L_k and M_n in (10) satisfy the Lax equation $[L_k, M_n] = 0$. Let functions φ and ψ satisfy the equations*

$$L_k\{\varphi\} = \varphi\Lambda, \quad L_k^\tau\{\psi\} = \psi\tilde{\Lambda}, \quad \Lambda, \tilde{\Lambda} \in \mathrm{Mat}_{(K\times K)}(\mathbb{C}), \quad M_n\{\varphi\} = 0, \quad M_n^\tau\{\psi\} = 0. \tag{46}$$

Then the transformed operators $\tilde{L}_k = W_m L_k W_m^{-1}$ (see (42) with $\beta_k = 0$) and

$$\tilde{M}_n = W_m M_n W_m^{-1} = \alpha_n \partial_{t_n} - D^n - \sum_{i=1}^{n-1}\tilde{u}_i D^i \tag{47}$$

via the transformation W_m (41) also satisfy the Lax equation $[\tilde{L}_k, \tilde{M}_n] = 0$.

Proof. It can be checked directly that

$$[\tilde{L}_k, \tilde{M}_n] = [W_m L_k W_m^{-1}, W_m M_n W_m^{-1}] = W_m[L_k, M_n]W_m^{-1} = 0.$$

The exact form of the operators $\tilde{L}_k$ and $\tilde{M}_n$ follows from Theorem 2.

The following corollary follows from Corollary 1 and Theorem 2:

Corollary 2. *Suppose that the functions φ and ψ satisfy equations (46) with operators L_k and M_n defined by (23). Then the transformed operators have the form*

$$\tilde{L}_k = B_k - \tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^\top D + \tilde{v} + \beta D^{-1}\tilde{u},$$
$$\tilde{M}_n = \alpha_n\partial_{t_n} - \tilde{B}_n, \quad \tilde{B}_n = D^n + \sum_{i=1}^{n-1}\tilde{u}_i D^i, \tag{48}$$

where

$$\mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C, \quad \tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1},$$
$$\tilde{\Psi} = D^{-1}\{W_m^{\tau,-1}\{\psi\}\}C^{-1,\top} = D^{-1}\{\psi\}\Delta^{-1,\top}, \quad \tilde{\mathbf{q}} = W_m\{\mathbf{q}\},$$
$$\tilde{\mathbf{r}} = W_m^{-1,\tau}\{\mathbf{r}\} \quad , \tilde{\Delta} = C + D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}, \quad \Delta = C + D^{-1}\{\psi^\top\varphi\}, \tag{49}$$
$$\tilde{u} = W_m^{-1,\tau}\{D^{-1}\{u\}\}, \quad \tilde{v} = W_m\{v\} + \beta D^{-1}W_m^{-1,\tau}\{u\} - \beta W_m\{D^{-1}\{u\}\}.$$

As it was shown in previous sections, the most interesting systems arise from the k-cmKP hierarchy (10) and its reduction (23) after a Hermitian conjugation reduction. Our aim is to show that under additional restrictions, the BDT W_m (41) preserves this reduction.

Proposition 1. *(1) Let $\psi = \bar{\varphi}_x$ and $C = -C^*$ in the dressing operator W_m (41). Then the operator W_m is D-unitary ($W_m^{-1} = D^{-1}W_m^* D$).*

(2) Let the operator L_k (10) be D-Hermitian, $L_k^* = DL_kD^{-1}$ (D-skew-Hermitian, $L_k^* = -DL_kD^{-1}$) and M_n (10) be D-Hermitian (D-skew-Hermitian). Then the operator $\hat{L}_k = W_mL_kW_m^{-1}$ (see (42)) transformed via the D-unitary operator W_m is D-Hermitian (D-skew-Hermitian) and $\hat{M}_n := W_mM_nW_m^{-1}$ (47) is D-Hermitian (D-skew-Hermitian).

(3) Assume that the conditions of items 1 and 2 hold. Let $\tilde{\Lambda} = \bar{\Lambda}$ in the case of D-Hermitian L_k ($\tilde{\Lambda} = -\bar{\Lambda}$ in D-skew-Hermitian case). We shall also assume that the function φ satisfies the corresponding equations in formulae (46). Then $\mathcal{M} = \mathcal{M}^*$ ($\mathcal{M} = -\mathcal{M}^*$) and $\tilde{\Psi} = \bar{\tilde{\Phi}}$ (see formulae (39)).

In Subsection 4.1 we will show how one can use the methods described in Theorem 2 and its corollaries in order to obtain solutions of KdV equation (33) and its generalization (30).

4.1. Solution generating technique for system (30) and KdV equation (33). We shall consider equation (30) in the case where the dimension of the vector $\mathbf{q}$ and the matrix $\mathcal{M}_0$ is even, i.e., $m = 2\tilde{m}$, $\tilde{m} \in \mathbb{N}$ (in this situation, the skew-symmetric matrix $\mathcal{M}_0$ can be nondegenerate). Assume that the skew-symmetric matrix $\mathcal{M}_0$ in (30) and the vector-function $\mathbf{q}$ has the form

$$\mathcal{M}_0 = \begin{pmatrix} 0_{\tilde{m}} & I_{\tilde{m}} \\ -I_{\tilde{m}} & 0_{\tilde{m}} \end{pmatrix}, \quad \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) = (q_{11}, q_{12}, \ldots, q_{1\tilde{m}}, q_{21}, q_{22}, \ldots, q_{2\tilde{m}}), \tag{50}$$

where $0_{\tilde{m}}$ is a $(\tilde{m} \times \tilde{m})$-dimensional matrix consisting of zeros, $I_{\tilde{m}}$ is an identity matrix with the dimension $\tilde{m} \times \tilde{m}$. Equation (30) with the notation $\tilde{u} := u$ can be rewritten in the following form:

$$\begin{aligned} \alpha_3 \mathbf{q}_{1,t_3} &= \mathbf{q}_{1,xxx} - 3(\mathbf{q}_{1,x}\mathbf{q}_2^\top - \mathbf{q}_{2,x}\mathbf{q}_1^\top)\mathbf{q}_{1,x} + 3\beta\tilde{u}\mathbf{q}_{1,x}, \\ \alpha_3 \mathbf{q}_{2,t_3} &= \mathbf{q}_{2,xxx} - 3(\mathbf{q}_{1,x}\mathbf{q}_2^\top - \mathbf{q}_{2,x}\mathbf{q}_1^\top)\mathbf{q}_{2,x} + 3\beta\tilde{u}\mathbf{q}_{2,x}, \\ \alpha_3 \tilde{u}_{t_3} &= \tilde{u}_{xxx} - 3(\tilde{u}(\mathbf{q}_{1,x}\mathbf{q}_2^\top - \mathbf{q}_{2,x}\mathbf{q}_1^\top))_x + 6\beta\tilde{u}\tilde{u}_x. \end{aligned} \tag{51}$$

In this subsection our aim is to consider the case $\tilde{m} = 1$ (although the corresponding solution generating technique can be generalized to the case of an arbitrary natural $\tilde{m}$). In this situation $\mathbf{q}_1 = q_1$ and $\mathbf{q}_2 = q_2$ are scalars. We shall suppose that $K = 2\tilde{K}$ is an even natural number. Assume that the function φ is a $(1 \times K)$-vector solution of the system

$$\begin{aligned} L_{10}\{\varphi\} &= \varphi_x + \beta D^{-1}\{u\varphi\} = \varphi\Lambda, \quad \Lambda \in Mat_{K\times K}(\mathbb{C}), \quad \beta \in \mathbb{R}, \\ M_{30}\{\varphi\} &= \alpha_3\varphi_{t_3} - \varphi_{xxx} - 3\beta u\varphi_x = 0, \end{aligned} \tag{52}$$

with a number $u \in \mathbb{R}$.

Using Theorem 2 and Proposition 1 we obtain that the dressed operators $\tilde{L}_{10}$ and $\tilde{M}_{30}$ via the operator W_m (41) with the skew-Hermitian matrix C and $\psi = \bar{\varphi}_x$ has the form

$$\begin{aligned} \tilde{L}_{10} &= W_m L_{10} W_m^{-1} = D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Phi}^* D + \beta D^{-1}\tilde{u} + \tilde{v}, \\ \tilde{M}_{30} &= W_m M_{30} W_m^{-1} = \alpha_3\partial_{t_3} - D^3 - (\tilde{v} + \tilde{\Phi}\mathcal{M}\tilde{\Phi}^*)D^2 - \\ &\quad - 3\left((\tilde{\Phi}\mathcal{M}\tilde{\Phi}^* + \tilde{v})^2 + \tilde{\Phi}_x\mathcal{M}\tilde{\Phi}^* + \tilde{v}_x + \beta\tilde{u}\right)D, \end{aligned} \tag{53}$$

where $\mathcal{M} = C\Lambda - \Lambda^* C^*$, $\tilde{\Phi} = \varphi\tilde{\Delta}^{-1}$, $\tilde{u} = uD\{\bar{\varphi}\bar{\tilde{\Delta}}^{-1}D^{-1}\{\varphi^\top u\}\}$, $\tilde{v} = \beta(\tilde{\Phi}D^{-1}\{\varphi^* u\} - D^{-1}\{u\varphi\}\tilde{\Phi}^*)$, $\tilde{\Delta} = -C + D^{-1}\{\varphi^*\varphi_x\}$. It has to be pointed out that the function $\tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1}$ satisfies the equation $\tilde{M}_{30}\{\tilde{\Phi}\} = 0$ because

$$\tilde{M}_{30}\{\tilde{\Phi}\} = W_m M_{30} W_m^{-1}\{W_m\{\varphi\}C^{-1}\} = 0.$$

Now we assume that the function φ and the matrices C and Λ are real. In this case, $\tilde{v} = \tilde{v}^\top = \beta(\tilde{\Phi}D^{-1}\{\varphi^\top u\} - D^{-1}\{u\varphi\}\tilde{\Phi}^\top)^\top = -\tilde{v} = 0$.

Let us put

$$\Lambda = \operatorname{diag}(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{\tilde{K}1}, \lambda_{\tilde{K}2}), \lambda_{ij} \in \mathbb{R},$$

$$C = \begin{pmatrix} C_{11} & C_{12} & \ldots & C_{1\tilde{K}} \\ C_{21} & C_{22} & \ldots & C_{2\tilde{K}} \\ \vdots & \vdots & \ldots & \vdots \\ C_{\tilde{K}1} & C_{\tilde{K}2} & \ldots & C_{\tilde{K}\tilde{K}} \end{pmatrix}, \tag{54}$$

where the elements C_{ij} are (2×2)-matrices of the form

$$C_{ij} = \begin{pmatrix} 0 & -\dfrac{1}{\lambda_{j2} + \lambda_{i1}} \\ \dfrac{1}{\lambda_{j1} + \lambda_{i2}} & 0 \end{pmatrix}. \tag{55}$$

Under such a choice of C (55) and Λ (54) we obtain that the $(2\tilde{K} \times 2\tilde{K})$-dimensional matrix $\mathcal{M} = C\Lambda - \Lambda^\top C^\top$ has the block form, $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1}^{\tilde{K}}$, where $\mathcal{M}_{ij} = \mathcal{M}_0$ (see formula (50) in case $\tilde{m} = 1$). Let us denote by $\mathbf{1}_{\tilde{K}} = (I_2, \ldots, I_2)$ a matrix that consists of $\tilde{K}$ (2×2)-dimensional identity matrices I_2. Then $\mathcal{M} = -\mathbf{1}_{\tilde{K}}^\top \mathcal{M}_0 \mathbf{1}_{\tilde{K}}$.

Let us put $u = \mathrm{const}$ and choose a solution of system (52) in the form

$$\varphi = \left(\varphi_{11}, \varphi_{12}\varphi_{21}, \varphi_{22}, \ldots, \varphi_{\tilde{K}1}, \varphi_{\tilde{K}2}\right), \quad \varphi_{ij} = \exp\left\{\left(\frac{1}{2}\lambda_{ij} + \gamma_{ij}\right)x + a_{ij}t\right\},$$

where

$$\gamma_{ij} = \sqrt{\frac{1}{4}\lambda_{ij}^2 - \beta u}, \quad a_{ij} = \left\{\left(\frac{1}{2}\lambda_{ij} + \gamma_{ij}\right)^3 + 3\beta u\left(\frac{1}{2}\lambda_{ij} + \gamma_{ij}\right)\right\}/\alpha_3.$$

The $(2\tilde{K} \times 2\tilde{K})$-matrix $\tilde{\Delta}$ then takes the block form

$$\tilde{\Delta} = -C + D^{-1}\{\varphi^\top \varphi_x\} = \left(\tilde{\Delta}_{ij}\right)_{i,j=1}^{\tilde{K}} =$$

$$= \begin{pmatrix} \frac{\alpha_{i1}}{\alpha_{i1}+\alpha_{j1}} e^{(\alpha_{i1}+\alpha_{j1})x+(a_{i1}+a_{j1})t} & \frac{\alpha_{i2}}{\alpha_{i2}+\alpha_{j1}} e^{(\alpha_{i2}+\alpha_{j1})x+(a_{i2}+a_{j1})t} + \frac{1}{\lambda_{j2}+\lambda_{i1}} \\ \frac{\alpha_{i1}}{\alpha_{i1}+\alpha_{j2}} e^{(\alpha_{i1}+\alpha_{j2})x+(a_{i1}+a_{j2})t} - \frac{1}{\lambda_{j1}+\lambda_{i2}} & \frac{\alpha_{i2}}{\alpha_{i2}+\alpha_{j2}} e^{(\alpha_{i2}+\alpha_{j2})x+(a_{i2}+a_{j2})t} \end{pmatrix}_{i,j=1}^{\tilde{K}}, \tag{56}$$

where $\alpha_{ij} = \dfrac{1}{2}\lambda_{ij} + \gamma_{ij}$. The functions $\mathbf{q} = (q_1, q_2) = \varphi\tilde{\Delta}^{-1}\mathbf{1}_{\tilde{K}}^\top$ and $\tilde{u} = u - D\left\{\varphi\tilde{\Delta}^{-1}D^{-1}\{\varphi^\top u\}\right\}$ will be solutions of system (51).

We shall point out that in case $\beta = 0$, $\tilde{K} = 1$, $\alpha_3 = 1$, we obtain the following solution of the real version of the mKdV-type equation (equation (51) with $\tilde{u} = 0$):

$$\mathbf{q} = (q_1, q_2), \quad q_1 = -\frac{2(\lambda_{11} + \lambda_{12})\varphi_{12}}{(\lambda_{11} - \lambda_{12})\varphi_{11}\varphi_{12} - 2}, \quad q_2 = \frac{2(\lambda_{11} + \lambda_{12})\varphi_{11}}{(\lambda_{11} - \lambda_{12})\varphi_{11}\varphi_{12} - 2},$$

$$\varphi_{1j} = e^{\lambda_{1j}x + \lambda_{1j}^3 t_3}, \quad \lambda_{1j} > 0, \quad j = \overline{1, 2}.$$

It is also possible to choose other types of matrices C and Λ in (54) and (55). In particular the following remark holds.

Remark 1. In case $\tilde{K} = 1$, the vector of the functions

$$\varphi = (\varphi_1, \varphi_2), \quad \varphi_1 = \cos\left(x\lambda_{12} + (3\lambda_{11}^2\lambda_{12} - \lambda_{12}^3)t + \frac{\pi}{4}\right) e^{x\lambda_{11} + (\lambda_{11}^3 - 3\lambda_{11}\lambda_{12}^2)t},$$

$$\varphi_2 = \sin\left(x\lambda_{12} + (3\lambda_{11}^2\lambda_{12} - \lambda_{12}^3)t + \frac{\pi}{4}\right) e^{x\lambda_{11}+(\lambda_{11}^3-3\lambda_{11}\lambda_{12}^2)t}$$

will be a solution of system (52) with $u = 0$ and $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ -\lambda_{12} & \lambda_{11} \end{pmatrix}$. The corresponding solution generating technique given by (54) – (56) in case $\tilde{K} = 1, C_{\tilde{K}} = C_1 = \begin{pmatrix} 0 & \frac{1}{2\lambda_{11}} \\ -\frac{1}{2\lambda_{11}} & 0 \end{pmatrix}$ gives us a solution of mKdV-type equation (51) with $\tilde{u} = 0$ that coincides with a solution obtained in [36].

Now we will consider solution generating technique for KdV (33). For this purpose we assume that the function φ, the matrices $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_{\tilde{K}})$ and $C = \operatorname{diag}(C_1 \ldots, C_{\tilde{K}})$ are real and have the form

$$\Lambda_j = \begin{pmatrix} 0 & \lambda_j \\ \lambda_j & 0 \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & -c_j \\ c_j & 0 \end{pmatrix}. \tag{57}$$

In this case we obtain that the matrix $\mathcal{M} = C\Lambda - \Lambda^\top C^\top$ consists of zeros in (53). Consider the following solution of system (52):

$$\varphi = \left(\varphi_{11}, \varphi_{12}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{\tilde{K}1}, \varphi_{\tilde{K}2}\right),$$

$$\varphi_{j1} = e^{\gamma_j x + a_j t} \cosh\left(\frac{\lambda_j}{2}x + b_j t\right), \quad \varphi_{j2} = e^{\gamma_j x + a_j t} \sinh\left(\frac{\lambda_j}{2}x + b_j t\right),$$

where

$$\gamma_j = \sqrt{\frac{1}{4}\lambda_j^2 - \beta u}, \quad a_j = \left(\gamma_j^3 + \frac{3}{4}\gamma_j\lambda_j^2 + 3\beta u\gamma_j\right)/\alpha_3, \quad b_j = \left(3\gamma_j^2\frac{\lambda_j}{2} + \frac{\lambda_j^3}{8} + \frac{3}{2}\beta u\lambda_j\right)/\alpha_3$$

and λ_j, α_3, β, $u \in \mathbb{R}$. Thus, $\tilde{v} = 0$ and we obtain a Lax pair for the KdV equation in (53), $\tilde{L}_{10} = D + \beta D^{-1}\tilde{u}$, $\tilde{M}_{30} = \alpha_3\partial_{t_3} - D^3 - 3\beta\tilde{u}D$.

The formula

$$\tilde{u} = u - D\left\{\varphi\tilde{\Delta}^{-1}D^{-1}\{\varphi^\top u\}\right\} := u + \hat{u} \tag{58}$$

gives us a finite density solution of equation (33). In particular, if $\tilde{K} = 1$ and $c_1 = \frac{1}{8}\frac{\lambda_1}{\gamma_1}$ we obtain the following solution:

$$\tilde{u} = u + \frac{2\gamma_1^2}{\beta\cosh^2(\gamma_1 x + a_1 t)}. \tag{59}$$

Now we shall substitute $\tilde{u}$ (58) in KdV equation (33),

$$\alpha_3\hat{u}_{t_3} = \hat{u}_{xxx} + 6\beta\hat{u}\hat{u}_x + 6\beta u\hat{u}_x. \tag{60}$$

The corresponding pair of operators has the form $L_1 = D + \beta D^{-1}(\hat{u} + u)$, $M_3 = \alpha_3\partial_{t_3} - D^3 - 3\beta\hat{u}D - 3\beta uD$. We have two ways to obtain soliton solutions (that are rapidly decreasing at both infinities in contrast to finite density solutions (58) that tend to an arbitrary real number u) for KdV from formula (58):

(1) By taking the limit $u \to 0$ in (58)–(60).

(2) By making a change of the independent variables: $\tilde{x} := x + 6\alpha_3^{-1}\beta ut_3$, $\tilde{t}_3 := t_3$ and $\hat{v}(\tilde{x}, \tilde{t}_3) := \hat{u}(x, t_3)$ in equation (60) and solutions (58), (59). This change corresponds to the change of differential operators in the Lax pair for equation (60) consisting of the operators L_1 and M_3, $\alpha_3\partial_{\tilde{t}_3} = \alpha_3\partial_{t_3} - 3\beta uD$.

5. Conclusions. In this paper we obtain new generalizations (23) of the modified k-cKP (k-cmKP) hierarchy (10). The obtained hierarchy also generalizes the BKP hierarchy [36–38] which is a special case of the k-cmKP hierarchy. Dressing methods elaborated via BDT-type operators (Section 4) give rise to exact solutions of the integrable systems that hierarchy (23) contains. In particular, soliton solutions for generalization of mKdV-type equation (51) and finite density solutions as well as regular soliton solutions were constructed for the KdV equation using the proposed dressing methods. These methods also allow to obtain rational and singular multisoliton solutions of the corresponding nonlinear systems under a special choice of spectral matrix Λ in the linear system (52). In order to minimize the size of this article we do not include those results here. We shall point out that the special case of equation (51) ($\tilde{u} = 0$) and its solutions were considered in [36]. Generalizations (23) of the k-cmKP hierarchy (10) together with different extensions of k-cKP hierarchy is a good basis for construction of other hierarchies of nonlinear integrable equations with corresponding dressing methods. In particular in our forthcoming papers we plan to introduce a (2+1)-BDk-cmKP hierarchy and investigate solution generating technique for the corresponding integrable systems. Consider as an example the Lax pair from the (1+1)-BDk-cKP hierarchy that was investigated in [31],

$$\begin{aligned} P_{1,1} &= D + c_1 M_2\{\mathbf{q}\}\mathcal{M}_0 D^{-1}\mathbf{r}^\top + c_1\mathbf{q}\mathcal{M}_0 D^{-1}(M_2^\tau\{\mathbf{r}\})^\top + c_0\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top = \\ &= D + c_1\Big(\alpha_2\mathbf{q}_{t_2}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \alpha_2\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_{t_2}^\top - \mathbf{q}_{xx}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \\ &\quad - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_{xx}^\top - u\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top u\Big) + c_0\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top, \\ M_2 &= \alpha_2\partial_{t_2} - D^2 - u \end{aligned} \tag{61}$$

with vector-functions $\mathbf{q}$ and $\mathbf{r}$ satisfying $c_1 M_2^2\{\mathbf{q}\} + c_0 M_2\{\mathbf{q}\} = 0$, $c_1(M_2^\tau)^2\{\mathbf{r}\} + c_0 M_2^\tau\{\mathbf{r}\} = 0$. It was shown in [31] that the Lax equation $[P_{1,1}, M_2] = 0$ in (61) is equivalent to the system

$$[P_{1,1}, M_2]_{\geq 0} = 0, \quad c_1 M_2^2\{\mathbf{q}\} + c_0 M_2\{\mathbf{q}\} = 0, \quad c_1(M_2^\tau)^2\{\mathbf{r}\} + c_0 M_2^\tau\{\mathbf{r}\} = 0 \tag{62}$$

that is equivalent to a generalization of the AKNS system. In case $c_0 = 1$, $c_1 = 0$ we obtain the AKNS system in (62),

$$\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} - u\mathbf{q} = 0, \quad -\alpha_2\mathbf{r}_{t_2} - \mathbf{r}_{xx} - u\mathbf{r} = 0, \quad u = 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top.$$

Assume that the scalar function f satisfies the equations $P_{1,1}\{f\} = f\lambda$, $M_2\{f\} = 0$. We shall introduce the notations $\tilde{M}_2 := f^{-1}M_2 f$, $\hat{M}_2 := D\tilde{M}_2 D^{-1}$, $\tilde{P}_{1,1} := f^{-1}P_{1,1}f$, $\tilde{\mathbf{q}} := f^{-1}\mathbf{q}$,

$\tilde{\mathbf{r}}^\top := D^{-1}\{\mathbf{r}^\top f\}$ and consider the following gauge transformations:

$$\tilde{M}_2 = f^{-1}M_2 f = \alpha_2\partial_{t_2} - D^2 - 2\tilde{u}D, \quad \tilde{u} = f^{-1}f_x,$$

$$\begin{aligned}\tilde{P}_{1,1,} &= f^{-1}P_{1,1}f = D + f^{-1}f_x + c_1 f^{-1}M_2\{\mathbf{q}\}\mathcal{M}_0 D^{-1}\mathbf{r}^\top f + \\ &+ c_1 f^{-1}\mathbf{q}\mathcal{M}_0 D^{-1}(M_2^\tau\{\mathbf{r}\})^\top f + c_0 f^{-1}\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top f = \\ &= D - c_1\tilde{M}_2\{\tilde{\mathbf{q}}\}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D - c_1\tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}(\hat{M}_2^\tau\{\tilde{\mathbf{r}}\})^\top D - c_0\tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D.\end{aligned}$$

The equation $[\tilde{M}_2, \tilde{P}_{1,1}] = 0$ is equivalent to the following system:

$$[\tilde{P}_{1,1}, \tilde{M}_2]_{>0} = 0, \quad c_1\tilde{M}_2^2\{\tilde{\mathbf{q}}\} + c_0\tilde{M}_2\{\tilde{\mathbf{q}}\} = 0, \quad c_1(\hat{M}_2^\tau)^2\{\tilde{\mathbf{r}}\} + c_0\hat{M}_2^\tau\{\tilde{\mathbf{r}}\} = 0$$

or in an equivalent form (after notation $\mathbf{q}_0 := \tilde{\mathbf{q}}$, $\mathbf{r}_0 := \tilde{\mathbf{r}}$),

$$\begin{aligned}&[\tilde{P}_{1,1}, \tilde{M}_2]_{>0} = 0, \quad \mathbf{q}_1 = \tilde{M}_2\{\mathbf{q}_0\}, \quad \mathbf{r}_1 = \hat{M}_2^\tau\{\mathbf{r}_0\}, \\ &c_1\tilde{M}_2\{\mathbf{q}_1\} + c_0\tilde{M}_2\{\mathbf{q}_0\} = 0, \quad c_1\hat{M}_2^\tau\{\mathbf{r}_1\} + c_0\hat{M}_2^\tau\{\mathbf{r}_0\} = 0.\end{aligned} \tag{63}$$

System (63) is a generalization of the Chen – Lee – Liu system (case $c_1 = 0, c_0 = 1$). In case of the additional reduction $\alpha_2 \in i\mathbb{R}$, $c_0 = 0$, $c_1 \in \mathbb{R}$, $\mathcal{M}_0^* = -\mathcal{M}_0$, $\mathbf{r} = \bar{\mathbf{q}}$, (63) reads as follows:

$$\alpha_2\mathbf{q}_{0,t_2} - \mathbf{q}_{0,xx} + 2c_1(\mathbf{q}_1\mathcal{M}_0\mathbf{q}_0^* + \mathbf{q}_0\mathcal{M}_0\mathbf{q}_1^*)\mathbf{q}_{0,x} - \mathbf{q}_1 = 0,$$

$$\alpha_2\mathbf{q}_{1,t_2} - \mathbf{q}_{1,xx} + 2c_1(\mathbf{q}_1\mathcal{M}_0\mathbf{q}_0^* + \mathbf{q}_0\mathcal{M}_0\mathbf{q}_1^*)\mathbf{q}_{1,x} = 0.$$

We shall also point out that the extension of the k-cmKP hierarchy (23) can also be generalized to the matrix case. It leads to matrix generalizations of integrable systems that hierarchy (23) contains (including Chen – Lee – Liu (15) and modified-type KdV equation (18)). In particular, the matrix generalization of the modified KdV-type equation (18) differs from the well-known matrix mKdV equation that was investigated by the inverse scattering method in [39].

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