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CONFIGURATIONS OF INVARIANT STRAIGHT LINES OF CUBIC DIFFERENTIAL SYSTEMS WITH DEGENERATE INFINITY

Some properties of cubic systems with invariant straight lines are determined and are classified all the systems with the infinite line filled up with singularities (i.e. with the degenerated infinity) and having exactly five and exactly six straight lines three of which are parallel. It is proved that there are 15 affine classes of such systems with different configurations of invariant straight lines. For every class was carried out the qualitative investigation in the Poincaré disc.

1. Introduction

We consider the real cubic differential system

$$\begin{cases} \frac{dx}{dt} = \sum\limits_{r=0}^{3} P_r\left(x,y\right) \equiv P\left(x,y\right), \\ \frac{dy}{dt} = \sum\limits_{r=0}^{3} Q_r\left(x,y\right) \equiv Q\left(x,y\right), \end{cases} \quad (1)$$

where P_r, Q_r are homogeneous polinomials of degree r, $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$ and $GCD\,(P,Q) = 1$.

The curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ (the function $f(x,y) = exp(\frac{g}{h})$, $g,h \in \mathbb{C}[x,y]$) is said to be an *invariant algebraic curve* (*invariant exponential function*) of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \leq 2$ such that the identity

$$\frac{\partial f}{\partial x}P(x,y) + \frac{\partial f}{\partial y}Q(x,y) \equiv f(x,y)K_f(x,y)$$

holds.

In [5] it is shown that if $f = \exp(\frac{g}{h})$ is an invariant exponential factor, then $h(x,y) = 0$ is an invariant algebraic curve for (1).

We say that an invariant algebraic curve $f(x,y) = 0$ has the *degree of invariance* equal to m, if m is the greatest positive integer such that f^{m-1} divides K_f [7]. If $f(x,y) = 0$ has the degree of invariance equal to $m \geq 2$, then $exp(1/f), ..., exp(1/f^{m-1})$ are exponential functions.

We say that the system (1) is *Darboux integrable* if there exists a non-constant function of the form $f = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, where f_j is an invariant algebraic curve or an invariant exponential function and $\lambda_j \in \mathbb{C}$, $j = \overline{1,s}$, such that either $f = const$ is a first integral or f is an integrating factor for (1).

We will be interested in invariant algebraic curves of degree one, that is invariant straight lines $\alpha x + \beta y + \gamma = 0$, $(\alpha, \beta) \neq (0,0)$.

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimation the number of invariant straight lines which can have a polynomial differential system was considered in [1]; the problem of coexistence of the invariant straight lines and limit cycles in [8, 9, 15]; the problem of coexistence of the invariant straight lines and singular points of a center type for cubic system in [7], [17]. An interesting relation between the number of invariant straight lines and the possible number of directions for them is established in [2].

A qualitative investigation of quadratic systems ($P_3 \equiv 0, Q_3 \equiv 0$) with degenerate infinity is given in [16]. For cubic differential systems with degenerate infinity the problems:of integrability, of the center and of his isochronicity were studied in [3], [11], [4].

In this paper a qualitative investigation of cubic systems with degenerate infinity and exactly five (exactly six) invariant straight lines (real or complex) of which three are parallel is given.

2. Properties of the cubic systems with straight lines

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By *configuration of straight lines* we understand the $\mathbb{R}^2$ plane with a certain number of straight lines.To each bidimensional differential system (with invariant straight lines) we can associate a configuration consisting of invariant straight lines of this system. It easy to show that reciprocal affirmation is not always true.

The problem arise to determine such properties for invariant straight lines which will allow to construct configurations of straight lines realizable for (1). The proof of these properties is not complicated and will be not given in this paper.

2.1. Singular points and invariant straight lines

2.1) *In the finite part of the phase plane the system* (1) *has at most nine singular points.*

2.2) *In the finite part of the phase plane on any straight line there are at most 3 singular points of the system* (1) .

2.3) *In the finite part of the phase plane the system* (1) *has no more than eight invariant straight lines* [1], [18].

2.4) *If system* (1) *has complex invariant straight lines then they occur in complex conjugated pairs* (l *and* $\bar{l}$).

2.5) *The intersection point* (x_0, y_0) *of two invariant straight lines* l_1 *and* l_2 *of the system* (1) *is a singular point for this system. Moreover, if* $l_1, l_2 \in \mathbb{R}[x, y]$ *or* $l_2 \equiv \bar{l_1}$, *then* $x_0, y_0 \in \mathbb{R}$.

2.6) *A complex straight line* l *can pass through at most one point with real coordinates.*

2.7) *If a straight line passes through two distinct real points or through two complex conjugated points, then this straight line is real.*

Unlike the complex straight lines, a real straight line $ax + by + c = 0$, $a, b, c \in \mathbb{R}$, $a^2 + b^2 \neq 0$, passes through an infinite number of real points and through an infinite number of points with at least one complex coordinate. Indeed, if $x_0, y_0 \in \mathbb{R}$ and $ax_0 + by_0 + c = 0$, then this straight line passes through complex points $(x_0 + \alpha b, y_0 - \alpha a)$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

A complex straight line passing through a real point will be called *a complex straight line with a real point*, and a complex straight line not passing through a real point - *a purely imaginary complex straight line.*

2.8) *A complex straight line with a real point intersect transversally the coordinate axis.*

2.9) *Through one and the same point of a purely imaginary straight line can pass at most one real straight line.*

2.2. The parallel invariant straight lines

2.10) *A complex invariant straight line of the system* (1) *is purely imaginary iff this straight line is parallel with his conjugate* ($l \parallel \bar{l}$).

2.11) *Any purely imaginary invariant straight line by a linear transformation can be brought to a straight line parallel to one of the axes of coordinate.*

Regarding two parallel invariant straight lines we have the following property:

2.12) *If* l_1 *and* l_2 *are two parallel invariant straight lines of the system* (1), *then only one of the following properties occurs:*

a) $l_1, l_2 \in \mathbb{R}[x, y]$,

b) l_1 *is real and* l_2 *is purely imaginary,*

c) l_1 *and* l_2 *are purely imaginary,*

d) l_1 *and* l_2 *are complex straight lines with a real point.*

2.13) *The system* (1) *can not have more than three straight lines parallel among themselves.*

2.14) *The system* (1) *can not have more than two triplets of parallel invariant straight lines.*

2.3. The cubic systems with degenerate infinity

The cubic system (1) has *degenerate infinity* if the following identity holds

$$yP_3(x, y) - xQ_3(x, y) \equiv 0. \tag{2}$$

If (2) holds, then infinity consists only of singular points.

2.15) *The identity* (2) *is invariant under affine transformation of the system* (1).

2.16) *The invariant straight lines of the cubic system* (1) *with degenerate infinity passing through one point* $M_0(x_0, y_0)$, $x_0, y_0 \in \mathbb{C}$ *have at most three slopes.*

2.17) *Through one and the same point of a complex invariant straight line of the cubic system with degenerate infinity can not pass more than one real straight line.*

2.18) *The straight line passing through three distinct singular points of system (1) with degenerate infinity is invariant for (1).*

Let $\psi(x,y)$ is the polynomial

$$\psi(x,y) = P \cdot (P \cdot \partial Q/\partial x + Q \cdot \partial Q/\partial y) - Q \cdot (P \cdot \partial P/\partial x + Q \cdot \partial P/\partial y) .$$

If $\alpha x + \beta y + \gamma = 0$ is an invariant straight line of the system (1), then $\alpha x+\beta y+\gamma$ divides $\psi(x,y)$ (see [6]); furthemore, the condition (2) limits the degree of the polynomial $\psi(x,y)$ to at most six, so it follows next two properties:

2.19) *The maximum number of the invariant straight lines for a differential cubic system with degenerate infinity is equal to six..*

2.20) *The system* (1) *with degenerate infinity has invariant straight lines along at most six different directions* [2].

2.21) *If the cubic differential system* (1) *with degenerate infinity has a triplet of parallel invariant straight lines l_1, l_2, l_3, then only one case of following two occurs:*

a) $l_1, l_2, l_3 \in \mathbb{R}[x,y]$,

b) $l_1 \in \mathbb{R}[x,y]$, $l_{2,3} \in \mathbb{C}[x,y] \setminus \mathbb{R}[x,y]$ *and* $l_3 = \overline{l_2}$.

2.22) *If $l_{1,2,3}$ are a triplet of parallel invariant straight lines of the cubic system with degenerate infinity* [(1), (2)], *then all singular points of* [(1), (2)] *lie on these straight lines.*

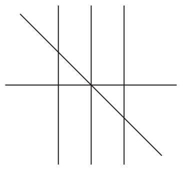
2.23) *Let l_j, $j = \overline{1,5}$ are five invariant straight lines of the cubic system with degenerate infinity, where $l_{1,2,3}$ compose a triplet of parallel straight lines, the straight lines $l_{4,5}$ are distinct and intersect transversally the triplet $l_{1,2,3}$. Then $l_4 \not\parallel l_5$ and $l_4, l_5 \in \mathbb{R}[x,y]$, or $l_{4,5}$ are complex with a real point with $l_5 = \overline{l_4}$.*

3. The cubic system with degenerate infinity and five (six) straight lines three of which are parallel

The purpose of this section is to give a classification of the cubic systems with degenerate infinity having invariant straight lines with total degree of invariance equal to five (six), where three of them are parallel. If the straight line l has the degree of invariance equal to m, then l will be counted as m parallel straight lines.

3.1. Configurations of straight lines

In this section, taking into account the properties **2.1)-2.23)**, we will construct all configurations with five (six) straight lines, three of which are parallel. The number near the straight line (see Conf.10-Conf.15) indicates how many times this line is counted. If a straight line is invariant, then this number is equal to degree of invariance of the line. All other straight lines are counted once and their invariance degree is considered equal to one. Thus, we get the following 15 configurations of the straight lines (real straight lines are represented by continuous lines; complex straight lines are represented by dashed lines):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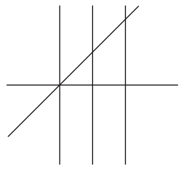
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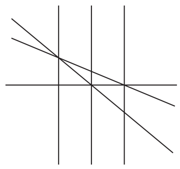
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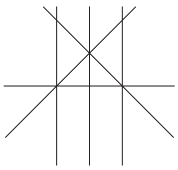
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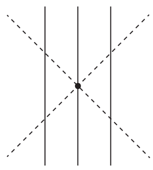
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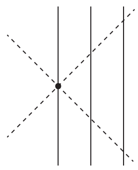
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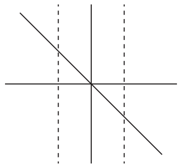
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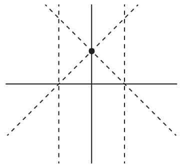
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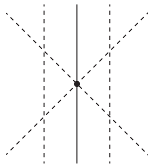
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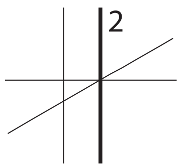

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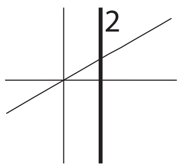

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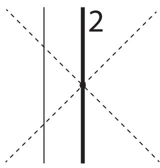

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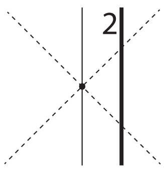

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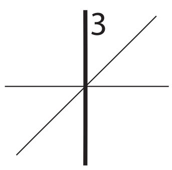

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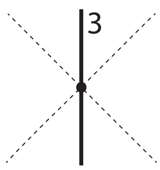

Conf.15

3.2. The classification of systems

We will use the following notations: $5r$ means five real invariant straight lines; $2c-$ two complex invariant straight lines; $5(2)r-$ four distinct real invariant straight lines of which one has the degree of invariance equal to two; and so on.

Using the above configurations of the straight lines we can state the following theorem.

Theorem. *Any cubic system with degenerate infinity having invariant straight lines with total degree of invariance 5 (6), three of which are parallel, via affine transformation and time rescaling can be written as one of the following* 15 *systems. In the square brackets is given the number and type of invariant straight lines; the existence of Darboux first integral ($\mathcal{F}$) or Darboux integrating factor (μ). In the figure associated to each system is presented the phase portrait in the Poincaré disc.*

3.1) $\begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,, \ a>0, \ c\neq 2, \\ \dot{y} = y\left(-a+c\,x-y+x^2\right), \ a+c>1 \\ [5r;\ \mathcal{F};\ \text{Fig. 3.1}]\,; \end{cases}$

3.2) $\begin{cases} \dot{x} = x\,(x+1)\,(x-a)\,, \ a>0,\ b>0, \\ \dot{y} = y\left(b+(b-a)\,x-y+x^2\right), \ b-a\neq 0 \\ [5r;\ \mathcal{F};\ \text{Fig. 3.2}]\,; \end{cases}$

3.3) $\begin{cases} \dot{x} = x(x+1)(x-a), \ a>0, \\ \dot{y} = y(-a+2x-y+x^2) \\ [6r;\ \mathcal{F};\ \text{Fig. 3.3}]\,; \end{cases}$

3.4) $\begin{cases} \dot{x} = x(x+1)(x-a), \ a>0, \\ \dot{y} = y(a-y+x^2) \\ [6r;\ \mathcal{F};\ \text{Fig. 3.4}]\,; \end{cases}$

3.5) $\begin{cases} \dot{x} = x(x+1)(x-a), \ a>0, \\ \dot{y} = y(x+1)(x-a)+x^2+y^2 \\ [3r+2c;\ \mu;\ \text{Fig. 3.5}]\,; \end{cases}$

3.6) $\begin{cases} \dot{x} = x(x+1)(x-a), \ a>0, \\ \dot{y} = (x+1)^2+xy(x-a)+by^2, \ b>0 \\ [3r+2c;\ \mu;\ \text{Fig. 3.6}]\,; \end{cases}$

3.7) $\begin{cases} \dot{x} = (x-a)(x^2+1), \ a\in\mathbb{R}, \\ \dot{y} = y(1-ac+cx-y+x^2), \ c\neq 0 \\ [3r+2c;\ \mu;\ \text{Fig. 3.7}]\,; \end{cases}$

3.8) $\begin{cases} \dot{x} = (x-a)(x^2+1), \ a\in\mathbb{R}, \\ \dot{y} = y(-1-2ax-y+x^2) \\ [2r+4c;\ \mu;\ \text{Fig. 3.8}]\,; \end{cases}$

3.9) $\begin{cases} \dot{x} = (x-a)(x^2+1), \ a\in\mathbb{R}, \\ \dot{y} = (x-a)^2+y+\frac{1}{b}y^2+x^2y, \ b>0 \\ [1r+4c;\ \mu;\ \text{Fig. 3.9}]\,; \end{cases}$

3.10) $\begin{cases} \dot{x} = x^2(x+1), \ a>0 \\ \dot{y} = y\left((a+1)x-y+x^2\right) \\ [5(2)r;\ \mathcal{F};\ \text{Fig. 3.10}]\,; \end{cases}$

3.11) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y(a+ax-y+x^2), \ a\neq 0 \\ [5(2)r;\ \mathcal{F};\ \text{Fig. 3.11}]\,; \end{cases}$

3.12) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = ax^2+xy+ay^2+x^2y, \ a\neq 0 \\ [3(2)r+2c;\ \mu;\ \text{Fig. 3.12}]\,; \end{cases}$

3.13) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = a(x+1)^2+ay^2+x^2y, \ a\neq 0 \\ [3(2)r+2c;\ \mu;\ \text{Fig. 3.13}]\,; \end{cases}$

3.14) $\begin{cases} \dot{x} = x^3, \ a>0 \\ \dot{y} = y(ax-y+x^2) \\ [5(3)r;\ \mathcal{F};\ \text{Fig. 3.14}]\,; \end{cases}$

3.15) $\begin{cases} \dot{x} = x^3, \ a>0 \\ \dot{y} = ax^2+ay^2+x^2y \\ [3(3)r+2c;\ \mu;\ \text{Fig. 3.15}]\,. \end{cases}$

3.3. Darboux integrability and invariant straight line of the systems 3.1)-3.15)

The systems 3.1)-3.15) have the following invariant straight lines and Darboux first integrals (integrating factors), respectively:

3.1): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_4=y$, $l_5=(a+c-1)x-y$; $(l_1/l_3)^{a+c-1}(l_4/l_5)^{a+1}=const$;

3.2): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_4=y$, $l_5=b(x+1)-y$; $l_2^b l_3^{-b} l_4^a l_5^{-a}=const$;

3.3): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_4=y$, $l_5=(a+1)x-y$, $l_6=x-y-a$; $l_1l_4/(l_3l_5)=const$;

3.4): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_4=y$, $l_5=a(x+1)-y$, $l_6=x+y-a$; $l_2l_4/(l_3l_5)=const$.

3.5): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_{4,5}=y\pm ix$; $\mu(x,y)=1/(l_1l_3l_4l_5)$;

3.6): $l_1=x+1$, $l_2=x$, $l_3=x-a$, $l_{4,5}=\sqrt{b}\,y\pm i(x+1)$; $\mu(x,y)=1/(l_2l_3l_4l_5)$;

3.7): $l_1=x-i$, $l_2=x-a$, $l_3=x+i$, $l_4=y$, $l_5=cx-y-ac$; $\mu(x,y)=1/(l_1l_3l_4l_5)$, $y\exp(-c\cdot arctan(x))/l_5=const$;

3.8): $l_1=x-i$, $l_2=x-a$, $l_3=x+i$, $l_4=y$, $l_{5,6}=(a\mp i)x+y+1\pm ai$; $\mu(x,y)=1/(l_1l_3l_5l_6)$;

3.9): $l_1=x-i$, $l_2=x-a$, $l_3=x+i$, $l_{4,5}=y\pm i\sqrt{b}\,(x-a)$; $\mu(x,y)=1/(l_1l_3l_4l_5)$;

3.10): $l_1=x+1$, $l_2\equiv l_3=x$, $l_4=y$, $l_5=ax-y$; $l_l^a l_2^{-a} l_4 l_5^{-1}=const$;

3.11): $l_1 = x+1$, $l_2 \equiv l_3 = x$, $l_4 = y$, $l_5 = a + ax - y$; $y \exp(a/x)/(a + ax - y) = const$;

3.12): $l_1 = x+1$, $l_2 \equiv l_3 = x$, $l_{4,5} = y \pm ix$; $\mu(x,y) = 1/(l_1 l_2 l_4 l_5)$;

3.13): $l_1 = x+1$, $l_2 \equiv l_3 = x$, $l_{4,5} = y \pm i(x+1)$; $\mu(x,y) = 1/(l_2^2 l_4 l_5)$;

3.14): $l_{1,2,3} = x, l_4 = y, l_5 = ax - y; y \exp(a/x)/(ax - y) = const$;

3.15): $l_{1,2,3} = x, l_{4,5} = y \pm ix; \mu(x,y) = 1/(l_1^2 l_4 l_5)$.

3.4. Phase portraits of the systems 3.1)-3.15)

The qualitative investigation of the systems 3.1)-3.15) is presented, respectively, in the following figures:

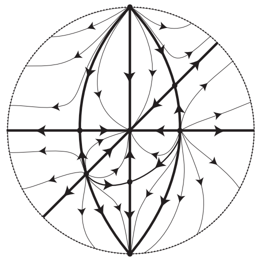

Fig. 3.1

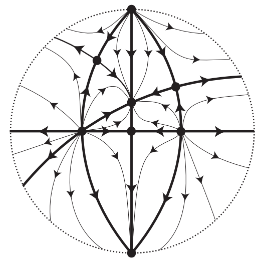

Fig. 3.2

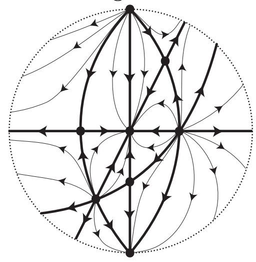

Fig. 3.3

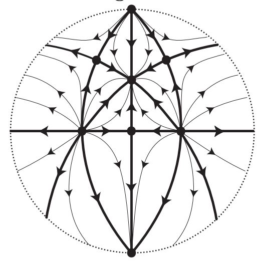

Fig. 3.4

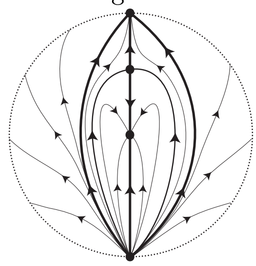

Fig. 3.5

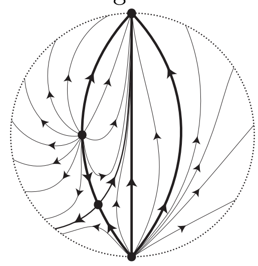

Fig. 3.6

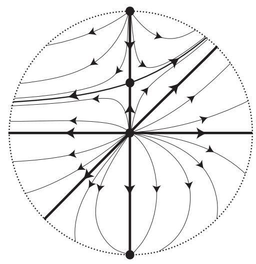

Fig. 3.7

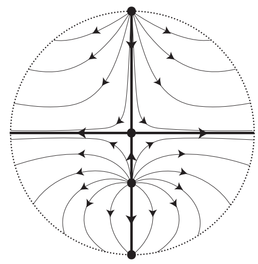

Fig. 3.8

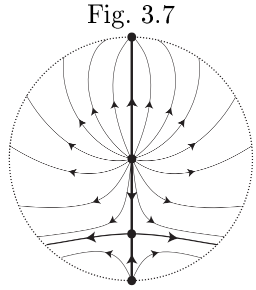

Fig. 3.9

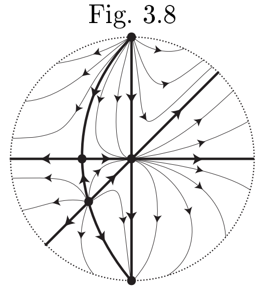

Fig. 3.10

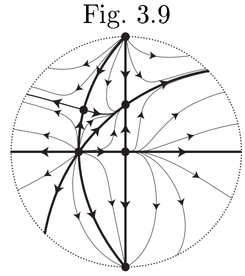

Fig. 3.11

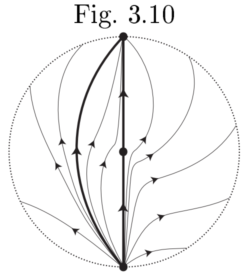

Fig. 3.12

We remark that Fig. 3.1 and Fig. 3.4 (Fig. 3.8 and Fig. 3.9) represent two topologically equivalent phase portraits with different numbers of invariant straight lines (one of the separatrices of the saddle singular point from Fig. 3.1 (Fig. 3.8) becomes a straight line in Fig. 3.4 (Fig. 3.9)).

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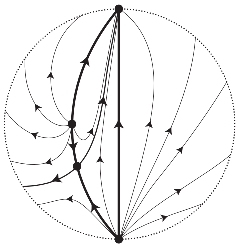

Fig. 3.13

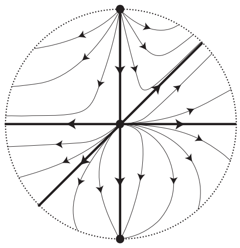

Fig. 3.14

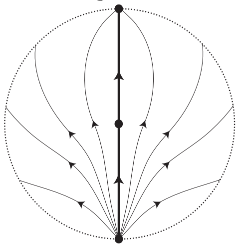

Fig. 3.15

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