UDC 539.3, 534.1

https://doi.org/10.33271/nvngu/2020-1/046

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TWO-DIMENSIONAL ELASTIC THEORY METHODS FOR DESCRIBING THE STRESS STATE AND THE MODES OF ELASTIC BORING

Purpose. To find the stressed state and describe the operating regimes of an important element of mining industrial equipment – an elastic drilling bit – based on the method of finding the solutions of problems of solid mechanics using holomorphic functions of two complex variables.

Methodology. Methods for constructing the basic solutions of three-dimensional boundary value problems of the theory of elasticity are based on the representation of the fundamental solution of the Lame equations in the Papkovich-Neuber form using a scalar spatial harmonic function and a vectoral spatial harmonic function. This made it possible to formulate the boundary value problem in terms of holomorphic functions of complex variables. Based on the representation of the above holomorphic functions in the form polynomial of the n^{th} order in accordance with degrees of the complex variables, corresponding boundary conditions are formulated for basic solutions and integral conditions for the principle moment of the stress vector on the lateral surface being equal to zero are additionally specified.

Findings. The paper formulates complex-conjugate boundary value problems of spatial theory of elasticity in terms of holomorphic functions of two complex variables. We have considered the case when there is one of the spatial coordinates on which the stress tensor does not depend on, for a body whose surface is described by standard curves. For the given finite elastic cylindrical solid (a drilling bit) of canonical cross-section, a set of the basic complex solutions of the n^{th} order and of corresponding vectors of external loads has been constructed.

Originality. For the first time, a scheme and a method for constructing the basic solutions of static boundary value problems of spatial theory of elasticity have been suggested and the corresponding boundary conditions have been imposed; the real and imaginary parts of the solutions of the basic boundary value problems for a cylindrical drilling bit have been constructed and an analysis of these solutions has been carried out.

Practical value. An example of using the developed methods for constructing solutions of boundary value problems of spatial theory of elasticity has been considered for finding an exact analytical solution of a two-dimensional boundary value problem that describes the distribution of stresses and corresponding external loads on the lateral surface of a cylindrical drilling bit of a canonical cross-section. Such mathematical models and analysis of the external load structure can be effectively used to describe the safe operating modes of mining boring equipment.

Keywords: *theory of elasticity, stress tensor, displacement vector, external load vector, holomorphic function*

Introduction. Literature review. One of the main scientific directions of the study on boundary value problems of mechanics of a deformable solid is the spatial problems of the theory of elasticity. For fundamental and applied research studies, the relevance of this topic is determined by the fact that almost always a stress-strain state of a body is of three-dimensional character.

Approaches of mechanics of a deformable solid are basic for calculating and evaluating the parameters of strength and reliability of elements of engineering structures and devices that are under the influence of external force load. A great deal of scientific monographs and publications, in particular, the classical works by S. Tymoshenko, J. N. Goodier, and A. Lurie, are devoted to the formulation of boundary value problems of the theory of elasticity and to the development of their research methods. At present, numerical [1], qualitative [2, 3] and analytical [4, 5] approaches to the construction of solutions of boundary value problems of dynamic and static theory of elasticity have been developed.

The Fourier method, the potential method, and the method of theory of functions of a complex variable made it possible to substantiate the correct formulation of the basic boundary value problems of the theory of elasticity. Moreover, in some cases, these methods allowed us to construct spatial static and dynamic problems for canonical domains.

In recent decades, experimental [6, 7] and approximate [8, 9] approaches to solving boundary value problems of the theory of elasticity have been actively developing.

The methods for solving spatial problems of static theory of elasticity known from literature are based on representing solutions of the homogeneous equations of equilibrium in terms of displacements using harmonic and biharmonic functions. In the case of a plane static problem of the theory of elasticity of an isotropic body, the general integral of the equations of equilibrium is represented in terms of a biharmonic function.

In the classical papers by Muskhelishvili, for the first time the stress tensor and the displacement vector were represented through the analytic functions of a complex variable. These representations were rigorously substantiated for the case of a multiply-connected domain, and appropriate methods were developed for solving the plane problems of the theory of elasticity. Important results concerning application of a mathematical apparatus of functions of a complex variable are also used when considering the spatial problems of the theory of elasticity. The representation of solutions of the basic equations in the form of integral operators of analytical functions of a complex variable and the variant of formulation of the problem of spatial theory of elasticity with the use of functions of two complex variables, which are introduced in the four-dimensional space of real variables, have been suggested. Three-dimensional problems of the theory of elasticity are considered as some partial class of the problems of the four-dimensional theory of elasticity. The equilibrium equations in terms of displacements for a four-dimensional elastic medium in the absence of bulk forces are represented by the equations

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_i}+\mu\Delta_4 u_i=0;\quad \theta=\frac{\partial u_i}{\partial x_i},\quad i=\overline{1,4},$$

where $\vec{u} = u_i \vec{e}_i$, is the displacement vector, l, μ are the Lame parameters, $\Delta_4 \equiv \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator $\left(i = \overline{1,4}\right)$. There is considered a class of solutions of such equations which are determined by the equality

$$\theta = \frac{\partial u_i}{\partial x_4} = 0, \quad \left(i = \overline{1,4}\right).$$

For the class of solutions determined in this way, the constitutive equation of equilibrium is broken down into three equations of equilibrium of the three-dimensional theory of elasticity and the Laplace equation for the function $u_4(x_1, x_2, x_3)$. The integral functions of displacements and the complex variables are introduced

$$W_1 = u_1 + iu_2; \quad W_2 = u_3 + iu_4;$$

$$z_1 = x_1 + ix_2; \quad z_2 = x_3 + ix_4.$$

As a result, the constitutive equations of equilibrium take the following form

$$\left(\lambda + \mu\right)\frac{\partial\theta}{\partial \bar{z}_i} + 2\mu\frac{\partial^2 W_i}{\partial z_1 \partial \bar{z}_1} + 2\mu\frac{\partial^2 W_i}{\partial z_2 \partial \bar{z}_2} = 0;$$

$$\theta = \frac{\partial W_i}{\partial z_i} + \frac{\partial \overline{W}_i}{\partial \bar{z}_i}, \quad \left(i = \overline{1,2}\right);$$

$$\frac{\partial W_i}{\partial z_2} + \frac{\partial \overline{W}_i}{\partial \bar{z}_2} = 0.$$

The solution of the constructed equations can be obtained in terms of two arbitrary holomorphic functions $\Phi(z_1, z_2)$, $\Psi(z_1, z_2)$ in the form of formal series

$$W_i = \sum_{m,n=0}^{\infty} \frac{\left(-1\right)^m}{m!n!}\left[z_1^m z_2^n \bar{J}^{m-k} \frac{\partial^{l-k}}{\partial \bar{z}_2^{l-k}} \left\{ \frac{\partial \overline{\Psi}\left(z_1, z_2\right)}{\partial \bar{z}_2} \right\} + \right.$$

$$+\left(\frac{m}{4\left(1-\nu\right)} + i - 1\right)\overline{\Phi}\left(z_1, z_2\right) + \left(-1\right)^m \bar{z}_1^{m} \bar{z}_2^{n} \times$$

$$\left. \times \frac{m+i-1}{4\left(1-\nu\right)} J^{m+k} \frac{\partial^{l+k}\Phi\left(z_1, z_2\right)}{\partial z_2^{l+k}} \right],$$

where $l = 2m + n$, $k = 2 - i$, $i = \overline{1,2}$. Here J is the operator inverse to the operator $\dfrac{\partial}{\partial z_1}$. To determine the holomorphic functions $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$, like in the case of a plane problem of the theory of elasticity, we can use either their integral representations in terms of their boundary values or their series expansion in terms of a complete system of holomorphic functions in a two-dimensional complex domain.

In the papers [10, 11], by means of the method of complex potentials the spatial contact problems of thermoelastic behavior of bimaterial bodies with heat-permeable interphase frictions under thermal and force loads are solved, as well as the problems of interaction of bodies with surface inhomogeneities.

A mathematical body of the theory of functions of a complex variable is widely used for constructing solutions of boundary value problems of theory of plastic deformation [12], fracture mechanics for piezoelectric solids [13, 14], piezoceramic solids [15], and bimaterials [16] with interfacial cracks under the action of mechanical loads, electromagnetic and thermal fields [17, 18].

In this paper, the boundary value problem of a two-dimensional theory of elasticity in terms of the holomorphic functions $G_0(z_1, z_2)$, $\vec{G}\left(z_1, z_2\right)$ has been considered, provided that the stress tensor does not depend on one of the spatial variables for a finite cylindrical elastic drill geometrically described by a standard surface.

Basic relations and statement of the boundary value problems in terms of holomorphic functions of two complex variables. We consider a finite elastic cylindrical solid of the height $2h$, which occupies the domain X of Euclidean space delimited by the surface ∂X, the cross section of which is described by the canonical curve $f(x_1, x_2) = 0$ referred to the Cartesian coordinate system, the origin of which is in the geometrical center of the solid (Fig. 1). The solid is under the action of a stationary force load applied to the lateral surface $\partial X = \partial X_3 \cup \partial X_{-3} \cup \partial X_2$.

The solution construction of a linear problem of elasticity in a static formulation reduces to finding a solution of the equations of equilibrium

$$\mu\Delta\vec{u} + \left(\lambda + \mu\right)\vec{\nabla} \otimes \left(\vec{\nabla} \cdot \vec{u}\right) = 0. \qquad (1)$$

The stress tensor $\hat{\sigma}$ is represented in terms of the vector of displacement $\vec{u}$ by the relationship

$$\hat{\sigma} = \lambda\left(\vec{\nabla} \cdot \vec{u}\right)\hat{I} + \mu\left(\vec{\nabla} \otimes \vec{u} + \vec{u} \otimes \vec{\nabla}\right) = 0, \qquad (2)$$

and on the lateral surface ∂X it satisfies the boundary condition

$$\left(\vec{n} \cdot \hat{\sigma}\right)\Big|_{\partial X} \equiv \vec{\sigma}_n \equiv \vec{n} \cdot \left[\lambda\left(\vec{\nabla} \cdot \vec{u}\right)\hat{I} + \mu\left(\vec{\nabla} \otimes \vec{u} + \vec{u} \otimes \vec{\nabla}\right)\right]_{\partial X} = \vec{\sigma}_n^+, \qquad (3)$$

where $\hat{I}$ is a unit tensor; $\partial X_{\pm 3}$ are the upper and lower surfaces of the cylindrical solid, for which the normal vector $\vec{n}_{\pm 3} = \pm\vec{e}_3$; ∂X_2 is the lateral surface of the cylindrical solid for which the normal vector $\vec{n}_2 = n_i \vec{e}_i$, and

$$n_i = \frac{\partial f}{\partial x_i} \Bigg/ \sqrt{\sum_i \left(\frac{\partial f}{\partial x_i}\right)^2} = 0, \quad \left(i = \overline{1,2}\right),$$

where $\vec{u} = u_i\left(x_1, x_2, x_3\right)\vec{e}_i$ is the vector of displacement $\left(i = \overline{1,3}\right)$; $\{x_i\}$ are Cartesian coordinates of the arbitrarily chosen material point $x \in X$; $\vec{e}_i$ are the base unit vectors of the chosen Cartesian system $(0, x_1, x_2, x_3)$; $\vec{\nabla} \equiv \dfrac{\partial}{\partial \vec{r}} = \vec{e}_i \dfrac{\partial}{\partial x_i}$ is the Hamiltonian $\left(i = \overline{1,3}\right)$; $\Delta \equiv \vec{\nabla} \cdot \vec{\nabla} = \dfrac{\partial^2}{\partial x_i^2}$ is the Laplacian; $\vec{r}\left(x_1, x_2, x_3\right) = x_i \vec{e}_i$ is the radius vector of the arbitrarily chosen point of the solid; $\otimes$ is the operation of dyadic product; λ, μ are the Lame elastic constants; $\vec{\sigma}_n\left(x_1, x_2, x_3\right)$ is the stress vector; $\vec{\sigma}_n^+\left(x_1, x_2, x_3\right)$ is the given vector of superficial applied forces that satisfies the integral conditions of self-equilibrium of external loading on the lateral surface of the solid ∂X

$$\int_{\partial X} \vec{\sigma}_n^+ d\Sigma = 0; \quad \int_{\partial X} \left(\vec{\sigma}_n^+ \times \vec{r}\right) d\Sigma = 0.$$

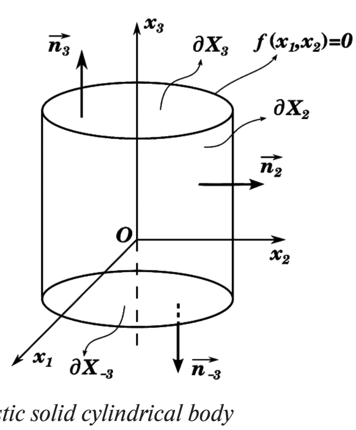

Fig. 1. Elastic solid cylindrical body

In [19, 20] on the basis of representation of the displacement vector in the Papkovich-Neuber form

$$\vec{u} = \vec{\nabla}\left(\varphi_0 + \vec{r}\cdot\varphi\right) - 4\left(1-\nu\right)\vec{\varphi}$$

in terms of the scalar $\varphi_0 = \varphi_0(x_1, x_2, x_3)$ and vectorial $\vec{\varphi} = \vec{\varphi}\left(x_1, x_2, x_3\right)$ harmonic functions, the basic boundary value problem of theory of elasticity (1–3) is reduced to finding the holomorphic functions $G_0(z_1, z_2)$, $\vec{G}\left(z_1, z_2\right)$ satisfying the equation

$$\Delta G_0(z_1, z_2); \quad \Delta\vec{G}\left(z_1, z_2\right) = 0, \tag{4}$$

the boundary conditions

$$\vec{T}_n\left(z_1,z_2,z_3\right) \equiv \left(\vec{n}\cdot\hat{T}\right)\Big|_{\partial X} = \Big\{\vec{n}\cdot\Big[\vec{\nabla}\otimes\vec{\nabla}G_0\left(z_1,z_2\right) + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{G}\left(z_1,z_2\right)\right)\cdot\vec{r}\left(z_1,z_2,z_3\right) - \left(1-2\nu\right)\left(\vec{\nabla}\otimes\vec{G}\left(z_1,z_2\right) + \vec{G}\left(z_1,z_2\right)\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{G}\left(z_1,z_2\right)\right)\hat{I}\Big]\Big\}\Big|_{\partial X} = \frac{\vec{T}_n^+}{2\mu},$$

as well as the corresponding integral conditions of the static equilibrium of an elastic body

$$\int_{\partial X}\left(\vec{r}\times\vec{T}_n^+\right)d\Sigma = 0,$$

where

$$\vec{\nu}\left(z_1,z_2,z_3\right) = \left(\vec{\nabla}\otimes\vec{G}\left(z_1,z_2\right)\right)\cdot\vec{r}\left(z_1,z_2,z_3\right) - \left(3-4\nu\right)\vec{G}\left(z_1,z_2\right) + \vec{\nabla}G_0\left(z_1,z_2\right)$$

is a complex vector of displacements;

$$\hat{T}\left(z_1,z_2,z_3\right) = 2\mu\Big[\vec{\nabla}\otimes\vec{\nabla}G_0\left(z_1,z_2\right) + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{G}\left(z_1,z_2\right)\right)\cdot\vec{r}\left(z_1,z_2,z_3\right) - \left(1-2\nu\right)\left(\vec{\nabla}\otimes\vec{G}\left(z_1,z_2\right) + \vec{G}\left(z_1,z_2\right)\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{G}\left(z_1,z_2\right)\right)\hat{I}\Big]$$

is a complex stress tensor;

$$\vec{T}_n^+ \equiv \vec{\sigma}_n\left(x_1,x_2,x_3\right) + i\vec{\sigma}_n^+\left(x_1,x_2,x_3\right)$$

is a complex vector of superficial applied forces;

$$\vec{r}\left(z_1,z_2,z_3\right) = \frac{(1+i)}{2}\Big[\left(z_1 - iz_2 - z_3\right)\vec{e}_1 + \left(z_2 - iz_3 - z_1\right)\vec{e}_2 + \left(z_3 - iz_1 - z_2\right)\vec{e}_3\Big];$$

$$\vec{\nabla} = \vec{e}_1\frac{\partial}{\partial z_1} + \vec{e}_2\left(i\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) + \vec{e}_3\left(i\frac{\partial}{\partial z_2}\right);$$

$\Delta \equiv \vec{\nabla}^* \cdot \vec{\nabla}^* = 2i\dfrac{\partial^2}{\partial z_1 \partial z_2}$ is the Laplacian.

Note that the representation of the fundamental solution of the basic boundary value problem in terms of the displacement vector and in the form of Papkovich-Neuber ensures the satisfaction of the integral condition of the equality of the main vector of external loading to zero.

General scheme of construction of solutions by means of the method of expansion in terms of the base states. Basic representations for the formulation of the basic boundary value problems of spatial theory of elasticity are the representations of holomorphic functions $G_0(z_1, z_2)$, $\vec{G}\left(z_1, z_2\right)$ in the form of polynomials of the n^{th} degree in terms of powers of the complex variables z_1 and z_2

$$G_0^{(n)}\left(z_1,z_2\right) = \sum_{k=0}^{n} M_0^{(k)}\left(z_1,z_2\right); \quad \vec{G}^{(n)}\left(z_1,z_2\right) = \sum_{k=0}^{n}\vec{M}^{(k)},$$

where

$$M_0^{(k)}\left(z_1,z_2\right) = \sum_{j=0}^{k} C^{((k-j)j)} z_1^{k-j} z_2^{j};$$

$$\vec{M}^{(k)}\left(z_1,z_2\right) = \sum_{j=0}^{k}\vec{D}^{((k-j)j)} z_1^{k-j} z_2^{j}, \tag{5}$$

are homogeneous polynomials of the k^{th} degree of the complex variables z_1 and z_2;

$$C^{((k-j)j)} = a_{(k-j)j} + ib_{(k-j)j};$$

$$\vec{D}^{((k-j)j)} = D_m^{((k-j)j)}\vec{e}_m = \left(c_m^{((k-j)j)} + id_m^{((k-j)j)}\right)\vec{e}_m;$$

$$a_{(k-j)j},\quad b_{(k-j)j},\quad c_m^{((k-j)j)},\quad d_m^{((k-j)j)},$$

are real numbers; $m = \overline{1,3}$; $j,k = \overline{0,n}$; $n \in N$.

Within the representation (5), the homogeneous polynomials $M_0^{(k)}, \vec{M}^{(k)}$ of the k^{th} degree satisfy Eq. (4), i.e.

$$\frac{\partial^2 M_0^{(k)}}{\partial z_1 \partial z_2} = 0; \quad \frac{\partial^2 \vec{M}^{(k)}}{\partial z_1 \partial z_2} = 0.$$

In the general case, for construction of a complex stress tensor $\hat{T}^{(n)}$ of the n^{th} order within the framework of the adopted approach, the basis is a scalar holomorphic function $G_0(z_1, z_2)$ in the form of a homogeneous polynomial $M_0^{(n)}\left(z_1,z_2\right)$ of the $n + 2^{th}$ degree and a holomorphic vector function $\vec{G}\left(z_1,z_2\right)$ in the form of a vector homogeneous polynomial $\vec{M}^{(n)}\left(z_1,z_2\right)$ of the $n + 1^{th}$ degree, respectively. Thus, we denote

$$G_0^{(n+2)} \equiv M_0^{(n+2)} = C^{((n+2)0)} z_1^{n+2} + C^{(0(n+2))} z_2^{n+2};$$

$$\vec{G}^{(n+1)} \equiv \vec{M}^{(n+1)} = \vec{D}^{((n+1)0)} z_1^{n+1} + \vec{D}^{(0(n+1))} z_2^{n+1}. \tag{6}$$

Then the complex stress tensor of the n^{th} order is presented in the way

$$\hat{T}^{(n)} = 2\mu\Big[\vec{\nabla}\otimes\vec{\nabla}M_0^{(n+2)} + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{M}^{(n+1)}\right)\cdot\vec{r} - \left(1-2\nu\right)\left(\vec{\nabla}\otimes\vec{M}^{(n+1)} + \vec{M}^{(n+1)}\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{M}^{(n+1)}\right)\hat{I}\Big]. \tag{7}$$

For each basic stressed state $\hat{T}^{(k)}$ of the k^{th} order, the expression of the stress vector $\vec{T}_n^{(k)}$ will be initial for the formation of the corresponding structure of boundary conditions on the lateral surface of the body ∂X with the normal $\vec{n} = n_i\left(x_1,x_2,x_3\right)\vec{e}_i$

$$\vec{T}_n^{(k)} \equiv \left(\vec{n}\cdot\hat{T}^{(k)}\right)\Big|_{\partial X} = 2\mu\Big\{\vec{n}\cdot\Big[\vec{\nabla}\otimes\vec{\nabla}M_0^{(k+2)} + \left(\vec{\nabla}\otimes\vec{\nabla}\otimes\vec{M}^{(k+1)}\right)\cdot\vec{r} + \left(2\nu-1\right)\times \left(\vec{\nabla}\otimes\vec{M}^{(k+1)} + \vec{M}^{(k+1)}\otimes\vec{\nabla}\right) - 2\nu\left(\vec{\nabla}\cdot\vec{M}^{(k+1)}\right)\hat{I}\Big\}\Big|_{\partial X}.$$

Moreover, on the lateral surface of the body ∂X, the vector of the complex external load $\vec{T}_n^{(k)(+)}$ is to be equal to the stress vector $\vec{T}_n^{(k)}$, i.e. $\vec{T}_n^{(k)(+)} = \vec{T}_n^{(k)}$.

This method of finding solutions of basic boundary value problems of the theory of elasticity will be applied to a cylindrical elastic body in the case when the complex stress tensor does not depend on the spatial coordinate x_3.

The specified condition, which is given by the relation

$$\frac{\partial\hat{T}\left(z_1,z_2,z_3\right)}{\partial x_3} \equiv \left(i\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\right)\hat{T}\left(z_1,z_2,z_3\right) = 0, \tag{8}$$

imposes relations between the coefficients of scalar $G_0(z_1, z_2)$ and vectorial $\vec{G}\left(z_1,z_2\right)$ holomorphic functions.

Let us apply the condition to the basic solutions of the k^{th} order of the complex stress tensor $\hat{T}^{(k)}$, whose structure is determined by the holomorphic functions $G_0(z_1, z_2)$, $\vec{G}\left(z_1,z_2\right)$ in the form of homogeneous polynomials (6).

As an example, consider the solution of the boundary value problem of theory of elasticity, which is characterized by the complex second-order tensor.

Taking into account the relation (7) and the condition (8), we obtain the recurrence formula for the basic solution $\hat{T}^{(k)}$ of the k^{th} order

$$\hat{T}^{(k)} = 2\mu z_1^{k-1}\left[\hat{K}^{(1)}z_1 + \hat{K}^{(3)}\left(z_2 - iz_3\right)\right],$$

where $\hat{K}^{(1)}$, $\hat{K}^{(3)}$ are the constant tensor coefficients, which are expressed in terms of coefficients of the holomorphic functions $G_0^{(k+2)}\left(z_1,z_2\right)$, $\vec{G}^{(k+1)}\left(z_1,z_2\right)$.

Let us represent a structure of the stress vectors on the lateral surface of the body ∂X with the normal $\vec{n} = n_i\left(x_1,x_2,x_3\right)\vec{e}_i$. For each basic solution $\hat{T}^{(k)}$ of the k^{th} order, the stress vector is represented as follows

$$\vec{T}_n^{(k)} \equiv \left(\vec{n}\cdot\hat{T}^{(k)}\right)\Big|_{\partial X} = 2\mu\left(\vec{n}\cdot\left[z_1^{(k-1)}\left(\hat{K}^{(1)}z_1 + \hat{K}^{(3)}\left(z_2 - iz_3\right)\right)\right]\right)\Big|_{\partial X} \equiv$$
$$\equiv T_{(n)m}^{(k)}\vec{e}_m,\quad \left(m=\overline{1,3}\right),\quad \left(k=\overline{1,n}\right).$$

For the given cylindrical solid, the stress vectors to the surfaces ∂X_2 and $\partial X_{\pm 3}$ have the following forms:

- on the surface ∂X_2, where $\vec{n}_2 = n_{2i}\vec{e}_i$ $\left(i=\overline{1,2}\right)$

$$\vec{T}_{n_2}^{(k)} \equiv \left(\vec{n}_2\cdot\hat{T}^{(k)}\right)\Big|_{\partial X_2} = 2\mu\left(\vec{n}_2\cdot\left[z_1^{(k-1)}\left(\hat{K}^{(1)}z_1 + \hat{K}^{(3)}\left(z_2 - iz_3\right)\right)\right]\right)\Big|_{\partial X_2};$$

- on the surface $\partial X_{\pm 3}$, where $\vec{n}_{\pm 3} = \pm\vec{e}_3$ we have

$$\vec{T}_{n_{\pm 3}}^{(k)} \equiv \left(\vec{n}_{\pm 3}\cdot\hat{T}^{(k)}\right)\Big|_{\partial X_{\pm 3}} = 2\mu\left(\left(\pm\vec{e}_3\right)\cdot\left[z_1^{(k-1)}\left(\hat{K}^{(1)}z_1 + \hat{K}^{(3)}\left(z_2 - iz_3\right)\right)\right]\right)\Big|_{\partial X_{\pm 3}} =$$
$$= 2\mu\left(\left[z_1^{(k-1)}\left(K_{3m}^{(1)}z_1 + K_{3m}^{(3)}\left(z_2 - iz_3\right)\right)\right]\vec{e}_m,\quad \left(m=\overline{1,2}\right)\right).$$

In this connection, the boundary conditions for the base stressed state $\hat{T}^{(k)}$ of the k^{th} order on the lateral surfaces of the body ∂X_2, $\partial X_{\pm 3}$ are the conditions of balancing of the stress vectors $\vec{T}_{n_2}^{(k)}$, $\vec{T}_{n_{\pm 3}}^{(k)}$ by the given external load vectors $\vec{T}_{n_2}^{(k)(+)}$, $\vec{T}_{n_{\pm 3}}^{(k)(+)}$

$$\vec{T}_{n_2}^{(k)(+)} = \vec{T}_{n_2}^{(k)};\quad \vec{T}_{n_{\pm 3}}^{(k)(+)} = \vec{T}_{n_{\pm 3}}^{(k)}.$$

Let us analyze the basic solutions of complex-conjugate problems. As an example, consider a second-order basic solution $\hat{T}^{(2)}$

$$\hat{T}^{(2)} \equiv \mathrm{Re}\,\hat{T}^{(2)} + i\,\mathrm{Im}\,\hat{T}^{(2)} = \left(\mathrm{Re}\,T_{ij}^{(2)} + i\,\mathrm{Im}\,T_{ij}^{(2)}\right)\vec{e}_i\otimes\vec{e}_j.$$

Accordingly, the components of the stress tensor for the real part of the solution of the problem in the basis of the Cartesian coordinate system $\left\{\vec{e}_m\right\}$ are given in terms of the real and imaginary parts of the coefficients of the holomorphic functions $a_{(k-j)j}$, $b_{(k-j)j}$, $c_m^{((k-j)j)}, d_m^{((k-j)j)}$, $m=\overline{1,3}$, $j,k=\overline{0,4}$. For example, the component of the stress tensor $T_{11}^{(2)}$ is written as follows:

- for the real part of the solution $\mathrm{Re}\,T_{11}^{(2)}$

$$\mathrm{Re}\,T_{11}^{(2)} = 6\mu\Big\{\left[4a^{40} + 2\nu c_1^{30} - d_1^{30} + 4c_2^{30}\right]x_1^2 +$$
$$+\left[-4a^{40} + \left(1-2\nu\right)c_1^{30} - 3c_2^{30} - d_2^{30} + 6\nu d_3^{03}\right]x_2^2 + \qquad (9)$$
$$+\left[-4b^{40} - c_1^{30} + c_2^{30} + \left(1-4\nu\right)d_1^{30} - 7d_2^{03}\right]x_1x_2\Big\};$$

- for the imaginary part of the solution $\mathrm{Im}\,T_{11}^{(2)}$

$$\mathrm{Im}\,T_{11}^{(2)} = 6\mu\Big\{\left[4b^{40} + 4c_1^{30} + 2\nu d_1^{30} + 4d_2^{30}\right]x_1^2 +$$
$$+\left[-4b^{40} + c_2^{30} - 6\nu c_3^{03} + \left(1-2\nu\right)d_1^{30} - 3d_2^{30}\right]x_2^2 + \qquad (10)$$
$$+\left[8a^{40} - \left(1-4\nu\right)c_1^{30} + d_2^{30} - d_1^{30}\right]x_1x_2\Big\}.$$

Application of the method of holomorphic functions for constructing a solution for an elastic cylindrical drill. As an example, we consider the boundary value problem of theory of elasticity, which describes the distribution of stresses and corresponding external loads on the lateral surface of a cylindrical body (drill) of an elliptical cross-section, geometrically delimited by the surfaces

$$\partial X_2:\quad f\left(x_1,x_2\right) \equiv \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0;$$
$$\partial X_{\pm 3}:\ x_3 = \pm h.$$

For the given body, the normal vectors are of the following forms:

- on the surface ∂X $\vec{n}_2 = n_{2i}\vec{e}_i$, where

$$n_{21} = 1\Big/\sqrt{1 + \frac{a^4}{b^4}\frac{x_2^2}{x_1^2}};\quad n_{22} = 1\Big/\sqrt{1 + \frac{b^4}{a^4}\frac{x_1^2}{x_2^2}};$$

- on the surface $\partial X_{\pm 3}$ $\vec{n}_{\pm 3} = \pm\vec{e}_3$.

Since the coefficients of holomorphic functions (11) are degrees of freedom, in the relations (9, 10) for the solution $\hat{T}^{(2)}$ of the second order we assume a_{30}, b_{30}, a_{03}, b_{03}, c_i^{30}, d_i^{30}, $c_i^{03}, d_i^{03} = 0$ and $c_1^{03}, d_3^{30} \neq 0$ $(i=\overline{0,3})$.

Then the components of the stress tensor are represented as follows. For the real part of the solution $\mathrm{Re}\,T_{ij}^{(2)}$ $\left(i,j=\overline{1,3}\right)$.

$$\mathrm{Re}\,T_{11}^{(2)} = \mathrm{Re}\,T_{33}^{(2)} = \mathrm{Re}\,T_{12}^{(2)} = 0;$$
$$\mathrm{Re}\,T_{22}^{(2)} = -2\mu\nu c_1^{03}x_1^2;\qquad \mathrm{Re}\,T_{31}^{(2)} = -12\mu d_3^{30}x_1x_2;$$
$$\mathrm{Re}\,T_{23}^{(2)} = 12\mu\left(1-2\nu\right)d_3^{30}x_1x_2.$$

Accordingly, the stress vector on the surface of the body for the real part takes the form

$$\left(\mathrm{Re}\,\vec{T}_{n_2}^{(2)}\right)\Big|_{\partial X_2} = \left(\mathrm{Re}\,T_{n_{22}}^{(2)}\vec{e}_2 + \mathrm{Re}\,T_{n_{23}}^{(2)}\vec{e}_3\right)\Big|_{\partial X_2} =$$
$$= \left(\left(-2\mu\nu c_1^{03}x_1^2\right)\Big/\sqrt{1 + \frac{b^4}{a^4}\frac{x_1^2}{x_2^2}}\right)\vec{e}_2 +$$
$$+12\mu d_3^{30}\left(-x_1x_2\Big/\sqrt{1 + \frac{a^4}{b^4}\frac{x_2^2}{x_1^2}} + \left(1-2\nu\right)x_1x_2\Big/\sqrt{1 + \frac{b^4}{a^4}\frac{x_1^2}{x_2^2}}\right)\vec{e}_3\Bigg|_{\partial X_2};$$

$$\left(\mathrm{Re}\,\vec{T}_{n_3}^{(2)}\right)\Big|_{\partial X_{\pm 3}} = \left(\mathrm{Re}\,T_{n_{31}}^{(2)}\vec{e}_1 + \mathrm{Re}\,T_{n_{32}}^{(2)}\vec{e}_2\right)\Big|_{\partial X_{\pm 3}=\pm h} =$$
$$= \pm 12\mu\nu d_3^{30}\ \left(-x_1x_2\vec{e}_1 + \left(1-2\nu\right)x_1x_2\vec{e}_2\right)\Big|_{\partial X_{\pm 3}=\pm h}.$$

For the imaginary part of the solution $\mathrm{Im}\,T_{ij}^{(2)}$, we obtain the following components of the stress tensor

$$\mathrm{Im}\,T_{11}^{(2)} = \mathrm{Im}\,T_{33}^{(2)} = \mathrm{Im}\,T_{12}^{(2)} = 0;$$
$$\mathrm{Im}\,T_{22}^{(2)} = 2\mu c_1^{03}\left(x_2^2 - 3x_1x_2\right);$$
$$\mathrm{Im}\,T_{13}^{(2)} = 6\mu\left(1-2\nu\right)d_3^{30}\left(x_1^2 - x_2^2\right);$$
$$\mathrm{Im}\,T_{23}^{(2)} = 12\mu d_3^{30}\left(\nu x_1^2 + \left(1-2\nu\right)x_2^2\right).$$

Then the stress vector for the imaginary part $\mathrm{Im}\,T_{ij}^{(2)}$ of the solution takes the form

$$\left(\operatorname{Im}\vec{T}_{n_2}^{(2)}\right)\Big|_{\partial X_2}=\left(\operatorname{Im}T_{n_{22}}^{(2)}\vec{e}_2+\operatorname{Im}T_{n_{23}}^{(2)}\vec{e}_3\right)\Big|_{\partial X_2}=$$

$$=\left(\left(2\mu c_1^{03}\left(x_2^2-3x_1x_2\right)\right)\vec{e}_2\Big/\sqrt{1+\frac{b^4}{a^4}\frac{x_1^2}{x_2^2}}+\right.$$

$$+6\mu d_3^{30}\left(\left(1-2\nu\right)\left(x_1^2-x_2^2\right)\Big/\sqrt{1+\frac{\dot{a}^4}{b^4}\frac{x_2^2}{x_1^2}}+\right.$$

$$\left.\left.+2\left(\nu x_1^2+\left(1-2\nu\right)x_2^2\right)\Big/\sqrt{1+\frac{b^4}{a^4}\frac{x_1^2}{x_2^2}}\right)\vec{e}_3\right)\Bigg|_{\partial X_2};$$

$$\left(\operatorname{Im}\vec{T}_{n_3}^{(2)}\right)\Big|_{\partial X_{\pm 3}}=\left(\operatorname{Im}T_{n_{31}}^{(2)}\vec{e}_1+\operatorname{Im}T_{n_{32}}^{(2)}\vec{e}_2\right)\Big|_{\partial X_{\pm 3}=\pm h}=$$

$$=\pm 6\mu d_3^{30}\left(\left(1-2\nu\right)\left(x_1^2-x_2^2\right)\vec{e}_1+2\left(\nu x_1^2+\left(1-2\nu\right)x_2^2\right)\vec{e}_2\right)\Big|_{\partial X_{\pm 3}=\pm h}.$$

In Fig. 2, there are schematically depicted vectors of external load that correspond to the real part of the basic solution $\operatorname{Re}T_{ij}^{(2)}\ \left(i,j=\overline{1,3}\right)$. The vectors of external load can be specified by the integral characteristics (force and momentum) that implement this solution.

In particular, on the surface ∂X_2, the component of the stress vector $\operatorname{Re}T_{n_{22}}^{(2)}\vec{e}_2$ describes the normal component of external load, and the component of the stress vector $\operatorname{Re}T_{n_{23}}^{(2)}\vec{e}_3$ describes the tangential one, respectively. On the surface $\partial X_{\pm 3}$, the components of the stress vector $\operatorname{Re}\vec{T}_{n_3}^{(2)}=\operatorname{Re}T_{n_{31}}^{(2)}\vec{e}_1+\operatorname{Re}T_{n_{32}}^{(2)}\vec{e}_2$ are described only by momentum integral characteristics of external load.

Conclusions. The results obtained in this paper have shown that using the method of constructing the basic solutions of spatial problems of theory of elasticity on the basis of representation of holomorphic functions in the terms of polynomials of the *n-th* degree with respect to the complex variables z_1 and z_2, it is possible to form a structure and a set of exact analytic solutions and corresponding external loads, which are distributed on the lateral surface of a cylindrical drilling bit. The coefficients $a^{((i-j)j)}$, $\vec{b}^{((i-j)j)}$ $\left(i,j=\overline{0,n}\right)$ of the aforesaid holomorphic functions are the degrees of freedom that make it possible to concretize the integral characteristics (force and momentum) of the external load. Such an approach allows us to obtain a superposition of exact analytic solutions for an elastic deformable body. In this paper, an example of using the developed methods for constructing solutions of boundary value problems of spatial theory of elasticity has been considered for finding an exact solution of a two-dimensional boundary value problem. This boundary value problem simulates the distribution of stresses and of corresponding external loads on the lateral surface of a cylindrical solid (drill) of canonical cross-section. As follows from the results of the study on the mathematical model considered in this paper, the suggested approach allows us to construct and analyze the structure of the external load, which, in turn, can be used to synthesize the parameters of the boring equipment which provide efficient modes of its operation.

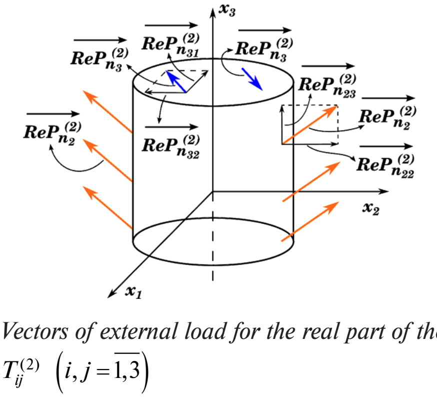

Fig. 2. Vectors of external load for the real part of the solution $\operatorname{Re}T_{ij}^{(2)}\ \left(i,j=\overline{1,3}\right)$

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Методи двовимірної теорії пружності для опису напруженого стану та режимів роботи пружного бура

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Мета. Знаходження напруженого стану та опис режимів роботи важливого елемента гірничого промислового обладнання – пружного вала на основі методики побудови базових розв'язків крайових задач статичної тривимірної теорії пружності з використанням голоморфних функцій двох комплексних змінних.

Методика. Методика побудови базових розв'язків просторових задач теорії пружності ґрунтується на основі представлення фундаментального розв'язку рівнянь Ляме у формі Папковича-Нейбера через скалярну й векторну гармонічні функції та формулювання відповідних комплексно спряжених крайових задач у голоморфних функціях комплексних змінних. На основі подання вищезгаданих голоморфних функцій у вигляді многочленів порядку за степенями комплексних змінних сформульовані відповідні крайові умови для базових розв'язків і конкретизовані додатково інтегральні умови рівності нулеві головного моменту вектора напружень на бічній поверхні тіла.

Результати. У роботі сформульовані комплексно спряжені крайові задачі просторової теорії пружності в голоморфних функціях двох комплексних змінних. Розглянута постановка задачі у випадку, коли тензор напружень не залежить від однієї із просторових координат для тіл, обмежених канонічними кривими. Для заданого скінченного пружного циліндричного тіла (бура) канонічного поперечного перетину побудована структура базових комплексних розв'язків порядку *n* і відповідних векторів зовнішніх навантажень.

Наукова новизна. У роботі вперше запропонована схема та методика побудови розв'язків базових крайових задач просторової теорії пружності та сформульовані відповідні крайові умови, побудовані дійсна та уявна складові розв'язків базових крайових задач для циліндричного бура та проведено аналіз цих розв'язків.

Практична значимість. Розглянуто приклад використання розроблених методів побудови розв'язків крайових задач просторової теорії пружності для знаходження точного аналітичного розв'язку двовимірної крайової задачі, що описує розподіл напружень і відповідних зовнішніх навантажень на бічній поверхні циліндричного бура канонічного поперечного перетину. Такі математичні моделі та аналіз структури зовнішнього навантаження можуть бути ефективно використані для опису безпечних режимів роботи гірничого бурового обладнання.

Ключові слова: *теорія пружності, тензор напружень, вектор переміщень, вектор зовнішніх навантажень, голоморфна функція*

Методы двумерной теории упругости для описания напряженного состояния и режимов работы упругого бура

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Цель. Нахождение напряженного состояния и описание режимов работы важного элемента горного промышленного оборудования – упругого вала на основе методики построения базовых решений краевых задач статической трехмерной теории упругости с использованием голоморфных функций двух комплексных переменных.

Методика. Методика построения базовых решений пространственных задач теории упругости основывается на основе представления фундаментального решения уравнений Ляме в форме Папковича-Нейбера через скалярную и векторную гармонические функции и формулирование соответствующих комплексно сопряженных краевых задач в голоморфных функциях комплексных переменных. На основе представления вышеупомянутых голоморфных функций в виде многочленов порядка по степеням комплексных переменных сформулированы соответствующие граничные условия для базовых решений и конкретизированы дополнительно интегральные условия равенства нулю главного момента вектора напряжений на боковой поверхности тела.

Результаты. В работе сформулированы комплексно сопряженные краевые задачи пространственной теории упругости в голоморфных функциях двух комплексных переменных. Рассмотрена постановка задачи в случае, когда тензор напряжений не зависит от одной из пространственных координат для тел, ограниченных каноническими кривыми. Для заданного конечного упругого цилиндрического тела (бура) канонического поперечного сечения построена структура базовых комплексных решений порядка *n* и соответствующих векторов внешних нагрузок.

Научная новизна. В работе впервые предложена схема и методика построения решений базовых краевых задач пространственной теории упругости и сформулированы соответствующие краевые условия, построены действительная и мнимая составляющие решений базовых краевых задач для цилиндрического бура и проведен анализ этих решений.

Практическая значимость. Рассмотрен пример использования разработанных методов построения решений краевых задач пространственной теории упругости для нахождения точного аналитического решения двумерной краевой задачи, описывающей распределение напряжений и соответствующих внешних нагрузок на боковой поверхности цилиндрического тела (бура) канонического поперечного сечения. Такие математические модели и анализ структуры внешней нагрузки могут быть эффективно использованы для описания безопасных режимов работы горного бурильного оборудования.

Ключевые слова: *теория упругости, тензор напряжений, вектор перемещений, вектор внешних нагрузок, голоморфная функция*

Recommended for publication by R. S. Musii, Doctor of Physical and Mathematical Sciences. The manuscript was submitted 26.10.18.