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OPTIMAL CONTROL IN NON-SELF-ADJOINT ELLIPTIC BOUNDARY VALUE PROBLEM WITH TERMINAL CRITERION

We obtain precise solution of the optimal control problem for elliptic equation with nonlocal boundary conditions in a circular sector with terminal quadratic cost functional in the class of controls that depend only on the angular variable.

В роботі одержано точний розв'язок задачі оптимального керування для еліптичного рівняння з нелокальними крайовими умовами в круговому секторі та з квадратичним термінальним критерієм якості, в класі керувань, що залежать лише від кутової змінної.

Introduction.

The theory of linear-quadratic optimal control problems for distributed systems is well researched [1, 2] and for many cases with the help of Fourier method it can be reduced to countable number of finite-dimensional problems [3]. In this paper we consider control problem for elliptic equation with non-local boundary conditions in circular sector [4, 5] with terminal quadratic cost functional. This problem does not allow total splitting and using L^2-theory. For resolving this problem in the class of controls that depend only on the angular variable we use apparatus of specially constructed biorthonormal basis systems of function [6].

1. Setting of the problem.

In circular sector $Q = \{(r,\theta) | r \in (0,1),\ \theta \in (0,\pi)\}$ we consider the optimal control problem

$$\begin{cases} \Delta y := \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial y}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2} = u(\theta),\ (r,\theta) \in Q, \\ y(1,\theta) = p(\theta),\ p(0) = 0, \\ y(r,0) = 0,\ r \in (0,1), \\ \frac{\partial y}{\partial \theta}(r,0) = \frac{\partial y}{\partial \theta}(r,\pi),\ r \in (0,1), \end{cases} \tag{1}$$

$$J(y,u) = \|y(\alpha)\|_D^2 + \|u\|_D^2 \to \inf, \tag{2}$$

where $p \in C^1([0,\pi])$ is given function, $\alpha \in (0,1)$ is given number, $\|\cdot\|_D$ is norm in $L^2(0,\pi)$, which is equivalent to standard one and is given by the equality

$$\|v\|_D = \left(\sum_{n=1}^{\infty} v_n^2\right)^{1/2},\ \text{where}\ \ v_n = \int_0^\pi v(\theta)\psi_n(\theta)d\theta,$$

$$\psi_0(\theta) = \frac{2}{\pi^2},\ \psi_{2n}(\theta) = \frac{4}{\pi^2}(\pi-\theta)\sin 2n\theta,\ \psi_{2n-1}(\theta) = \frac{4}{\pi^2}\cos 2n\theta.$$

The aim of the paper is to find optimal process of the problem (1), (2) in classical sense, that is, to find optimal among admissible processes

$$\{u,y\} \in C([0,\pi]) \times \left(C(\bar{Q}) \cap C^2(Q)\right).$$

For the application of the spectral method we use biorthonormal and complete in $L^2(0,\pi)$ well-known Samarsky-Ionkin systems of functions [6] $\Psi = \{\psi_n\}_{n=1}^{\infty}$ and

$$\Phi = \{\varphi_0(\theta) = \theta,\ \varphi_{2n}(\theta) = \sin 2n\theta,\ \varphi_{2n-1}(\theta) = \theta\cos 2n\theta\}_{n=1}^{\infty}. \quad (3)$$

Then $\forall u \in L^2(0,\pi)$

$$u(\theta) = \sum_{n=0}^{\infty} u_n \cdot \varphi_n(\theta), \quad (4)$$

where $u_n = \int_0^\pi u(\theta)\psi_n(\theta)d\theta$. So we seek solution of the problem (1) in the form

$$y(r,\theta) = y_0(r)\theta + \sum_{n=1}^{\infty} \left(y_{2n-1}(r)\theta\cos 2n\theta + y_{2n}(r)\sin 2n\theta\right), \quad (5)$$

where functions $\{y_k(r)\}_{k=0}^{\infty}$ are solutions of the system of ordinary differential equations

$$\frac{d}{dr}(r\frac{dy_0}{dr}) = r\cdot u_0,\ y_0(1) = p_0, \quad (6)$$

$$r\cdot\frac{d}{dr}\left(r\cdot\frac{dy_{2k-1}}{dr}\right) - (2k)^2 y_{2k-1} = r^2\cdot u_{2k-1},\ y_{2k-1}(1) = p_{2k-1}, \quad (7)$$

$$r\frac{d}{dr}\left(r\cdot\frac{dy_{2k}}{dr}\right) - (2k)^2 y_{2k} - 4k\cdot y_{2k-1} = r^2\cdot u_{2k},\ y_{2k}(1) = p_{2k}, \quad (8)$$

where $p_k = \int_0^\pi p(\theta)\cdot\psi_k(\theta)d\theta$.

Thus the original problem (1), (2) is reduced to the following one: among admissible pairs $\{u_n(r), y_n(r)\}_{n=0}^{\infty}$ of the problem (6) - (8) one should minimize the cost functional

$$J(y,u) = y_0^2(\alpha) + u_0^2 + \sum_{k=1}^{\infty}\left(y_{2k-1}^2(\alpha) + y_{2k}^2(\alpha) + u_{2k-1}^2 + u_{2k}^2\right) = J_0 + \sum_{k=1}^{\infty} J_k, \quad (9)$$

and for obtained process $\{\tilde{u}_n, \tilde{y}_n(r)\}_{n=0}^{\infty}$ one should prove that the formula (4) defines function from $C([0,\pi])$, and the formula (5) defines function from $C(\bar{Q})\cap C^2(Q)$.

2. The main result.

For fixed set $\{u_k\}_{k=0}^{\infty}$ after integration of (6) - (8) and using conditions at $r=1$ and conditions $\lim_{r\to 0} y_n(r) = 0$ we obtain the following formula

$$y_0(r) = p_0 - \frac{u_0}{4} + \frac{r^2}{4}u_0, \quad (10)$$

$$y_1(r) = p_1 r^2 + \frac{u_1}{4}r^2\ln r. \quad (11)$$

$$y_2(r) = p_2 r^2 + r^2\left(\frac{u_1}{8}\ln^2 r + (\frac{u_2}{4} + p_1 - \frac{u_1}{16})\ln r\right), \quad (12)$$

and for $k \geq 2$:

$$y_{2k-1}(r) = \left(p_{2k-1} - \frac{u_{2k-1}}{4-(2k)^2}\right)r^{2k} + r^2\frac{u_{2k-1}}{4-(2k)^2}, \quad (13)$$

$$y_{2k}(r) = p_{2k}r^{2k} - \frac{1}{4-(2k)^2}\left(u_{2k} + \frac{4k\cdot u_{2k-1}}{4-(2k)^2}\right)r^{2k} +$$
$$+\frac{1}{4-(2k)^2}\left(u_{2k} + \frac{4k\cdot u_{2k-1}}{4-(2k)^2}\right)r^2 + \left(p_{2k-1} - \frac{u_{2k-1}}{4-(2k)^2}\right)r^{2k}\ln r. \qquad (14)$$

Then admissible set $\{\tilde{y}_k(r), \tilde{u}_k\}_{k=0}^{\infty}$ minimizes (9) if and only if when $\tilde{u}_0$ is solution of

$$J_0 \to \inf, \qquad (15)$$

and for $\forall k \geq 1$ $\{\tilde{u}_{2k-1}, \tilde{u}_{2k}\}$ is solution of the problem

$$J_k \to \inf. \qquad (16)$$

From formula (10) – (14) we can deduce that J_0 and J_k are quadratic forms on variables u_0 and $\{u_{2k-1}, u_{2k}\}$, and, additionally, $J_0 \geq u_0^2$, $J_k \geq u_{2k-1}^2 + u_{2k}^2$. So the problems (15), (16) have unique solution $\{\tilde{u}_k\}_{k=0}^{\infty}$, where for $k \geq 2$

$$\tilde{u}_{2k-1} = \Delta_k^{-1}\Big(-(a_k^2+1)(a_k p_{2k-1}\alpha^{2k} + d_k(a_k b_k - c_k)) + a_k^2 d_k(a_k b_k - c_k)\Big), \qquad (17)$$

$$\tilde{u}_{2k} = \Delta_k^{-1}\Big(-a_k d_k(a_k^2 + (a_k b_k - c_k)^2 + 1) +$$
$$+a_k(a_k b_k - c_k)(a_k p_{2k-1}\alpha^{2k} + d_k(a_k b_k - c_k))\Big), \qquad (18)$$

where

$$\Delta_k = (1 + a_k^2)^2 + (a_k b_k - c_k)^2,$$
$$a_k = \frac{\alpha^2 - \alpha^{2k}}{4-4k^2},\ b_k = \frac{4k}{4-4k^2},\ c_k = \frac{\alpha^{2k} ln\alpha}{4-4k^2},\ d_k = \alpha^{2k}(p_{2k-1} ln\alpha + p_{2k}).$$

As $\Delta_k \sim 1$, $k \to \infty$, so for all sufficiently large $k \geq 1$ we have

$$|\tilde{u}_{2k-1}| + |\tilde{u}_{2k}| \leq \alpha^{2k-1}k^{-1}(|p_{2k-1}| + |p_{2k}|). \qquad (19)$$

Functions $\{\varphi_k\}_{k=0}^{\infty}$ from (3) are bounded, $|\varphi_k'(\theta)| \leq M \cdot k$, so formula

$$\tilde{u}(\theta) = \sum_{k=0}^{\infty} \tilde{u}_k \cdot \varphi_k(\theta) \qquad (20)$$

defines function from the class $C^1([0,\pi])$.

The following theorem guarantees, that the formula (20) defines optimal control of our problem in classical sense and, moreover, the class of admissible controls includes smooth on $[0,\pi]$ functions.

Theorem 1. *For every $u \in C^1([0,\pi])$, $u(0) = 0$ the formula (5) with coefficients $\{y_k(r)\}_{k=0}^{\infty}$ from (10) – (14) defines classical solution of the problem (1).*

Proof Let us prove that the formula (5) defines function $y(r,\theta)$, for which

$$y \in C([0,1]\times[0,\pi]),\ y \in C^2([0,1]\times[0,\pi]). \qquad (21)$$

Let us denote

$$F_1(r,\theta) = \sum_{k=2}^{\infty}\left(p_{2k-1}\cdot r^{2k}\cdot\theta\cos 2k\theta + p_{2k}\cdot r^{2k}\sin 2k\theta + p_{2k-1}\cdot r^{2k}\ln r\sin 2k\theta\right).$$

Then F_1 satisfies condition (21). Indeed, functions $r^{2k}\cdot\sin 2k\theta$ and $r^{2k}(\theta\cos 2k\theta+\ln r\sin 2k\theta)$ are harmonic, so for (21) it is sufficient to prove the uniform convergence of series F_1 on $[0,1]\times[0,\pi]$, which follows from [5]. For remainder of the series $\sum_{k=2}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot r^{2k}\cdot\theta\cos 2k\theta$ due to Bessel inequality $\sum_{k=2}^{\infty}u_k^2<\infty$ and Cauchy-Schwarz inequality we have

$$\left|\sum_{k=N}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}r^{2k}\cdot\theta\cos 2k\theta\right|\le$$

$$\le\pi\left(\sum_{k=N}^{\infty}u_{2k-1}^2\right)^{1/2}\cdot\left(\sum_{k=N}^{\infty}\frac{1}{((2k)^2-4)^2}\right)^{1/2}<\varepsilon, \tag{22}$$

beginning from some $N\ge 1$ uniformly on $[0,1]\times[0,\pi]$.

Moreover, because of the multiplier $\frac{r^{2k}}{4-(2k)^2}$ the partial derivatives of this series on r and θ up to second order are uniformly convergent series on every compact in $(0,1)\times(0,\pi)$.

Let us conduct a similar argument for the series $\sum_{k=2}^{\infty}\frac{1}{4-(2k)^2}\left(u_{2k}+\frac{4k\cdot u_{2k-1}}{4-(2k)^2}\right)\cdot r^{2k}\sin 2k\theta$.

For remainder of the series $\sum_{k=2}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot r^{2k}\cdot\ln r\sin 2k\theta$ we have:

$$\left|\sum_{k=N}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot r^{2k}\cdot\ln r\sin 2k\theta\right|\le\left(\sum_{k=N}^{\infty}\frac{u_{2k-1}^2}{(4-(2k)^2)^2}\right)^{1/2}\times$$

$$\times\left(\sum_{k=N}^{\infty}r^{4k}\cdot\ln^2 r\right)^{1/2}<\left(\sum_{k=N}^{\infty}u_{2k-1}^2\right)^{1/2}\cdot\left(\frac{r^{4N}\cdot\ln^2 r}{1-r^4}\right)^{1/2}$$

for every $\theta\in[0,\pi]$, $r\in(0,1)$. Then $\forall\varepsilon>0\ \ \exists N\ge 1$

$$\sup_{r\in[0,1],\theta\in[0,\pi]}\left|\sum_{k=N}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot r^{2k}\cdot\ln r\sin 2k\theta\right|<\varepsilon.$$

Let us consider the series $F_2(r,\theta)=\sum_{k=2}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot r^k\cdot\theta\cos 2k\theta$. It is uniformly convergent on $[0,1]\times[0,\pi]$ due to Cauchy-Schwarz inequality.

In the same way one can prove convergence of the series $\frac{\partial F_2}{\partial r}$, $\frac{\partial F_2}{\partial\theta}$, $\frac{\partial^2 F_2}{\partial r^2}$, $\frac{\partial^2 F_2}{\partial r\partial\theta}$. Convergence of the series $\frac{\partial^2 F_2}{\partial\theta^2}$ will follow from convergence of the series $\sum_{k=2}^{\infty}\frac{u_{2k-1}}{4-(2k)^2}\cdot(2k)^2\cdot r^2\cdot\theta\cos 2k\theta$, which is convergent with the series

$$\sum_{k=2}^{\infty}u_{2k-1}\cdot r^2\cdot\theta\cos 2k\theta. \tag{23}$$

For $u\in C^1([0,\pi])$ we obtain

$$u_{2k-1}=\int\limits_0^{\pi}u(\theta)\cdot\frac{4}{\pi^2}\cos 2k\theta d\theta=-\frac{2}{\pi^2}\cdot\frac{1}{k}\int_0^{\pi}u'(\theta)\sin 2k\theta d\theta=-\frac{2}{\pi^2}\cdot\frac{1}{k}v_{2k}.$$

As for $v = u' \in C([0,\pi])$ $\sum\limits_{n=0}^{\infty} v_n^2 < \infty$, $v_n = \int\limits_0^{\pi} v(\theta)\varphi_n(\theta)d\theta$, then $\sum\limits_{k=2}^{\infty} v_{2k}^2 < \infty$ and from Cauchy-Schwarz inequality the series (23) converges uniformly on $[0,1]\times[0,\pi]$.

Applying the previous discussion to the series

$$F_3(r,\theta) = \sum_{k=2}^{\infty} \frac{1}{4-(2k)^2}\left(u_{2k} + \frac{4k\cdot u_{2k-1}}{4-(2k)^2}\right)\cdot r^2 \sin 2k\theta,$$

we need to prove the convergence of the series

$$\sum_{k=2}^{\infty} u_{2k}\cdot r^2 \cdot \sin 2k\theta. \tag{24}$$

For $u \in C^1([0,\pi])$, $u(0) = 0$ we have:

$$u_{2k} = \frac{4}{\pi^2}\int\limits_0^{\pi} u(\theta)(\pi-\theta)\sin 2k\theta d\theta = \frac{2}{\pi^2}\cdot\frac{1}{k}\int\limits_0^{\pi} u'(\theta)(\pi-\theta)\cos 2k\theta d\theta -$$

$$-\frac{2}{\pi^2}\cdot\frac{1}{k}\int\limits_0^{\pi} u(\theta)\cos 2k\theta d\theta = \frac{2}{\pi}\cdot\frac{1}{k}\int\limits_0^{\pi} u'(\theta)\cos 2k\theta d\theta -$$

$$-\frac{2}{\pi^2}\cdot\frac{1}{k}\int\limits_0^{\pi} u'(\theta)\theta\cos 2k\theta d\theta - \frac{2}{\pi^2}\cdot\frac{1}{k}\int\limits_0^{\pi} u(\theta)\cos 2k\theta d\theta = \frac{1}{k}(\alpha_k+\beta_k+\gamma_k),$$

where $\sum\limits_{k=0}^{\infty}(\alpha_k^2+\beta_k^2+\gamma_k^2) < \infty$, as $u' \in C([0,\pi])$. Then from Cauchy-Schwarz inequality the series (24) converges uniformly on $[0,1]\times[0,\pi]$. Theorem is proved.

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