UDC 517.5
DOI: 10.37069/1683-4720-2019-33-5

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BOUNDARY VALUE PROBLEMS FOR THE GENERALIZED ANALYTIC AND HARMONIC FUNCTIONS

The study of the Dirichlet problem with arbitrary measurable data for harmonic functions is due to the famous dissertation of Luzin. Later on, the known monograph of Vekua has been devoted to boundary value problems (only with Hölder continuous data) for the generalized analytic functions, i.e., continuous complex valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form $\partial_{\bar{z}}h + ah + bh = c$, where $\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right)$, and it was assumed that the complex valued functions a, b and c belong to the class L^p with some $p > 2$ in the corresponding domains $D \subset \mathbb{C}$. The present paper is a natural continuation of our articles on the Riemann, Hilbert, Dirichlet, Poincare and, in particular, Neumann boundary value problems for quasiconformal, analytic, harmonic and the so-called $A-$harmonic functions with boundary data that are measurable with respect to logarithmic capacity. Here we extend the corresponding results to the generalized analytic functions $h : D \to \mathbb{C}$ with the sources $g : \partial_{\bar{z}}h = g \in L^p$, $p > 2$, and to generalized harmonic functions U with sources $G : \triangle U = G \in L^p$, $p > 2$. It was also given relevant definitions and necessary references to the mentioned articles and comments on previous results. This paper contains various theorems on the existence of nonclassical solutions of the Riemann and Hilbert boundary value problems with arbitrary measurable (with respect to logarithmic capacity) data for generalized analytic functions with sources. Our approach is based on the geometric (theoretic-functional) interpretation of boundary values in comparison with the classical operator approach in PDE. On this basis, it is established the corresponding existence theorems for the Poincare problem on directional derivatives and, in particular, for the Neumann problem to the Poisson equations $\triangle U = G$ with arbitrary boundary data that are measurable with respect to logarithmic capacity. These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

MSC: Primary 31A05, 31A20, 31A25, 31B25, 30C62; Secondary 30E25, 31C05, 35J61, 34M50, 35F45, 35Q15.

Keywords: *Poisson equations; Riemann, Hilbert, Dirichlet, Neumann and Poincare problems; generalized analytic and harmonic functions; logarithmic capacity.*

Dedicated to the 100th anniversary of the birth of Georgii Dmitrievich Suvorov

1. Introduction.

The well–known monograph of Vekua [46] has been devoted to the theory of the **generalized analytic functions**, i.e., continuous complex valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form

$$\partial_{\bar{z}}h + ah + bh = c , \qquad \partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right) , \tag{1}$$

This work was partially supported by grants of Ministry of Education and Science of Ukraine, project number is 0119U100421.

where it was assumed that the complex valued functions a, b and c belong to the class L^p with some $p > 2$ in the corresponding domains $D \subseteq \mathbb{C}$.

The present paper is a natural continuation of the articles [6]– [8], [15]– [20], [47] and [48] devoted to the Riemann, Hilbert, Dirichlet, Poincare and, in particular, Neumann boundary value problems for quasiconformal, analytic, harmonic and the so–called $A-$harmonic functions with boundary data that are measurable with respect to logarithmic capacity. Here we extend the corresponding results to **generalized** analytic and harmonic functions, see relevant definitions with history notes in the mentioned articles and necessary comments on previous results below.

The first part of the paper is devoted to the proof of existence of nonclassical solutions of Riemann, Hilbert and Dirichlet boundary value problems with arbitrary measurable boundary data with respect to logarithmic capacity for the equations

$$\partial_{\bar{z}} h(z) \ = \ g(z) \tag{2}$$

with the real valued function g in the class L^p, $p > 2$. We will call continuous solutions h of the equation (2) with the generalized first partial derivatives by Sobolev **generalized analytic functions with sources g**.

The second part of the paper contains the proof of existence of nonclassical solutions to the Poincare problem on the directional derivatives and, in particular, to the Neumann problem with arbitrary measurable boundary data with respect to logarithmic capacity for the Poisson equations

$$\triangle U(z) \ = \ G(z) \tag{3}$$

with real valued functions G of a class $L^p(D)$, $p > 2$, in the corresponding domains $D \subset \mathbb{C}$. For short, we will call continuous solutions to (3) of the class $W^{2,p}_{\rm loc}(D)$ **generalized harmonic functions with the source G**. Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [44], such functions belong to the class C^1.

The research of boundary value problems with arbitrary measurable data is due to the famous dissertation of Luzin, see its original text [30], and its reprint [31] with comments of his pupils Bari and Men'shov. Namely, he has established that, for each measurable a.e. finite $2\pi-$periodic function $\varphi(\vartheta) : \mathbb{R} \to \mathbb{R}$, there is a harmonic function U in the unit disk $\mathbb{D}$ such that $U(z) \to \varphi(\vartheta)$ for a.e. ϑ as $z \to \zeta := e^{i\vartheta}$ along all nontangential paths to $\partial\mathbb{D}$. The latter was based on his other deep result on the antiderivatives stated that, for any measurable function $\psi : [0, 1] \to \mathbb{R}$, there is a continuous function $\Psi : [0, 1] \to \mathbb{R}$ with $\Psi' = \psi$ a.e., see e.g. his papers [29] and [32].

Later on, the Luzin theorem on harmonic functions was strengthened in the paper [39], Corollary 5.1, see also [40], by the statement that, for each (Lebesgue) measurable function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, the space of all harmonic functions $u : \mathbb{D} \to \mathbb{R}$ with the angular limits $\varphi(\zeta)$ for a.e. $\zeta \in \partial\mathbb{D}$ has the infinite dimension. Recall, it is well–known the uniqueness theorem to the Dirichlet problem in terms of the angular limits e.g. for bounded harmonic functions u, see Corollary IX.1.1 and Theorem IX.2.3 in [37]. However, in general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under a.e. zero boundary data, see e.g. Theorem 2.1 in [40].

The Luzin theorem was key to establish the corresponding result on the Hilbert boundary value problem in [39], Theorems 2.1 and 5.2: for arbitrary measurable functions $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D} \to \mathbb{R}$, the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ with angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)} \cdot f(z)\} \;=\; \varphi(\zeta) \qquad \text{for a.e.} \quad \zeta \in \partial\mathbb{D} \tag{4}$$

has the infinite dimension. Then this theorem was extended to arbitrary Jordan domains with rectifiable boundaries in terms of the natural parameter, see Theorem 3.1 in [39].

In turn, these results have been applied in the paper [41] to the study of the Poincare problem on directional derivatives and, in particular, of the Neumann problem for harmonic functions with arbitrary boundary data that are measurable with respect to natural parameter in arbitrary Jordan domains with rectifiable boundaries. Similarly, the results on the Hilbert and Riemann problems for analytic functions along the so–called **Bagemihl–Seidel systems** of Jordan arcs terminating at the boundary in [42] can be applied to the Poincare and Neumann problems for harmonic functions.

Moreover, a series of the corresponding results have been formulated and proved in terms of logarithmic capacity, see its definition and properties e.g. in [16]. The base is the following analog of the Luzin theorem in [7], see also [48], where the abbreviation **q.e. means quasi–everywhere** with respect to logarithmic capacity.

Theorem A. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e.*

Furthermore, the function Φ can be chosen in such a way that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ for any prescribed $\varepsilon > 0$ and all $x \in [a, b]$.

On the basis of Theorem A, it was proved the analog of the second Luzin theorem:

Theorem B. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, measurable with respect to logarithmic capacity and finite q.e. Then a space of harmonic functions u in $\mathbb{D}$ with the angular limits $u(z) \to \varphi(\vartheta)$ as $z \to e^{i\vartheta}$ q.e. on $\mathbb{R}$ has the infinite dimension.*

In turn, on the basis of Theorem B, it was obtain the result on the Hilbert problem:

Theorem C. *Let $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be of bounded variation and $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there is a space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)} \cdot f(z)\} \;=\; \varphi(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}\,. \tag{5}$$

Then this result was extended to domains with the so–called quasiconformal boundaries and, in particular, to arbitrary smooth (C^1) domains, see [7] and [48], and it was also applied to the Poincare and Neumann problems for harmonic and $A-$harmonic functions, see [47]. Moreover, it was proved in [19] the following result:

Theorem D. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be of countable bounded variation* *and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there is a space of analytic functions $f : D \to \mathbb{C}$ of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)}f(z)\} \;=\; \varphi(\zeta) \qquad q.e.\ on\ \partial D. \tag{6}$$

See the next section for definitions. As usual, this theorem on the Hilbert problem for analytic functions implies the corresponding theorems on the Poincare and Neumann problems for harmonic functions. Finally, notice a wide circle of the corresponding results on boundary value problems in terms of the Bagemihl–Seidel systems in [17].

2. Hilbert problem and angular limits.

In this section, we prove the existence of nonclassical solutions of the Hilbert boundary value problem for generalized analytic functions with arbitrary boundary data that are measurable with respect to logarithmic capacity. The result is formulated in terms of the angular limit that is a traditional tool of the geometric function theory, see e.g. monographs [5, 26, 31, 36] and [37].

Recall that the classic boundary value **problem of Hilbert**, see [24], was formulated as follows: To find an analytic function $f(z)$ in a domain D bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\ f(z)\} \;=\; \varphi(\zeta) \qquad \forall\, \zeta\in C\ , \tag{7}$$

where the **coefficient** λ and the **boundary date** φ of the problem are continuously differentiable with respect to the natural parameter s and $\lambda \neq 0$ everywhere on C. The latter allows to consider that $|\lambda| \equiv 1$ on C. Note that the quantity $\mathrm{Re}\,\{\overline{\lambda}\, f\}$ in (7) means a projection of f into the direction λ interpreted as vectors in $\mathbb{R}^2$.

The reader can find a rather comprehensive treatment of the theory in the new excellent books [2, 3, 23, 45]. We also recommend to make familiar with the historic surveys contained in the monographs [10, 33, 46] on the topic with an exhaustive bibliography and take a look at our recent papers, see Introduction.

Next, recall that a straight line L is **tangent** to a curve Γ in $\mathbb{C}$ at a point $z_0 \in \Gamma$ if

$$\limsup_{z\to z_0, z\in\Gamma} \frac{\mathrm{dist}\,(z,L)}{|z-z_0|} \;=\; 0\ . \tag{8}$$

Let D be a Jordan domain in $\mathbb{C}$ with a tangent at a point $\zeta \in \partial D$. A path in D terminating at ζ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle with the vertex at ζ. The limit along all nontangential paths at ζ is called **angular** at the point.

Following [19], we say that a Jordan curve Γ in $\mathbb{C}$ is **almost smooth** if Γ has a tangent q.e. In particular, Γ is almost smooth if Γ has a tangent at all its points except a countable collection. The nature of such a Jordan curve Γ can be complicated enough because this countable collection can be everywhere dense in Γ.

Recall that the **quasihyperbolic distance** between points z and z_0 in a domain $D \subset \mathbb{C}$ is the quantity

$$k_D(z,z_0) \ := \ \inf_{\gamma} \int\limits_{\gamma} ds/d(\zeta,\partial D) \ ,$$

where $d(\zeta,\partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D, see [12].

Further, it said that a domain D satisfies the **quasihyperbolic boundary condition** if there exist constants a and b and a point $z_0 \in D$ such that

$$k_D(z,z_0) \ \leq \ a \ + \ b \ \ln \frac{d(z_0,\partial D)}{d(z,\partial D)} \qquad \forall \ z \in D \ . \tag{9}$$

The latter notion was introduced in [11] but, before it, was first implicitly applied in [4]. By the discussion in [20], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition.

Note that it is well–known the so–called $(A)-$condition by Ladyzhenskaya–Ural'tseva, which is standard in the theory of boundary value problems for PDE, see e.g. [28]. Recall that a domain D in $\mathbb{R}^n$, $n \geq 2$, is called satisfying **(A)-condition** if

$$\mathrm{mes}\ D \cap B(\zeta,\rho) \ \leq \ \Theta_0 \, \mathrm{mes}\ B(\zeta,\rho) \qquad \forall \ \zeta \in \partial D \ , \ \rho \leq \rho_0 \tag{10}$$

for some Θ_0 and $\rho_0 \in (0,1)$, where $B(\zeta,\rho)$ denotes the ball with the center $\zeta \in \mathbb{R}^n$ and the radius ρ, see 1.1.3 in [28].

Recall also that a domain D in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every boundary point of D from the completion of D after its suitable rotations and shifts. It is clear that the outer cone condition implies (A)–condition.

Probably one of the simplest examples of an almost smooth domain D with the quasihyperbolic boundary condition and without (A)–condition is the union of 3 open disks with the radius 1 centered at the points 0 and $1 \pm i$. It is clear that this domain has zero interior angle at its boundary point 1.

Given a Jordan domain D in $\mathbb{C}$, we call $\lambda : \partial D \to \mathbb{C}$ a **function of bounded variation**, write $\lambda \in \mathcal{BV}(\partial D)$, if

$$V_{\lambda}(\partial D) \ : \ = \ \sup \sum_{j=1}^{k} \ |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| \ < \ \infty \tag{11}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial D$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_{\lambda}(\partial D)$ is called the **variation of the function** λ.

Now, we call $\lambda : \partial D \to \mathbb{C}$ a function of **countable bounded variation**, write $\lambda \in \mathcal{CBV}(\partial D)$, if there is a countable collection of mutually disjoint arcs γ_n of ∂D,

$n = 1, 2, \ldots$ on each of which the restriction of λ is of bounded variation and the set $\partial D \setminus \cup\gamma_n$ has logarithmic capacity zero. In particular, the latter holds true if the set $\partial D \setminus \cup\gamma_n$ is countable. It is clear that such functions can be singular enough.

Theorem 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $g : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re} \left\{ \overline{\lambda(\zeta)} \cdot h(z) \right\} = \varphi(\zeta) \qquad q.e.\ on\ \ \partial D\ . \tag{12}$$

Furthermore, the space of such functions h has the infinite dimension.

Later on, we often apply the **logarithmic (Newtonian) potential $\mathcal{N}_G$ of sources** $G \in L^p(\mathbb{C})$, $p > 2$, with compact supports given by the formula:

$$\mathcal{N}_G(z) := \frac{1}{2\pi} \int\limits_{\mathbb{C}} \ln|z - w|\, G(w)\, dm(w)\ . \tag{13}$$

By Lemma 3 in [16], $\mathcal{N}_G \in W^{2,p}_{\mathrm{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$, $\alpha := (p-2)/p$, and $\triangle\mathcal{N}_G = G$ a.e.

Proof. Extending the function g by zero outside of D and setting $P = \mathcal{N}_G$ with $G = 2g$, $U = P_x$ and $V = -P_y$, we have that $U_x - V_y = G$ and $U_y + V_x = 0$. Thus, elementary calculations show that $H := U + iV$ is just a generalized analytic function with the source g. Moreover, the function

$$\varphi_*(\zeta) := \lim_{z\to\zeta} \mathrm{Re} \left\{ \overline{\lambda(\zeta)} \cdot H(z) \right\} = \mathrm{Re} \left\{ \overline{\lambda(\zeta)} \cdot H(\zeta) \right\}\ , \qquad \forall\, \zeta \in \partial D\ , \tag{14}$$

is measurable with respect to logarithmic capacity because the function H is continuous in the whole plane $\mathbb{C}$.

By Theorem 2 in [19], see also Theorems 5.1 and 6.1 in [21], there exist analytic functions $\mathcal{A}$ in D with the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\, \{\overline{\lambda(\zeta)} \cdot \mathcal{A}(z)\} = \Phi(\zeta) \qquad \text{q.e. on } \partial D \tag{15}$$

for the function $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$, $\zeta \in \partial D$. The space of such analytic functions $\mathcal{A}$ has the infinite dimension, see e.g. Corollary 8.1 in [21].

Finally, it is clear that the functions $h := \mathcal{A} + H$ are desired generalized analytic functions with the source g satisfying the Hilbert condition (12). Thus, the space of such functions h has really the infinite dimension. $\Box$

Remark 1. As it follows from the proof of Theorems 1, the generalized analytic functions h with a source $g \in L^p$, $p > 2$, satisfying the Hilbert boundary condition (12) q.e. in the sense of the angular limits can be represented in the form of the sums $\mathcal{A}+H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (15) and a generalized analytic function $H = U + iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential $\mathcal{N}_G$ with $G = 2g$ in the class $W^{2,p}_{\rm loc}(\mathbb{C}) \cap C^{1,\alpha}_{\rm loc}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\triangle P = G$.

In particular, for the case $\lambda \equiv 1$, we obtain the following consequence of Theorem 1 on the Dirichlet problem for the generalized analytic functions.

Corollary 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity and let $g : D \to \mathbb{R}$ be in $L^p(D)$ for some $p > 2$.*

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta} \ {\rm Re}\ h(z) \ = \ \varphi(\zeta) \qquad q.e.\ on\ \ \partial D\ . \tag{16}$$

Furthermore, the space of such functions h has the infinite dimension.

3. Hilbert problem and Bagemihl–Seidel systems.

Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite collection of mutually disjoint Jordan curves. A family of mutually disjoint Jordan arcs $J_\zeta : [0,1] \to \overline{D}$, $\zeta \in \partial D$, with $J_\zeta([0,1)) \subset D$ and $J_\zeta(1) = \zeta$ that is continuous in the parameter ζ is called a **Bagemihl–Seidel system** or, in short, of **class** $\mathcal{BS}$.

Theorem 2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, and let functions $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, $\varphi : \partial D \to \mathbb{R}$ and $\psi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$ for some $p > 2$. Then there is a generalized analytic function $f : D \to \mathbb{C}$ with the source g such that

$$\lim_{z\to\zeta} \ {\rm Re}\ \{\overline{\lambda(\zeta)} \cdot h(z)\} \ = \ \varphi(\zeta)\ , \tag{17}$$

$$\lim_{z\to\zeta} \ {\rm Im}\ \{\overline{\lambda(\zeta)} \cdot h(z)\} \ = \ \psi(\zeta) \tag{18}$$

along γ_ζ q.e. on ∂D.

Proof. As in the proof of Theorem 1, the function $H = U + iV$ with $U = P_x$ and $V = -P_y$, where $P = \mathcal{N}_G$ with $G = 2g$ is a generalized analytic function with the source g. Moreover, the functions

$$\varphi_*(\zeta) \ := \ \lim_{z\to\zeta} \ {\rm Re}\ \left\{\overline{\lambda(\zeta)} \cdot H(z)\right\} \ = \ {\rm Re}\ \left\{\overline{\lambda(\zeta)} \cdot H(\zeta)\right\}\ , \quad \forall\, \zeta \in \partial D\ , \tag{19}$$

$$\psi_*(\zeta) \ := \ \lim_{z\to\zeta} \ {\rm Im}\ \left\{\overline{\lambda(\zeta)} \cdot H(z)\right\} \ = \ {\rm Im}\ \left\{\overline{\lambda(\zeta)} \cdot H(\zeta)\right\}\ , \quad \forall\, \zeta \in \partial D\ , \tag{20}$$

are measurable with respect to logarithmic capacity because the function H is continuous in the whole plane $\mathbb{C}$.

Next, by Theorem 3 in [17] there is an analytic function $\mathcal{A}$ in D that has along γ_ζ q.e. on ∂D the limits

$$\lim_{z\to\zeta} \text{Re}\,\{\overline{\lambda(\zeta)}\cdot\mathcal{A}(z)\} \;=\; \Phi(\zeta)\;, \tag{21}$$

$$\lim_{z\to\zeta} \text{Im}\,\{\overline{\lambda(\zeta)}\cdot\mathcal{A}(z)\} \;=\; \Psi(\zeta) \tag{22}$$

for the functions $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$ and $\Psi(\zeta) := \psi(\zeta) - \psi_*(\zeta)$, $\zeta \in \partial D$. Thus, the function $h := \mathcal{A} + H$ is a desired generalized analytic function with the source g. $\Box$

Remark 2. As it follows from the proof of Theorems 2, the generalized analytic functions h with a source $g \in L^p$, $p > 2$, satisfying the Hilbert boundary condition (17) q.e. in the sense of the limits along γ_ζ can be represented in the form of the sums $\mathcal{A}+H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (21) and a generalized analytic function $H = U + iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential $\mathcal{N}_G$ with $G = 2g$ in the class $W^{2,p}_{\text{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\triangle P = G$.

The space of all solutions h of the Hilbert problem (17) in the given sense has the infinite dimension for any such prescribed φ, λ and $\{\gamma_\zeta\}_{\zeta\in D}$ because the space of all functions $\psi : \partial D \to \mathbb{R}$ which are measurable with respect to the logarithmic capacity has the infinite dimension.

The latter is valid even for its subspace of continuous functions $\psi : \partial D \to \mathbb{R}$. Indeed, by the Riemann theorem every Jordan domain G can be mapped with a conformal mapping g onto the unit disk $\mathbb{D}$ and by the Caratheodory theorem g can be extended to a homeomorphism of $\overline{G}$ onto $\overline{\mathbb{D}}$. By the Fourier theory, the space of all continuous functions $\tilde{\psi} : \partial\mathbb{D} \to \mathbb{R}$, equivalently, the space of all continuous 2π-periodic functions $\psi_* : \mathbb{R} \to \mathbb{R}$, has the infinite dimension.

Corollary 2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, and $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the logarithmic capacity.*

Suppose also that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$, $p > 2$.

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the limits (17) along γ_ζ q.e. on ∂D. Furthermore, the space of such functions h has the infinite dimension.

In particular, for the case $\lambda \equiv 1$, we obtain the corresponding consequence on the Dirichlet problem for the generalized analytic functions with the source g along any prescribed Bagemihl–Seidel system:

Corollary 3. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves and $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity.*

Suppose also that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$, $p > 2$.

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g such that

$$\lim_{z\to\zeta} \operatorname{Re} h(z) \ = \ \varphi(\zeta) \qquad \text{along } \gamma_\zeta \text{ q.e. on } \partial D\ . \tag{23}$$

Furthermore, the space of such functions h has the infinite dimension.

4. Riemann problem and Bagemihl–Seidel systems.

Recall that the classical setting of the **Riemann problem** in a smooth Jordan domain D of the complex plane $\mathbb{C}$ is to find analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ that admit continuous extensions to ∂D and satisfy the boundary condition

$$f^+(\zeta) \ = \ A(\zeta) \cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\, \zeta \in \partial D \tag{24}$$

with prescribed Hölder continuous functions $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$.

Recall also that the **Riemann problem with shift** in D is to find analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ satisfying the condition

$$f^+(\alpha(\zeta)) \ = \ A(\zeta) \cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\, \zeta \in \partial D \tag{25}$$

where $\alpha : \partial D \to \partial D$ was a one-to-one sense preserving correspondence having the non-vanishing Hölder continuous derivative with respect to the natural parameter on ∂D. The function α is called a **shift function**. The special case $A \equiv 1$ gives the so–called **jump problem** and $B \equiv 0$ gives the **problem on gluing** of analytic functions.

Arguing similarly to the proof of Theorem 1, we obtain by Theorem 8 in [17] on the Riemann problem for analytic functions the following statement.

Theorem 3. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$ be functions that are measurable with respect to the logarithmic capacity and let $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ be families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C} \setminus \overline{D}$, correspondingly.*

Suppose that $g : \mathbb{C} \to \mathbb{R}$ is a function with compact support in the class $L^p(\mathbb{C})$ with some $p > 2$. Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ with the source g that satisfy (24) q.e. on $\zeta \in \partial D$, where $f^+(\zeta)$ and $f^-(\zeta)$ are limits of $f^+(z)$ and $f^-(z)$ az $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.

Furthermore, the space of all such couples (f^+, f^-) has the infinite dimension for every couple (A, B) and any collections γ^+_ζ and γ^-_ζ, $\zeta \in \partial D$.

Theorem 3 is a special case of the following lemma based on Lemma 3 in [17] on the Riemann problem with shift that may have of independent interest.

Lemma 1. *Under the hypotheses of Theorem 3, let in addition $\alpha : \partial D \to \partial D$ be a homeomorphism keeping components of ∂D such that α and α^{-1} have the $(N)-$property of Luzin with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ with the source g that satisfy (25) for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity, where $f^+(\zeta)$ and $f^-(\zeta)$ are limits of $f^+(z)$ and $f^-(z)$ az $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for every couple* (A, B) *and any collections* γ_ζ^+ *and* γ_ζ^-, $\zeta \in \partial D$.

Remark 3. Some investigations were devoted also to the nonlinear Riemann problems with boundary conditions of the form

$$\Phi(\,\zeta,\, f^+(\zeta),\, f^-(\zeta)\,) \;=\; 0 \qquad \forall\, \zeta \in \partial D\ . \tag{26}$$

It is natural as above to weaken such conditions to the following

$$\Phi(\,\zeta,\, f^+(\zeta),\, f^-(\zeta)\,) \;=\; 0 \qquad \text{q.e. on } \zeta \in \partial D\ . \tag{27}$$

It is easy to see that the proposed approach makes possible also to reduce such problems to the algebraic measurable solvability of the relations

$$\Phi(\,\zeta,\, v,\, w\,) \;=\; 0 \tag{28}$$

with respect to complex-valued functions $v(\zeta)$ and $w(\zeta)$, cf. e.g. [13].

Later on, we sometimes say in short "$C-$measurable"instead of the expression "measurable with respect to the logarithmic capacity".

Example 1. For instance, correspondingly to the scheme given above, special nonlinear problems of the form

$$f^+(\zeta) \;=\; \varphi(\,\zeta,\, f^-(\zeta)\,) \qquad \text{q.e. on} \quad \zeta \in \partial D \tag{29}$$

are always solved if the function $\varphi : \partial D \times \mathbb{C} \to \mathbb{C}$ satisfies the **Caratheodory conditions** with respect to the logarithmic capacity, that is if $\varphi(\zeta, w)$ is continuous in the variable $w \in \mathbb{C}$ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity and it is $C-$measurable in the variable $\zeta \in \partial D$ for all $w \in \mathbb{C}$.

Furthermore, the spaces of solutions of such problems always have the infinite dimension. Indeed, by the Egorov theorem, see e.g. Theorem 2.3.7 in [9], see also Section 17.1 in [27], the function $\varphi(\zeta, \psi(\zeta))$ is $C-$measurable in $\zeta \in \partial D$ for every $C-$measurable function $\psi : \partial D \to \mathbb{C}$ if the function φ satisfies the Caratheodory conditions, and the space of all $C-$measurable functions $\psi : \partial D \to \mathbb{C}$ has the infinite dimension, see e.g. arguments in Remark 2 above.

5. On mixed boundary value problems.

Remark 3 makes possible to formulate a series of nonlinear boundary value problems in terms of Bagemihl–Seidel systems for generalized analytic functions including mixed boundary value problems. In order to demonstrate the potentiality of our approach, we give here a couple of results. Namely, arguing similarly to the proof of Theorem 1, see also Theorem 1.10 in [46], we obtain for instance by Theorem 10 and Lemma 5 in [17] the following statement on mixed boundary value problems.

Theorem 4. *Let* D *be a domain in* $\mathbb{C}$ *whose boundary consists of a finite number of mutually disjoint Jordan curves,* $\varphi : \partial D \times \mathbb{C} \to \mathbb{C}$ *satisfy the Caratheodory conditions and* $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, *be measurable with respect to the logarithmic capacity.*

Suppose also that $g : \mathbb{C} \to \mathbb{R}$ *is in* $C^{\alpha}(\mathbb{C})$, $\alpha \in (0,1)$, *with compact support,* $\{\gamma_{\zeta}^{+}\}_{\zeta\in\partial D}$ *and* $\{\gamma_{\zeta}^{-}\}_{\zeta\in\partial D}$ *are families of Jordan arcs of class* $\mathcal{BS}$ *in* D *and* $\mathbb{C}\setminus\overline{D}$, *correspondingly.*

Then there exist generalized analytic functions $f^{+} : D \to \mathbb{C}$ *and* $f^{-} : \mathbb{C}\setminus\overline{D} \to \mathbb{C}$ *with the source* g *such that*

$$f^{+}(\zeta) \;=\; \varphi\left(\zeta,\, \left[\frac{\partial f}{\partial \nu}\right]^{-}(\zeta)\right) \qquad q.e. \text{ on } \partial D\ , \tag{30}$$

where $f^{+}(\zeta)$ *and* $\left[\frac{\partial f}{\partial \nu}\right]^{-}(\zeta)$ *are limits of the functions* $f^{+}(z)$ *and* $\frac{\partial f^{-}}{\partial \nu}(z)$ *as* $z \to \zeta$ *along* γ_{ζ}^{+} *and* γ_{ζ}^{-}, *correspondingly.*

Furthermore, the space of all such couples (f^{+}, f^{-}) *has the infinite dimension for any such prescribed functions* g, φ, ν *and collections* γ_{ζ}^{+} *and* γ_{ζ}^{-}, $\zeta \in \partial D$.

Theorem 4 is a special case of the following lemma on the mixed problem with shift.

Lemma 2. *Under the hypotheses of Theorem 4, let in addition* $\beta : \partial D \to \partial D$ *be a homeomorphism keeping components of* ∂D *such that* β *and* β^{-1} *have the* $(N)-$*property of Luzin with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^{+} : D \to \mathbb{C}$ *and* $f^{-} : \overline{\mathbb{C}}\setminus\overline{D} \to \mathbb{C}$ *with the source* g *such that*

$$f^{+}(\beta(\zeta)) \;=\; \varphi\left(\zeta,\, \left[\frac{\partial f}{\partial \nu}\right]^{-}(\zeta)\right) \qquad q.e. \text{ on } \partial D\ , \tag{31}$$

where $f^{+}(\zeta)$ *and* $\left[\frac{\partial f}{\partial \nu}\right]^{-}(\zeta)$ *are limits of the functions* $f^{+}(z)$ *and* $\frac{\partial f^{-}}{\partial \nu}(z)$ *as* $z \to \zeta$ *along* γ_{ζ}^{+} *and* γ_{ζ}^{-}, *correspondingly.*

Furthermore, the space of all such couples (f^{+}, f^{-}) *has the infinite dimension for any such prescribed* g, φ, ν, β *and collections* $\{\gamma_{\zeta}^{+}\}_{\zeta\in\partial D}$ *and* $\{\gamma_{\zeta}^{-}\}_{\zeta\in\partial D}$.

6. Poincare and Neumann problems in terms of angular limits.

In this section, we consider the Poincare boundary value problem on the directional derivatives and, in particular, the Neumann problem for the Poisson equations

$$\triangle U(z) \;=\; G(z) \tag{32}$$

with real valued functions G of classes $L^{p}(D)$ with $p > 2$ in the corresponding domains $D \subset \mathbb{C}$. Recall that a continuous solution U of (32) in the class $W^{2,p}_{\rm loc}$ is called a **generalized harmonic function with the source G** and that by the Sobolev embedding theorem such a solution belongs to the class C^{1}.

Theorem 5. *Let* D *be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent q.e.,* $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, *be in* $\mathcal{CBV}(\partial D)$ *and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to logarithmic capacity.*

Suppose that $G : D \to \mathbb{R}$ *is in* $L^{p}(D)$, $p > 2$. *Then there exist generalized harmonic functions* $U : D \to \mathbb{R}$ *with the source* G *that have the angular limits*

$$\lim_{z\to\zeta} \frac{\partial U}{\partial \nu}(z) \;=\; \varphi(\zeta) \qquad q.e. \text{ on } \partial D\ . \tag{33}$$

Furthermore, the space of such functions U has the infinite dimension.

Proof. Indeed, let us extend the function G by zero outside of D and let P be the logarithmic potential $\mathcal{N}_G$ with the source G, see (13). Then by Lemma 3 in [16] $P \in W^{2,p}_{\rm loc}(\mathbb{C}) \cap C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$ and $\triangle P = G$ a.e. in $\mathbb{C}$. Set

$$\varphi_*(\zeta) \ = \ {\rm Re}\,\nu(\zeta)H(\zeta) \ , \qquad \zeta \in \partial D \ , \tag{34}$$

where

$$H(z) := \overline{\nabla P(z)} \ , \ z \in \mathbb{C} \ , \qquad \nabla P := P_x + iP_y \ , \ z = x + iy \ . \tag{35}$$

Then by Theorem 1 with $g = G/2$ in D and $\lambda = \overline{\nu}$ on ∂D, there exist generalized analytic functions h with the source g that have the angular limits

$$\lim_{z\to\zeta} \ {\rm Re}\,\nu(\zeta)h(z) \ = \ \varphi(\zeta) \qquad \text{q.e. on } \ \partial D \tag{36}$$

and, moreover, by Remark 4 the given functions h can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ in D that have the angular limits

$$\lim_{z\to\zeta} \ {\rm Re}\ \nu(\zeta)\mathcal{A}(z) \ = \ \Phi(\zeta) \qquad \text{q.e. on } \partial D \tag{37}$$

with $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$, $\zeta \in \partial D$, and the space of such analytic functions $\mathcal{A}$ has the infinite dimension.

Note that any indefinite integral $\mathcal{F}$ of such $\mathcal{A}$ in the simply connected domain D is also a single-valued analytic function and the harmonic functions $u := {\rm Re}\ \mathcal{F}$ and $v := {\rm Im}\ \mathcal{F}$ satisfy the Cauchy-Riemann system $u_x = v_y$ and $u_y = -v_x$. Hence

$$\mathcal{A} \ = \ \mathcal{F}' \ = \ \mathcal{F}_x \ = \ u_x \ + \ i\cdot v_x \ = \ u_x \ - \ i\cdot u_y \ = \ \overline{\nabla u} \ . \tag{38}$$

Consequently, setting $U_* = u + P$, we see that U_* is a generalized harmonic function with the source G and, moreover, by the construction $h = \overline{\nabla U_*}$.

Note also that the directional derivative of U_* along the unit vector ν is the projection of its gradient ∇U_* into ν, i.e., the scalar product of ν and ∇U_* interpreted as vectors in $\mathbb{R}^2$ and, consequently,

$$\frac{\partial U_*}{\partial \nu} \ = \ (\nu, \nabla U_*) \ = \ {\rm Re}\ \nu\cdot\overline{\nabla U_*} \ = \ {\rm Re}\ \nu\cdot h \ . \tag{39}$$

Thus, (36) implies (33) and the proof is complete. □

Remark 4. We are able to say more in the case of ${\rm Re}\ n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν into n. In view of (33), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$U(z) \ = \ U(z_0) \ - \ \int\limits_0^1 \frac{\partial U}{\partial \nu} \ (z_0 + \tau(z - z_0)) \ d\tau \ . \tag{40}$$

Thus, at each point with condition (33), there is the directional derivative

$$\frac{\partial U}{\partial \nu}\,(\zeta) \;:=\; \lim_{t\to 0}\;\frac{U(\zeta+t\cdot\nu)-U(\zeta)}{t} \;=\; \varphi(\zeta)\ . \tag{41}$$

In particular, in the case of the Neumann problem, Re $n(\zeta)\overline{\nu(\zeta)} \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have by Theorem 5 and Remark 4 the following significant result.

Corollary 4. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta \in \partial D$, belong to the class $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then one can find generalized harmonic functions $U : D \to \mathbb{R}$ with the source G such that q.e. on ∂D there exist:

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) \;:=\; \lim_{z\to\zeta}\; U(z)\ ,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}\,(\zeta) \;:=\; \lim_{t\to 0}\;\frac{U(\zeta+t\cdot n(\zeta))-U(\zeta)}{t} \;=\; \varphi(\zeta)\ ,$$

3) the angular limit

$$\lim_{z\to\zeta}\;\frac{\partial U}{\partial n}\,(z) \;=\; \frac{\partial U}{\partial n}\,(\zeta)\ .$$

Furthermore, the space of such functions U has the infinite dimension.

7. Poincare and Neumann problems and Bagemihl–Seidel systems.

Arguing similarly to the last section, we obtain by Theorem 6 in [17], as well as Theorem 2 and Remark 2 above, the following statement.

Theorem 6. *Let D be a Jordan domain in $\mathbb{C}$, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{C}$ be measurable functions with respect to the logarithmic capacity and let $\{\gamma_\zeta\}_{\zeta\in\partial D}$ be a family of Jordan arcs of class $\mathcal{BS}$ in D.*

Suppose that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then there exist generalized harmonic functions $U : D \to \mathbb{C}$ with the source G that have the limits along γ_ζ

$$\lim_{z\to\zeta}\;\frac{\partial U}{\partial \nu}\,(z) \;=\; \varphi(\zeta) \qquad q.e.\ on\ \ \partial D\ . \tag{42}$$

Furthermore, the space of such functions U has the infinite dimension.

Remark 5. As it follows from the proofs of Theorems 5 and 6, the generalized harmonic functions U with a source $G \in L^p$, $p > 2$, satisfying the Poincare boundary conditions can be represented in the form of the sums $\mathcal{N}_G + \mathcal{H}$ of the logarithmic (Newtonian) potential $\mathcal{N}_G$ that is a generalized harmonic function with the source G and harmonic functions $\mathcal{H}$ satisfying the corresponding Poincare boundary conditions.

8. Other consequences in terms of Bagemihl–Seidel systems.

Finally, arguing similarly to the proofs of Corollaries 9 and 10 in [17] and supporting on Lemmas 1 and 2 from Sections 4 and 5, correspondingly, we obtain the following consequences.

Corollary 5. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $B : \partial D \to \mathbb{R}$ and $C : \partial D \to \mathbb{R}$ be functions that are measurable with respect to the logarithmic capacity and let $\alpha : \partial D \to \partial D$ be a homeomorphism keeping components of ∂D such that α and α^{-1} have the $(N)-$property of Luzin with respect to the logarithmic capacity.*

Suppose that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$, $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ are families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C}\setminus\overline{D}$, correspondingly. Then there exist generalized harmonic functions $u^+ : D \to \mathbb{R}$ and $u^- : \mathbb{C}\setminus\overline{D} \to \mathbb{R}$ with the source G such that

$$u^+(\alpha(\zeta)) \ = \ B(\zeta)\cdot u^-(\zeta) \ + \ C(\zeta) \qquad q.e. \ on \ \ \partial D \ , \tag{43}$$

where $u^+(\zeta)$ and $u^-(\zeta)$ are limits of $u^+(z)$ and $u^-(z)$ az $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.

Furthermore, the space of all such couples (u^+, u^-) has the infinite dimension for any such prescribed functions G, B, C, α and collections $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$.

In particular, we are able to obtain from the following corollary solutions of the problem on gluing of the Dirichlet problem in the unit disk $\mathbb{D}$ and the Neumann problem outside of $\mathbb{D}$ in the class of generalized harmonic functions with the source G.

Corollary 6. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be a measurable function, $\beta : \partial D \to \partial D$ be a homeomorphism such that β and β^{-1} have the $(N)-$property of Luzin and $\varphi : \partial D \times \mathbb{R} \to \mathbb{R}$ satisfy the Caratheodory conditions with respect to the logarithmic capacity.*

Suppose that $G : \mathbb{C} \to \mathbb{R}$ is in $C^\alpha(\mathbb{C})$, $\alpha \in (0, 1)$, with compact support, $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ are families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C}\setminus\overline{D}$, correspondingly. Then there exist generalized harmonic functions $u^+ : D \to \mathbb{R}$ and $u^- : \mathbb{C}\setminus\overline{D} \to \mathbb{R}$ with the source G such that

$$u^+(\beta(\zeta)) \ = \ \varphi\left(\zeta, \left[\frac{\partial u}{\partial \nu}\right]^-(\zeta)\right) \qquad q.e. \ on \ \ \partial D \ , \tag{44}$$

where $u^+(\zeta)$ and $\left[\frac{\partial u}{\partial \nu}\right]^-(\zeta)$ are limits of the functions $u^+(z)$ and $\frac{\partial u^-}{\partial \nu}(z)$ as $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.

Furthermore, the space of all such couples (u^+, u^-) has the infinite dimension for any such prescribed functions G, ν, β, φ and collections γ^+_ζ and γ^-_ζ, $\zeta \in \partial D$.

The corresponding results on the boundary value problems for semi–linear equations of mathematical physics in anisotropic and inhomogeneous media with arbitrary measurable data can be proved on the basis of the factorization theorem in the paper [14], too.

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Крайові задачі для узагальнених аналітичних і гармонічних функцій.

Вивчення задачи Діріхле з довільними вимірними даними для гармонічних функцій сходить до знаменитої дисертації Лузіна. Пізніше, відома монографія Вєкуа була присвячена крайовим задачам (тільки з неперервними по Гельдеру граничними даними) для узагальнених аналітичних функцій, а саме для неперервних комплекснозначних функцій комплексної змінної з узагальненими першими частинними похідними по Соболєву, які задовольняють лінійним рівнянням першого порядку, чиї коефіцієнти інтегровані порядку більше 2 у відповідних областях комплексної площини. Подана стаття є природнім продовженням наших статей, присвячених крайовим задачам Рімана, Гільберта, Діріхле, Пуанкаре і, зокрема, Неймана для квазіконформних, аналітичних, гармонічних і, так званих, A-гармонічних функцій з крайовими умовами, вимірними відносно логарифмічної ємності. В цій роботі ми поширюємо відповідні результати на узагальнені аналітичні функції з витоками інтегрованими порядку більше 2, а також на узагальнені гармонічні функції з витоками інтегрованими порядку більше 2. Також ми даємо відповідні визначення з необхідними посиланнями на згадані статті та коментарі до попередніх результатів. Стаття містить різноманітні теореми існування некласичних розв'язків крайових задач Гільберта та Рімана з довільними вимірними відносно логарифмічної ємності даними для узагальнених аналітичних функцій з витоками. Наш підхід грунтується на геометричній (теоретико-функциональной) інтерпретації граничних значень в порівнянні з класичним операторним підходом в теорії диференційних рівнянь з частинними похідними. На цій основі встановлені відповідні теореми існування для задачі Пуанкаре про похідні за напрямами і, зокрема, для задачі Неймана до рівняння Пуасона з довільними крайовими умовами, вимірними відносно логарифмічної ємності. Ці результати також можуть бути застосовані для напівлінійних рівнянь математичної фізики в анізотропних та неоднорідних середовищах.

Ключові слова: *рівняння Пуасона; задачі Рімана, Гільберта, Діріхле, Неймана та Пуанкаре; узагальнені аналітичні та гармонічні функції; логарифмічна ємність.*

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Received 24.11.19