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FIRST BASIC ELASTICITY THEORY PROBLEM IN A HALF-SPACE WITH SEVERAL PARALLEL ROUND CYLINDRICAL CAVITIES

A three-dimensional problem of the theory of elasticity is calculated, when a voltage is specified in an elastic half-space on the boundaries of parallel cylindrical cavities and on the boundary of a half-space. The solution is obtained by the generalized Fourier method relative to a system of the Lame equations in the cylindrical coordinates associated with the cylinders and the Cartesian coordinates related to the half-space. An infinite system of linear algebraic equations, to which the problem is reduced, is solved by the truncation method. As a result, displacements and stresses in an elastic body are found. Numerical results are presented for the case of a half-space and two cylinders.

Introduction

When designing different kinds of structures in rock and geotechnical mechanics, one has to know the stress-strain state of a half-space with cavities if loads are specified on the boundaries of the cavities and the half-space. Such information can be obtained by solving the relevant elasticity theory problem. Presently there are no ready-made solutions to such problems for a spatial variant, and hence, the problem of solving them is actual.

Review of publications

Spatial (3-dimensional) elasticity theory problems for bodies bounded by canonical surfaces (second-order surfaces: cylinder, cone, sphere, ellipsoid, paraboloid, and others) were investigated in [1–4]. The exact solutions in those studies were obtained exclusively by using both the method of separation of variables and the Fourier method. Problems for elastic bodies with several boundary surfaces cannot be solved within a classical framework. Such problems required that the generalised Fourier method [5–8] be used. That was why the approach to solving the problem indicated in the title of this paper was developed on that very method.

This article is a continuation of the author's work: the first basic problem for space with round cylindrical cavities, based on the generalised Fourier method, was considered in [9]. A mixed problem for space with cylindrical cavities, when displacements are specified on the boundaries of some parallel cylindrical cavities; stresses on others; tangential forces and displacements on third ones, is considered in [10]. In this work, unlike the author’s previous ones, a half-space is considered. In studies [11–15] the generalised Fourier method was also used for transversely-isotropic bodies bounded by coordinate surfaces in cylindrical and parabolic coordinates with a force applied to a half-space boundary. Paper [16] considered the first basic problem in the theory of elasticity for a half-space with one cylindrical cavity and the following conditions on the half-space boundary $\sigma_y = f(x)\cdot\cos(\lambda z),\ \ \tau_{yx}=\tau_{yz}=0$.

Objective of the paper-

The objective of this paper is to analyze the stress-strain state of a half-space with several cylindrical cavities parallel both to each other and the half-space boundary, as well as their mutual influence. To achieve this objective, the generalised Fourier method is used. The problem is being considered for the first time.

Problem statement

An elastic homogeneous half-space has *N* number of round cylindrical parallel cavities non-intersecting each other and the half-space boundary. The cavities will be considered in the cylindrical system of coordinates (ρ_p, φ_p, z, where *p* is cylinder number) and the half-space − in the Cartesian reference system (*x, y, z*), which is oriented identically and linked to the system of coordinates of the cylinder with number *p*=1. The half-space boundary is located at distance *y*=*h*, with the equalities for the half-space boundary S_d: *y*=*h*, and those for the surfaces of cylinders: S_p: $\rho_p = R_p$. Stresses are specified both on the half-space boundary and on the boundaries of the cavities which are assumed to decay rapidly to zero far from the origin of coordinates. One has to find a solution to the Lame equation $\Delta\vec{u}+(1-2\sigma)^{-1}\nabla\,\mathrm{div}\,\vec{u}=0$ provided that the stresses $\vec{f}_p(\varphi_p,z)$ are specified on the boundaries of cylindrical cavities, $p=1,\ 2,\ ...,\ N$; (ρ_p,φ_p,z) ρ is a system of local cylindrical coordinates, and the stresses $\vec{f}_d(x,z)$ are specified on the half-space boundary. The vectors $\vec{f}_p(\varphi_p,z)$ and $\vec{f}_d(x,z)$ are assumed to decay rapidly to zero along coordinate z far from the origin of coordinates.

Method of solution

In the given systems of coordinates, we take the basic solutions to the Lame equation:

$$\vec{u}_k^{\pm}(M_d;\lambda,\mu)=N_k^{(d)}u^{\pm}(M_d;\lambda,\mu),\quad (k=1,2,3),\tag{1}$$

$$\vec{R}_{k,m}(M_p;\lambda)=N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\varphi)},\quad \vec{S}_{k,m}(M_p;\lambda)=N_k^{(p)}\left[s_m(\rho_p;\lambda)\cdot e^{i(\lambda z+m\varphi)}\right],\quad k=1,2,3,\tag{2}$$

$$N_1^{(d)}=\frac{1}{\lambda}\nabla,\quad N_2^{(d)}=\frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)}+\frac{1}{\lambda}\nabla(y\,\cdot),\quad N_3^{(d)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(1)}\right),\ u^{\pm}(M_d;\lambda,\mu)=e^{i(\lambda z+\mu x)\pm\gamma y},$$

$$N_1^{(p)}=\frac{1}{\lambda}\nabla,\quad N_2^{(p)}=\frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right)+4(\sigma-1)\left(\nabla-\vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right],\quad N_3^{(p)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(2)}\right),$$

$$s_m(\rho_p;\lambda)=(\mathrm{sign}\,\lambda)^m K_m(|\lambda|\rho_p),\quad \gamma=\sqrt{\lambda^2+\mu^2},\ -\infty<\lambda,\mu<\infty,$$

where $M_d=(x,y,z)$ is a point in space in the Cartesian reference system linked to a half-space; $M_p=(\rho_p,\varphi_p,z)$ is a point in space in the cylindrical system of coordinates linked to the *p*-th cylinder; $\vec{e}_j^{(k)},\ (j=1,2,3)$ are the unit coordinate vectors of the Cartesian (*k*=1) and cylindrical (*k*=2) systems of coordinates; σ is the Poisson ratio; $I_m(\lambda\rho)$, $K_m(|\lambda|\rho)$ are the modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, $(k=1,2,3)$ are respectively the internal and external solutions to the Lame equation for a cylinder; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ are solutions to the Lame equation for a half-space.

The solution is presented as follows:

$$\vec{U}=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(M_d;\lambda,\mu)d\mu d\lambda+\sum_{p=1}^{N}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}^{(p)}(\lambda)\cdot\vec{S}_{k,m}(M_p;\lambda)d\lambda,\tag{3}$$

where $\vec{S}_{k,m}(M_p;\lambda)$ and $\vec{u}_k^{(+)}(M_d;\lambda,\mu)$ are the basic solutions specified by the formulas (1) and (2), and the unknown functions $H_k(\lambda,\mu)$ and $B_{k,m}^{(p)}(\lambda)$ are to be found from the edge conditions.

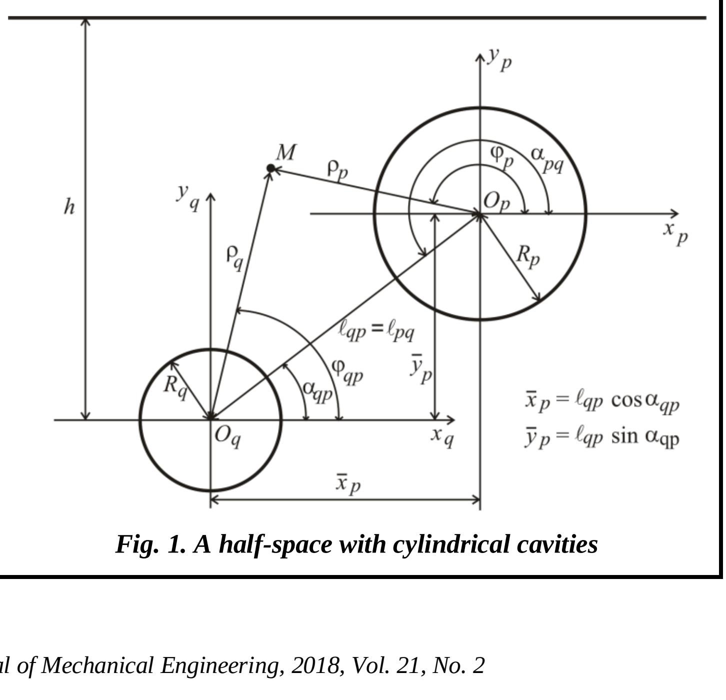


Fig. 1. A half-space with cylindrical cavities

The following formulas are used for transition between the coordinate systems (Fig. 1):

– for transition from the coordinates of cylinder with number p to the half-space coordinates

$$\vec{S}_{k,m}\left(M_p;\lambda\right)=\frac{\left(-i\cdot\operatorname{sign}(\lambda)\right)^m}{2}\int\limits_{-\infty}^{\infty}\omega^m\cdot\vec{u}_k^{(-)}\cdot e^{-i\mu\bar{x}_p+\gamma\bar{y}_p}\frac{d\mu}{\gamma},\quad k=1,3,$$

$$\vec{S}_{2,m}\left(M_p;\lambda\right)=\frac{\left(-i\cdot\operatorname{sign}(\lambda)\right)^m}{2}\int\limits_{-\infty}^{\infty}\omega^m\cdot\left(\left(m\cdot\mu-\frac{\lambda^2}{\gamma}+\lambda^2 y_p\right)\vec{u}_1^{(-)}-\lambda^2\vec{u}_2^{(-)}+4\mu(1-\sigma)\vec{u}_3^{(-)}\right)\frac{e^{-i\mu\bar{x}_p+\gamma\bar{y}_p}d\mu}{\gamma^2},\tag{4}$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$, $\omega(\lambda,\mu)=\dfrac{\mu-\gamma}{\lambda}$, $y>0$, $m=-\infty\ldots\infty$;

– for transition from the half-space coordinates to the coordinates of cylinder with number p

$$\vec{u}_k^{(+)}=\sum_{m=-\infty}^{\infty}(i\cdot\omega)^m\vec{R}_{k,m}\cdot e^{i\mu\bar{x}_p+\gamma\bar{y}_p},\quad (k=1,3),$$

$$\vec{u}_2^{(+)}=e^{i\mu\bar{x}_p+\gamma\bar{y}_p}\sum_{m=-\infty}^{\infty}\left[(i\cdot\omega)^m\cdot\lambda^{-2}\left(m\cdot\mu\cdot\vec{R}_{1,m}+\gamma\cdot\vec{R}_{2,m}+4\mu(1-\sigma)\vec{R}_{3,m}\right)+\bar{y}_p\cdot(i\cdot\omega)^m\vec{R}_{1,m}\right],\tag{5}$$

where $\vec{R}_{k,m}=\tilde{\vec{b}}_{k,m}\left(\rho_p,\lambda\right)\cdot e^{i\left(n\varphi_p+\lambda z\right)}$; $\bar{x}_p,\bar{y}_p$ are the coordinates of the cylinder p relative to the first cylinder.

$$\tilde{\vec{b}}_{1,n}\left(\rho_p,\lambda\right)=\vec{e}_\rho\cdot I_n'\left(\lambda\rho_p\right)+i\cdot I_n\left(\lambda\rho_p\right)\cdot\left(\vec{e}_\varphi\frac{n}{\lambda\rho_p}+\vec{e}_z\right),$$

$$\tilde{\vec{b}}_{2,n}\left(\rho_p,\lambda\right)=\vec{e}_\rho\cdot\left[(4\sigma-3)\cdot I_n'\left(\lambda\rho_p\right)+\lambda\rho_p I_n''\left(\lambda\rho_p\right)\right]+\vec{e}_\varphi i\cdot m\left(I_n'\left(\lambda\rho_p\right)+\frac{4(\sigma-1)}{\lambda\rho_p}I_n\left(\lambda\rho_p\right)\right)+\vec{e}_z i\lambda\rho_p I_n'\left(\lambda\rho_p\right),$$

$$\tilde{\vec{b}}_{3,n}\left(\rho_p,\lambda\right)=-\left[\vec{e}_\rho\cdot I_n\left(\lambda\rho_p\right)\frac{n}{\lambda\rho_p}+\vec{e}_\varphi\cdot i\cdot I_n'\left(\lambda\rho_p\right)\right];$$

– for transition from the coordinates of cylinder with number p to the coordinates of the cylinder q

$$\vec{S}_{k,m}\left(M_p;\lambda\right)=\sum_{n=-\infty}^{\infty}\vec{b}_{k,pq}^{mn}\left(\rho_q\right)\cdot e^{i\left(n\varphi_q+\lambda z\right)},\quad k=1,2,3;$$

$$\vec{b}_{1,pq}^{mn}\left(\rho_q\right)=(-1)^n\tilde{K}_{m-n}\left(\lambda\ell_{pq}\right)\cdot e^{i(m-n)\alpha_{pq}}\cdot\tilde{\vec{b}}_{1,n}\left(\rho_q,\lambda\right),$$

$$\vec{b}_{3,pq}^{mn}\left(\rho_q\right)=(-1)^n\tilde{K}_{m-n}\left(\lambda\ell_{pq}\right)\cdot e^{i(m-n)\alpha_{pq}}\cdot\tilde{\vec{b}}_{3,n}\left(\rho_q,\lambda\right),\tag{6}$$

$$\vec{b}_{2,pq}^{mn}\left(\rho_q\right)=(-1)^n\left\{\tilde{K}_{m-n}\left(\lambda\ell_{pq}\right)\cdot\tilde{\vec{b}}_{2,n}\left(\rho_q,\lambda\right)-\frac{\lambda}{2}\ell_{pq}\cdot\left[\tilde{K}_{m-n+1}\left(\lambda\ell_{pq}\right)+\tilde{K}_{m-n-1}\left(\lambda\ell_{pq}\right)\right]\cdot\tilde{\vec{b}}_{1,n}\left(\rho_q,\lambda\right)\right\}\cdot e^{i(m-n)\alpha_{pq}}.$$

where α_{pq} is an angle between the coordinate axes and the segment ℓ_{qp}, $\tilde{K}_m(x)=\left(sign(x)\right)^m\cdot K_m\left(|x|\right)$.

Having satisfied the edge conditions on the boundary S_q of the cylinder q in the solution (3), we rewrite $\vec{u}_k^{(+)}\left(M_d\right)$ in the system of coordinates of the cylinder $p=q$ using the formulas (5). For each cylinder $p\neq q$, we rewrite $\vec{S}_{k,m}\left(M_p;\lambda\right)$ using the formulas (6). Taking into account the basic solutions (2) for the cylinder $p=q$, for this equation we apply the stress operator

$$F\vec{U}=2\cdot G\cdot\left[\frac{\sigma}{1-2\cdot\sigma}\vec{n}\cdot\operatorname{div}\vec{U}+\frac{\partial}{\partial n}\vec{U}+\frac{1}{2}\left(\vec{n}\times\operatorname{rot}\vec{U}\right)\right],\tag{7}$$

where $\vec{n}$ is unit normal vector to surface S_q; G is shear modulus; σ is the Poisson ratio. This yields the equation for $H_s(\lambda,\mu)$ and $B_{s,m}^{(p)}(\lambda)$ on the cylinder q, which we designate as

$$F\vec{U}\big|_{\rho_q=R_q} = \vec{f}_q(\varphi_q, z). \tag{8}$$

Satisfying the edge conditions on the boundary of the half-space S_d in the solution (3), we use the formulas (4) to rewrite $\vec{S}_{k,m}(M_p;\lambda)$ for each cylinder in the half-space system of coordinates. Taking into account the basic solution for a half-space (1), we apply to this equation the stress operator (7) to obtain equations for $H_s(\lambda,\mu)$ and $B_{s,m}^{(p)}(\lambda)$ in the half-space, which we denote as

$$F\vec{U}\big|_{y=h} = \vec{f}_d(x, z). \tag{9}$$

We free (9) from integrals and series, and find the expression for $H_s(\lambda,\mu)$. Having freed (8) from integrals and series, we substitute it into the expression for $H_s(\lambda,\mu)$. This yields a vector equation. If now we project this vector equation onto the axis of coordinates, this will yield a set of $3\cdot N$ infinite systems of algebraic equations for $B_{s,m}^{(p)}(\lambda)$. For the systems obtained, one can prove their unique solvability. Moreover, one can solve these systems, using the truncation method, and converge approximate solutions to an exact one [16]. The unknowns $B_{s,m}^{(p)}(\lambda)$ found from the systems of equations are substituted into (9) to yield the unknowns $H_s(\lambda,\mu)$.

Numerical results for two cylinders

We have two parallel cylindrical cavities in a half-space (Fig. 1). The space is an isotropic material, the Poisson ratio σ=0.35, the elasticity modulus $E = 2\ \text{kN}/\text{cm}^2$. The half-space boundary is located $\ell_{qp} = R_p + h$ from the cylinder q. On the cylinder boundaries, the stresses $\sigma_\rho^{(p)} = \sigma_\rho^{(q)} = 0$; $\tau_{\rho\varphi}^{(p)} = \tau_{\rho\varphi}^{(q)} = 0$; $\tau_{\rho z}^{(p)} = \tau_{\rho z}^{(q)} = 0$ are specified, on the half-space boundary $\sigma_y^{(d)} = -\left(10^{-4}\cdot\left(z^2+10^2\right)^2\right)\cdot\left(10^{-4}\cdot\left(x^2+10^2\right)^2\right)$; $\tau_{yx}^{(d)} = \tau_{yz}^{(d)} = 0$.

The obtained infinite system of equations was reduced to a finite one for the parameter m – system order (the parameter m is investigated in [14]), the integration limits for the given function having been taken from -1 to 1. The integrals were calculated using the Philo and Simpson quadrature formulas. The accuracy achieved to satisfy the boundary conditions due to these parameters was brought to 10^{-4}.

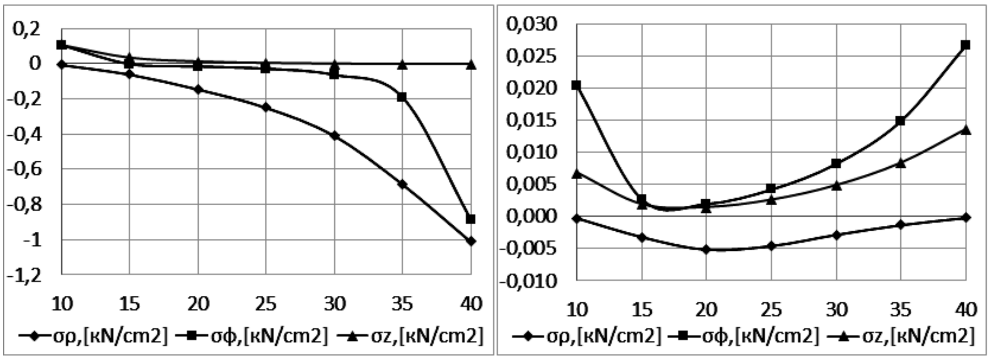
To analyze the mutual influence of the half-space and cylindrical cavities, 6 variants of geometric parameters were calculated.

Variant 1. Cavity radius R_p=50 cm, cavity radius R_q=10 cm, α_{qp}=3π/2, h=40 cm. Fig. 2 shows normal stresses in the bridges: in fig. 2, a – between the cylinder q and the half-space boundary, and in fig. 2, b – between the cylinders. Fig. 2, a depicts the graph of normal stresses in the bridge between the cylinder q and the half-space boundary, whence it can be seen that in moving away from the half-space to the cylinder, the stresses σ_ρ and σ_φ decrease gradually, and in the end σ_φ and σ_z become tensile ones. Fig. 2, b shows that, from the cylinder q to the cylinder p, the stresses σ_φ and σ_z first decrease and then, as a result of the space being weakened by the cavity p, they increase and on the boundary of the cylinder p become even greater than on that of the cylinder q, the stress σ_ρ in this interval, due to redistribution of stresses, demonstrating a slight increase.

Variant 2. Cavity radius R_p=30 cm, cavity radius R_q=10 cm , α_{qp}=0, h=40 cm. Normal stresses in the bridges are shown in fig. 3. In this case, the cylindrical cavity p is located on the horizontal axis together with the cylinder q and, as compared to the first variant, has no effect on the stress in the bridge between the cylinder q and the half-space boundary (Fig. 3, a). The stresses between the cylinders (Fig. 3, b) decay as they approach the cylinder p.

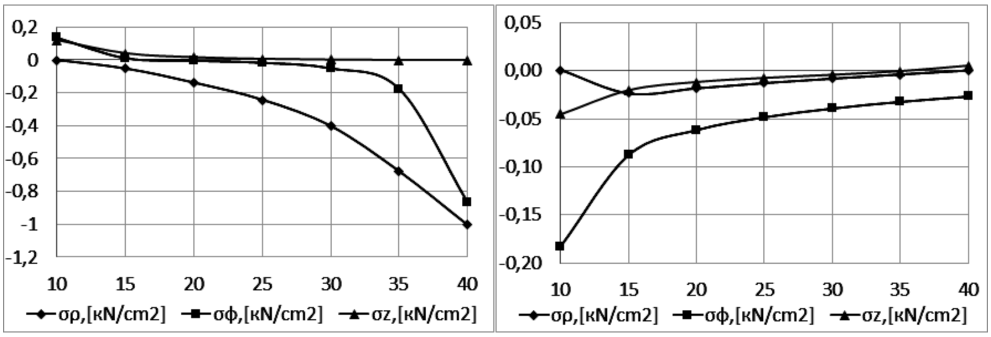
Variant 3. Cavity radius R_p=10 cm, cavity radius R_q=10 cm, α_{qp}=3π/2, h=40 cm. Normal stresses in the bridge are shown in fig. 4.

As compared to variant 1, the radius of the cylinder p is reduced, which affected the stresses in the bridge between the cylinders (Fig. 3, b): on the cylinder q both σ_φ and σ_z increased; on the cylinder p they decreased, and σ_ρ increased, becoming more pronounced.



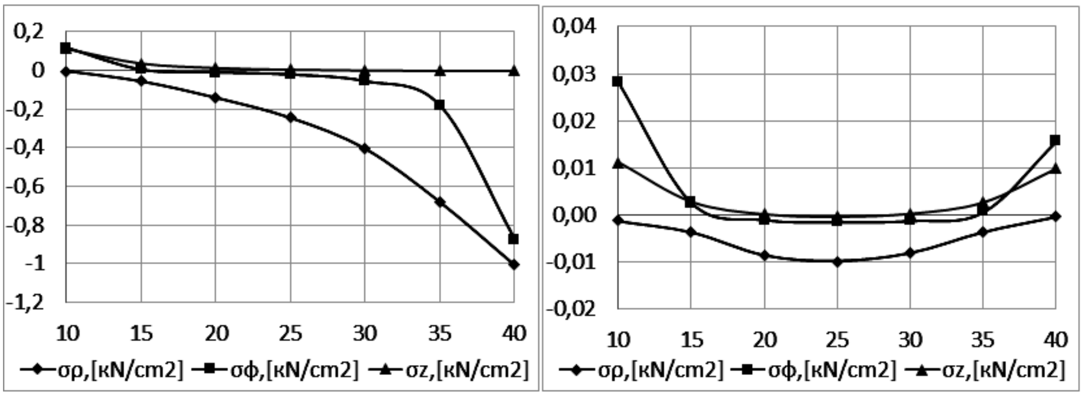
a b

Fig. 2. Normal stresses for the cylinder q in the bridges in plane z=0:
a – between the cylinder *q* and the half-space boundary; b – between the cylinders



a b

Fig. 3. Normal stresses for the cylinder q in the bridges in plane z=0:
a – between the cylinder *q* and the half-space boundary; b – between the cylinders



a b

Fig. 4. Normal stresses for the cylinder q in the bridges in plane z=0:
a – between the cylinder *q* and the half-space boundary; b – between the cylinders

Variant 4. Cavity radius R_p=10 m, cavity radius R_q=10 m, $\alpha_{qp}=3\pi/2$, h=20 m. Normal stresses in the bridges are shown in fig. 5.

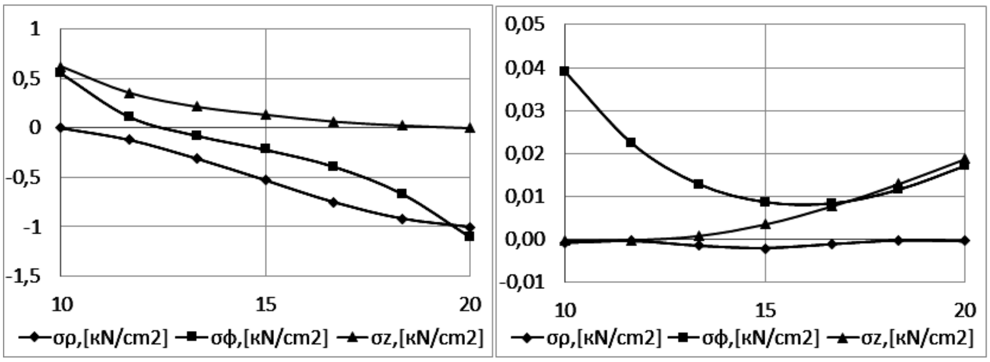
As compared to variant 3, both the distance between the cylinder *q* and the half-space as well as that between the cylinders decreased, which had a substantial effect on the stresses between the cylinder *q* and the half-space boundary (Fig. 5, a): on the cylinder *q* the stresses σ_φ and σ_z increased, and in the half-space the stresses

σ_φ increased. In the bridge between the cylinders σ_ρ decreased, σ_z decreased on the cylinder q and increased on the cylinder p, σ_φ increased slightly.

Variant 5. Cavity radius R_p=200 cm, cavity radius R_q=10 cm, α_{qp}=3π/2, h=40 cm. Normal stresses in the bridges are shown in fig. 6.

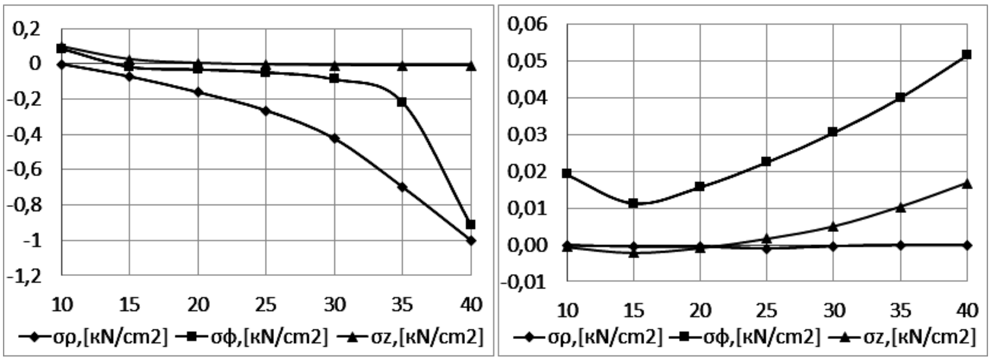
As compared to variants 1 and 3, the radius of the cylinder p increased, imitating the half-space. These changes have little effect in the bridge between the cylinder q and the half-space boundary (Fig. 6, a).

In the bridge between the cylinders (Fig. 6, b), as the radius R_p increases the stresses σ_ρ decrease, on the cylinder q, the stresses σ_φ and σ_z decrease, however they increase on the cylinder p.



a b

Fig. 5. Normal stresses for the cylinder q in the bridges in plane z=0:
a – between the cylinder q and the half-space boundary; b – between the cylinders



a b

Fig. 6. Normal stresses for the cylinder q in the bridges in plane z=0:
a – between the cylinder q and the half-space boundary; b – between the cylinders

Conclusion

With the help of the generalised Fourier method the spatial problem in the theory of elasticity when stresses are specified on the half-space boundary and on the boundaries of several parallel round cylindrical cavities is solved. The problem is reduced to an infinite set of systems of linear algebraic equations.

The numerical analysis of the algebraic system for two cylinders and a half-space allows asserting that its solution can be found with any degree of accuracy using the reduction method. This is confirmed by the high accuracy of satisfying the boundary conditions.

The graphs show the pattern of stresses in the body in the most interesting zones. They also give a representation of the mutual influence of cylindrical cavities on each other in the half-space, and the mutual influence of the half-space and cylindrical cavities.

Within the topic of the study there is a possibility of a further consideration of problems with mixed boundary conditions, when stresses are specified on some boundaries and displacements are specified on the others.

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