Low Temperature Physics/Fizika Nizkikh Temperatur, 2020, vol. 46, No. 11, pp. 1276–1286

Dynamics for pair of coupled nonlinear systems. II. Dicrete self-trapped model

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Received May 25, 2020, published online September 21, 2020

In the framework of the discrete self-trapped model and its generalizations, the dynamics of two nonlinear elements of different physical origin is considered. The influence on the dynamics of their own nonlinearity, various types of interaction nonlinearity and nonequivalence of subsystems is investigated. Exact solutions of dynamic equations are found and investigated. Particular attention is paid to the study of essentially nonlinear inhomogeneous states with different levels of excitation for identical subsystems as a discrete analogue for different solitons.

Keywords: stationary states, main nonlinear oscillations, inhomogeneous states, integrals of motion, phase portrait, bifurcations.

1. Introduction

In solving numerous problems of nonlinear physics, a small set of well-studied models and the corresponding equations are often used such as: the sin-Gordon equation, the Korteweg–de Vries, Boussinesq, Landau–Lifshitz equations, the nonlinear Schrödinger equation [1, 2]. These partial differential evolution equations are completely integrable and widely used for the analysis of nonlinear excitations in various media (optics [3], hydrodynamics [4], electrodynamics [5] and crystal lattice theory [6]), in particular, for the study of soliton excitations. In integrable systems (usually one-dimensional), soliton solutions have a simple form and solitons have no internal dynamics. In contrast, in non-integrable systems (mainly multidimensional and discrete), exact soliton solutions are absent, and approximate and obtained numerical solutions demonstrate internal dynamics. But usually, due to the complexity of the problem, the simplest "stationary" one-frequency states are investigated. On the other hand, as was first shown in Ref. 7, inhomogeneous, localized excitations are also possible in the systems with a finite number of the degrees of freedom. These "quasi-soliton" excitations have many properties of solitons in the systems with distributed parameters.

Their study makes it possible to better understand the dynamics of soliton excitations in non-integrable distributed systems (in particular, the internal dynamics of solitons). In this context, it is especially important to study the finite-dimensional exactly integrable systems and the entire set of their exact solutions. That is why, recently, much attention, has been paid to the study of discrete systems and systems with the finite number of degrees of freedom and finite-dimensional analogues of these models [8, 9]. First, we are talking about discrete media (for example, molecular crystals) under conditions of strong localization of excitations on several particles or elements of the system [7]. Secondly, the case of the interaction of several nanoelements, for example, a pair of qubits [10] or coupled magnetic bilayers, can be attributed to such systems. Finally, the interaction of several nonlinear wave modes in the same object is possible (the interaction of waves of different polarization in optical waveguides [11, 12]). Just as in systems with distributed parameters the problem often reduces to the popular nonlinear Schrödinger equation, in discrete nonlinear systems the so-called DSTM (discrete self-trapped model) and DNSE (discrete nonlinear Schrodinger equation) [13] and their modifications are often used. In the traditional form [8], this model is described by the equation

$$i\dot{\psi}_n = \omega_0\psi_n - \alpha|\psi_n|^2\psi_n + \varepsilon(\psi_n - \psi_m). \qquad (1)$$

It is used to describe nonlinear (in particular, soliton) dynamics in layered media, quasi-two-dimensional magnets, and the propagation of coherent optical pulses (solitons) in waveguide systems and photonic crystals [3]. Modifications of this equation of the form

$$i\dot{\psi}_n = \omega_0\psi_n - \alpha|\psi_n|^2\psi_n + F_n(\psi_n, \psi_m)$$

studied below for some specific types of the function F are used for description of the coupled qubits [10], spatially homogeneous interacting Bose–Einstein condensates [14], and the interaction of optical pulses with different polarizations [12]. Usually, within the framework of these models, the soliton dynamics of systems is studied. However, as was previously shown [7], in the systems with a small number of degrees of freedom, it appears some properties of nonlinear dynamics which are characteristic for systems with a large number of degrees of freedom and systems with distributed parameters. Moreover, in systems with two degrees of freedom the possibility of states similar to soliton states in systems with distributed parameters appears. This was demonstrated in [15] for coupled magnetic systems as an example. In this paper, the nonlinear dynamics of DSTM for two degrees of freedom is investigated and exact solutions of the corresponding equations are found. Also the dynamics was analyzed in the model, generalizing DSTM.

2. DSTM for two coupled elements with nonlinear on-site potentials

We begin the consideration of the problem with the standard DSTM in the framework of the NLSE for two particles, the dynamics of which are described by the system of equations (1). This system of equations describes, for example, a magnetic double layers used in magnetic valves, a pair of optical fibers at the point of contact in the optical switches, a pair of magnetic nanodots. At low excitation levels, the Landau–Lifshitz equations considered in [15] approximate to the system of equations (1). But at high excitation levels, the dynamics of equations (1) has a number of properties that are absent in magnetic systems. It is convenient to describe the dynamics of a linear oscillator in terms of complex quantity $\psi = (\omega_0 x + ip/m)/\sqrt{2\omega_0}$, which plays the role of a classic analogue of the annihilation operator for the quantum of the excitation for the oscillator with frequency ω_0. By taking into account the weak (linear) interaction between oscillators and the simplest form of self nonlinearity (of on-site potential), we get for the two oscillators in a resonant approximation the system of two equations (1), with $n, m = 1, 2$. This is a special case of the well-known discrete self-trapped model for two particles. The system of equations (1) is fully integrable, with two integrals of motion: total energy

$$E = \sum\left(\omega_0|\psi_n|^2 - \alpha|\psi_n|^4/2\right) + \varepsilon|\psi_1 - \psi_2|^2 \quad (2)$$

and additional integral

$$N = \sum|\psi_n|^2, \quad (3)$$

corresponding to adiabatic invariant and represented the number of excitations under the quasi-classical interpretation of the dynamics of the coupled oscillators. Since $N = \text{const}$, it is convenient to introduce the following new variables:

$$\psi_1 = \sqrt{N}\cos\vartheta\exp(i\varphi_1), \quad \psi_2 = \sqrt{N}\sin\vartheta\exp(i\varphi_2), \quad (4)$$

in terms of which equations (1) will be rewritten as

$$\dot{u} = 2\varepsilon\sin\psi, \quad (5)$$

$$\dot{\psi} = 2\varepsilon\,\text{ctg}\,u\,\cos\psi - \alpha N\cos u, \quad (6)$$

where $u = 2\vartheta$ and $\psi = \varphi_2 - \varphi_1$. This closed system for the variables u and ψ is supplemented by the equation for the variable $\varphi = \varphi_2 + \varphi_1$:

$$\dot{\varphi} = 2(\omega_0 + \varepsilon - \alpha N/2) + 2\varepsilon\cos\psi/\sin u. \quad (7)$$

In new variables the energy is written as

$$E = \left(\omega_0 + \varepsilon(1 - \sin u\cos\psi)\right)N - \alpha\left(1 + \cos^2 u\right)N^2/4. \quad (8)$$

Equations (1) allow single-frequency solutions corresponding to stationary states with the following relation between the amplitudes of oscillators:

$$(a_1 - a_2)(a_1 + a_2)(a_1 a_2 - \varepsilon/\alpha) = 0. \quad (9)$$

Thus the in-phase (s), anti-phase (a) and nonuniform (n) stationary states with the following dependences of frequency oscillations on the solution norm are possible

$$\omega_s = \omega_0 - \alpha N/2, \quad (10)$$

$$\omega_a = \omega_0 + 2\varepsilon - \alpha N/2, \quad a_2 = -a_1, \quad (11)$$

$$\omega_n = \omega_0 + \varepsilon - \alpha N, \quad a_2 = \varepsilon a_1/\alpha. \quad (12)$$

These dependences demonstrate two important properties of nonlinear oscillations: the dependence of the frequency of oscillations on their energy (or, which is the same — on their norm), and the emergence of additional to the "main nonlinear" oscillations the dynamic state of motion at the critical level of the excitation $N > N_b = 2\varepsilon/\alpha$ in bifurcation way: the motion with different amplitudes of oscillations for the different elements of the system $a_1 \neq a_2$, i. e., the localization of energy on one of the two identical oscillators. These features of the dynamics are illustrated in Fig. 1(a). Unfortunately, only the single-frequency solutions can be depicted on the plane (ω, N), while the vibrations of the general type are two-frequency with incommensurate frequencies. Their properties are convenient to discuss on the plane of the integrals of motion (E, N).

The dependences of the energy of single-frequency oscillations on their norms (adiabatic invariants) are determined by the usual mechanical ratio $\omega = dE/dN$. Therefore, it is easy to get the relation between the motion

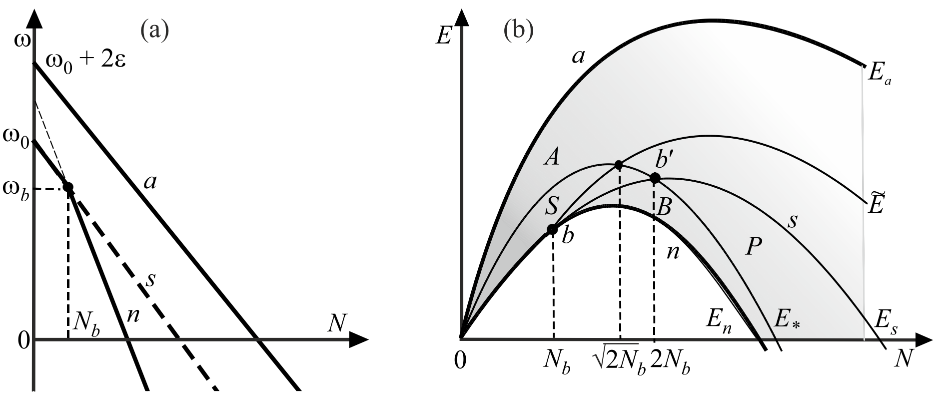

Fig. 1. Dependences of the stationary state frequencies on the norms of the solutions (a) and (b): the area of existence for solutions with different dynamics on the plane (E, N) of integrals of motion (shaded in figure).

integrals for single-frequency vibrations, corresponding to dependences (10)–(12):

$$E_s = \omega_0 N - \alpha N^2 / 4, \quad (13)$$

$$E_a = (\omega_0 + 2\varepsilon) N - \alpha N^2 / 4, \quad (14)$$

$$E_n = (\omega_0 + \varepsilon) N - \alpha N^2 / 2 - \varepsilon^2 / \alpha, \quad (15)$$

$$E_* = (\omega_0 + \varepsilon) N - \alpha N^2 / 2, \quad (16)$$

$$\tilde{E} = (\omega_0 + \varepsilon) N - \alpha N^2 / 4 - 2\varepsilon^2 / \alpha. \quad (17)$$

Here the important (as it will be seen later) dependences (16), (17) are given, in which there appear the important changes in the dynamics of the system. The dependences (13)–(17) are depicted in Fig. 1(b).

In Fig. 1(b), the line a corresponds to anti-phase oscillations (phases of which differ by π), line s — $0bb'sE_s$ — to the in-phase oscillations, line n — bnE_n — to localized states with different levels of excitation of the oscillators (localization of energy on one element of the system), and the line $0b'E_*$ corresponds to the dependence from Eq. (15). Equations (5), (6) correspond to the effective system with one degree of freedom and the integral of motion (8), which can be integrated in quadratures, and its dynamics is depicted on the «phase plane» (u, ψ) — Fig. 2.

In Fig. 2(a), the special points such as «centers» s and a correspond to single-frequency in-phase and anti-phase oscillations and to the lines $0b$ and $0a$ in Fig. 1(b), while the detached separatrix (C) separates the oscillations close to the in-phase one from those close to the anti-phase oscil-

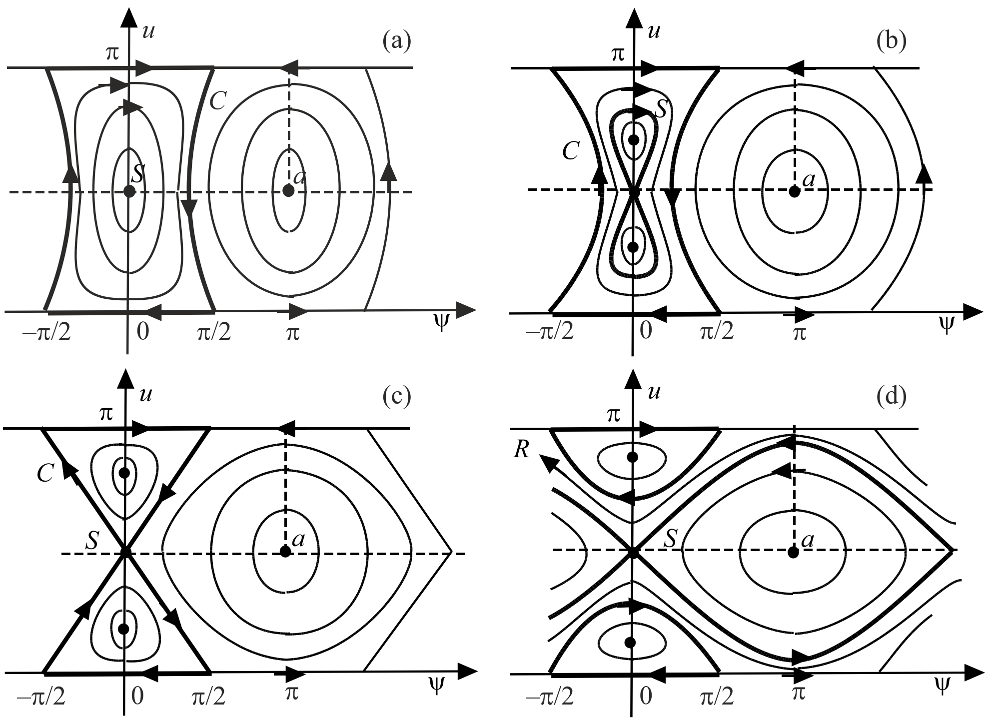

Fig. 2. Phase portraits of the system (5), (6) on the plane of variables (u, ψ) for different values of the complete norm of the solution: $N < N_b$ (a), $N_b < N < 2N_b$ (b), $N = 2N_b$ (c) and $N > 2N_b$ (d).

lations. It corresponds to the dependence (15) and to the line $0b'$ in Fig. 1(b). Special points of the type of «centeres» are stable, which indicates the stability of in-phase and anti-phase excitations in this range of norm values. At the point of bifurcation b there appear two new states with different values of oscillators excitations: two new centers in Fig. 2(b). They correspond to the line n in Fig. 1a and the line bnE_n in Fig. 1(b). The saddle point $(u = \psi = 0)$ is now associated with in-phase oscillations. This denotes that the in-phase oscillations become unstable for $N > N_b$. They correspond to the dashed line s in Fig. 1(a) and the line $(bb'sE_s)$ in Fig. 1(b). The last line simultaneously corresponds to the separate loops S in Fig. 2(b). The areas inside the separatrix S, between the separatrices S and C and outside of them, correspond, respectively, to the domains B, S and A in Fig. 1(b). The second bifurcation takes place at $N = 2N_b$, i. e., in the point of merging of separatices S and C on the phase portrait [triangles in Fig. 2(c)]. Finally, at $N > 2N_b$ the two separatrices are separated again [Fig. 2(d)], but the separatrix which going out of the saddle point of the in-phase oscillations does not have a form of closed separation loop.

The states investigated by qualitative methods in the phase plane allow exact analytical solutions. From Eq. (7) it results the relation $\psi = \psi(u, E, N)$:

$$\cos\psi = \frac{\left(E_* - E\right) + \left(\alpha N^2/4\right)\sin^2 u}{\varepsilon N \sin\left(u\right)}, \tag{18}$$

and for the new function $f = (\alpha N/2)\cos(u)$ Eq. (5) can be reduced to a closed equation for the function f:

$$\left(df/dt\right)^2/\alpha = \left(A_1 - f^2\right)\left(f^2 - A_2\right), \tag{19}$$

with

$$A_{1,2} = \left(\left(\tilde{E} - E\right) \pm 2\varepsilon\sqrt{E - E_n}/\sqrt{\alpha}\right). \tag{20}$$

2.1. Low-amplitude excitations

In the domain with low-amplitude excitation for $N < N_b$, in which $\tilde{E} < E_n < E_s < E < E_a$ [Fig. 2(a)],

$$A_1 = -\left(E - \tilde{E}\right) + 2\varepsilon\sqrt{E - E_n}/\sqrt{\alpha} = a^2 < b^2, \tag{21}$$

$$A_2 = -\left(E - \tilde{E}\right) - 2\varepsilon\sqrt{E - E_n}/\sqrt{\alpha} = -b^2 \tag{22}$$

and the solution to the Eq. (19) reads

$$f = a\,\mathrm{cn}\left(\sqrt{a^2 + b^2}\sqrt{\alpha}\,t, k\right), \quad k = a/\sqrt{a^2 + b^2}, \tag{23}$$

where $\mathrm{cn}(z, k)$ denotes Jacobi's elliptical cosine. So $a = 0$, $b^2 = 2\varepsilon\left(N + N_b\right)$ for $E = E_a$ and $a = 0$, $b^2 = 2\varepsilon(N_b - N)$ for $E = E_s$. The frequency of the oscillation for the relative amplitude of subsystems is equal to

$$\Omega\left(E, N\right) = \frac{\pi\sqrt{\alpha}}{2kK\left(k\right)}a, \tag{24}$$

where $K(k)$ is the complete elliptical integral of the first kind. On the boundaries for the area of the solution existing (a and s in Fig. 1(b)) $k \to 0$, $a \to 0$ and $b_{s,a} \to \sqrt{\alpha N_b\left(N_b \mp N\right)}$. That is, the frequency of periodic energy transfer between the oscillators is equal to

$$\Omega_{s,a} = \alpha\sqrt{N_b\left(N_b \mp N\right)} = 2\varepsilon\sqrt{1 \mp N/N_b}\,. \tag{25}$$

In the linear limit it turns out the well-known result $\Omega = 2\varepsilon$. The elliptical integral module reaches the maximum value $k_m = N/2N_b$ on the line $E = E_*(N)$ (16) on which $a_m^2 = \alpha N^2/4$. This line corresponds to the separatrix C in the phase portrait in Fig. 2(a).

The formula (25) describes the relative oscillations of the amplitudes of the two oscillators and they are determined by the interaction of subsystems. In addition, the system demonstrates a joint «rotation» (similar to the joint rotation of two related magnetic moments in [15]) with frequencies close to the oscillators' own frequency ω_0. This movement resembles the oscillations of two bound linear oscillators, which vibrate with the frequencies of their own modes (in-phase and anti-phase) and simultaneously demonstrate the wobbling with the frequency of the order of the magnitude of the interaction between them. In this case, there is also the transfer of energy between oscillators with frequency (25), but now the principle of superposition and normal nodes do not exist. The complete dynamics of the system is described by the following formulas:

$$\psi_{1,2} = \sqrt{\frac{N}{2}}\sqrt{1 \pm \cos^2 u}\,\exp\left(\frac{i}{2}\int\dot{\varphi}\,dt \mp \frac{i}{2}\int\dot{\psi}\,dt\right), \tag{26}$$

in which the values $\dot{\psi}$ and $\dot{\varphi}$ are defined by Eqs. (6), (7). The azimuthal rotation of oscillators is determined by the exponential expression and is expressed through the elliptical integrals of the third kind. This movement consists of average rotation with frequency $\nu = \left\langle\dot{\varphi} \mp \dot{\psi}\right\rangle/2$, where angular brackets mean the averaging over the period of motion, and additional azimuthal oscillation with frequency Ω.

Integral in expression (26) also gives an additional linear in time contribution, and the total formula in the limits $E \to E_s$ and $E \to E_a$ leads to synchronous in-phase and anti-phase rotations with frequencies (10), (11). The amplitude of modulation for the excitations of individual oscillators is equal to $\delta N = 4a^2/\alpha N$, and on the line $E = E_*\left(N\right)$ we have $\delta N_{\max} = N$, i. e., the periodic complete transfer of energy between oscillators is observed.

Thus, in the system of coupled anharmonic oscillators, the dynamics of the system is two-frequency, but it does not represent the superposition of normal modes: the frequency of energy exchanging between the oscillators (radial movement) does not represent a difference between the frequencies of azimuth rotations (main nonlinear oscillations).

2.2. Large-amplitude excitations

In the domain of large excitations with $N_b < N < 2N_b$, where $E_n < E_s < \tilde{E} < E_a$ [Fig. 2(b)], we have

$$A_1 = -\left(E-\tilde{E}\right) + 2\varepsilon\sqrt{E-E_n}/\sqrt{\alpha} = a^2, \quad E_n < E < E_a, \tag{27}$$

$$A_2 = -\left(E-\tilde{E}\right) - 2\varepsilon\sqrt{E-E_n}/\sqrt{\alpha} = -b^2, \quad E_s < E < E_a, \tag{28}$$

$$A_2 = -\left(E-\tilde{E}\right) - 2\varepsilon\sqrt{E-E_n}/\sqrt{\alpha} = c^2 < a^2, \quad E_n < E < E_s. \tag{29}$$

Thus, the area for the valid parameters N and E values is divided into two parts with $E > E_s$ and $E < E_s$. [The domain (28) corresponds to A in Fig. 1(b)]. From the same picture one can see that now the dependence $\tilde{E}(N)$ is inside the domain A, and while passing through over the parameters (E, N) the solution slightly changes: at $\tilde{E} < E < E_a$ as in the previous case $a^2 < b^2$, and for $E_s < E < \tilde{E}$ we have $a^2 > b^2$. For $a^2 > b^2$ in this area at the border $E = E_s$ parameters $b = 0$ and $a^2 = 2\varepsilon(N - N_b)$. Thus, on the border $E = E_s$, corresponding to the separatrix S in Fig. 2(b), the module $k = 1$ and the solution is aperiodic:

$$\cos u = 2\sqrt{N_b/N\left(1-N_b/N\right)}\,\mathrm{cosech}\left(\sqrt{N/N_b-1}2\alpha t\right). \tag{30}$$

[The line $E = \tilde{E}$ still corresponds to the separatrix C in Fig. 1(b)]. In the region $E_n < E < E_s$ the solution is being radically transformed:

$$f = a\,\mathrm{dn}\left(a\sqrt{\alpha}\,t, k\right), \quad k = \sqrt{1-c^2/a^2}. \tag{31}$$

At the border $E = E_s$ the parameters of the solution are equal to $a^2 = 2\varepsilon(N - N_b)$, $c = 0$ and $k = 1$. On the other border $E = E_n$ they take the values $a^2 = c^2 = \alpha(N^2 - N_b^2)/4$ and $k = 0$. Modulation of the levels of excitations of two oscillators is determined by the expression

$$N_{1,2} = N/2 \pm a\,\mathrm{dn}\left(a\sqrt{\alpha}\,t, k\right)/\sqrt{\alpha}. \tag{32}$$

It is important that although in this area of the system parameters $N > N_b$ and $E_n < E < E_s$, as above, there is the periodic exchange of energy between oscillators with the frequency

$$\Omega = \pi\sqrt{\alpha}\,a\left(E, N\right)/K\left(k\right). \tag{33}$$

But now the average levels of the excitations of individual subsystems are not equal:

$$\left\langle N_{1,2}\right\rangle = N\left(1 \pm \pi a/\sqrt{\alpha}NK\right)/2. \tag{34}$$

At the border of the area of the existence of solutions with the maximum difference of the excitations of oscillators we have $\left\langle N_{1,2}\right\rangle = N(1 \pm \sqrt{1-N_b^2/N^2})/2$. In the limit $N \to \infty$ the total energy is concentrated on one of the oscillators: $\left\langle N_1\right\rangle \to N$, $\left\langle N_2\right\rangle \to 0$. Such the spatial localization of energy in the system of identical oscillators is the nature of the solitary localization in nonlinear systems with distributed parameters.

From Fig. 2 it is clear that the purely in-phase oscillations correspond to a special point of the «saddle» point. Therefore the in-phase oscillations become unstable [see dotted line in Fig. 1(a)]. From this point separatrix loops «come out», which correspond to the aperiodic evolution for the amplitudes of oscillators. (But they are accompanied by periodic in-phase rotation of oscillators with a large frequency of order of ω_0.) The specific type of separatrices depends on the norm of the solution N. While $N_b < N < 2N_b$ two separatrices S and C in Fig. 2(b) [E_* and E_s in Fig. 1(b)] separate quasi-in-phase and quasi-antiphase oscillations and quasi-in-phase and heterogeneous oscillations, respectively. For $N = 2N_b$ [Fig. 2(c)] these separatrices merge and the aperiodic component of the movement looks particularly simple: $u = \pi/2 \pm \psi$ and $N_1 = N/2\mathrm{ch}^2(2\varepsilon t)$. Oscillators rotate with frequency $\Omega = \omega_0 - \varepsilon$ and with the total phase shift equal to π. This phase shift is also preserved at $N > 2N_b$ for one of the separatrices [E_* on Fig. 1(b)].

3. DSTM for two coupled elements with nonlinear interaction

So far, we have considered a pair of coupled nonlinear dynamical systems with linear interaction between the elements. At the same time, the situations often arise in physical applications in which the interaction between two fields is substantially nonlinear. The two degrees of freedom of a dynamic system in these cases are, for example, the amplitudes of two modes in one optical waveguide [3, 16, 17] or the superconducting phases of two modes of superconducting resonators in contact with Josephson junctions [10].

1. In the simple case of a waveguide with two modes propagating with different carrier frequencies but with spatially uniform amplitudes of two modes slowly varying with time, the equations have the form [3, 16]

$$i\dot{\psi}_1 = \omega_0\psi_1 - \alpha\left|\psi_1\right|^2\psi_1 - \beta\left|\psi_2\right|^2\psi_1, \tag{35}$$

$$i\dot{\psi}_2 = \omega_0\psi_2 - \alpha\left|\psi_2\right|^2\psi_2 - \beta\left|\psi_1\right|^2\psi_2, \tag{36}$$

where the interaction between nonlinear elements is essentially nonlinear, but has a specific character. (The linear interaction is absent.) With positive values of the parameters of anharmonism $\alpha, \beta > 0$, elementary excitations in each subsystem are attracted. Excitations from different subsystems are also attracted.

2. More complicated is the interaction in the system of interacting superconducting phases of superconducting

resonators connected to SQUID. It contains both nonlinear and linear (asymmetric) in field variables terms [10]:

$$i\dot{\psi}_1 = \mu\psi_1 - \varepsilon\psi_2 - \alpha|\psi_1|^2\psi_1 - \beta|\psi_2|^2\psi_1, \quad (37)$$

$$i\dot{\psi}_2 = -\mu\psi_2 - \varepsilon\psi_1 - \alpha|\psi_2|^2\psi_2 - \beta|\psi_1|^2\psi_2. \quad (38)$$

3. A similar dynamic system arises for the description of the propagation of two optical modes with orthogonal polarization in an optical waveguide with weak optical anisotropy. In this case, the dynamics of spatially homogeneous modes is described by the system of equations [10, 16]

$$i\dot{\psi}_1 = \mu\psi_1 - \alpha|\psi_1|^2\psi_1 - \beta|\psi_2|^2\psi_1 - \delta\bar{\psi}_1\psi_2^2, \quad (39)$$

$$i\dot{\psi}_2 = -\mu\psi_2 - \alpha|\psi_2|^2\psi_2 - \beta|\psi_1|^2\psi_2 - \delta\bar{\psi}_2\psi_1^2. \quad (40)$$

We begin by studying the system (35), (36), which can be considered as a simple finite-dimensional analogue of the Manakov system.

3.1. Systems with only nonlinear interaction

The dynamics of the systems described by equations (35), (36) are quite simple. The nonlinear interaction appearing in it does not lead to the transfer of energy between the two subsystems. It is seen from (35), (36) that the norms of each of the two effective oscillators are conserved: $N_1 = |\psi_1|^2$ and $N_2 = |\psi_2|^2$ are two independent integrals of motion and the conserved total energy is not an independent integral and is expressed in terms of N_i:

$$E = \mu N - \alpha N^2/2 + (\alpha - \beta)N_1N_2, \quad (41)$$

where $N = N_1 + N_2$ is the total "power" of excitation. From (41) the influence of nonlinear interaction between oscillators on their dynamics is seen. With weak nonlinear interaction $\beta < \alpha$ at a fixed level of total excitation, the energy is minimal when only one oscillator is excited [Fig. 3(a)]. With a strong nonlinear interaction $\beta > \alpha$, the energy is minimal when the excitations are equally distributed between two degrees of freedom. Nonlinear interaction between oscillators manifests itself in the fact that the frequencies of each of them depend on the level of excitation of the other:

$$\omega_1 = \omega_0 - \alpha N_1 - \beta N_2, \quad \omega_2 = \omega_0 - \alpha N_2 - \beta N_1. \quad (42)$$

(Naturally, mechanical requirements are fulfilled $dE/dN_i = \omega_i$.) The regions of existence of solutions for different interactions of oscillators on the plane of the integrals of motion E, N are shown in Figs. 3(a), (b). Here, the insets show the change in the total energy during the redistribution of the total number of elementary excitations between oscillators while conservation of their total number.

3.2. Interacting nonequivalent systems

The systems described by equations (37), (38) are more complicated. Equations (37), (38) include both linear and nonlinear interactions between the two subsystems. In addition, the difference in the first terms in the right-hand sides leads to the fact that the subsystems become nonequivalent and the degeneracy that was present in the systems described by Eqs. (1) is removed. Due to the linear part of the interaction, the norms of individual effective oscillators are not preserved, but the total norm

$$N = |\psi_1|^2 + |\psi_2|^2 \quad (43)$$

remains an integral of motion. The total energy is also conserved. It becomes an independent integral of motion:

$$E = \mu(N_1 - N_2) - \alpha N^2/2 + $$
$$+ (\alpha - \beta)N_1N_2 - \varepsilon(\bar{\psi}_1\psi_2 + \psi_1\bar{\psi}_2). \quad (44)$$

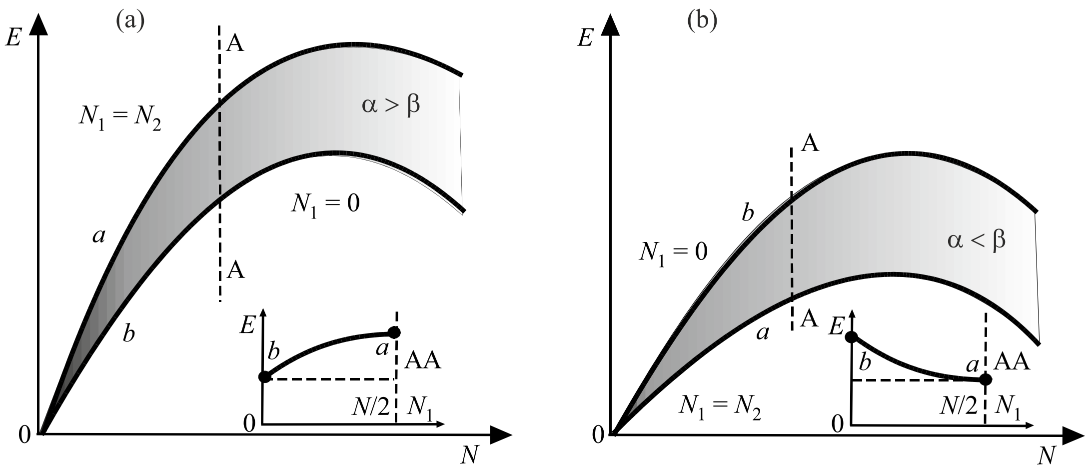

Fig. 3. The domains of an existence for the solutions on the plane of the integrals of motion with (a) small interparticle interaction and (b) in the case when the interparticle interaction exceeds the "internal" interactions of elementary excitations. The insets show the energy dependence on the distribution of excitations between subsystems for a fixed total number *N*.

As above, it is convenient to represent the solution in the form (4) in spherical coordinates by introducing the variables $u = 2\vartheta$ and $\psi = \varphi_2 - \varphi_1$. In this case, Eq. (5) will retain its form, and Eq. (6) will slightly change:

$$\dot{\psi} = 2\mu + 2\varepsilon \operatorname{ctg} u \cos\psi - (\alpha - \beta) N \cos u . \qquad (45)$$

Expression (8) for energy also changes and transforms to the form

$$E = (\mu \cos u - \varepsilon \sin u \cos\psi) N -$$

$$-(\beta \sin^2 u + \alpha \cos^2 u) N^2 / 4. \qquad (46)$$

It differs from (8) in the form of the first term. As in the first section, expressing the variable ψ with the help of (46) as a dependence $\psi = \psi(u, E, N)$ and substituting it into Eq. (5), we obtain a closed equation for $u = u(t)$, which in terms of the variable $f = (N/2)\cos u$ has the form

$$\dot{f}^2 = a_0 + a_1 f + a_2 f^2 + a_3 f^3 - (\alpha - \beta)^2 f^4, \qquad (47)$$

$$a_0 = N^2\varepsilon^2 - \left(E + \beta N^2/4\right)^2,$$

$$a_2 = -4(\mu^2 + \varepsilon^2) - 2(\alpha - \beta)\left(E + \beta N^2/4\right), \qquad (48)$$

$$a_1 = 4\mu\left(E + \beta N^2/4\right), \; a_3 = 4\mu(\alpha - \beta). \qquad (49)$$

For $\omega_0 \to 0$ and $\beta \to 0$, Eq. (47) transforms into Eq. (19) considered earlier, but also for nonzero μ and β is solved in terms of Jacobi elliptic functions. However, the dynamics of a system is easily analyzed by the methods of the qualitative theory of dynamical systems. As in the previous case, we consider the single-frequency solutions of the form $\psi_i = a_i \exp(-i\omega t)$ for which the dependence of the frequencies of such stationary states on the norm of solutions $\omega = \omega(N)$ follows from (37), (38), (43). Implicitly, it has the form

$$(\omega - \omega_a)(\omega - \omega_s)(\omega - \omega_n)^2 = \mu^2\left(\omega - (\omega_a + \omega_s)/2\right)^2, \qquad (50)$$

where $\omega_a = \varepsilon - N(\alpha + \beta)/2$, $\omega_s = -\varepsilon - N(\alpha + \beta)/2$ and $\omega_n = -N\alpha$ are the dependences of anti-phase, in-phase, and inhomogeneous single-frequency oscillations on the total excitation norm in the limit $\mu \to 0$. They are similar to the dependences (10)–(12) from the first part of the paper and are shown by dashed lines a, s and n in Fig. 4. For small values of the parameter μ (for example, for the weak anisotropy of the refractive index in optical waveguides), the dependence (50) is easily analyzed. The obtained dependences are shown in bold lines in Fig. 4 with the same indices (capital letters).

The Fig. 4 shows that taking into account the linear interaction of the subsystems with a difference in their characteristics leads to the removal of the degeneracy of the frequency dependence for inhomogeneous states. Typically, the frequency dependences for inhomogeneous states are split off from the dependence for in-phase oscillations in the case of "soft" nonlinearity, when the frequencies decrease with increasing energy, and the dependence for antiphase oscillations in the case of "hard" nonlinearity, when the frequencies increase with increasing excitation level of the system.

In this example, both nonlinearities (in the interaction of elementary excitations in each subsystem and between them) are soft, but the dependences for the inhomogeneous states can also be split off from the line of antiphase vibrations at $\beta > \alpha$. In the limit $\mu = 0$ (identical subsystems), the bifurcation value of the norm is equal to $N_b = 2\varepsilon/|\alpha - \beta|$. When $\mu \neq 0$, in the limit of weak excitation of the system, the frequencies of the main nonlinear oscillations are "repelled"

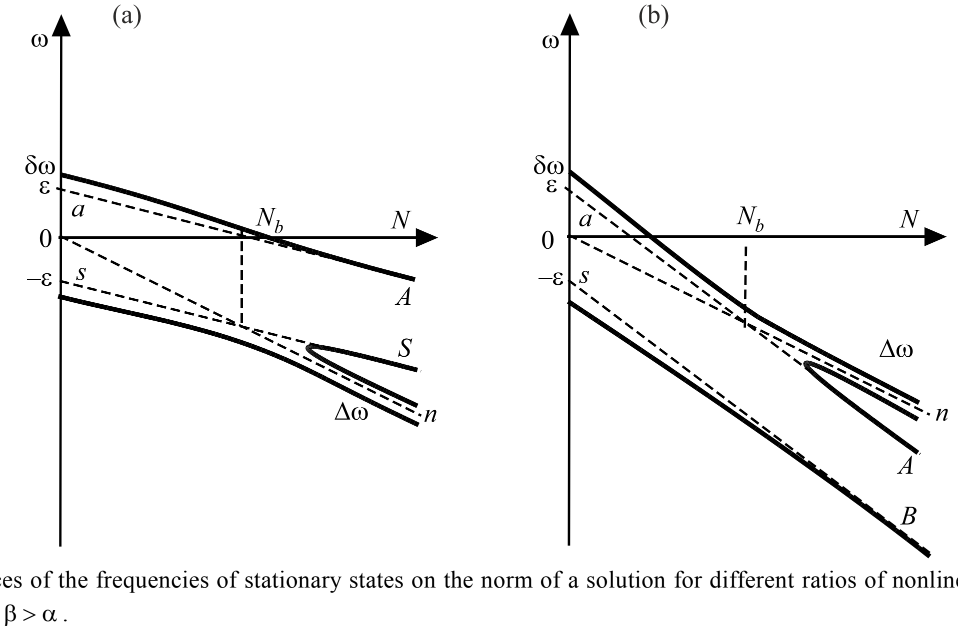

Fig. 4. Dependences of the frequencies of stationary states on the norm of a solution for different ratios of nonlinearity parameters: (a) $\beta < \alpha$ and (b) $\beta > \alpha$.

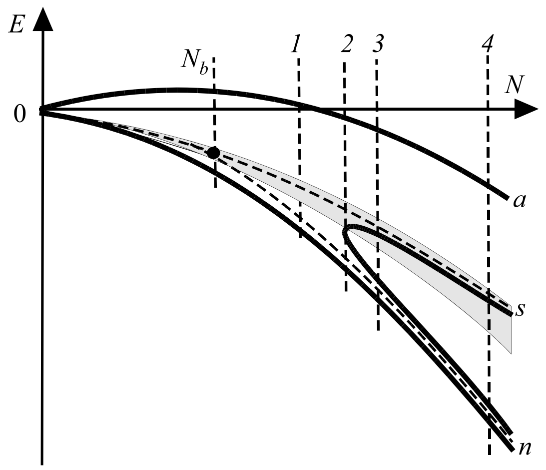

Fig. 5. The region of existence for two-frequency excitations of a general form on the plane of the integrals of motion (E, N) is between the lines of antiphase oscillations a and the line n of inhomogeneous single-frequency oscillations with a minimum energy.

by an amount $2\delta\omega \approx \mu^2 / \varepsilon$, and at high levels of excitation $N >> N_b$, the removal of the degeneracy of the frequency dependences of two inhomogeneous states is $\Delta\omega \approx 2\mu$. Figure 4(a) is similar to Fig. 1(a) for the standard DSTM discussed above. We also give an image of the region for the existence of various solutions on the plane (E, N) for the integrals of motion, similar to that shown in Fig. 1(b) for DSTM. It is shown in Fig. 5 for the case $\alpha > \beta$.

To get an idea of the nature of the dynamics for different ratios of the integrals of motion, we consider this dynamics on the "phase plane" in the framework of Eqs. (5), (45). Figure 6 shows the phase portraits of the system for the different values of the norm corresponding to lines *1*, *2*, *3*, and *4* in Fig. 5.

Figure 6(a) corresponds to section 1 in Fig. 5. The centers s and a describe in-phase and antiphase oscillations. Those the phases of the two oscillators coincide or differ by π, but in contrast to the case of the usual DSTM, the amplitudes a_i now do not coincide and the special points are shifted from the value $u = \pi / 2$. In addition, the separatrix C in Fig. 2(a) "splits", forming a region of the phase plane [hatched in Fig. 6a], in which the phase between the oscillations of individual oscillators monotonically increases. This is due to the nonidentity of the subsystems. With an increase of the norm N at a critical value N corresponding to Section 2 in Fig. 5, a saddle-focus pair appears on the phase plane in a bifurcation manner. For large N (section 3 in Fig. 5), three special points in Fig. 6(b) (S, N, S) correspond to in-phase oscillations, but in the solution S the amplitudes of the two oscillators practically coincide, and this is an analog of unstable in-phase oscillations after bifurcation in the previous model. Two stable solutions N correspond to inhomogeneous in-phase oscillations with a large difference in amplitudes (an analog of inhomogeneous states). In this case, they are different due to the inequality of the subsystems. The next bifurcation occurs at a value N at which the energies of the two saddle points S and σ in the phase portrait of Fig. 6(b) coincide, which corresponds to the portrait in Fig. 6(c). Finally, at even higher levels of the norm [Fig. 6(d)], there exist regions with the monotonous relative rotation of the oscillators in opposite

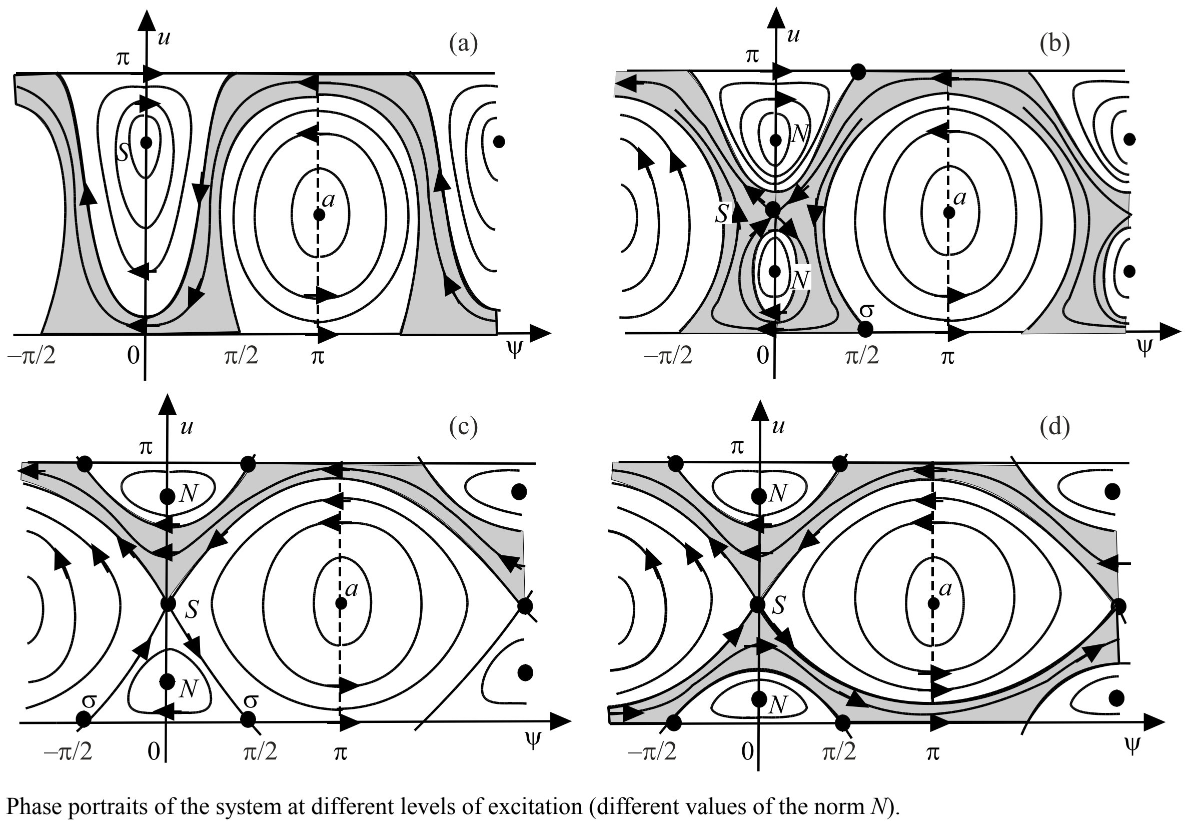

Fig. 6. Phase portraits of the system at different levels of excitation (different values of the norm N).

directions. The regions of the monotonous relative rotation of the oscillators, marked by shading on the phase portraits, are marked by shading on the plane of the integrals of motion in Fig. 5 as well.

Thus, the main point in the dynamics of two nonlinear systems within the framework of the model (37), (38) is the localization of excitations in one of the coupled subsystems, starting from a certain level of the excitation. In this case, in contrast to the standard DSTM, due to the nonequivalence of the subsystems, the excitation is localized in one quite certain subsystem. This suppresses uncertainty in symmetric systems and can be useful in technical applications of this problem.

3.3. Systems with parametrical nonlinear interaction

In conclusion, we consider the systems described by Eqs. (39), (40). As in the systems considered above, two integrals of motion are conserved in it: the total norm (43) and the total energy

$$E = \mu\left(N_1 - N_2\right) - \alpha N^2/2 + \left(\alpha - \beta\right) N_1 N_2 - \\ -\delta\left(\bar{\psi}_1^2\psi_2^2 + \psi_1^2\bar{\psi}_2^2\right)/2 . \quad (51)$$

Introducing the new variables $\psi_1 = \sqrt{N}\cos\vartheta\exp(i\varphi_1)$, $\psi_2 = \sqrt{N}\sin\vartheta\exp(i\varphi_2)$, $u = 2\vartheta$, and $\psi = \varphi_2 - \varphi_1$, we obtain equations for them that replace (5) and (45):

$$\dot{u} = \delta N \sin u \sin 2\psi , \quad (52)$$

$$\dot{\psi} = 2\mu - N\left(\left(\alpha - \beta\right) - \delta\cos 2\psi\right)\cos u , \quad (53)$$

and the expression for energy

$$E = \mu N\cos u - \alpha\frac{N^2}{2} + \left(\alpha - \beta\right)\frac{N^2}{4}\sin^2 u - \\ -\delta\frac{N^2}{4}\sin^2 u\cos 2\psi . \quad (54)$$

As in the previous case, the dynamics of the system depends on the ratio of the constants of the model. For definiteness and for comparison with the previous results, we will consider the case $\alpha < \beta + \delta$, i. e., the opposite sign of the inequality.

Let us consider the single-frequency stationary states of the type $\psi_i = a_i\exp(-i\omega t) = \sqrt{N_i}\exp(-i\omega t + i\Phi_i)$ with $N_i = a_i\bar{a}_i$. For them, from (39), (40) it follows the relations:

$$a_1 a_2\left(2\mu + \left(\alpha - \beta\right)\left(N_2 - N_1\right) + \delta\left(a_1^2\bar{a}_2/a_2 - a_2^2\bar{a}_1/a_1\right)\right) = 0, \quad (55)$$

$$\omega = -\left(\alpha + \beta\right)N/2 - \delta/2\cdot\left(a_1^2\bar{a}_2/a_2 + a_2^2\bar{a}_1/a_1\right). \quad (56)$$

Thus, the system allows four different stationary states: two obvious states with the complete localization of excitation in one of the subsystems. These are the states with $a_1 = 0$ and $N = N_2$ ($n2$), and with $a_2 = 0$ and $N = N_1$ ($n1$). For the other two solutions, from the reality of (55), (56) it follows that in them $\Phi_1 = \Phi_2$ (s) and $\Phi_1 = \Phi_2 + \pi/2$ (a). (In the last solution, the doubled phase shifts by π, as in the problem of parametric resonance [2].) From (55) for these solutions it follows the relation $2\mu + (\alpha - \beta \mp \delta)(N_2 - N_1) = 0$. Taking into account that $N_1 + N_2 = N$, we finally obtain for four stationary states:

$$(n1)\text{: } N_2 = 0,\ N_1 = N,\ \omega = \mu - \alpha N,\ E = -\mu N - \alpha N^2/2, \quad (57)$$

$$(n2)\text{: } N_2 = N,\ N_1 = 0,\ \omega = -\mu - \alpha N,\ E = \mu N - \alpha N^2/2, \quad (58)$$

$$(s)\text{: } N_{1,2} = N/2 \pm \frac{\mu}{\alpha - \beta - \delta},\ \omega = -\frac{N\left(\alpha + \beta + \delta\right)}{2}, \\ E = \frac{\mu^2}{\alpha - \beta - \delta} - \left(\alpha + \beta + \delta\right)\frac{N^2}{4}, \quad (59)$$

$$(a)\text{: } N_{1,2} = N/2 \pm \frac{\mu}{\alpha - \beta + \delta},\ \omega = -\frac{N\left(\alpha + \beta - \delta\right)}{2}, \\ E = \frac{\mu^2}{\alpha - \beta + \delta} - \left(\alpha + \beta - \delta\right)\frac{N^2}{4}. \quad (60)$$

Figure 7(a) shows the frequency dependences on the total excitation norm for these stationary states, and Fig. 7b demonstrates the dependences for the total energy of the system.

Two bifurcation points, which correspond to $N_{b1} = 2\mu/\left(\beta - \alpha + \delta\right)$ and $N_{b2} = 2\mu/\left(\beta - \alpha - \delta\right)$, are observed on the dependences of the frequency and energy on the norm of the stationary state $(n1)$, in which the excitation is completely localized in the first subsystem. At these points the states in which energy is distributed between the subsystems are split off. For a qualitative understanding of the dynamics of the system, let us consider the transformation of its phase portrait of it on the phase plane (u, ψ).

The phase portrait has a simple structure without special points for $N < N_{b1}$ [Fig. 8(a)].

For small N we have $\psi \approx 2\mu t$, which corresponds to $\psi_2 = \sqrt{N_2}\exp(i\mu t)$ and $\psi_1 = \sqrt{N_1}\exp(-i\mu t)$. Those two moments rotate in the opposite direction with frequencies $\pm\mu$. The minimum energy corresponds to a straight line $u = \pi$ for the state $n2$ with the first element in rest. The maximum energy corresponds to a straight line $u = 0, 2\pi$ on which only the first oscillator is excited $n1$. At intermediate energies, a periodic transfer of energy occurs between two subsystems: $u \approx u_0 + (\delta N/2\mu)\sin u_0\cos 2\psi$ as shown in Fig. 8(a). This scenario corresponds to Section 1 in Fig. 7(b).

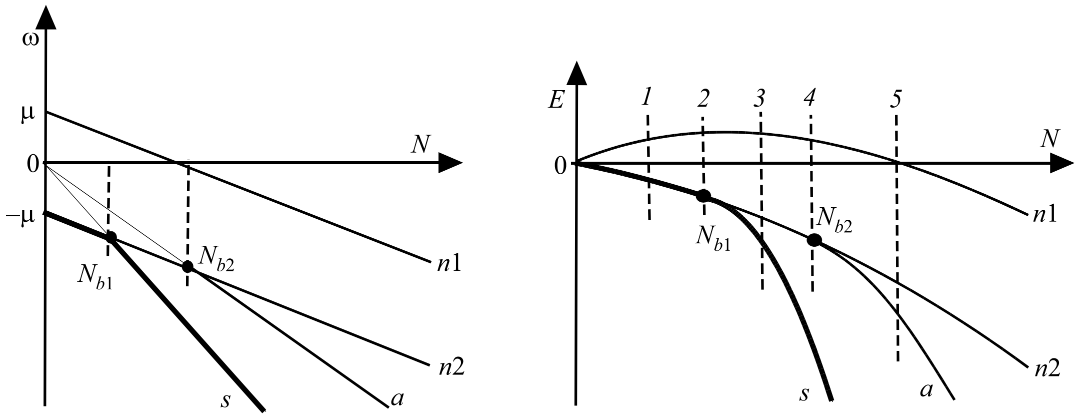

Fig. 7. The dependences of the frequencies (a) and energies (b) on the norm for the stationary states.

The first bifurcation occurs at $N_{b1} = 2\mu/(\beta-\alpha+\delta)$ and the special point $(u = \pi,\ \psi = 0)$ forms in the phase portrait [in section 2 in Fig. 7(b)]. When the norm N exceeds the value N_{b1}, it splits into four special points: two centers and two saddles: the stable centers s at points $\psi = 0$, $u_{b1} = \pi \pm \sqrt{2(N - N_{b1})/N}$, and unstable saddles z at points $u = \pi$, $\psi_{b1} = \pm\sqrt{4\mu(N - N_{b1})/\delta N N_{b1}}$ [see section 3 in Figs. 7(b) and 8(b)]. A stable center corresponds to a state s in which a nonzero excitation of the first subsystem appears. In this case, both «oscillators» rotate synchronously in the same phase with a frequency close to the value μ.

With growth of N, the saddle points z move along the line $u = \pi$ to the points $\psi = \pm\pi/2$. The second bifurcation takes place at $N_{b2} = 2\mu/(\beta-\alpha-\delta)$ [Section 4 in Fig. 7(b)] when in the phase portrait a "reconnection" of the saddle points occurs: two saddle points z transform into two other saddles c with coordinates $\psi = \pi/2$, $u_{b2} = \pi \pm \sqrt{2(N - N_{b2})/N}$ [see Fig. 8(d)]. These solutions correspond to the dependence a in Fig. 7(b) for an unstable

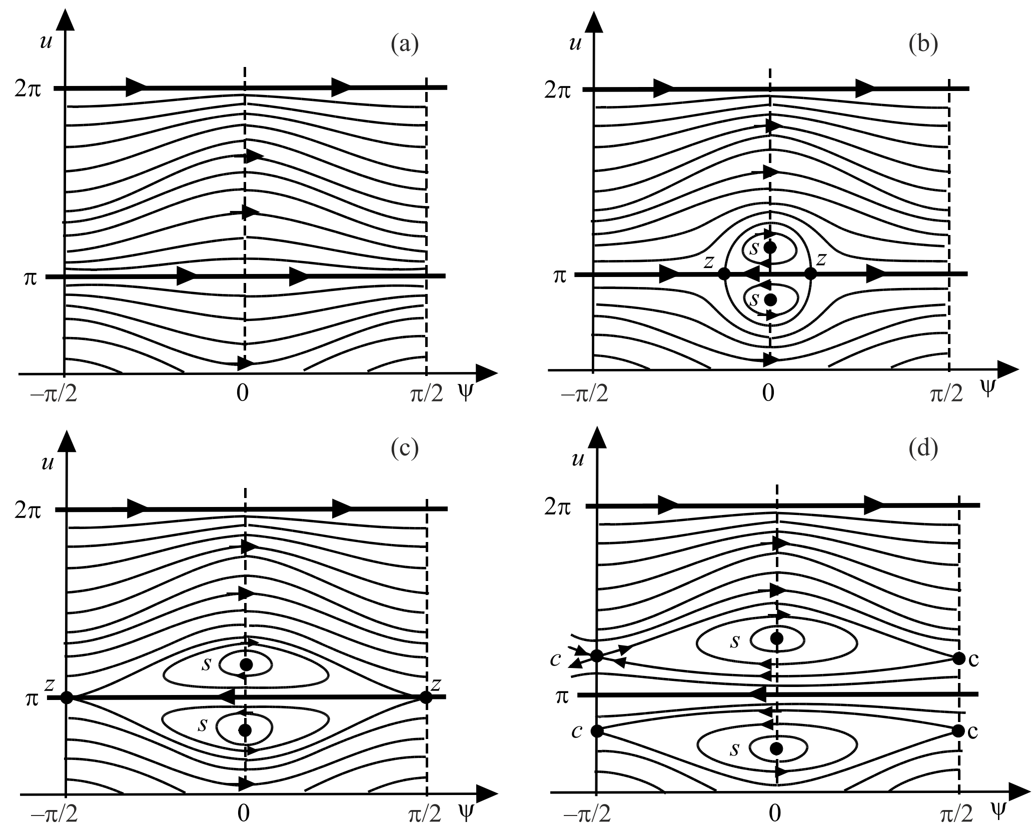

Fig. 8. The phase portrait of the system for different values of the excitation norm.

stationary state. The state *s* with in-phase rotation of two effective oscillators becomes the ground state with minimum energy. Moreover, in the limit of large excitations with $N >> N_b$, the energy is practically equally distributed between the subsystems: $N_1 \approx N_2$.

Thus, in this system the dynamics is fundamentally different from those considered in the previous model.

4. Conclusions

It is well known that in nonlinear systems with a finite number of degrees of freedom the principle of superposition is absent. But some spatial single-frequency stationary states corresponding to the so-called "main nonlinear oscillations" and specific states with unequal energy distribution between the separate subsystems are easily found. The latter are the finite-dimensional analogues of the soliton excitations in systems with distributed parameters. But in some cases related to real physical objects (coupled optical fibers, interacting optical modes, magnetic dots, and quantum resonators), corresponding systems with two degrees of freedom allow exact solutions. They include, as special cases, the generally discussed main nonlinear oscillations and localized states. But general solutions describe in addition such important manifestations of nonlinear dynamics as the changing in time of the energy distribution between individual objects of the system. This phenomenon can be associated, for example, with the appearance of internal soliton excitations in non-integrable physical systems.

We investigated the nonlinear dynamics for some integrable systems of two identical and nonidentical coupled nonlinear elements with linear and nonlinear interparticle interaction of different types and pay attention to some common features of this dynamics. (i) Although the principle of the superposition is absent in nonlinear systems, nevertheless the spectrum for the integrable one contains the frequencies of two definite quasilinear modes with the periodical transfer of the energy between them. (ii) The most interesting facts consists in the appearance of the additional states with the average nonuniform distribution of the energy between the degrees of freedom. This nonuniform nonlinear mode appears in the bifurcation way at the critical value of the total energy. (iii) These states can be treated as the soliton analogous in the system with the finite numbers of the degrees of freedom. But from the exact solutions, it can be seen that in the general case there is a partial exchange of energy between the subsystems and between the four main single-frequency states. Some solutions can be interpreted as the internal dynamics of localized states.

This work was supported by the scientific project of the National Academy of Sciences of Ukraine No. 4. 17-N and the Scientific program 1.4.10. 26/F-26-4.

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1. R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equaions*, Academic Press. Inc., London (1984).
2. A. M. Kosevich, A. S. Kovalev, *Introduction to nonlinear physical mechanics*, Naukova Dumka, Kiev (1989) (in Russian).
3. Yu. S. Kivshar and G. P. Agraval, *Optical Solitons*, Academic Press, London (2003).
4. J. W. Miles, *J. Fluid Mech*. **106**, 186 (1981).
5. M. Q. Tran, *Phys. Scr*. **20**, 317 (1979).
6. O. M. Braun and Yu. S. Kivshar, *The Frenkel–Kontorova Model*, Springer, London (2004).
7. J. C. Eilbeck and M. Johanssen, *arXiv:nlin/0211049* (2002).
8. P. G. Kevrekidis, *IMA J. Appl. Math*. **76** (3), 389 (2011).
9. A. A. Ovchinnikov, *JETP* **57**, 263 (1969).
10. W. Wustmann and V. Shumeiko, *Fiz. Nizk. Temp*. **45**, 995 (2019) [*Low Temp. Phys*. **45**, 848 (2019)].
11. B. Crosignani and P. Di Porto, *Opt. Lett*. **6**, 329 (1981).
12. C. R. Menyuk, *IEEE J. Quantum Electron*. **25**, 2674 (1989).
13. A. C. Newell, *Solitons in Mathematics and Physics*, Soc. Industrial and Applied Math., USA (1985).
14. *Emergent Nonlinear Phenomena in Bose–Einshtein Condensates: Theory and Experiment*, P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzalez (eds.), Springer Series on Physics, **45**, 3 (2007).
15. A. S. Kovalev, *Fiz. Nizk. Temp*. **46**, 837 (2020) [*Low Temp. Phys*. **46**, 704 (2020)].
16. G. P. Agrawal, *Nonlinear Fiber Optics*, Academic, San Diego (2001).
17. A. Villeneuve, J. U. Kang, J. S. Aitchison, and G. I. Stegeman, *Appl. Phys. Lett*. **67**, 760 (1995).

Динаміка пари зв'язаних нелінійних систем. II. Дискретна модель самозахвату

О. С. Ковальов, Я. Є. Прилепський

В межах дискретної моделі самозахвату (DSTM) та її узагальнень розглянуто нелінійну динаміку двох зв'язаних елементів у різних фізичних додатках, стосовно до магнітних, оптичних та надпровідних систем. Знайдено та досліджено точні розв'язки відповідних рівнянь. Особливу увагу приділено дослідженню суттєво нелінійних неоднорідних станів з різним рівнем збудження ідентичних підсистем, які можна розглядати в якості дискретних аналогів солітонних збуджень в системах з розподіленими параметрами.

Ключові слова: стаціонарні стани, головні нелінійні коливання, неоднорідні стани, інтеграли руху, фазовий портрет, біфуркації.