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Stochastic dynamics of a nonlinear oscillator driven by periodic force with slowly varying frequency

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The topology of phase portraits for nonlinear oscillators driven by a periodic force undergoes significant changes within a narrow interval of the driving force frequencies ν. This property leads to nonintegrability of the equations of motion, and stochastization of their solutions when ν is periodically modulated. Such behavior is due to the violation of adiabaticity and destruction of the integral manifolds, accompanied by topological rearrangements of the integral curves. We study specific features of such stochastic dynamics in a wide range of modulation periods and damping decrements.

Keywords: nonlinear oscillator, periodic modulation, stochastic dynamics.

Introduction

The Hamiltonian of a nonlinear oscillator

$$\mathcal{H}=\frac{1}{2}\left(\dot{x}^2+\omega^2x^2\right)+\frac{1}{4}Vx^4+xf\cos\left(\nu t\right) \quad (1)$$

can be used to describe a variety of mechanical, electronic, and physical systems. In particular, in [1] it was used to study the nonlinear ferromagnetic resonance. In the absence of a driving force ($f=0$) this Hamiltonian gives integrable equations of motion. When $\nu(t)$ in the driving term oscillates itself, with a modulation frequency that is amplitude-dependent, equations of motion become nonintegrable at $f\neq 0$.

The first study of nonlinear oscillations driven by a periodic force of low amplitude *f*, slowly varying frequency $\left|\dot{\nu}(t)/\nu^2\right|<<1$, and low damping, $\gamma<<\omega$ (where γ is the decrement of damping), is due to Mitropolsky [2,3]. He investigated the resonant oscillator dynamics by numerically solving the equations in the first approximation of asymptotic expansion using the Krylov–Bogolyubov–Mitropolsky method. (It's worth mentioning that at that time the only calculating tool available to him was a mechanical integrator.) Mitropolsky found hysteretic behavior of the oscillation amplitude as $\nu(t)$ slowly crossed the resonant domain $|\nu-\omega|<<\omega$.

What happens if $\nu(t)$ crosses that domain back and forth many times? The dynamic behavior of the oscillator (1) at the slow periodic modulation of $\nu(t)$ was not investigated till now. In this communication, we are aiming to solve this problem.

Hamiltonian dynamics at slow varying driving force frequency

To obtain the equations of the first approximation in asymptotic expansions one has to substitute an ansatz $x=a\cos(\nu t+\theta)$, $a\geq 0$ in (1), and average it, assuming a and θ to be slowly varying quantities.

The fact that a^2 and θ form a pair of canonic variables of the averaged Hamiltonian [4,5] leads to the following equations:

$$\frac{da}{dt}=-\varepsilon(t)\sin\theta;\quad \frac{d\theta}{dt}=\frac{\omega^2(a)-\nu(t)^2}{2\nu(t)}-\frac{1}{a}\varepsilon(t)\cos\theta;$$

$$\varepsilon(t)=\frac{f}{2\nu(t)};\quad \omega(a)=1+\frac{3}{8}Va^2. \quad (2)$$

Here $\nu(t)$ is a periodic function. The frequency of its oscillation, Ω, is much less than ν.

Poincare proved that periodic perturbation of a nonlinear Hamiltonian oscillatory system rules out the existence of analytical integrals if there is a separatrix in the phase portrait of the non-perturbed motion [6]. In fact, this general property of dynamic systems is the sufficient condition for the existence of random integral curves [6,7]. Therefore, in order to search for random solutions of Eqs. (2) we start by considering the topology of the phase portraits of the system (2) at constant ν [8]. In this case, the integral of motion W (energy) determines the phase trajectories $L(W)$

$$\frac{1}{2}Va^4+(1-\nu)a^2-2\varepsilon a\cos\theta=W\,. \quad (3)$$

Let us introduce a variable action

$$J(W) = \oint_{L(W)} a^2(\theta, W)\, d\theta \qquad (4)$$

and the period of motion along the trajectory $L(W)$

$$T(W) = \oint_{L(W)} \left(\frac{d\theta}{dt}\right)^{-1} d\theta. \qquad (5)$$

There exist three open domains of ν in which the topologies of the phase portraits (determined by Eq. (3)) are essentially different.

At $\nu<\nu_{cs}$ (Fig. 1a), there is one center c_1 and two saddle points $s_{1\pm}$ which are connected by two branches of the separatrix S_1. Degenerated singular point "center-saddle", appearing at $\nu=\nu_{cs}$, splits into a center c_2 and a saddle point s_2 at $\nu>\nu_{cs}$. At that moment the separatrix S_2 appears (Fig. 1b).

There is frequency ν^* belonging to semiaxis $\nu>\nu_{cs}$, at which the two separatrices S_1 and S_2 are merging into a joint separatrix S^* (Fig. 1c). An infinitesimal increase of ν at this point leads to the splitting of S^* into two separated separatrices S_3 and S_4 as shown in Fig. 1d. At $\nu=\nu^*$ both period's centers are equal to $\varpi(0)=4.5\varepsilon^{-2/3}\approx 100$, and logarithmically diverge on separatrix. The period $T(J)$ within the resonance domain R1 is shown in Fig. 2.

The loci of the fixed points, centers and saddle points, determined from the equations $\dot{a}=\dot{\theta}=0$ are shown in Fig. 3 (thin lines, $\gamma=0$). The resonance curve of a dissipative oscillator (which is considered below in brief (thick lines, $\gamma=10^{-2}$) is shown in this figure as well. At $\varepsilon\to 0$ the branches of the resonance curve are approaching the so-called skeleton curve $\omega(a)=\nu$, which is shown by a dash-dotted line.

The phase trajectories are plane sections of the cylindrical invariant manifolds of the system (2). The pitch of a helical integral curve on the cylinder with plain section (3) is equal to the period $T(W)$.

Let us perturb the system by a slow modulation of the frequency $\nu=\bar{\nu}+\delta\nu\sin\Omega t$; $\Omega \ll \delta\nu \ll \bar{\nu}$. At $\Omega \ll \varpi(J) = 2\pi/T(J)$ there are approximate integrals of motion, i.e., adiabatic invariants J. However, as it follows from Kolmogorov–Arnold–Moser (KAM) theory [7], in the vicinity of a resonant cylinder, $n\varpi(J)+m\bar{\nu}=0$ (n and m are integers) the integral manifolds $J=\text{const}$ are destroyed. The resonant domains, filled by the destroyed cylinders, are separated by the extant isolated non-resonant manifolds. The countable set of the open resonant domains is everywhere dense but its measure is small.

In the vicinity of a separatrix, where $\Omega \ge \varpi(J)$, the adiabaticity is violated, and a stochastic layer is formed. Chirikov has shown that this layer appears due to the overlap of the resonant domains [9,10]. The measure of this

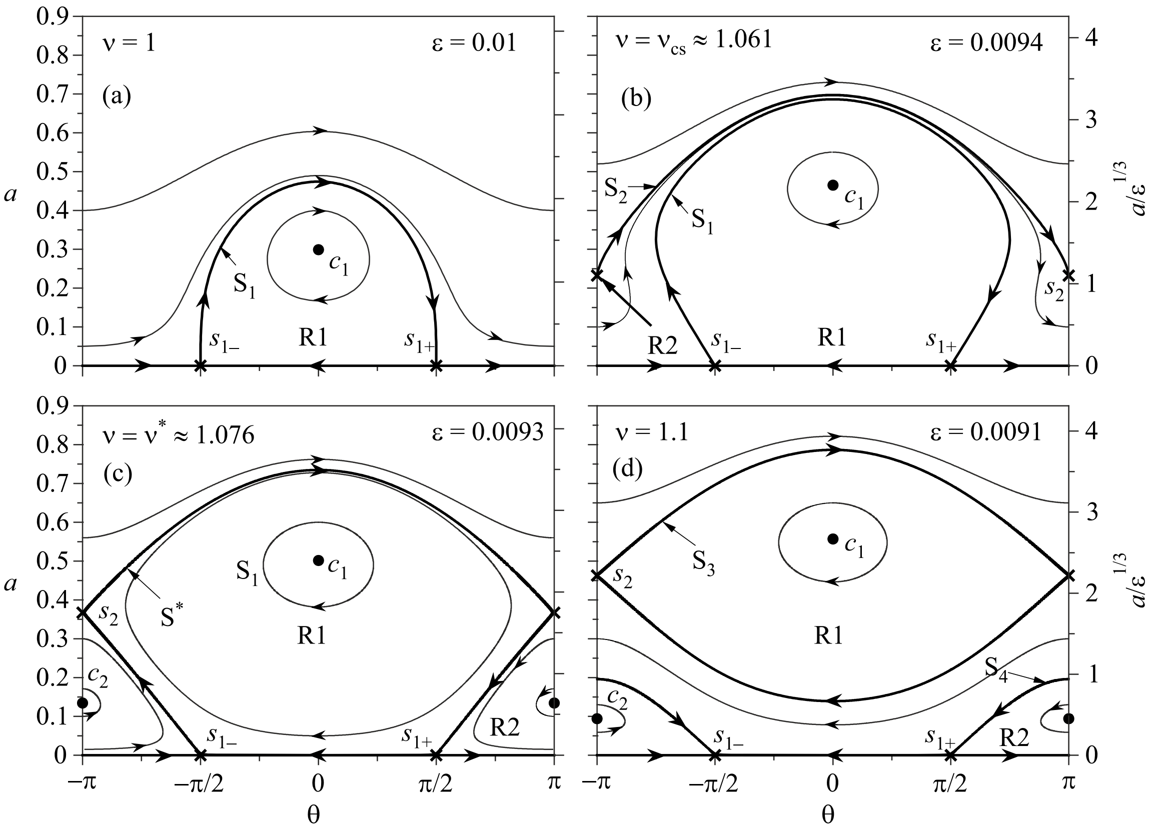

Fig. 1. The phase portraits of the system (2) at various values of ν. At $\nu=\nu^*$ two separatrices are merging.

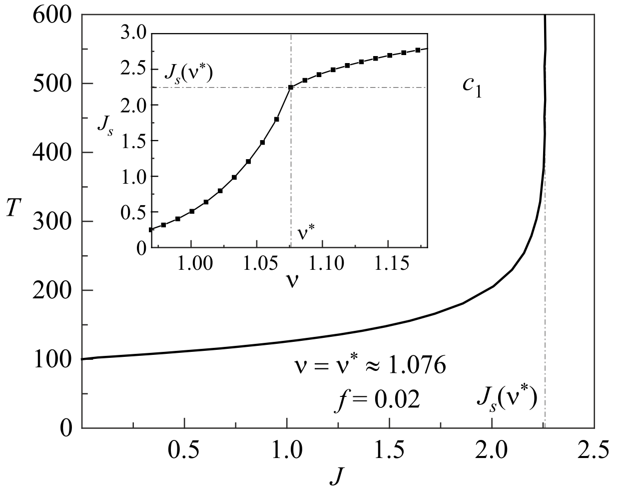

Fig. 2. The period of motion T vs J within the resonance domain R1. Inset depicts the dependence of the separatrix action value J_s on frequency ν.

layer at $(\bar{\nu}+\delta\nu) < \nu_{cs}$ and $(\bar{\nu}-\delta\nu) > \nu^*$ is exponentially small. A much stronger mixing of the integral curves, and an increase of related stochasticity, happens at $\bar{\nu} = \nu^*$ and $(\nu^* - \delta\nu) > \nu_{cs}$ due to the essential changes of the phase portrait topology. Here we investigate the motion randomization in this case.

The topology of an integral curve is changing as it crosses the joint separatrix S^*. Integral curves, initially belonging to the same integral manifold before crossing the S^*, enter different manifolds after the crossing. At the crossing J experiences jumps with jump magnitude, ΔJ, essentially depending on the cross point phase θ. Overall, multiple crossings of S^* due to the periodic modulation of $\nu(t)$ lead to the randomization of integral curves. Their autocorrelation time is $\sim \tau = 2\pi/\Omega$. Qualitatively, the appearance of a stochastic layer due to successive crossings of S^* at frequency modulation is similar to that at periodic modulation of amplitude [11]. Quantitative difference is caused by the specific topology of the separatrices S_1 and S_2, as well as by the appearance/disappearance of separatrix S_2 at ν_{cs}.

Let us introduce a parameter $\varepsilon_\nu(J) = \Omega/\varpi(J)$ determining the perturbation of the phase trajectory per period. The action J, being adiabatic invariant, is conserved with the accuracy $\sim \varepsilon_\nu^2(J)$ during the time $\sim 1/\varepsilon_\nu(J)$. It is known that the method of asymptotic expansions [12,13] allows one to obtain adiabatic invariants in all orders on the parameter $\varepsilon_\nu(J)$. However, this procedure fails at $\varepsilon_\nu(J) \sim 1$. In this case, the perturbed phase trajectory reaches the separatrix S^* during one period, and gets into the nonintegrability domain where the adiabatic invariants are destroyed, and the integral curve is randomized. Let us denote by J_ν^* the crossover value of J at which an integral curve leaves the invariant manifolds and enters the stochasticity domain. The quantity $\Delta J_\nu = J_s(\nu) - J_\nu^*$ can be evaluated using the following equation:

$$2\pi\Delta J_\nu = \Omega T(J_\nu^*)\frac{dJ_s(J_\nu^*)}{d\nu}. \qquad (6)$$

Function $J_s(\nu)$ is shown in the inset in Fig. 2. Numerical solutions of Eq. (6) are shown in Fig. 4. Let us note that there are two asymptotic dependences of ΔJ_ν on $\varepsilon_\nu^* = \Omega T(J_\nu^*)$: $\Delta J_\nu \sim (\varepsilon_\nu^*)^2$ at $\varepsilon_\nu^* < 10^{-4}$ and $\Delta J_\nu \sim (\varepsilon_\nu^*)^{1/3}$ at $\varepsilon_\nu^* > 10^{-2}$.

Solutions of Eqs. (2) for the varying frequency ν can be obtained numerically. The Poincare map of the recurrent stochastic trajectories in phase plane (a, θ, ν^*) at $\Omega = 10^{-4}$; 10^{-3}; $3\cdot 10^{-2}$ ($\varepsilon_\nu^* = 0.022$; 0.17; 3.3) is shown in Fig. 5. Quasiperiodic KAM curves surrounding the stochastic layer are not drawn there. The measure of the stochastic layer vs ε_ν^*, shown in Fig. 4 with open boxes, is compatible with the solution of Eq. (6).

The distributions $F(\Delta_\tau)$ of the phase point jump per one period $\tau = 2\pi/\Omega$,

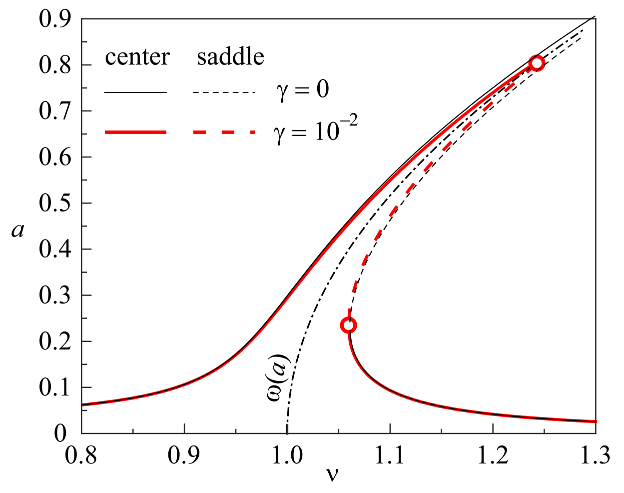

Fig. 3. The resonant curve showing loci of the fixed points of the dissipativeless (thin lines) and dissipative (thick lines) system.

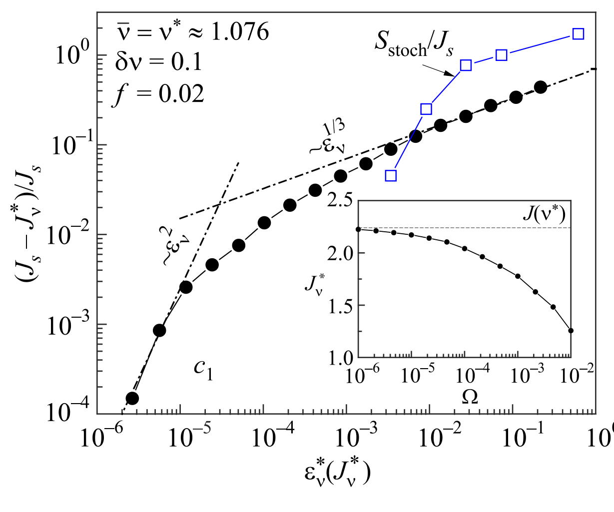

Fig. 4. The dependence of ΔJ_ν on parameter ε_ν^*. The crossover values J_ν^* vs Ω are shown in the inset.

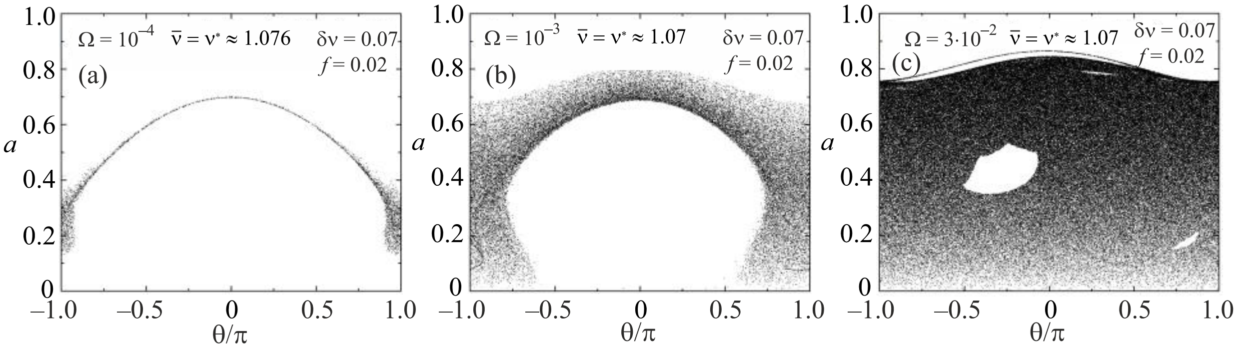

Fig. 5. The Poincare maps of the stochastic trajectories on the phase plane (a, q, ν^*) at various values of Ω.

$$\Delta_\tau = \sqrt{(a_n \cos\theta_n - a_{n-1}\cos\theta_{n-1})^2 + (a_n \sin\theta_n - a_{n-1}\sin\theta_{n-1})^2},$$

$$n = 1, 2, \ldots, \qquad (7)$$

where $a_n = a(n\tau)$ and $\theta_n = \theta(n\tau)$ for Poincare maps, are shown in Fig. 6. As one can see, the maximum of $F(\Delta_\tau)$ existing at $\Omega = 10^{-2}\varpi(0)$ disappears at $\Omega \ge 10^{-1}\varpi(0)$. These results show that the jump of J at the separatrix crossing can be treated as a random quantity.

Non-Hamiltonian stochasticity

The Hamiltonian dynamics is applicable while interactions of the system with its environment can be ignored. In a "mean-field" approximation, the impact of the environment on the dynamics is described by damping forces and random disturbances in the equations of motion:

$$\frac{da}{dt} = -\gamma a - \varepsilon(t)\sin\theta + \eta_a(t)$$

$$\frac{d\theta}{dt} = \omega(a) - \nu(t) - \frac{1}{a}\varepsilon(t)\cos\theta + \eta_\theta(t). \qquad (8)$$

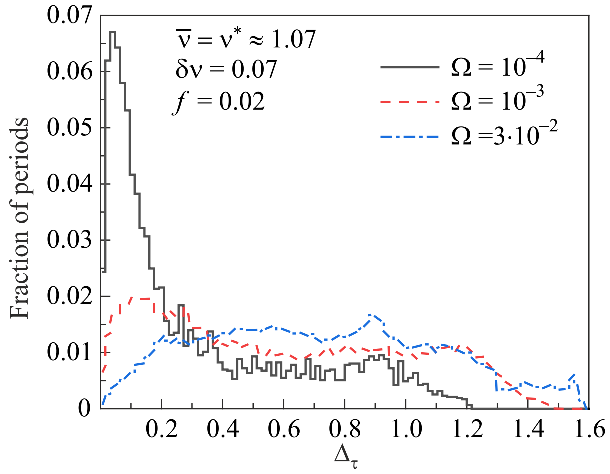

Fig. 6. The distributions of the phase point jumps Δ_τ per one period $\tau = 2\pi/\Omega$.

Here $\eta_a(t)$, $\eta_\theta(t)$ are the projections of random driving force with small (as compared to ν^{-1}) correlation time on a and θ axes [8].

To simplify the treatment of random disturbances, we simulate them by introducing a random jump at each period τ of the phase point, $\langle\delta a\rangle = 0$, $[\langle\delta a^2\rangle]^{1/2} = 10^{-2}$, in the Poincare mapping as in [10]. As an example, the Poincare map of damped dynamics impacted by a random force at (a) $\gamma = 10^{-4}$ and (b) $\gamma = 10^{-3}$ is shown in Fig. 7. The dash line depicts the joint separatrix S^* of the unperturbed system. The scale of probability is shown in Fig. 7a. As seen, a stable distribution of randomly walking points on a map is formed. At $\gamma = 10^{-4}$ the lifetime of a phase point within the domain filled by KAM curves at $\gamma = \delta a = 0$ (depicted by short-dash line) is much smaller than the time spent within the stochasticity domain. At $\gamma = 10^{-3}$ the domain of stochastic dynamics disappears, and the probability distribution has a maximum at the attracting focus, located not too far from the center c_1.

A more detailed account of the stochastic dynamics of a nonlinear resonant system driven by a periodic force with modulated frequency will be published elsewhere [14].

Discussion

On a qualitative level, the investigated randomization of a set of integral curves of nonlinear oscillator driven by resonance periodic force with slowly varying frequency is in accord with the basic understanding of nonlinear dynamics [6,7]. Equation (6) allows one to evaluate the stochastic trajectories measure and connect it with the parameters of equations of motion of the first approximation of asymptotic expansion (2), (8). Equations of this type can be used to describe a variety of nonlinear resonant mechanical, physical and electronic systems [2,3] but their skeleton curves $\omega(a) = \nu$ are different. The intrinsic frequency $\omega(a)$ can be, e.g., a smooth or a piecewise-smooth function, allowing one to perform a qualitative analysis of the rearrangement of separatrices for a range of ν near $\omega(0)$, and to find the domains of topological instability. In particular, the phase portraits topology of Hamiltonian sys-

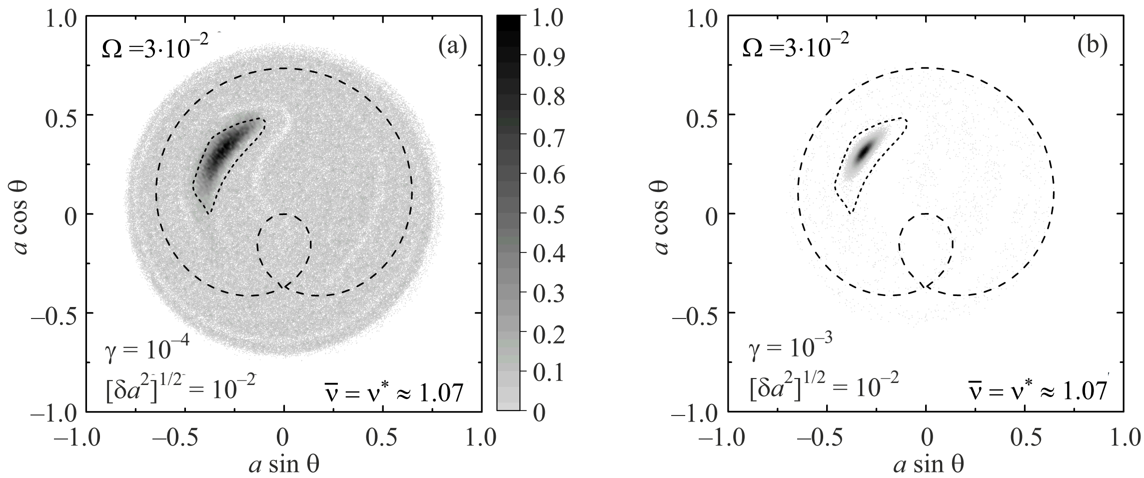

Fig. 7. The Poincare map of the dissipative dynamics impacted by a random force at $\gamma = 10^{-4}$ (a) and 10^{-3} (b).

tems with smooth monotonic functions $\omega(a)$ is topologically equivalent to those shown in Fig. 1.

Notably, the breakdown of adiabaticity due to frequency modulation at $\Omega \sim \varpi(0)$ leads to a formation of strong stochastic layer of measure $\sim \varepsilon_\nu^{1/3}$ due to the integral curves mixing near the point of separatrix merger at $\nu = \nu^*$.

Conclusions

Modulation of the driving force frequency near resonance gives rise to the strong stochasticity of nonlinear oscillatory motion if the frequency range contains the domain where separatrices merge and split.

The behavior of the measure of stochastic trajectories set is varying with the adiabaticity parameter $\varepsilon_\nu = \Omega / \varpi(0)$. At $\varepsilon_\nu > 10^{-4}$ it is nearly proportional to $\varepsilon^{1/3}$ while at $\varepsilon_\nu < 10^{-4}$ it decreases as ε_ν as ε_ν^2.

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1. A.I. Akhiezer, A.S. Bakai, *Ukr. J. Phys*. **13**, 355 (1969).
2. Y.A. Mitropolsky, *Nestatsionrnie processy v nelineynyh kolebatelnyh sistemah*, Izdat. Acad. Nauk UkrSSR, Kyiv (1955).
3. N.N. Bogoliubov, Y.A. Mitropolsky, *Asimptoticheskie metody v teorii nelineynyh kolebaniy*, Nauka, Moscow (1974) [*Asymptotic Methods in the Theory of Non-Linear Oscillations*, Gordon & Breach, New York (1961)].
4. A.S. Bakai, *Matematicheskaya fizika*, Naukova dumka, Kyiv 4, 3 (1968).
5. A.S. Bakai, *Fiz. Nizk. Temp*. **36**, 994 (2010) [*Low Temp. Phys.* **36**, 792 (2010)]
6. H. Poincare, *New Methods in Celestial Mechanics,* AIP, Bristol (1992).
7. V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt, *Matematicheskie aspekty klassicheskoy i nebesnoy mehaniki*, VINITI, Moscow (1985) [*Mathematical Aspects of Classical and Celestial Mechanics*, Springer-Verlag, Berlin (2006)].
8. A.S. Bakai, *Differentsial'nie uravneniya* **2**, 1428 (1966)
9. B.V. Chirikov, *Phys. Rep.* **52**, 263 (1979)
10. G.M. Zaslavsky, *Stohastichnost dinamicheskih system*, Nauka, Moscow (1984) [*Chaos in Dynamic Systems*, Harwood Academic Publishers, New York (1985)].
11. A.V. Artemyev, A.I. Neishtadt, and L.M. Zelenyi, *Chaos* **21**, 043120 (2011)
12. M. Kruskal, *Adiabatic Invariants*, Princeton University Press, Princeton (1961); russian trans.: *Adiabatičeskie invarianty*, Izdat. Inostr. Literatury, Moscow (1962).
13. A.S. Bakai and Ju.P. Stepanovsky, *Adiabaticheskie invarianty*, Naukova dumka, Kyiv (1981).
14. O. Bakai and M. Bratchenko, (in preparation).

Стохастична динаміка нелінійного осцилятора під дією періодичної сили, частота якої повільно змінюється

О.С. Бакай, М.І. Братченко

Топологія фазових портретів нелінійного осцилятора, який збуджується періодичною силою, зазнає значних змін у вузькому інтервалі частот ν рушійної сили. Ця властивість призводить до неінтегрованості рівнянь руху та стохастизації осцилятора при періодичній модуляції ν через порушення адіабатичності та руйнування інтегральних многовидів, викликаного перебудовою топології інтегральних кривих. Особливості стохастичної динаміки вивчено в широкому діапазоні періодів модуляції частоти та декрементів згасання коливань.

Ключові слова: нелінійний осцилятор, періодична модуляція, стохастична динаміка.