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PROPERTIES OF DECELERATING NON-DIFFRACTIVE ELECTROMAGNETIC AIRY PULSES

A.G. NERUKH, D.A. ZOLOTARIOV, D.A. NERUKH

The existence of electromagnetic pulses in time domain with the Airy function envelope is shown. The pulses satisfy an equation similar to the Schrodinger equation but in which the time and space variables play opposite roles. The pulses are generated by an Airy time varying field at a source point and propagate in vacuum preserving their shape and magnitude. The pulse motion is according to a quadratic law with the velocity changing from infinity at the source point to zero in infinity. Properties of such pulses are investigated in detail.

Keywords: Airy pulse, paraxial approximation, slowing propagation.

An Airy beam or Airy wave packet is a wave described by the Airy function [1]. The Airy beams are characterised by very special properties: they are non-diffractive (remain invariant during propagation) and accelerating (increase their envelope velocity with time) [2-9]. Recently there has been active development in the theory and experimental applications of optical Airy beams. Among the striking applications of the optical Airy beams are the transport of small particles and living cells along a parabolic trajectory and the self-healing property of the beam, when the beam form is restored after passing an obstacle [10]. A new way of generating Airy beams by using three wave mixing processes in nonlinear media has been examined experimentally in [11].

The detailed analysis of the mathematical aspects as well as physical interpretation of electromagnetic Airy beams is done by considering the wave as a function of *spatial coordinates only* and assuming that their time dependence is harmonic, $\exp(i\omega t)$, [2-7]. Yet, the idea of electromagnetic Airy beams comes from the analogy of the paraxial equation describing these beams with the time dependent Schrodinger equation [2, 12], where the time variable is replaced with a spatial coordinate. The solution of the Schrodinger equation produces time dependent Airy wave packets in free space [12, 13]. Their features such as the diffraction free form and continuous acceleration has been explained on the basis of the semi-classical approximation. (It is worth to note that the Airy wave function is known in quantum mechanics for a long time [14] as a solution to the stationary Schrodinger equation.) As for the time dependent solution of the three-dimensional electromagnetic problem, the possibility of the existence of non-diffractive Bessel (not Airy) waves has been pointed out in [2, 15]. However, the three dimensional solutions to the paraxial equations containing the time variable do not include the parabolic variable responsible for the accelerating feature of the beams.

Therefore, it is important to investigate the *explicitly time dependent* solutions of the electromagnetic problem in the form of an Airy pulse and deduce whether it possesses the unique features described above. We show that it is not only possible to find the Airy pulse solution starting from the first principles, rather than by exploiting the analogy with the paraxial equation, but also that the obtained beam has the same property of non-diffractive propagation and velocity change without any external influences (in vacuum). There are, however, important conceptual differences that lead to the pulse deceleration, rather than acceleration as in quantum mechanics.

We consider here the role of the time variable in the solution of a 'paraxial' equation including explicit presence of time. We start with the wave equation, followed from the Maxwell equations,

$$\partial_{zz}^2 E(t,z) - c^{-2}\partial_{tt}^2 E(t,z) = 0 \qquad (1)$$

which describes the electric field of a wave propagating along the z axis. Substitution of the field in the form $E(t,z) = F(t,z)e^{\pm ikz}$, $k = \omega / c$ and under the assumption that $\left|F_{zz}^{''}\right| << \left|2ikF_z^{'}\right|$, typical for the paraxial approximation [16-18], the wave equation is reduced to the form

$$\mp i2\partial_\xi F + \partial_{\tau\tau}^2 F + \kappa^2 F = 0 , \qquad (2)$$

where the normalized dimensionless variables are $\xi = z/(kc^2 t_0{}^2)$, $\tau = t/t_0$ with t_0 being the temporal scale and the dimensionless parameter is $\kappa = kct_0$. Comparing this equation with the commonly considered spatial paraxial equation in the x, z coordinates

$$i2\partial_\xi \Phi + \partial_{ss}^2 \Phi = 0 \qquad (3)$$

we see that the longitudinal spatial variables ξ are the same and the transverse variable $s = x/x_0$ ($x_0 = ct_0$) corresponds to the temporal variable τ in (2). The equation (3) is considered in the literature as the analogue to the Schrodinger equation

$$2^{-1} m^{-1}\hbar^2 \partial_{xx}^2 \Psi(x,t) + i\hbar\partial_t \Psi(x,t) = 0 \qquad (4)$$

from which the Airy wave packet originated in [12] if the temporal variable t in (4) is replaced by the longitudinal variable z (ξ in (3)). Thus, the variable z (ξ) along which an electromagnetic wave propagates plays the role of time in the electromagnetic phenomenon. As it was shown in [12] equation (4) has a solution in the form of a non-spreading wave packet with the envelope as the Airy function (designations as in [12])

$$\begin{gathered}\Psi(x,t) = Ai\left[B\hbar^{-2/3}\left(x - B^3t^2m^{-2}/4\right)\right]\times \\ exp\left[iB^3(2m\hbar)^{-1}\left(tx - B^3t^3m^{-2}/6\right)\right]\end{gathered}. \qquad (5)$$

This function describes the accelerating wave packet which moves uniformly with the velocity $\dot{x} = B^3t/2m^2$ and the constant acceleration $\ddot{x} = B^3/2m^2$. Contrary to the equation (3), which describes a beam harmonically oscillating in time, the function (5) represents the pulse with a complicated time varying envelope enclosed in the Airy function. The Airy function in the solution to (3)

$$\Phi = Ai[s - \xi^2/4]\exp\left\{i\left[-s\xi/2 + \xi^3/12\right]\right\} \qquad (6)$$

describes the inhomogeneous distribution with respect to the spatial coordinates s and ξ of the wave paraxial propagating along the z axis but with harmonic temporal variation $E = \Phi(x,z)e^{ikz - i\omega t}$.

Our equation (2), derived from the first principle rather than by the analogy with the Schrodinger equation, shows that the roles of the time and space variables in the electromagnetic time paraxial equation (2) are opposite to those of the Schrodinger equation (4). This destroys the analogy between the equations (2) and (4)and, therefore, the direct correspondence between the time and space variables of the Schrodinger equation and the space variables of the spatial paraxial equation(3). Thus, we need to solve the equation (2) in order to find the time-spatial pulse originating from it.

The solution to the equation (2) can be constructed following the procedure described in [8]. The sought function is represented as $F = W(\eta)e^{i\Theta(\eta,\xi)}$, where $W(\eta)$ and $\Theta(\eta,\xi)$ are real functions of the argument ξ and the quadratic variable $\eta = -a\tau + \tau_0 - \xi^2/4 + b\xi$. The parameter $a = \pm 1$ determines the movement forward or backward along the time axes. The parameters τ_0 and b in η allow changing the model. This representation leads to the equation for the phase

$$\begin{gathered}\Theta(\eta,\xi) = \pm a^{-2}(-\xi/2 + b)\eta \mp \\ 2^{-1}a^{-2}\left(\xi^3/12 - b\xi^2/2 + (b^2 + a^2\kappa^2)\xi\right)\end{gathered}. \qquad (7)$$

and to the equation for the envelope which is the Airy equation $W''(\eta) - \eta W(\eta) = 0$. Its solution is the Airy function

$$W(\eta) = Ai[-\left(\xi/2 - b\right)^2 - a\tau + \tau_0 + b^2]. \qquad (8)$$

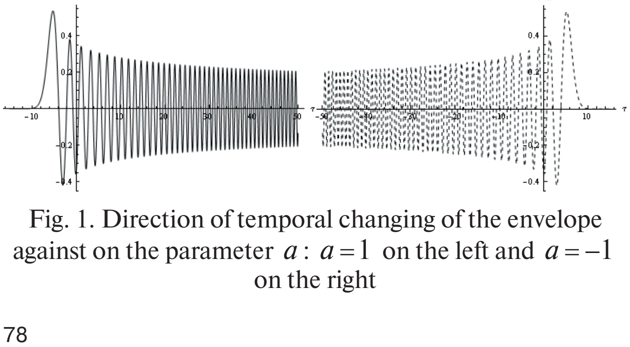


Fig. 1. Direction of temporal changing of the envelope against on the parameter a: $a = 1$ on the left and $a = -1$ on the right

Dependence of the envelope (8) on time for a fixed value of a spatial coordinate is shown in Fig. 1. As one can see the parameter $a = 1$ corresponds to the envelope movement with a main lobe ahead (it appears at first and then a tail does) whereas $a = -1$ corresponds to the opposite movement. The trajectory of the main lobe movement is given by the equation

$$-\left(\xi/2 - b\right)^2 - a\tau + \tau_0 + b^2 = 0 \qquad (9)$$

following from (8) and it is shown in Fig. 2 where $a = 1$ corresponds to a left parabola.

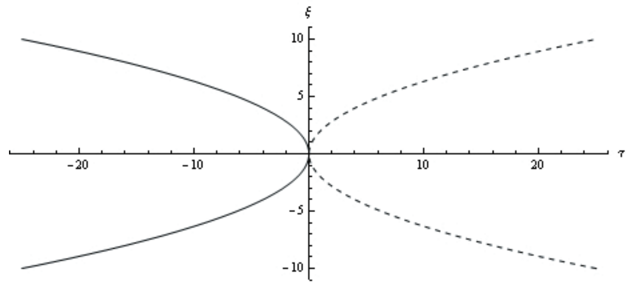


Fig. 2. The trajectories of the envelope: the left hand side for $a = 1$, the right for $a = -1$

Evidently, such movement is impossible because for a given initial condition at $t = 0$ in Fig. 2 realization of this movement means a backward time course. On the other hand, if the initial condition corresponds to a certain negative time moment the movement along the left parabola will lead to stop of a time at $t = 0$. It is consequence of a parabolic kind of the equation (2). It is known that an equation of a parabolic kind describes solutions with the infinite value of an influence velocity.

So, the physical sense can have only the solution with $a = -1$ that corresponds to the trajectory on the right-hand side in Fig. 2. In this case the solution to the equation (2) is

$$\begin{gathered}F(\tau,\xi) = Ai[-\left(\xi/2 - b\right)^2 + \tau + \tau_0 + b^2]\times \\ \times\exp\left\{\pm i\left[-b^3 + (2b^2 - \kappa^2)\xi/2 - \right.\right. \\ \left.\left. -b\xi^2/2 + \xi^3/12 - (\tau + \tau_0 + b^2)\left(\xi/2 - b\right)\right]\right\}\end{gathered}. \qquad (10)$$

This function satisfies the boundary condition

$$F(\tau, \xi = 2b) = Ai[\tau + \tau_0 + b^2]\exp\left\{\pm i\left[-b(\kappa^2 + b^2/3)\right]\right\}$$

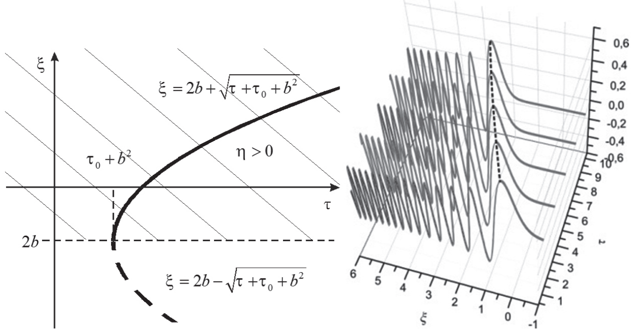
that can be interpreted as a time varying source located at the point $\xi = 2b$. Therefore, the solution (10) describes the propagation of this source radiation

$$\begin{gathered}E = F(\tau,\xi)e^{\pm i\kappa^2\xi} = Ai\left[-\left(\xi/2 - b\right)^2 + \tau + \tau_0 + b^2\right]\times \\ \times\exp\left\{\pm i\left[-(\tau + \tau_0 + b^2)\left(\xi/2 - b\right) + \right.\right. \qquad (11) \\ \left.\left. + \xi^3/12 - b\xi^2/2 + (2b^2 + \kappa^2)\xi/2 - b^3\right]\right\}.\end{gathered}$$

This field is uniquely defined in the half-space $\xi \geq 2b$, Fig. 3a.

Starting from the source point the field profile propagates according to the quadratic law $\tau + \tau_0 - (\xi/2 - b)^2 + b^2 = const$ preserving its form. Fig. 3*a* illustrates the lines of propagation of the field equal values determined by the parabola

$\tau + \tau_0 - \xi^2 / 4 + b\xi = const$ (one of the branches for $const = 0$ is shown using the solid line in the figure). The quadratic variable η is positive inside the region bounded by the parabola and negative outside of it. It determines that the main lobe of the Airy envelope, corresponding to nearly zero argument of the Airy function at the source point, comes off the source leaving the space free of the field, Fig. 3*b*. The velocity of this movement decreases with distance, $\dot{\xi} = 2/(\xi - 2b)$ and time $\dot{\xi} = 1/\sqrt{\tau + \tau_0 + b^2}$, therefore the acceleration $\ddot{\xi} = -4/(\xi - 2b)^3$ or $\dot{\xi} = -\left(\tau + \tau_0 + b^2\right)^{-3/2}/2$ is negative. Such a slowing motion leads to a complete stop as its velocity and acceleration tend to zero at the infinite distance from the source. The contours of the envelope constant values in the time-spatial diagram are shown in Fig. 4.



a *b*

Fig. 3. The region of definition of the electric field (hatched region) (*a*). The movement of the field distribution given by the Airy function envelope in (the magnitudes of this envelope is shown on the vertical axes) (*b*)

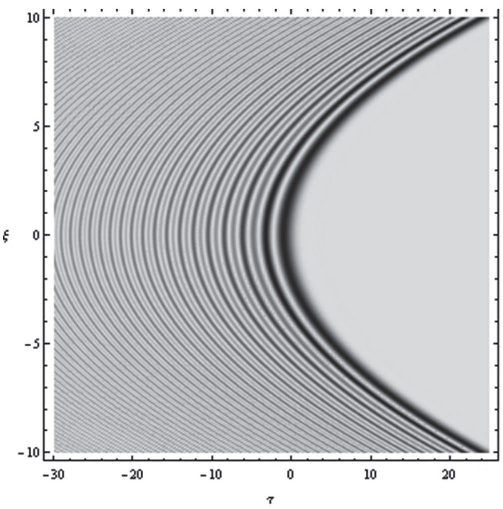


Fig. 4. The contours of the envelope constant values

The presence of the exponent factor in (11) does not change substantially the main feature of the trajectory but adds property of interference between oscillatory character of the Airy function of the envelope and oscillatory character of the exponent factor, Fig. 5. It is seen here that homogeneity of the field distribution in space and time is broken and regions of more sophisticated field changing appear in the time-spatial diagram of the function (11) in Fig. 5.

A swinging character of the field oscillatory appears in the vicinity of the point $\xi = 0$ and the time moment $\tau_k = -\left(\tau_0 + b^2\right)$. This swinging propagates with the infinite velocity, that is nature for equations of parabolic kind, to other values of ξ. It is seen very well in Fig. 6.

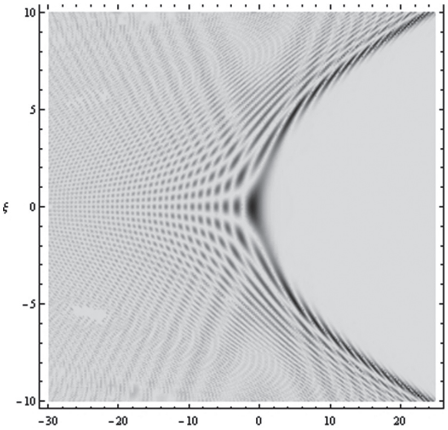


Fig. 5. The contour plot for the whole filed

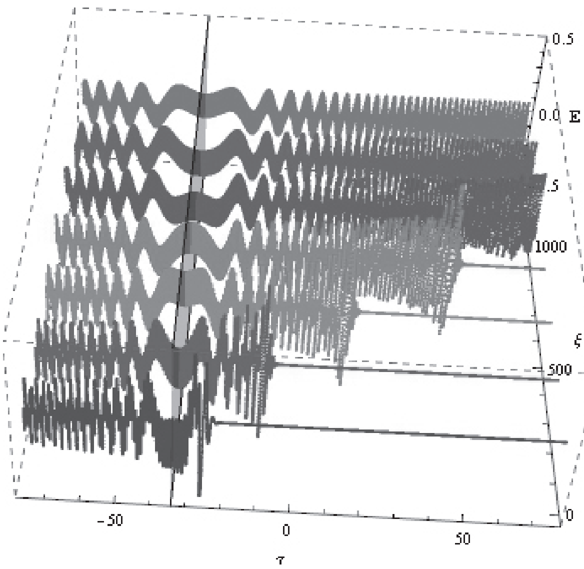


Fig. 6. The picture of interference in the neighborhood of the moment τ_k

Fig. 7 illustrates details of this interaction and shows that such behaviour appears only due to interference of two multipliers.

The considered Airy pulses are of little practical importance because of their infinite energy. To overcome this deficiency it was suggested in [2, 3] to consider the exponentially decaying version at the input of the system. Following this suggestion we consider a different boundary condition

$$F(\tau, \xi = 2b) = Ai[\tau + \tau_0 + b^2] \times \exp\left\{\mp ib(\kappa^2 + b^2/3) + \alpha(\tau + \tau_0 + b^2)\right\} \qquad (12)$$

for obtaining the pulse with finite energy. To solve the equation (2) with the boundary condition (12) we represent the solution as the Fourier transform

$$F(\tau, \xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu e^{i\nu\tau} \bar{F}(\nu, \xi). \qquad (13)$$

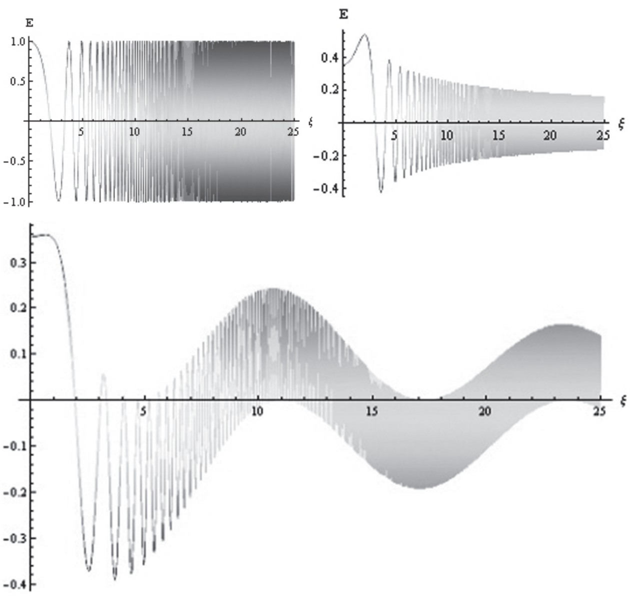


Fig. 7. A swinging character of the field oscillatory: the exponent multiplier (left) and the Airy function (right) are on the top; their product is on the bottom

The equation for this Fourier transform follows from (2)

$$\mp i2\partial_{\xi}\bar{F}(\nu,\xi)-(\nu^2-\kappa^2)\bar{F}(\nu,\xi)=0 \qquad (14)$$

and has the solution satisfying the condition (12)

$$\bar{F}(\nu,\xi)=\exp\Big[i\nu(\tau_0+b^2)\mp ib^3/3\pm i\nu^2(\xi/2-b)\mp i\kappa^2\xi/2+i(\nu+i\alpha)^3/3\Big]. \qquad (15)$$

Applying the Fourier transform (13) we obtain finally the electric field of the pulse

$$E(\tau,\xi)=Ai[\tau+\tau_0+b^2-(\xi/2-b)^2\mp i2\alpha(\xi/2-b)]\times \times\exp i\Big\{\pm b(\kappa^2-b^2/3)\pm\kappa^2(\xi/2-b)+2i\alpha(\xi/2-b)^2\pm 2(\xi/2-b)^3/3\mp(\xi/2-b\pm i\alpha)(\tau+\tau_0+b^2)\mp \alpha^2(\xi/2-b)\Big\} \qquad (16)$$

Introduction of the exponentially decaying blurs the lines of constant field values but preserve the main features of the phenomenon.

It is seen in Fig. 8, where influence of the parameter α is shown for two values of the parameter κ which characterises spatial changing of the field.

Fig. 8 shows also influence of the parameter κ on the solution of the equation considered. Indeed, the value of this parameter is crucial for correctness of the paraxial approximation $\left|F''_{\xi\xi}\right|/\left|2i\kappa F'_{\xi}\right|<<1$. As it seen in Fig. 9 the correctness of the paraxial approximation is valid for $\kappa=25$ and greater.

CONCLUSION

In conclusion, we derived the time dependent electromagnetic Airy pulses that satisfy the 'paraxial' equation similar to the Schrodinger equation in which the time and space variables interchange their roles. The solution to the electromagnetic equation is the Airy pulse which propagates with deceleration along its trajectory and stops at the infinite distance from the source. The realistic situation when an initial wave has a finite energy is considered also and it is shown that the main features of the phenomenon are preserved. Dependence of the derived Airy pulses on the phenomenon parameters is investigated in details.

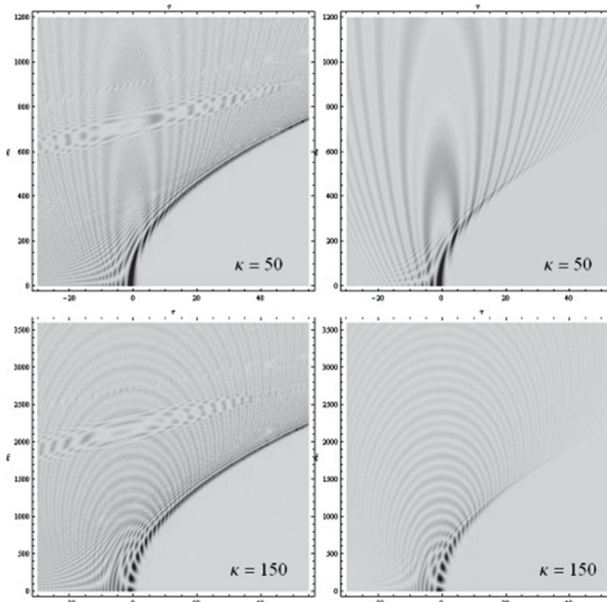


Fig. 8. The contour plots for the whole filed for $\alpha=0$ (left) and $\alpha=0.035$ (right)

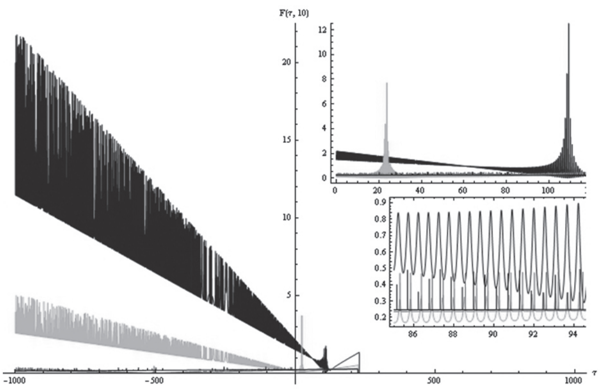


Fig. 9. Correctness of the paraxial approximation versus the parameter κ : from top to bottom $\kappa=5, 15, 25, 50$

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Властивості недифрагуючих імпульсів Ейрі, що уповільнюються / О.Г. Нерух, Д.О. Золотарев, Д.О Нерух // Прикладна радіоелектроніка: наук.-техн. журнал. – 2012. Том 11. № 1. – С. 77-81.

У статті показано існування електромагнітних імпульсів Ейрі у часовій області. Імпульси задовольняють рівнянню, що подібне рівнянню Шредінгера, але в якому часова та просторова координати грають протилежні ролі. Імпульс генерується джерелом, яке змінюється в часі як функція Ейрі, та розповсюджується в вакуумі, зберігаючи свою форму та величину. Імпульс рухається за квадратичним законом зі швидкістю, що змінюється від нескінченності в точці джерела до нуля на нескінченності. Детально вивчені властивості таких імпульсів.

Ключові слова: імпульс Ейрі, параксіальне наближення, розповсюдження, що уповільнюється

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Свойства замедляющихся недифрагирующих импульсов Эйри / А.Г. Нерух, Д.А. Золотарев, Д.А Нерух // Прикладная радиоэлектроника: науч.-техн. журнал. – 2012. Том 11. № 1. – С. 77-81.

В статье показано существование электромагнитных импульсов Эйри во временной области. Импульсы удовлетворяют уравнению, подобному уравнению Шредингера, но в котором временная и пространственная координаты играют противоположные роли. Импульс генерируется источником, изменяющимся во времени как функция Эйри, и распространяется в вакууме, сохраняя свою форму и величину. Импульс двигается согласно квадратичному закону со скоростью, изменяющейся от бесконечности в точке источника до нуля на бесконечности. Детально исследованы свойства таких импульсов.

Ключевые слова: импульс Эйри, параксиальное приближение, замедляющееся распространение.

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