Теорія ймовір. та матем. статист.
Вип. 89, 2013, стор. 115–129

Teor. Ĭmovir. ta Matem. Statyst.
No. 89, 2013, pp. 115–129

MINIMAX-ROBUST FILTERING PROBLEM FOR STOCHASTIC SEQUENCE WITH STATIONARY INCREMENTS

UDC 519.21

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Abstract. The problem of optimal estimation of the linear functional $A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)$ which depends on unknown values of a stochastic sequence $\xi(k)$ with stationary nth increments from observations of the sequence $\xi(k) + \eta(k)$ at points of time $k = 0, -1, -2, \dots$ is considered. Formulas for calculation the mean-square error and spectral characteristic of the optimal linear estimate of the functional are derived under the condition of spectral certainty, where spectral densities of the sequences $\xi(k)$ and $\eta(k)$ are exactly known. The minimax (robust) method of estimation is applied in the case where spectral densities are not known exactly, but sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed for some special sets of admissible spectral densities.

Анотація. Досліджується задача оптимального оцінювання функціонала $A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)$ від невідомих значень стохастичної послідовності $\xi(k)$ зі стаціонарними n-ми приростами за спостереженнями послідовності $\xi(k) + \eta(k)$ у моменти часу $k = 0, -1, -2, \dots$. Знайдені формули для обчислення середньоквадратичної похибки та спектральної характеристики оптимальної оцінки функціонала за умови спектральної визначеності, тобто коли спектральні щільності послідовностей $\xi(m)$ та $\eta(m)$ відомі. У тому випадку, коли спектральні щільності невідомі, а задані лише множини допустимих спектральних щільностей, застосовано мінімаксний метод оцінювання. Для заданих множин допустимих спектральних щільностей визначені найменш сприятливі спектральні щільності та мінімаксні спектральні характеристики оптимальної лінійної оцінки функціонала.

Аннотация. Исследуется задача оптимального оценивания функционала $A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)$ от неизвестных значений стохастической последовательности $\xi(k)$ со стационарными n-ми приращениями по наблюдениям последовательности $\xi(k) + \eta(k)$ в моменты времени $k = 0, -1, -2, \dots$. Найдены формулы для вычисления среднеквадратической ошибки и спектральной характеристики оптима-льной оценки функционала в том случае когда спектральные плотности последовательностей $\xi(m)$ и $\eta(m)$ точно известны. В том случае, когда спектральные плотности неизвестны, а заданы лишь множества допустимых спектральных плотностей, используется минимаксный метод оценивания. Для заданных множеств допустимых спектральных плотностей определены наименее благоприятные спектральные плотности и минимаксные спектральные характеристики оптимальной линейной оценки функционала.

1. Introduction

Traditional methods of solution of extrapolation, interpolation and filtering problems for stationary stochastic processes and sequences were developed by A. N. Kolmogorov [11], N. Wiener [26], A. M. Yaglom [28] under the condition of spectral certainty where spectral densities of the considered stochastic processes are exactly known. In the case where spectral densities are not exactly known, but a set of admissible spectral densities is given, we can apply the minimax method for solving extrapolation, interpolation and filtering problems, which allows us to determine estimates that minimize the value of the mean-square error for all densities from a given class.

2010 *Mathematics Subject Classification.* Primary 60G10, 60G25, 60G35; Secondary 62M20, 93E10, 93E11.

Key words and phrases. Sequences with stationary increments, robust estimate, mean-square error, least favorable spectral density, minimax spectral characteristic.

A survey of results in minimax (robust) methods of data processing is proposed by S. A. Kassam and H. V. Poor [10]. The paper by Ulf Grenander [7] should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was developed. J. Franke [8], J. Franke and H. V. Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. In the works by M. P. Moklyachuk [14]–[17] problems of extrapolation, interpolation and filtering for stationary processes and sequences were studied. Methods of solution the minimax-robust estimation problems for vector-valued stationary sequences and processes were developed by M. P. Moklyachuk and O. Yu. Masyutka [19]–[23]. Methods of solution the minimax-robust estimation problems (extrapolation, interpolation and filtering) for linear functionals which depend on unknown values of periodically correlated stochastic processes were proposed by I. I. Dubovets'ka and M. P. Moklyachuk [2]–[6]. M. M. Luz and M. P. Moklyachuk [12]–[13] investigated the minimax interpolation problem for stochastic sequences $\xi(m)$ with stationary n-th increments from observations of the sequence with an additive noise and from observations without noise.

In this paper we investigate the problem of optimal linear filtering of a functional $A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)$ which depends on unobserved values of a stochastic sequence $\xi(m)$ with nth stationary increments based on observations of the sequence $\xi(k)+\eta(k)$ at points $k=0,-1,-2,\dots$, where $\eta(k)$ is a stochastic sequence with stationary nth increments which is uncorrelated with the sequence $\xi(k)$. This filtering problem is solved in the case of spectral certainty where spectral densities of sequences $\xi(m)$ and $\eta(m)$ are exactly known as well as in the case of spectral uncertainty where spectral densities of sequences are not exactly known, but a set of admissible spectral densities is given. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics of the optimal linear estimate of the functional are proposed in the case of spectral uncertainty for concrete classes of admissible spectral densities.

2. Stochastic stationary increment sequence. Spectral representation

Stochastic processes with stationary n-th increments were introduced and investigated by A. M. Yaglom [27], M. S. Pinsker [25], A. M. Yaglom and M. S. Pinsker [24].

Definition 2.1. For a given stochastic sequence $\{\xi(m),m\in\mathbb{Z}\}$ a sequence

$$\xi^{(n)}(m,\mu)=(1-B_\mu)^n\xi(m)=\sum_{l=0}^{n}(-1)^lC_n^l\xi(m-l\mu), \tag{1}$$

where B_μ is a backward shift operator with step $\mu\in\mathbb{Z}$, such that $B_\mu\xi(m)=\xi(m-\mu)$, is called stochastic nth increment sequence with step $\mu\in\mathbb{Z}$.

For the stochastic nth increment sequence $\xi^{(n)}(m,\mu)$ the following relations hold true:

$$\xi^{(n)}(m,-\mu)=(-1)^n\xi^{(n)}(m+n\mu,\mu), \tag{2}$$

$$\xi^{(n)}(m,k\mu)=\sum\nolimits_{l=0}^{(k-1)n}A_l\xi^{(n)}(m-l\mu,\mu),\qquad k\in\mathbb{N}, \tag{3}$$

where coefficients $\{A_l,l=0,1,2,\dots,(k-1)n\}$ are determined by the representation

$$(1+x+\dots+x^{k-1})^n=\sum_{l=0}^{(k-1)n}A_lx^l.$$

Definition 2.2. The stochastic nth increment sequence $\xi^{(n)}(m,\mu)$ generated by stochastic sequence $\{\xi(m),m\in\mathbb{Z}\}$ is wide sense stationary if the mathematical expectations

$$\mathsf{E}\xi^{(n)}(m_0,\mu)=c^{(n)}(\mu),$$
$$\mathsf{E}\xi^{(n)}(m_0+m,\mu_1)\xi^{(n)}(m_0,\mu_2)=D^{(n)}(m,\mu_1,\mu_2)$$

exist for all m_0, μ, m, μ_1, μ_2 and do not depend on m_0. The function $c^{(n)}(\mu)$ is called the mean value of the nth increment sequence and the function $D^{(n)}(m,\mu_1,\mu_2)$ is called the structural function of the stationary nth increment sequence (or the structural function of nth order of the stochastic sequence $\{\xi(m), m\in\mathbb{Z}\}$).

The stochastic sequence $\{\xi(m), m\in\mathbb{Z}\}$ which determines the stationary nth increment sequence $\xi^{(n)}(m,\mu)$ by formula (1) is called sequence with stationary nth increments.

Theorem 2.1. *The mean value $c^{(n)}(\mu)$ and the structural function $D^{(n)}(m,\mu_1,\mu_2)$ of the stochastic stationary nth increment sequence $\xi^{(n)}(m,\mu)$ can be represented in the following forms:*

$$c^{(n)}(\mu) = c\mu^n, \tag{4}$$

$$D^{(n)}(m,\mu_1,\mu_2) = \int_{-\pi}^{\pi} e^{i\lambda m}\left(1-e^{-i\mu_1\lambda}\right)^n\left(1-e^{i\mu_2\lambda}\right)^n \frac{1}{\lambda^{2n}}\,dF(\lambda), \tag{5}$$

where c is a constant, $F(\lambda)$ is a left-continuous nondecreasing bounded function with $F(-\pi)=0$. The constant c and the function $F(\lambda)$ are determined uniquely by the increment sequence $\xi^{(n)}(m,\mu)$.

From the other hand, a function $c^{(n)}(\mu)$ which has the form (4) with a constant c and a function $D^{(n)}(m,\mu_1,\mu_2)$ which has the form (5) with a function $F(\lambda)$ which satisfies the indicated conditions are the mean value and the structural function of some stationary nth increment sequence $\xi^{(n)}(m,\mu)$.

Using representation (5) of the structural function of a stationary nth increment sequence $\xi^{(n)}(m,\mu)$ and the Karhunen theorem [1], we obtain the following spectral representation of the stationary nth increment sequence $\xi^{(n)}(m,\mu)$:

$$\xi^{(n)}(m,\mu) = \int_{-\pi}^{\pi} e^{im\lambda}\left(1-e^{-i\mu\lambda}\right)^n \frac{1}{(i\lambda)^n}\,dZ(\lambda), \tag{6}$$

where $Z(\lambda)$ is an orthogonal stochastic measure on $[-\pi,\pi)$ connected with the spectral function $F(\lambda)$ by the relation

$$\mathsf{E}Z(A_1)\overline{Z(A_2)} = F(A_1\cap A_2) < \infty. \tag{7}$$

Example 2.1. Consider an ARIMA(0,1,1) sequence defined by the equation

$$\xi_m = \xi_{m-1} + \varepsilon_m + a\varepsilon_{m-1},$$

where ε_m is a sequence of uncorrelated identically distributed random variables with mean value 0 and variance σ^2. If we take $\eta_m = \xi_m - \xi_{m-1}$ we obtain a moving average sequence $\eta_m = \varepsilon_m + a\varepsilon_{m-1}$. Thus, ξ_m is a stochastic sequence with stationary increments of the 1st order. The spectral function $F(\lambda)$ of the sequence ξ_m can be calculated as follows

$$F(\lambda) = \frac{\sigma^2}{4\pi}\int_{-\pi}^{\lambda}\frac{u^2}{1-\cos u}\left(1+2a\cos u + a^2\right)du.$$

Here are some values of the structural function;

$$D^{(1)}(0,1,1) = \sigma^2\left(1+a^2\right), \qquad D^{(1)}(0,1,2) = \sigma^2\left(1+a+a^2\right),$$
$$D^{(1)}(0,2,2) = 2\sigma^2(1+a+a^2),$$
$$D^{(1)}(m,1,1) = \begin{cases}\sigma^2(1+a^2), & m=0,\\ \sigma^2 a, & m=-1,1,\\ 0, & \text{otherwise,}\end{cases}$$

$$D^{(1)}(m,1,2)=\begin{cases}\sigma^2(1+a+a^2), & m=-1,0,\\ \sigma^2a^2, & m=-2,1,\\ 0, & \text{otherwise},\end{cases}$$

$$D^{(1)}(m,2,2)=\begin{cases}2\sigma^2(1+a+a^2), & m=0,\\ \sigma^2(1+2a+a^2), & m=-1,1,\\ \sigma^2a^2, & m=-2,2,\\ 0, & \text{otherwise}.\end{cases}$$

3. Filtering problem for the functional $A\xi$

Let a stochastic sequence $\{\xi(m), m\in\mathbb{Z}\}$ define a stationary nth increment $\xi^{(n)}(m,\mu)$ with an absolutely continuous spectral function $F(\lambda)$ which has spectral density $f(\lambda)$. Let $\{\eta(m), m\in\mathbb{Z}\}$ be a stochastic sequence, uncorrelated with the sequence $\xi(m)$, which determines a stationary nth increment $\eta^{(n)}(m,\mu)$ with an absolutely continuous spectral function $G(\lambda)$ whith has spectral density $g(\lambda)$. Without loss of generality we will assume that the mean values of the increment sequences $\xi^{(n)}(m,\mu)$ and $\eta^{(n)}(m,\mu)$ equal to 0. Let us suppose that we know values of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$. Consider the problem of mean-square optimal linear estimation of the functional

$$A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)$$

of unknown values of the sequence $\xi(m)$ from observation of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$. We will consider the case where the step $\mu>0$.

From (1) we can obtain the formal equation

$$\xi(-k)=\frac{1}{(1-B_\mu)^n}\xi^{(n)}(-k,\mu)=\sum_{i=k}^{\infty}d_\mu(i-k)\xi^{(n)}(-i,\mu), \tag{8}$$

where $\{d_\mu(i)\colon i\geq 0\}$ are coefficients from decomposition $\sum_{i=0}^{\infty}d_\mu(i)x^i=\left(\sum_{l=0}^{\infty}x^{\mu l}\right)^n$. From equation (8) one can find the following relations:

$$\sum_{k=0}^{\infty}a(k)\xi(-k)=\sum_{i=0}^{\infty}\xi^{(n)}(-i,\mu)\sum_{k=0}^{i}a(k)d_\mu(i-k),$$

$$\sum_{k=0}^{\infty}b_\mu(k)\xi^{(n)}(-k,\mu)=\sum_{i=0}^{\infty}\xi(-i)\sum_{l=0}^{\min\left\{n,\left[\frac{i}{\mu}\right]\right\}}(-1)^lC_n^lb_\mu(i-l\mu).$$

From the last two relations we obtain the following representation of the functional $A\xi$:

$$A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)=\sum_{k=0}^{\infty}b_\mu(k)\xi^{(n)}(-k,\mu)=B\xi,$$

$$b_\mu(k)=\sum_{m=0}^{k}a(m)d_\mu(k-m)=(\mathbf{D}^\mu\mathbf{a})_k,\qquad k\geq 0, \tag{9}$$

where $\mathbf{D}^\mu$ is a linear operator with elements $\mathbf{D}^\mu_{k,j}=d_\mu(k-j)$ if $0\leq j\leq k$ and $\mathbf{D}^\mu_{k,j}=0$ if $j>k$; $\mathbf{a}=(a(0),a(1),a(2),\dots)$. Let $\widehat{A}\xi$ denote the mean-square optimal linear estimate of the functional $A\xi$ from observations of stochastic sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2,\dots$ and let $\widehat{B}\xi$ denote the mean-square optimal linear estimate of the functional $B\xi$ from observations of the stochastic nth increment sequence $\xi^{(n)}(m,\mu)+\eta^{(n)}(m,\mu)$ at points $m=0,-1,-2,\dots$.

Let $\Delta(f,g,\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2$ be the mean-square error of the estimate $\widehat{A}\xi$ of the functional $A\xi$ and let $\Delta(f,g,\widehat{B}\xi)=\mathsf{E}|B\xi-\widehat{B}\xi|^2$ be the mean-square error of the estimate $\widehat{B}\xi$ of the functional $B\xi$. Since $A\xi=B\xi$, the following equality holds true:

$$\widehat{A}\xi=\widehat{B}\xi. \tag{10}$$

Therefore, the following relations hold true

$$\Delta(f,g,\widehat{A}\xi)=\mathsf{E}|A\xi-\widehat{A}\xi|^2=\mathsf{E}|B\xi-\widehat{B}\xi|^2=\Delta(f,g,\widehat{B}\xi).$$

To find the mean-square optimal estimate of the functional $B\xi$ we use the Hilbert space orthogonal projection method proposed by A. M. Kolmogorov [11]. Suppose that conditions

$$\sum_{k=0}^{\infty}|b_\mu(k)|<\infty, \qquad \sum_{k=0}^{\infty}(k+1)|b_\mu(k)|^2<\infty, \tag{11}$$

$$\sum_{k=0}^{\infty}|(\mathbf{D}^\mu\mathbf{a})_k|<\infty, \qquad \sum_{k=0}^{\infty}(k+1)|(\mathbf{D}^\mu\mathbf{a})_k|^2<\infty \tag{12}$$

are satisfied.

Let $H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$ be the closed linear subspace of the Hilbert space $H=L_2(\Omega,\mathfrak{F},\mathsf{P})$ of the second order random variables generated by values $\big\{\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)\colon k\leq 0\big\}$, $\mu>0$. Consider also a closed linear subspace $L_2^0(f+g)$ of the Hilbert space $L_2(f+g)$ generated by functions

$$\left\{e^{i\lambda k}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}: k\leq 0\right\}.$$

From the formula

$$\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)=\int_{-\pi}^{\pi}e^{i\lambda k}\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)$$

one can verify the existence of one to one correspondence between element

$$e^{i\lambda k}(1-e^{-i\lambda\mu})^n/(i\lambda)^n$$

from the space $L_2^0(f+g)$ and element $\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu)$ from the space $H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$. Every linear estimate $\widehat{B}\xi$ of the functional $B\xi$ admits representation

$$\widehat{B}\xi=\int_{-\pi}^{\pi}h_\mu(\lambda)\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda), \tag{13}$$

where $h_\mu(\lambda)$ is the spectral characteristic of the estimate $\widehat{B}\xi$. The optimal estimate $\widehat{B}\xi$ is a projection of the element $B\xi$ on the subspace $H^0(\xi^{(n)}_\mu+\eta^{(n)}_\mu)$. This estimate $\widehat{B}\xi$ is determined by the following conditions:

1) $\widehat{B}\xi\in H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$;
2) $(B\xi-\widehat{B}\xi)\perp H^0\big(\xi^{(n)}_\mu+\eta^{(n)}_\mu\big)$.

It follows from condition 2) that for all $k\leq 0$ the function $h_\mu(\lambda)$ satisfies the relation

$$\begin{aligned}
&\mathsf{E}(B\xi-\widehat{B}\xi)\overline{(\xi^{(n)}(k,\mu)+\eta^{(n)}(k,\mu))}\\
&\quad=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}-h_\mu(\lambda)\right)e^{-i\lambda k}\left(1-e^{i\lambda\mu}\right)^n\frac{1}{(-i\lambda)^n}f(\lambda)\,d\lambda\\
&\quad-\frac{1}{2\pi}\int_{-\pi}^{\pi}h_\mu(\lambda)e^{-i\lambda k}\left(1-e^{i\lambda\mu}\right)^n\frac{1}{(-i\lambda)^n}g(\lambda)\,d\lambda\\
&\quad=0.
\end{aligned}$$

From the previous relation we derive the following relations

$$\int_{-\pi}^{\pi}\left(B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{f(\lambda)}{(i\lambda)^n}-h_\mu(\lambda)(f(\lambda)+g(\lambda))\right)\frac{\left(1-e^{i\lambda\mu}\right)^n}{(-i\lambda)^n}e^{-i\lambda k}\,d\lambda=0,$$
$$k\le 0,$$

which yields

$$h_\mu(\lambda)=B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{(-i\lambda)^nC_\mu\left(e^{i\lambda}\right)}{\left(1-e^{i\lambda\mu}\right)^n(f(\lambda)+g(\lambda))},$$
$$B_\mu(e^{i\lambda})=\sum_{k=0}^{\infty}b_\mu(k)e^{-i\lambda k},\qquad C_\mu(e^{i\lambda})=\sum_{k=1}^{\infty}c_\mu(k)e^{i\lambda k}.$$

It follows from condition 1) we conclude that the spectral characteristic $h_\mu(\lambda)$ admits the representation

$$h_\mu(\lambda)=h(\lambda)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n},\qquad h(\lambda)=\sum_{k=-\infty}^{0}s(k)e^{i\lambda k},$$

where

$$\int_{-\pi}^{\pi}|h(\lambda)|^2\left|1-e^{i\lambda\mu}\right|^{2n}\frac{f(\lambda)+g(\lambda)}{\lambda^{2n}}\,d\lambda<\infty,$$
$$\frac{(i\lambda)^nh_\mu(\lambda)}{\left(1-e^{-i\lambda\mu}\right)^n}\in L_2^0,$$
$$\int_{-\pi}^{\pi}\left(B_\mu(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{\lambda^{2n}C_\mu\left(e^{i\lambda}\right)}{\left(1-e^{-i\lambda\mu}\right)^n\left(1-e^{i\lambda\mu}\right)^n(f(\lambda)+g(\lambda))}\right)e^{-i\lambda l}\,d\lambda=0,$$
$$l\ge 1. \tag{14}$$

Let the following conditions holds true:

$$\int_{-\pi}^{\pi}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda<\infty,\qquad \int_{-\pi}^{\pi}\frac{\lambda^{2n}}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))}\,d\lambda<\infty. \tag{15}$$

Set

$$R_{k,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i\lambda(j+k)}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}\,d\lambda,$$
$$P_{k,j}^{\mu}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-k)}\frac{\lambda^{2n}}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))}\,d\lambda,$$
$$Q_{k,j}^{\mu}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\lambda(j-k)}\frac{\left|1-e^{i\lambda\mu}\right|^{2n}f(\lambda)g(\lambda)}{\lambda^{2n}(f(\lambda)+g(\lambda))}\,d\lambda.$$

Then (14) is equivalent to the following linear system:

$$\sum_{m=0}^{\infty}R_{l,m}b_\mu(m)=\sum_{k=1}^{\infty}P_{l,k}^{\mu}c_\mu(k),\qquad l\ge 1.$$

These system can be rewritten as

$$\mathbf{R}\mathbf{b}_\mu=\mathbf{P}_\mu\mathbf{c}_\mu, \tag{16}$$

where $\mathbf{c}_\mu=(c_\mu(1),c_\mu(2),c_\mu(3),\dots)$, $\mathbf{b}_\mu=(b_\mu(0),b_\mu(1),b_\mu(2),\dots)$, $\mathbf{P}_\mu$, $\mathbf{R}$ are linear operators in the space ℓ_2 defined by $(\mathbf{P}_\mu)_{l,k}=P_{l,k}^{\mu}$, $l,k\ge 1$, $(\mathbf{R})_{l,m}=R_{l,m}$, $l\ge 1$, $m\ge 0$. A solution $\mathbf{c}_\mu$ of the last equation defines the linear estimate $\widehat{B}\xi$ which is a projection of the element $B\xi$ from the Hilbert space H on the subspace $H^0\big(\xi_\mu^{(n)}+\eta_\mu^{(n)}\big)$. Since the space $H^0\big(\xi_\mu^{(n)}+\eta_\mu^{(n)}\big)$ is closed and convex, the projection $B\xi$ is uniquely determined for arbitrary sequence $b_\mu(0), b_\mu(1), b_\mu(2), \ldots$ satisfying conditions (11). Thus equation (16) has a unique solution for an arbitrary $\mathbf{b}_\mu \neq 0$ and the linear operator $\mathbf{P}_\mu \colon \ell_2 \to X$, $X = \{\mathbf{x}_\mu \in \ell_2 \colon \mathbf{x}_\mu = \mathbf{R}\mathbf{b}_\mu, \text{ where } \mathbf{b}_\mu \text{ satisfies (11)}\}$, has the inverse $(\mathbf{P}_\mu)^{-1}$.

Consequently, the unknown coefficients can be calculated by the formula

$$c_\mu(k) = \left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k,$$

where $(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu)_k$ is the kth element of the vector $\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu$. Thus, spectral characteristics $h_\mu(\lambda)$ of the optimal estimate $\widehat{B}\xi$ of the functional $B\xi$ is calculated by the formula

$$h_\mu(\lambda) = B_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n \frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{(-i\lambda)^n \sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}}{\left(1-e^{i\lambda\mu}\right)^n (f(\lambda)+g(\lambda))}. \quad (17)$$

The mean-square error of the estimate is calculated by the formula

$$\begin{aligned}
\Delta(f,g;\widehat{B}\xi) &= \mathsf{E}|B\xi - \widehat{B}\xi|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|B_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} f(\lambda)\,d\lambda \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|B_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} g(\lambda)\,d\lambda \\
&= \left\langle \mathbf{R}\mathbf{b}_\mu, \mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{b}_\mu\right\rangle + \langle \mathbf{Q}_\mu\mathbf{b}_\mu, \mathbf{b}_\mu\rangle,
\end{aligned} \quad (18)$$

where $\mathbf{Q}_\mu$ is a linear operator in the space ℓ_2 defined by elements $(\mathbf{Q}_\mu)_{l,k} = Q^\mu_{l,k}$, $l,k \geq 0$.

Let us summarize our reasoning and present the results in the form of theorem.

Theorem 3.1. *Let stochastic sequences $\{\xi(m), m \in \mathbb{Z}\}$ and $\{\eta(m), m \in \mathbb{Z}\}$ determine stationary nth increment sequences $\xi^{(n)}(m,\mu)$ and $\eta^{(n)}(m,\mu)$ with absolutely continuous spectral functions $F(\lambda)$ and $G(\lambda)$ which have spectral densities $f(\lambda)$ and $g(\lambda)$ satisfying conditions* (15). *Let coefficients $\{b_\mu(k) \colon k \geq 0\}$ satisfy conditions* (11). *The optimal linear estimate $\widehat{B}\xi$ of the functional $B\xi$ of known elements $\xi^{(n)}(m,\mu)$, $m \leq 0$, $\mu > 0$ from observations of the sequence $\xi^{(n)}(m,\mu) + \eta^{(n)}(m,\mu)$ at points $m = 0, -1, -2, \ldots$ is calculated by formula* (13). *The spectral characteristic $h_\mu(\lambda)$ of the optimal estimate $\widehat{B}\xi$ is calculated by formula* (17). *The value of the mean-square error $\Delta(f,g;\widehat{B}\xi)$ is calculated by formula* (18).

As a corollary from theorem 3.1 we can obtain the optimal estimate of the unknown value of the increment $\xi^{(n)}(m,\mu)$, $m \leq 0$, from observations of the sequence $\xi(k)+\eta(k)$ at points $k = 0, -1, -2, \ldots$. Let us take a vector b_μ with element 1 at the $(-m)$th position and elements 0 at the remaining positions in (17). Then the spectral characteristic $\varphi_m(\lambda,\mu)$ of the estimate

$$\widehat{\xi}^{(n)}(m,\mu) = \int_{-\pi}^{\pi} \varphi_m(\lambda,\mu)\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda) \quad (19)$$

is calculated by the formula

$$\varphi_m(\lambda,\mu) = e^{i\lambda m}\left(1-e^{-i\lambda\mu}\right)^n \frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{(-i\lambda)^n \sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}}{\left(1-e^{i\lambda\mu}\right)^n (f(\lambda)+g(\lambda))}, \quad (20)$$

where $\mathbf{r}_m = (R_{1,-m}, R_{2,-m}, \dots)$. The mean-square error of the estimate is calculated by the formula

$$\Delta\left(f,g;\widehat{\xi}^{(n)}(m,\mu)\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|e^{i\lambda m}\left|1-e^{i\lambda\mu}\right|^{2n} g(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_{\mu}^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} f(\lambda)\,d\lambda \tag{21}$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|e^{i\lambda m}\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_{\mu}^{-1}\mathbf{r}_m\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2} g(\lambda)\,d\lambda.$$

Thus, we have the following statement.

Corollary 3.1. *The optimal linear estimate $\widehat{\xi}^{(n)}(m,\mu)$ of the unknown value of the stochastic increment sequence $\xi^{(n)}(m,\mu)$, $m \le 0$, $\mu > 0$, from observations of the sequence $\xi(k)+\eta(k)$ at points $k = 0,-1,-2,\dots$ can be calculated by formula* (19)*. The spectral characteristic $\varphi_m(\lambda,\mu)$ of the optimal estimate $\widehat{\xi}^{(n)}(m,\mu)$ is calculated by formula* (20)*. The value of mean-square error $\Delta(f,g;\widehat{\xi}^{(n)}(m,\mu))$ is calculated formula* (21)*.*

Consider now the smoothing problem for the stationary nth increment sequence $\xi^{(n)}(m,\mu)$ which consists of finding the mean-square optimal linear estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value of the increment $\xi^{(n)}(0,\mu)$, $\mu > 0$, from observations of the stochastic sequence $\xi(k)+\eta(k)$ at points $k = 0,-1,-2,\dots$.

Let $r(k) = R_{k,0}$, $k \in \mathbb{Z}$. Then $\{r(k)\colon k \in \mathbb{Z}\}$ are the Fourier coefficients of the function $\frac{f(\lambda)}{f(\lambda)+g(\lambda)}$ which have the property $r(k) = \overline{r}(-k)$, $k \in \mathbb{Z}$, where $\overline{r}(k)$ denotes a conjugate element to $r(k)$. Let $\{V^{\mu}_{k,j}\colon k,j \ge 1\}$ be the coefficients which determine a linear operator $\mathbf{V}_{\mu} = (\mathbf{P}_{\mu})^{-1}$. Then we have relations

$$\sum_{l\ge 1} V^{\mu}_{l,j} P_{k,l} = \delta_{k,j}, \qquad k,j \ge 1, \tag{22}$$

where $\delta_{k,j}$ is the Kronecker symbol. Using formulas (20) and (22) we obtain the spectral characteristic of the optimal estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value of the increment $\xi^{(n)}(0,\mu)$:

$$\varphi(\lambda,\mu) = \frac{(1-e^{-i\lambda\mu})^n}{(i\lambda)^n}\sum_{k=0}^{\infty}\overline{r}(k)e^{-i\lambda k}.$$

The optimal estimate of the increment $\xi^{(n)}(0,\mu)$ is calculated by the formula

$$\widehat{\xi}^{(n)}(0,\mu) = \sum_{k=0}^{\infty}\overline{r}(k)\xi^{(n)}(-k,\mu) = \sum_{j=0}^{\infty}(\xi(-j)+\eta(-j))\sum_{l=0}^{\min\{n,[\frac{j}{\mu}]\}}(-1)^l C_n^l\,\overline{r}(j-l\mu). \tag{23}$$

The mean-square error of the estimate $\widehat{\xi}^{(n)}(0,\mu)$ is calculated by the formula

$$\Delta\left(f,g;\widehat{\xi}^{(n)}(0,\mu)\right) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\overline{V}^{\mu}_{k,j}\overline{r}(j)r(k) + \sum_{l\in\mathbb{Z}} r(l)g_{\mu}(-l), \tag{24}$$

where $\{g_{\mu}(k)\colon k\in\mathbb{Z}\}$ are the Fourier coefficients of the function $|1-e^{i\lambda\mu}|^{2n}g(\lambda)\lambda^{-2n}$.

Corollary 3.2. *The optimal estimate $\widehat{\xi}^{(n)}(0,\mu)$ of the unknown value $\xi^{(n)}(0,\mu)$ of the stationary nth increment sequence $\xi^{(n)}(m,\mu)$, $\mu > 0$, from observations of the sequence $\xi(k)+\eta(k)$ at points $k = 0,-1,-2,\dots$ is calculated by formula* (23)*. The value of the mean-square error $\Delta(f,g;\widehat{\xi}^{(n)}(0,\mu))$ of the estimate $\widehat{\xi}^{(n)}(0,\mu)$ is calculated by formula* (24)*.*

Theorem 3.1 and corollaries 3.1, 3.2 determine solutions of the filtering problems for the nth increment sequence $\widehat{\xi}^{(n)}(m,\mu)$ and the linear functional $B\xi$ which are based on the Fourier coefficients of functions

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}, \qquad \frac{f(\lambda)}{f(\lambda)+g(\lambda)}, \qquad \frac{|1-e^{i\lambda\mu}|^{2n}f(\lambda)g(\lambda)}{\lambda^{2n}(f(\lambda)+g(\lambda))}.$$

However, the problem of finding the inverse operator $(\mathbf{P}_\mu)^{-1}$ to the operator $\mathbf{P}_\mu$ determined by the Fourier coefficients of the function $\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}$ is a complicated problem in most cases. Therefore, we propose a method of finding the operator $(\mathbf{P}_\mu)^{-1}$ under the condition that the functions

$$\frac{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}{\lambda^{2n}}, \qquad \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))} \tag{25}$$

admit the canonical factorizations

$$\frac{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))}{\lambda^{2n}} = \left|\sum_{k=0}^{\infty}\varphi_\mu(k)e^{-i\lambda k}\right|^2, \tag{26}$$

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f(\lambda)+g(\lambda))} = \left|\sum_{k=0}^{\infty}\psi_\mu(k)e^{-i\lambda k}\right|^2. \tag{27}$$

Using the coefficients $\varphi_\mu(k)$, $\psi_\mu(k)$, $k\geq 0$, from factorizations (26), (27), we define linear operators Φ_μ and Ψ_μ in the space ℓ_2. Let $(\Phi_\mu)_{k,j}=\varphi_\mu(k-j)$ and $(\Psi_\mu)_{k,j}=\psi_\mu(k-j)$ if $1\leq j\leq k$, $(\Phi_\mu)_{k,j}=0$ and $(\Psi_\mu)_{k,j}=0$ if $j>k$ and $k,j\geq 1$. The defined operators admit the following relation: $\Psi_\mu\Phi_\mu=\Phi_\mu\Psi_\mu=I$, where I is the identity operator. Moreover, the operator $\mathbf{P}_\mu$ allows the factorization $\mathbf{P}_\mu=\overline{\Psi}'_\mu\Psi_\mu$. Thus, $(\mathbf{P}_\mu)^{-1}=\Phi_\mu\overline{\Phi}'_\mu$ and the coefficients of the operator $\mathbf{V}_\mu=(\mathbf{P}_\mu)^{-1}$ are calculated by the formula

$$V^{\mu}_{k,j}=\sum_{p=1}^{\min(k,j)}\varphi_\mu(k-p)\overline{\varphi}_\mu(j-p), \qquad k,j\geq 1.$$

These observations can be summarized in the form of the following theorem.

Theorem 3.2. *Let functions* (25) *admit the canonical factorizations* (26) *and* (27) *respectively. Then the inverse operator $\mathbf{P}_\mu^{-1}$ to the operator $\mathbf{P}_\mu$ is calculated by the formula $\mathbf{P}_\mu^{-1}=\Phi_\mu\overline{\Phi}'_\mu$, where the linear operator Φ_μ in ℓ_2 space is determined by the coefficients $(\Phi_\mu)_{k,j}=\varphi_\mu(k-j)$ if $1\leq j\leq k$ and $(\Phi_\mu)_{k,j}=0$ if $j<k$, $k,j\geq 1$.*

Using theorem 3.1 we can find the optimal estimate

$$\widehat{A}\xi=\int_{-\pi}^{\pi}h^{(a)}_\mu(\lambda)\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda) \tag{28}$$

of the functional $A\xi$. The spectral characteristic of the estimate $\widehat{A}\xi$ is calculated by the formula

$$h^{(a)}_\mu(\lambda)=A_\mu\left(e^{i\lambda}\right)\left(1-e^{-i\lambda\mu}\right)^n\frac{1}{(i\lambda)^n}\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{(-i\lambda)^n\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}}{\left(1-e^{i\lambda\mu}\right)^n(f(\lambda)+g(\lambda))}, \tag{29}$$

where $A_\mu(e^{i\lambda})=\sum_{k=0}^{\infty}(\mathbf{D}^\mu\mathbf{a})_k e^{-i\lambda k}$. The mean-square error can be calculated by formula

$$
\begin{aligned}
&\Delta\left(f,g;\widehat{A}\xi\right)\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n}g(\lambda)+\lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2}f(\lambda)\,d\lambda\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n}f(\lambda)-\lambda^{2n}\sum_{k=1}^{\infty}\left(\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\left|1-e^{i\lambda\mu}\right|^{2n}(f(\lambda)+g(\lambda))^2}g(\lambda)\,d\lambda\\
&=\left\langle\mathbf{R}\mathbf{D}^\mu\mathbf{a},\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right\rangle+\left\langle\mathbf{Q}_\mu\mathbf{D}^\mu\mathbf{a},\mathbf{D}^\mu\mathbf{a}\right\rangle.
\end{aligned}
\tag{30}
$$

Theorem 3.3. *Let uncorrelated stochastic sequences $\{\xi(m),m\in\mathbb{Z}\}$ and $\{\eta(m),m\in\mathbb{Z}\}$ define stationary nth increment sequences $\xi^{(n)}(m,\mu)$ and $\eta^{(n)}(m,\mu)$ with absolutely continuous spectral functions $F(\lambda)$ and $G(\lambda)$ which have spectral densities $f(\lambda)$ and $g(\lambda)$ satisfying conditions* (15). *Let conditions* (12) *be satisfied. The optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ of unknown elements $\xi(m)$, $m\leq 0$, from observations of the sequence $\xi(m)+\eta(m)$ at points $m=0,-1,-2\dots$ is calculated by formula* (28). *The spectral characteristic $h_\mu^{(a)}(\lambda)$ of the optimal estimate $\widehat{A}\xi$ is calculated by formula* (29). *The value of the mean-square error $\Delta(f,g;\widehat{A}\xi)$ is calculated by formula* (30). *If the function $|1-e^{i\lambda\mu}|^{2n}\lambda^{-2n}(f(\lambda)+g(\lambda))$ admits the canonical factorization* (26), *the operator $\mathbf{P}_\mu^{-1}$ from formulas* (29) *and* (30) *can be represented as $\mathbf{P}_\mu^{-1}=\Phi_\mu\overline{\Phi}'_\mu$.*

Example 3.1. Consider an ARIMA(0,1,2) sequence $\{\xi(m),m\in\mathbb{Z}\}$. The first order increments of the sequence $\xi(m)$ are stationary and the increments with step $\mu=1$ form a one-sided moving average sequence of order 2. Let the sequence $\xi(m)$ have the spectral density

$$f(\lambda)=\frac{\lambda^2\left|1-\phi e^{-i\lambda}\right|^2\left|1-\psi e^{-i\lambda}\right|^2}{|1-e^{-i\lambda}|^2}.$$

Consider an other stochastic sequence $\{\eta(m),m\in\mathbb{Z}\}$ with stationary increments of order 1 uncorrelated with $\xi(m)$ such that increments of the sequence $\{\xi(m)+\eta(m),m\in\mathbb{Z}\}$ with step 1 form a moving average sequence of order 1 and the spectral density has the form

$$f(\lambda)+g(\lambda)=\frac{\lambda^2\left|1-\phi e^{-i\lambda}\right|^2}{|1-e^{-i\lambda}|^2}.$$

Consider a real number sequence $\{a(k)\colon k\geq 0\}$ which is defined as follows: $a(0)=1$, $a(k)=-2^{-k}$ for $k\geq 1$. This sequence satisfies conditions (12). The problem is to find the optimal mean-square linear estimate $\widehat{A}\xi$ of the functional $A\xi=\sum_{k=0}^{\infty}a(k)\xi(-k)$ of unknown values $\xi(k)$, $k\leq 0$, of the sequence $\xi(m)$ from observations $\xi(k)+\eta(k)$, $k=0,-1,-2,\dots$. To calculate the spectral characteristic of the optimal estimate $\widehat{A}\xi$ of the functional $A\xi$ we use formula (29). The operator $\mathbf{P}_\mu=\mathbf{P}$ is determined by coefficients $(\mathbf{P})_{l,k}=\frac{\psi^p}{1-\psi^2}$, $|k-l|=p$, $l,k\geq 1$. The inverse operator $\mathbf{V}=\mathbf{P}^{-1}$ is defined by coefficients $(\mathbf{V})_{1,1}=1$, $(\mathbf{V})_{l,l}=1+\phi^2$ if $l\geq 2$, $(\mathbf{V})_{l,k}=-\phi$ if $|l-k|=1$, $l,k\geq 1$, and $(\mathbf{V})_{l,k}=0$ otherwise. The operator $\mathbf{R}$ is defined by coefficients $(\mathbf{R})_{1,0}=1$ and $(\mathbf{R})_{l,k}=0$ if $l\geq 1$, $k\geq 0$, $(l,k)\neq(1,0)$. The operator $\mathbf{D}^\mu=\mathbf{D}$ is defined by coefficients $d_\mu(k)=1$, $k\geq 0$. The spectral characteristic $h_1(\lambda)$ of the estimate $\widehat{A}\xi$ is calculated by the formula $h_1(\lambda)=\sum_{k=0}^{\infty}s(k)e^{-i\lambda k}\frac{1-e^{-i\lambda}}{i\lambda}$, where $s(0)=1-\frac{1}{2}\psi+\psi^2+\phi\psi\frac{2-\phi^2}{1-\phi^2}$, $s(k)=2^{-k-1}(2-5\psi+2\psi^2)+\phi^{k+1}\psi$, $k\geq 1$.

Denote $A(j)=\min\{n,[j/\mu]\}$, $j\geq 0$. Then the estimate $\widehat{A}\xi$ of the functional $A\xi$ is calculated by the formula

$$\widehat{A}\xi=\sum_{k=0}^{\infty}s(k)\left(\xi^{(n)}(-k,\mu)+\eta^{(n)}(-k,\mu)\right)=\sum_{j=0}^{\infty}(\xi(-j)+\eta(-j))\sum_{l=0}^{A(j)}(-1)^l C_n^l s(j-l\mu).$$

4. Minimax-robust method of filtering

The value of the mean-square error $\Delta(h_\mu^{(a)}(f,g);f,g):=\Delta(f,g;\widehat{A}\xi)$ and the spectral characteristic $h_\mu^{(a)}(f,g)$ of the optimal linear estimate $\widehat{A}\xi$ of the functional $A\xi$ of unknown values $\xi(m)$ based on observations of the stochastic sequence $\xi(k)+\eta(k)$ are determined by formulas (29) and (30) under the condition that spectral densities $f(\lambda)$ and $g(\lambda)$ of stochastic sequences $\xi(m)$ and $\eta(m)$ are known. In the case where spectral densities are not exactly known, but a set $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities is given, the minimax (robust) approach to estimation of functionals of the unknown values of stochastic sequence with stationary increments is reasonable. In other words we are interesting in finding an estimate that minimizes the maximum of the mean-square error for all spectral densities from a given class $\mathcal{D}$ of admissible spectral densities simultaneously.

Definition 4.1. For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ spectral densities $f_0(\lambda)\in\mathcal{D}_f$, $g_0(\lambda)\in\mathcal{D}_g$ are called least favorable in the class $\mathcal{D}$ for the optimal linear filtering of the functional $A\xi$ if

$$\Delta(f_0,g_0)=\Delta(h(f_0,g_0);f_0,g_0)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 4.2. For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ a spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied conditions

$$h^0\left(e^{i\lambda}\right)\in H_{\mathcal{D}}=\bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}L_2^0(f+g),$$

$$\min_{h\in H_{\mathcal{D}}}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h;f,g)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta\left(h^0;f,g\right).$$

Using the derived formulas and the introduced definitions we can conclude that the following statement holds true.

Lemma 4.1. *Spectral densities $f_\mu^0\in\mathcal{D}_f(\lambda)$, $g_\mu^0\in\mathcal{D}_g(\lambda)$ which satisfy conditions* (15) *are least favorable in the class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ if operators $\mathbf{P}_\mu^0$, $\mathbf{R}^0$, $\mathbf{Q}_\mu^0$ constructed with the help of the Fourier coefficients of the functions*

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}(f_\mu^0(\lambda)+g_\mu^0(\lambda))},\qquad \frac{f_\mu^0(\lambda)}{f_\mu^0(\lambda)+g_\mu^0(\lambda)},\qquad \frac{\left|1-e^{i\lambda\mu}\right|^{2n}f_\mu^0(\lambda)g_\mu^0(\lambda)}{\lambda^{2n}(f_\mu^0(\lambda)+g_\mu^0(\lambda))}$$

determine a solution of the conditional extremum problem

$$\begin{aligned}\max_{f\in\mathcal{D}}&\left(\left\langle\mathbf{R}\mathbf{D}^\mu\mathbf{a},\mathbf{P}_\mu^{-1}\mathbf{R}\mathbf{D}^\mu\mathbf{a}\right\rangle+\left\langle\mathbf{Q}_\mu\mathbf{D}^\mu\mathbf{a},\mathbf{D}^\mu\mathbf{a}\right\rangle\right)\\&=\left\langle\mathbf{R}^0\mathbf{D}^\mu\mathbf{a},\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right\rangle+\left\langle\mathbf{Q}_\mu^0\mathbf{D}^\mu\mathbf{a},\mathbf{D}^\mu\mathbf{a}\right\rangle.\end{aligned}\tag{31}$$

The minimax spectral characteristic is determined as $h^0=h_\mu(f_\mu^0,g_\mu^0)$ if $h_\mu(f_\mu^0,g_\mu^0)\in H_{\mathcal{D}}$.

The function h^0 and the pair (f_μ^0,g_μ^0) form a saddle point of the function $\Delta(h;f,g)$ on the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta\left(h;f_\mu^0,g_\mu^0\right)\geq\Delta\left(h^0;f_\mu^0,g_\mu^0\right)\geq\Delta\left(h^0;f,g\right)\quad\forall f\in\mathcal{D}_f,\forall g\in\mathcal{D}_g,\forall h\in H_{\mathcal{D}}$$

hold true if $h^0 = h_\mu(f^0_\mu, g^0_\mu)$ and $h_\mu(f^0_\mu, g^0_\mu) \in H_{\mathcal{D}}$, where (f^0_μ, g^0_μ) is a solution of the following conditional extremum problem

$$\widetilde{\Delta}(f,g) = -\Delta(h_\mu(f^0_\mu, g^0_\mu); f, g) \to \inf, \qquad (f,g) \in \mathcal{D},$$

$$\Delta\left(h_\mu\left(f^0_\mu, g^0_\mu\right); f, g\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g^0_\mu(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)^2} f(\lambda)\,d\lambda$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f^0_\mu(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|^2}{\lambda^{2n}\left|1-e^{i\lambda\mu}\right|^{2n}\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)^2} g(\lambda)\,d\lambda.$$

This conditional extremum problem is equivalent to the unconditional extremum problem

$$\Delta_{\mathcal{D}}(f,g) = \widetilde{\Delta}(f,g) + \delta(f, g \mid \mathcal{D}_f \times D_g) \to \inf,$$

$\delta(f, g|\mathcal{D}_f \times D_g)$ is the indicator function of the set $\mathcal{D}_f \times \mathcal{D}_g$. Solution (f^0_μ, g^0_μ) to this unconditional extremum problem is characterized by the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0_\mu, g^0_\mu)$ [18].

5. Least favorable spectral densities in the class $\mathcal{D}_f \times \mathcal{D}_g$

Consider the problem of optimal linear filtering of the functional $A\xi$ for the set of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$, where

$$\mathcal{D}^0_f = \left\{ f(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)d\lambda \le P_1 \right\}, \qquad \mathcal{D}^0_g = \left\{ g(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)d\lambda \le P_2 \right\}.$$

Let us assume that densities $f^0_\mu \in \mathcal{D}_f$, $g^0_\mu \in \mathcal{D}_g$ and functions

$$h_{\mu,f}\left(f^0_\mu, g^0_\mu\right) = \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g^0_\mu(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)}, \tag{32}$$

$$h_{\mu,g}\left(f^0_\mu, g^0_\mu\right) = \frac{\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f^0_\mu(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)} \tag{33}$$

are bounded. In this case the functional $\Delta(h_\mu(f^0_\mu, g^0_\mu); f, g)$ is continuous and bounded in $\mathcal{L}_1 \times \mathcal{L}_1$ space. It comes from the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0_\mu, g^0_\mu)$ that the least favorable densities $f^0_\mu(\lambda) \in \mathcal{D}_f$, $g^0_\mu(\lambda) \in \mathcal{D}_g$ satisfy the equations

$$\begin{aligned}&\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g^0_\mu(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| \\ &\qquad = \alpha_1|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right),\end{aligned} \tag{34}$$

$$\begin{aligned}&\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f^0_\mu(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| \\ &\qquad = \alpha_2|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right),\end{aligned} \tag{35}$$

where $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ are constants such that $\alpha_1 \ne 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} f^0_\mu(\lambda)\,d\lambda = P_1$ and $\alpha_2 \ne 0$ if $\frac{1}{2\pi}\int_{-\pi}^{\pi} g^0_\mu(\lambda)\,d\lambda = P_2$. Thus, the following statements hold true.

Theorem 5.1. *Let spectral densities $f^0_\mu(\lambda) \in \mathcal{D}_f$ and $g^0_\mu(\lambda) \in \mathcal{D}_g$ satisfy conditions* (15) *and let functions $h_{\mu,f}(f^0_\mu, g^0_\mu)$, $h_{\mu,g}(f^0_\mu, g^0_\mu)$ determined by equations* (32), (33) *be bounded. The spectral densities $f^0_\mu(\lambda)$ and $g^0_\mu(\lambda)$ determined by relations* (34), (35) *are least favorable in the class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ for the optimal linear filtering problem for the functional*

$A\xi$ if they determine a solution of the extremum problem (31). The function $h_\mu(f^0_\mu, g^0_\mu)$ determined by (29) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

Theorem 5.2. *Let the spectral density $f(\lambda)$ be known, the spectral density $g^0_\mu(\lambda) \in \mathcal{D}_g$ and let conditions (15) be satisfied. Let the function $h_{\mu,g}(f, g^0_\mu)$ be bounded. The spectral density $g^0_\mu(\lambda)$ is least favorable in the class $\mathcal{D}_g$ for the optimal linear filtering of the functional $A\xi$ if it is of the form*

$$g^0_\mu(\lambda) = \max\left\{0, \frac{\left|A_\mu(e^{i\lambda})\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{\alpha_2|\lambda|^n|1-e^{i\lambda\mu}|^n} - f(\lambda)\right\}$$

and the pair (f, g^0_μ) determines a solution of the extremum problem (31). The function $h_\mu(f, g^0_\mu)$ determined by (29) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

6. Least favorable spectral densities in the class $\mathcal{D} = \mathcal{D}^v_u \times \mathcal{D}_\varepsilon$

Consider the problem of the optimal linear filtering of the functional $A\xi$ for the set of spectral densities $\mathcal{D} = \mathcal{D}^v_u \times \mathcal{D}_\varepsilon$, where

$$\mathcal{D}^v_u = \left\{ f(\lambda) \,\middle|\, v(\lambda) \le f(\lambda) \le u(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)d\lambda \le P_1 \right\},$$

$$\mathcal{D}_\varepsilon = \left\{ g(\lambda) \,\middle|\, g(\lambda) = (1-\varepsilon)g_1(\lambda) + \varepsilon w(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)d\lambda \le P_2 \right\}.$$

Here spectral densities $u(\lambda)$, $v(\lambda)$, $g_1(\lambda)$ are known and fixed, and spectral densities $u(\lambda)$, $v(\lambda)$ are bounded.

Let $f^0_\mu(\lambda) \in \mathcal{D}^v_u$, $g^0_\mu(\lambda) \in \mathcal{D}_\varepsilon$ be spectral densities such that functions $h_{\mu,f}(f^0_\mu, g^0_\mu)$, $h_{\mu,g}(f^0_\mu, g^0_\mu)$ determined by (32), (33) are bounded. From the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0_\mu, g^0_\mu)$ we find the following equations that determine the least favorable densities

$$\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} g^0_\mu(\lambda) + \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| = \alpha_1|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)\left(\gamma_1(\lambda)+\gamma_2(\lambda)+\alpha_1^{-1}\right), \tag{36}$$

$$\left|A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f^0_\mu(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}^0_\mu\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right| = \alpha_2|\lambda|^n\left|1-e^{i\lambda\mu}\right|^n\left(f^0_\mu(\lambda)+g^0_\mu(\lambda)\right)\left(\varphi(\lambda)+\alpha_2^{-1}\right), \tag{37}$$

where $\gamma_1 \le 0$ and $\gamma_1 = 0$ if $f^0_\mu(\lambda) \ge v(\lambda)$; $\gamma_2(\lambda) \ge 0$ and $\gamma_2 = 0$ if $f^0_\mu(\lambda) \le u(\lambda)$; $\varphi(\lambda) \le 0$ and $\varphi(\lambda) = 0$ when $g^0_\mu(\lambda) \ge (1-\varepsilon)g_1(\lambda)$. The following statements hold true.

Theorem 6.1. *Let spectral densities $f^0_\mu(\lambda) \in \mathcal{D}^v_u$, $g^0_\mu(\lambda) \in \mathcal{D}_\varepsilon$ satisfy conditions (15). Let functions $h_{\mu,f}(f^0_\mu, g^0_\mu)$ and $h_{\mu,g}(f^0_\mu, g^0_\mu)$ determined by (32), (33) be bounded. Spectral densities $f^0_\mu(\lambda)$ and $g^0_\mu(\lambda)$ determined by equations (36), (37) are least favorable in the class $\mathcal{D} = \mathcal{D}^v_u \times \mathcal{D}_\varepsilon$ for the optimal linear filtering of the functional $A\xi$ if they determine a solution of extremum problem (31). The minimax spectral characteristic $h_\mu(f^0_\mu, g^0_\mu)$ of the optimal estimate of the functional $A\xi$ is determined by (29).*

Theorem 6.2. *Let the spectral density $f(\lambda)$ be known, the spectral density $g^0_\mu(\lambda) \in \mathcal{D}_\varepsilon$ and let conditions (15) be satisfied. Let the function $h_{\mu,g}(f, g^0_\mu)$ determined by (29) be* *bounded. The spectral density $g_\mu^0(\lambda)$ is least favorable in the class $\mathcal{D}_\varepsilon$ for the optimal linear filtering of the functional $A\xi$ if it is of the form*

$$g_\mu^0(\lambda) = \max\left\{(1-\varepsilon)g_1(\lambda), f_1(\lambda)\right\},$$

$$f_1(\lambda) = \frac{\alpha_2 \left| A_\mu\left(e^{i\lambda}\right)\left|1-e^{i\lambda\mu}\right|^{2n} f(\lambda) - \lambda^{2n}\sum_{k=1}^{\infty}\left(\left(\mathbf{P}_\mu^0\right)^{-1}\mathbf{R}^0\mathbf{D}^\mu\mathbf{a}\right)_k e^{i\lambda k}\right|}{|\lambda|^n|1-e^{i\lambda\mu}|^n} - f(\lambda),$$

and the pair (f, g_μ^0) determines a solution of the extremum problem (31). *The function $h_\mu(f, g_\mu^0)$ determined by* (29) *is minimax spectral characteristic of the optimal estimate of the functional $A\xi$.*

7. Conclusions

In this article we found a solution of the filtering problem for linear functionals $A\xi = \sum_{k=0}^{\infty} a(k)\xi(-k)$ which depend on unobserved values of a stochastic sequence $\xi(m)$ with stationary nth increments at points $m = 0, -1, -2, \ldots$. Estimate is based on observations of a sequence $\xi(m) + \eta(m)$ at points $m = 0, -1, -2, \ldots$, where $\eta(m)$ is an uncorrelated with $\xi(m)$ sequence with stationary nth increments. We derived formulas for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional in the case where spectral densities of sequences are exactly known. In the case of spectral uncertainty, where spectral densities are not exactly known, but a set of admissible spectral densities is specified, the minimax-robust method is applied. Formulas that determine the least favorable spectral densities and minimax (robust) spectral characteristics are derived for some special sets of admissible spectral densities.

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Received 27/11/2012