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QUASI-STATIONARY DISTRIBUTIONS FOR PERTURBED DISCRETE TIME REGENERATIVE PROCESSES

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Abstract. Non-linearly perturbed discrete time regenerative processes with regenerative stopping times are considered. We define the quasi-stationary distributions for such processes and present conditions for their convergence. Under some additional assumptions, the quasi-stationary distributions can be expanded in asymptotic power series with respect to the perturbation parameter. We give an explicit recurrence algorithm for calculating the coefficients in these asymptotic expansions. Applications to perturbed alternating regenerative processes with absorption and perturbed risk processes are presented.

Анотація. У статті розглядаються процеси відновлення з дискретным часом із нелінійними збуреннями. Визначено квазі-стаціонарні розподіли для таких процесів та представлено умови для їх збіжності. При деяких додаткових умовах для квазі-стаціонарних розподілів можна виписати асимптотичні розклади у степеневі ряди відносно параметру збурення. Представлено точний рекуррентний алгоритм для обчислення коефіцієнтів цих асимптотичних розкладів. Представлено застосування результатів для процесів відновлення із збуреннями з поглинанням та для процесів ризику із збуреннями.

Аннотация. В статье рассматриваются процессы восстановления с дискретным временем с нелинейными возмущениями. Определены квази-стационарные распределения для таких процессов и представлены условия их сходимости. При некоторых дополнительных условиях, для квази-стационарных распределений могут быть выписаны асимптотические разложения в степенные ряды относительно параметра возмущения. Представлен точный рекуррентный алгоритм для вычисления коэффициентов этих асимптотических разложений. Представлены приложения результатов для процессов восстановления с возмущениями с поглощением и для процессов риска с возмущениями.

1. Introduction

Many stochastic systems has a random lifetime, the process is terminated due to some rare event. This means that the stationary distribution of such process will be degenerated. However, before the lifetime of the system goes to an end, one can often observe something that resembles a stationary distribution. It is often of interest to describe such behaviour, so-called quasi-stationary phenomena.

In this paper we study such phenomena for discrete time regenerative processes with regenerative stopping time. Roughly speaking, such a process $\xi(n)$, $n = 0, 1, \dots$, regenerates at random times $\tau_1, \tau_2, \dots$, and has random lifetime μ which regenerates jointly with the process.

In particular, such processes includes discrete time semi-Markov processes with absorption. For example, $\xi(n)$ can be a Markov chain, $\tau_1, \tau_2, \dots$, the return times to some fixed state and μ, the first hitting time of some fixed state.

As a special case, when $\mu = \infty$ almost surely, this class of processes includes regenerative processes without stopping time.

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Under some conditions, it can be shown that for such processes there exists a probability distribution $\pi(A)$ such that

$$\mathsf{P}\{\xi(n) \in A/\mu > n\} \to \pi(A) \quad \text{as } n \to \infty.$$

We call this distribution the *quasi-stationary* distribution and use it to describe the quasi-stationary phenomena of the process. In the case $\mu = \infty$ almost surely, $\pi(A)$ is the usual stationary distribution.

Quasi-stationary distributions have been studied intensively since the 1960's. Some of the important early works are Vere-Jones (1962), Kingman (1963), Darroch and Seneta (1965, 1967) and Seneta and Vere-Jones (1966).

In this paper, we consider the case when $\xi(n)$ is perturbed and that the perturbation is described by a small parameter ε. Furthermore, it is assumed that some continuity conditions hold at $\varepsilon = 0$ for certain characteristics of the process $\xi^{(\varepsilon)}(n)$, regarded as a function of ε. This allows us to interpret $\xi^{(\varepsilon)}(n)$ as a perturbed version of the process $\xi^{(0)}(n)$.

We want the quasi-stationary distribution $\pi^{(\varepsilon)}(A)$ of the process $\xi^{(\varepsilon)}(n)$ to be an approximation of the quasi-stationary distribution $\pi^{(0)}(A)$ of the process $\xi^{(0)}(n)$, that is $\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A)$ as $\varepsilon \to 0$.

We give conditions such that the quasi-stationary distribution can be expanded as

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k),$$

where the coefficients $f_1(A), \ldots, f_k(A)$ can be calculated from an explicit recurrence algorithm.

Theoretical results are illustrated by example to the model of an alternating regenerative process with absorption. Under perturbation conditions on distributions of sojourn times and absorption probabilities, we give explicit the asymptotic expansion for the quasi-stationary distribution for such a process.

It is also shown how the results can be used in order to obtain approximations of the ruin probability for a discrete time risk process. We describe how an asymptotic expansion of the ruin probability can be obtained under perturbation conditions on claim probabilities and claim distributions.

The results in the present paper continue the line of research studies of the perturbed renewal equation in discrete time in Gyllenberg and Silvestrov (1994), Englund and Silvestrov (1997), Silvestrov (2000) and Petersson and Silvestrov (2012, 2013).

Corresponding results for perturbed regenerative processes in continuous time can be found in the book Gyllenberg and Silvestrov (2008) where one can also find an extended bibliography of works in the area.

Some works related to asymptotic expansions for perturbed Markov chains are Kartashov (1988, 1996), Latouche (1988), Hassin and Haviv (1992), Khasminskii, Yin and Zhang (1996), Yin and Zhang (2003), Altman, Avrachenkov and Núñes-Queija (2004), Koroliuk and Limnios (2005) and Yin and Nguyen (2009).

2. Quasi-Stationary Distributions for Regenerative Processes

For every $\varepsilon \geq 0$, let $\xi^{(\varepsilon)}(n)$ be a regenerative process in discrete time with a measurable phase space (X, Γ) and regeneration times $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \ldots$, and let $\mu^{(\varepsilon)}$ be a random variable defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and taking values in the set $\{0, 1, \ldots, \infty\}$.

We call $\mu^{(\varepsilon)}$ a regenerative stopping time for the regenerative process $\xi^{(\varepsilon)}(n)$ if for any $A \in \Gamma$, the probabilities $P^{(\varepsilon)}(n, A) = \mathsf{P}\big\{\xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n\big\}$ satisfies the renewal equation,

$$P^{(\varepsilon)}(n,A) = q^{(\varepsilon)}(n,A) + \sum_{k=0}^{n} P^{(\varepsilon)}(n-k,A) f^{(\varepsilon)}(k), \qquad n = 0,1,\ldots, \tag{1}$$

where

$$q^{(\varepsilon)}(n,A) = \mathsf{P}\left\{\xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n, \tau_1^{(\varepsilon)} > n\right\}$$

and

$$f^{(\varepsilon)}(n) = \mathsf{P}\left\{\tau_1^{(\varepsilon)} = n, \mu^{(\varepsilon)} > \tau_1^{(\varepsilon)}\right\}.$$

Note that the defect $f^{(\varepsilon)}$ of the distribution $f^{(\varepsilon)}(n)$ is given by the stopping probability in one regeneration period for the process $\xi^{(\varepsilon)}(n)$, that is,

$$f^{(\varepsilon)} = 1 - \sum_{n=0}^{\infty} f^{(\varepsilon)}(n) = \mathsf{P}\left\{\mu^{(\varepsilon)} \le \tau_1^{(\varepsilon)}\right\}.$$

We consider the case where the stopping probability in one regeneration period for the limiting process may be positive, i.e., $f^{(0)} \in [0,1)$.

Assume that the distributions $f^{(\varepsilon)}(n)$ satisfy the following conditions:

A: (**a**) $f^{(\varepsilon)}(n) \to f^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0,1,\ldots$, where the limiting distribution is non-periodic and not concentrated in zero.
(**b**) $f^{(\varepsilon)} \to f^{(0)} \in [0,1)$ as $\varepsilon \to 0$.

B: There exists $\delta > 0$ such that
(a) $\overline{\lim}_{0 \le \varepsilon \to 0} \sum_{n=0}^{\infty} e^{\delta n} f^{(\varepsilon)}(n) < \infty$.
(b) $\sum_{n=0}^{\infty} e^{\delta n} f^{(0)}(n) > 1$.

Let us consider the characteristic equation

$$\sum_{n=0}^{\infty} e^{\rho n} f^{(\varepsilon)}(n) = 1. \tag{2}$$

The following result from Petersson and Silvestrov (2012, 2013) gives some basic properties of $\rho^{(\varepsilon)}$ that will be used in what follows.

Lemma 2.1. *Assume that* **A** *and* **B** *hold. Then there exists a unique non-negative solution $\rho^{(\varepsilon)}$ of the characteristic equation* (2) *for ε small enough and $\rho^{(\varepsilon)} \to \rho^{(0)} < \delta$ as $\varepsilon \to 0$.*

For the rest of the paper, assume that **A** and **B** hold so that $\rho^{(\varepsilon)}$ is well defined for ε small enough. Also, to avoid repetition, we assume that ε always is small enough to satisfy the statements of Lemma 2.1. If both sides in (1) are multiplied by $e^{\rho^{(\varepsilon)} n}$, we see that the transformed probabilities $\tilde{P}(n,A) = e^{\rho^{(\varepsilon)} n} P(n,A)$ satisfy

$$\tilde{P}^{(\varepsilon)}(n,A) = \tilde{q}^{(\varepsilon)}(n,A) + \sum_{k=0}^{n} \tilde{P}^{(\varepsilon)}(n-k,A) \tilde{f}^{(\varepsilon)}(k), \qquad A \in \Gamma, \tag{3}$$

where

$$\tilde{q}^{(\varepsilon)}(n,A) = e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n,A), \qquad \tilde{f}^{(\varepsilon)}(n) = e^{\rho^{(\varepsilon)} n} f^{(\varepsilon)}(n).$$

It follows from the definition of $\rho^{(\varepsilon)}$, that (3) is a proper renewal equation. In order to apply the classical discrete time renewal theorem, the following condition is imposed on the tail probabilities of $\tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)}$.

C: There exists $\gamma > 0$ such that

$$\overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} e^{(\rho^{(0)} + \gamma) n} q^{(\varepsilon)}(n,X) < \infty.$$

For any $\varepsilon \geq 0$, we define the *quasi-stationary distribution* of $\xi^{(\varepsilon)}(n)$ by

$$\pi^{(\varepsilon)}(A) = \frac{\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)}n} q^{(\varepsilon)}(n, A)}{\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)}n} q^{(\varepsilon)}(n, X)}, \qquad A \in \Gamma. \tag{4}$$

Under conditions **A**, **B** and **C** the quasi-stationary distribution is well defined for sufficiently small ε.

Let us denote

$$\Gamma_0 = \left\{ A \in \Gamma \colon q^{(\varepsilon)}(n, A) \to q^{(0)}(n, A) \text{ as } \varepsilon \to 0, n = 0, 1, \ldots \right\}$$

We assume the following:

D: $X \in \Gamma_0$.

Note that Γ_0 is an algebra but does not necessarily coincide with Γ.

The first part of the following result motivates why it is natural to call $\pi^{(\varepsilon)}(A)$ quasi-stationary distributions. The second part gives conditions for convergence of $\pi^{(\varepsilon)}(A)$ for sets $A \in \Gamma_0$.

Theorem 2.2. *Assume that* **A**, **B** *and* **C** *hold.*

(**i**) *Then there exists* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \leq \varepsilon_0$,

$$\mathsf{P}\left\{ \xi^{(\varepsilon)}(n) \in A / \mu^{(\varepsilon)} > n \right\} \to \pi^{(\varepsilon)}(A) \quad \text{as } n \to \infty, \ A \in \Gamma.$$

(**ii**) *If, in addition, condition* **D** *holds, then*

$$\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A) \quad \text{as } \varepsilon \to 0, \ A \in \Gamma_0.$$

Proof. First note that if the limiting distribution $f^{(0)}(n)$ is non-periodic, then there exists a finite positive integer N such that

$$\gcd \left\{ 1 \leq n \leq N \colon f^{(0)}(n) > 0 \right\} = 1.$$

It follows from condition **A** that the distributions $\tilde{f}^{(\varepsilon)}(n)$ are non-periodic for ε sufficiently small, say $\varepsilon \leq \varepsilon_1$. Let $\tilde{m}_1^{(\varepsilon)}$ denote the expectation of $\tilde{f}^{(\varepsilon)}(n)$. Since $\rho^{(0)} < \delta$ we can choose $\delta_0 > 0$ such that $\rho^{(0)} < \delta - \delta_0$. Let $C = \sup_{n \geq 0} n e^{-\delta_0 n}$. Since $\rho^{(\varepsilon)} \to \rho^{(0)}$ and condition **B** holds it follows that

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}}\, \tilde{m}_1^{(\varepsilon)} &= \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} n e^{\rho^{(\varepsilon)} n} f^{(\varepsilon)}(n) \leq \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} n e^{(\delta - \delta_0) n} f^{(\varepsilon)}(n) \\ &\leq C \, \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} e^{\delta n} f^{(\varepsilon)}(n) < \infty. \end{aligned}$$

It follows that $\tilde{m}_1^{(\varepsilon)}$ is finite for all ε small enough, say $\varepsilon \leq \varepsilon_2$. Condition **C** implies that for any $A \in \Gamma$

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n, A) &= \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} \, \mathsf{P}\left\{ \xi^{(\varepsilon)}(n) \in A, \mu^{(\varepsilon)} > n, \tau_1^{(\varepsilon)} > n \right\} \\ &\leq \overline{\lim_{\varepsilon \to 0}} \sum_{n=0}^{\infty} e^{(\rho^{(0)} + \gamma) n} \, \mathsf{P}\left\{ \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n \right\} < \infty, \end{aligned}$$

so there exists $\varepsilon_3 > 0$ such that $\sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n, A) < \infty$ for all $\varepsilon \leq \varepsilon_3$.

Define $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. It follows from the classical discrete time renewal theorem that for any $\varepsilon \leq \varepsilon_0$,

$$\tilde{P}^{(\varepsilon)}(n, A) \to \frac{1}{\tilde{m}_1^{(\varepsilon)}} \sum_{n=0}^{\infty} \tilde{q}^{(\varepsilon)}(n, A) \quad \text{as } n \to \infty, \ A \in \Gamma. \tag{5}$$

Part (**i**) follows from relation (5) and the following equality,

$$\mathsf{P}\left\{\xi^{(\varepsilon)}(n) \in A/\mu^{(\varepsilon)} > n\right\} = \tilde{P}^{(\varepsilon)}(n, A) \,/\, \tilde{P}^{(\varepsilon)}(n, X).$$

Lemma 2.1, condition **C** and the definition of Γ_0 implies that

$$\lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} q^{(\varepsilon)}(n, A) = \sum_{n=0}^{\infty} e^{\rho^{(0)} n} q^{(0)}(n, A) < \infty, \qquad A \in \Gamma_0. \tag{6}$$

Part (**ii**) now follows from relations (4) and (6), and condition **D**. □

3. Asymptotic Expansions of Quasi-Stationary Distributions

A problem with $\pi^{(\varepsilon)}(A)$ is that the expression defining it is rather complicated. Both numerator and denominator are represented as infinite sums and involves $\rho^{(\varepsilon)}$, which is only given as the solution to the nonlinear equation (2). However, under some perturbation conditions, $\pi^{(\varepsilon)}(A)$ can be expanded in an asymptotic power series with respect to ε.

In order to do this, we first need to expand $\rho^{(\varepsilon)}$. This can be done under some perturbation conditions on the following mixed power-exponential moments of the distributions $f^{(\varepsilon)}(n)$,

$$\phi^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} f^{(\varepsilon)}(n), \qquad \rho \geq 0,\ r = 0, 1, \ldots$$

To expand the quasi-stationary distribution, some perturbation conditions on the following mixed power-exponential moment type functionals of $q^{(\varepsilon)}(n, A)$ are also needed,

$$\omega^{(\varepsilon)}(\rho, r, A) = \sum_{n=0}^{\infty} n^r e^{\rho n} q^{(\varepsilon)}(n, A), \qquad \rho \geq 0,\ r = 0, 1, \ldots,\ A \in \Gamma.$$

The perturbation conditions are the following:

$\mathbf{P_1^{(k)}}$: $\phi^{(\varepsilon)}(\rho^{(0)}, r) = \phi^{(0)}(\rho^{(0)}, r) + a_{1,r}\varepsilon + \cdots + a_{k-r,r}\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, where $|a_{n,r}| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$.

$\mathbf{P_2^{(k)}}$: $\omega^{(\varepsilon)}(\rho^{(0)}, r, A) = \omega^{(0)}(\rho^{(0)}, r, A) + b_{1,r}(A)\varepsilon + \cdots + b_{k-r,r}(A)\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, where $A \in \Gamma_0$ and $|b_{n,r}(A)| < \infty$, $n = 1, \ldots, k-r$, $r = 0, \ldots, k$.

For convenience, we define $a_{0,r} = \phi^{(0)}(\rho^{(0)}, r)$ and $b_{0,r} = \omega^{(0)}(\rho^{(0)}, r, A)$ for $r = 0, \ldots, k$ and $A \in \Gamma_0$.

Now we are ready to give the expansion of $\pi^{(\varepsilon)}(A)$. The details are presented in the following theorem.

Theorem 3.1. *Suppose that* **A**, **B** *and* $\mathbf{P_1^{(k)}}$ *hold.*

(**i**) *Then the root $\rho^{(\varepsilon)}$ of the characteristic equation* (2) *has the asymptotic expansion*

$$\rho^{(\varepsilon)} = \rho^{(0)} + c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k).$$

The coefficients $c_1, \ldots, c_k$ are given by the recurrence formulas

$$\begin{aligned} c_1 &= -a_{1,0}/a_{0,1}, \\ c_n &= -\frac{1}{a_{0,1}}\Bigg(a_{n,0} + \sum_{q=1}^{n-1} a_{n-q,1}c_q \\ &\qquad + \sum_{m=2}^{n}\sum_{q=m}^{n} a_{n-q,m} \cdot \sum_{n_1,\ldots,n_{q-1}\in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!}\Bigg), \qquad n = 2, \ldots, k, \end{aligned} \tag{7}$$

where $D_{m,q}$ is the set of all nonnegative integer solutions to the system

$$n_1 + \cdots + n_{q-1} = m,\ n_1 + \cdots + (q-1)n_{q-1} = q.$$

(**ii**) *If, in addition,* **C**, **D** *and* $\mathbf{P_2^{(k)}}$ *hold, then for any* $A \in \Gamma_0$ *the following asymptotic expansion holds,*

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k).$$

The coefficients $f_1(A), \ldots, f_n(A)$ *are given by*

$$f_n(A) = \frac{1}{d_0(X)} \left(d_n(A) - \sum_{q=0}^{n-1} d_{n-q}(X) f_q(A) \right), \tag{8}$$

where

$$d_0(A) = \omega^{(0)}(\rho^{(0)}, 0, A)$$

and $f_0(A) = \pi^{(0)}(A)$. *The coefficients* $d_1(A), \ldots, d_k(A)$ *are given by*

$$\begin{aligned} d_1(A) &= b_{1,0}(A) + b_{0,1}(A)c_1, \\ d_n(A) &= b_{n,0}(A) + \sum_{q=1}^{n} b_{n-q,1}(A)c_q \\ &\quad + \sum_{m=2}^{n}\sum_{q=m}^{n} b_{n-q,m}(A) \cdot \sum_{n_1,\ldots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} \frac{c_p^{n_p}}{n_p!}, \qquad n = 2, \ldots, k. \end{aligned} \tag{9}$$

Proof. For the proof of part (**i**), see Petersson and Silvestrov (2012, 2013). Here we give the proof of part (**ii**).

Let $\Delta^{(\varepsilon)} = \rho^{(\varepsilon)} - \rho^{(0)}$. Using the Taylor expansion of the exponential function, we obtain for any $n = 0, 1, \ldots,$

$$e^{\rho^{(\varepsilon)} n} = e^{\rho^{(0)} n} \left(\sum_{r=0}^{k} \frac{n^r (\Delta^{(\varepsilon)})^r}{r!} + \frac{n^{k+1} (\Delta^{(\varepsilon)})^{k+1}}{(k+1)!} e^{|\Delta^{(\varepsilon)}| n} \theta_{k+1}^{(\varepsilon)}(n) \right),$$

where $0 \leq \theta_{k+1}^{(\varepsilon)}(n) \leq 1$. Since $\rho^{(\varepsilon)} \to \rho^{(0)}$, there exists $\beta < \rho^{(0)} + \gamma$ and $\varepsilon_1 = \varepsilon_1(\beta)$ such that

$$\rho^{(0)} + |\Delta^{(\varepsilon)}| < \beta, \ \varepsilon \leq \varepsilon_1.$$

Let $\tilde{C}_r = \sup_{n \geq 0} n^r e^{(\rho^{(0)} + \gamma - \beta) n}$. From condition **C** it follows that there exists $\varepsilon_2 > 0$ and a constant C_r such that

$$\begin{aligned} \omega^{(\varepsilon)}(\beta, r, A) &= \sum_{n=0}^{\infty} n^r e^{\beta n} q^{(\varepsilon)}(n, A) \\ &\leq \tilde{C}_r \sum_{n=0}^{\infty} e^{(\rho^{(0)} + \gamma) n} \mathsf{P}\left\{ \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n \right\} \leq C_r, \qquad \varepsilon \leq \varepsilon_2. \end{aligned}$$

Define $\varepsilon_0 = \varepsilon_0(\beta) := \min\{\varepsilon_1(\beta), \varepsilon_2\}$. Substituting the Taylor expansion of $e^{\rho^{(\varepsilon)} n}$ into the definition of $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$ yields

$$\begin{aligned} \omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) &= \omega^{(\varepsilon)}(\rho^{(0)}, 0, A) + \omega^{(\varepsilon)}(\rho^{(0)}, 1, A)\Delta^{(\varepsilon)} + \cdots \\ &\quad + \omega^{(\varepsilon)}(\rho^{(0)}, k, A)(\Delta^{(\varepsilon)})^k / k! + r_{k+1}^{(\varepsilon)} (\Delta^{(\varepsilon)})^{k+1}, \end{aligned} \tag{10}$$

where

$$r_{k+1}^{(\varepsilon)} = \frac{1}{(k+1)!} \sum_{n=0}^{\infty} n^{k+1} e^{(\rho^{(0)} + |\Delta^{(\varepsilon)}|) n} \theta_{k+1}^{(\varepsilon)}(n) q^{(\varepsilon)}(n, A).$$

If $\varepsilon \leq \varepsilon_0$, the right hand side of (10) is finite and

$$r_{k+1}^{(\varepsilon)} \leq \frac{1}{(k+1)!} \omega^{(\varepsilon)}(\beta, k+1, A) \leq \frac{C_{k+1}}{(k+1)!}.$$

It follows that there exists a finite constant M_{k+1} and numbers $0 \leq \theta_{k+1}^{(\varepsilon)} \leq 1$ such that

$$r_{k+1}^{(\varepsilon)} = M_{k+1}\theta_{k+1}^{(\varepsilon)}, \qquad \varepsilon \leq \varepsilon_0. \tag{11}$$

Since $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{P_1^{(k)}}$ hold, it follows from part (**i**) that

$$\Delta^{(\varepsilon)} = c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k). \tag{12}$$

Substituting (11), (12) and condition $\mathbf{P_2^{(k)}}$ into the right hand side of (10) when $k = 0$ we see that $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) \to \omega^{(0)}(\rho^{(0)}, 0, A)$ as $\varepsilon \to 0$, which means that we have the representation

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + \omega_0^{(\varepsilon)}(A), \tag{13}$$

where $\omega_0^{(\varepsilon)}(A) \to 0$ as $\varepsilon \to 0$.

Now assume that $k = 1$. If we substitute (11), (12), (13) and condition $\mathbf{P_2^{(k)}}$ into the right hand side of (10), divide by ε and let $\varepsilon \to 0$, it is found that

$$\frac{\omega_0^{(\varepsilon)}(A)}{\varepsilon} \to b_{1,0}(A) + b_{0,1}(A)c_1 \quad \text{as } \varepsilon \to 0. \tag{14}$$

Using (13) and (14) we obtain the asymptotic representation

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \omega_1^{(\varepsilon)}(A),$$

where $d_1(A) = b_{1,0}(A) + b_{0,1}(A)c_1$ and $\omega_1^{(\varepsilon)}(A)$ is of order $o(\varepsilon)$.

If $k \geq 2$, we can continue in this way and build an asymptotic expansion of order k for $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$. Once the existence of the expansion is proved, the coefficients can be found by collecting the coefficients of equal powers of ε in the expansion of the following expression,

$$\begin{aligned}
&\big(b_{0,0}(A) + \cdots + b_{k,0}(A)\varepsilon^k + o(\varepsilon^k)\big) \\
&\quad + \big(b_{0,1}(A) + \cdots + b_{k-1,1}(A)\varepsilon^{k-1} + o(\varepsilon^{k-1})\big) \\
&\qquad \times \big(c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k)\big) + \cdots \\
&\quad + (b_{0,k}(A) + o(1))\big(c_1\varepsilon + \cdots + c_k\varepsilon^k + o(\varepsilon^k)\big)^k/k! + o(\varepsilon^k).
\end{aligned}$$

This yields the expansion

$$\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A) = \omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \cdots + d_k(A)\varepsilon^k + o(\varepsilon^k), \tag{15}$$

where the coefficients $d_1(A), \ldots, d_k(A)$ are given according to (9).

The quasi-stationary distribution can be written as

$$\pi^{(\varepsilon)}(A) = \frac{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)}{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, X)}, \qquad A \in \Gamma.$$

For sets $A \in \Gamma_0$, the numerator can be expanded as in equation (15). By condition $\mathbf{D}$, we always have $X \in \Gamma_0$ so the denominator can also be expanded. Thus, for any $A \in \Gamma_0$,

$$\pi^{(\varepsilon)}(A) = \frac{\omega^{(0)}(\rho^{(0)}, 0, A) + d_1(A)\varepsilon + \cdots + d_k(A)\varepsilon^k + o(\varepsilon^k)}{\omega^{(0)}(\rho^{(0)}, 0, X) + d_1(X)\varepsilon + \cdots + d_k(X)\varepsilon^k + o(\varepsilon^k)}. \tag{16}$$

Using (16), we can build the expansion of $\pi^{(\varepsilon)}(A)$ similarly to how we built the expansion of $\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0, A)$. To do this, first note that with $k = 0$ in (16) it immediately follows that $\pi^{(\varepsilon)}(A) \to \pi^{(0)}(A)$, which means that we have the representation

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + \pi_0^{(\varepsilon)}(A), \tag{17}$$

where $\pi_0^{(\varepsilon)}(A) \to 0$ as $\varepsilon \to 0$.

Now put $k = 1$ in (16). Since $\omega^{(0)}(\rho^{(0)}, 0, X) > 0$, it follows that the denominator of (16) is positive for ε small enough. Substituting (17) into the left hand side of (16), rearranging and using the identity $\pi^{(0)}(A)d_0(X) = d_0(A)$ gives the following for sufficiently small ε,

$$\pi_0^{(\varepsilon)}(A)d_0(X) + d_1(X)f_0(A) + o(\varepsilon) = d_1(A)\varepsilon + o(\varepsilon).$$

Dividing both sides by ε and letting $\varepsilon \to 0$, we conclude that

$$\frac{\pi_0^{(\varepsilon)}(A)}{\varepsilon} \to \frac{1}{d_0(X)}\left(d_1(A) - d_1(X)f_0(A)\right) \quad \text{as } \varepsilon \to 0.$$

Using this and (17), the following asymptotic representation is obtained,

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \pi_1^{(\varepsilon)}(A),$$

where $f_1(A) = (d_1(A) - d_1(X)f_0(A))/d_0(X)$ and $\pi_1^{(\varepsilon)}(A)$ is of order $o(\varepsilon)$.

This proves part (**ii**) when $k = 1$.

If $k \geq 2$ we can continuing in this way and prove that the asymptotic expansion of $\pi^{(\varepsilon)}(A)$ exists. When we know that the expansion exists, the coefficients can be found in the following way. Consider the equation

$$\begin{aligned}
&\left(f_0(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k)\right) \\
&\qquad \times \left(d_0(X) + d_1(X)\varepsilon + \cdots + d_k(X)\varepsilon^k + o(\varepsilon^k)\right) \\
&\quad = \left(d_0(A) + d_1(A)\varepsilon + \cdots + d_k(A)\varepsilon^k + o(\varepsilon^k)\right).
\end{aligned}$$

The coefficients $f_k(A)$ are obtained by equating the coefficients of ε^k in both sides of this equation. This yields

$$\pi^{(\varepsilon)}(A) = \pi^{(0)}(A) + f_1(A)\varepsilon + \cdots + f_k(A)\varepsilon^k + o(\varepsilon^k),$$

where $f_1(A), \ldots, f_k(A)$ are given according to the recurrent relation in equation (8). □

4. Perturbed Alternating Regenerative Processes

In this section, we consider a perturbed alternating regenerative process with absorption. We assume that the process $\eta^{(\varepsilon)}(n)$ starts in state 1 and stays there for a time with distribution $g_1^{(\varepsilon)}(n)$ before it jumps down to state 0. Then the process remains in state 0 for a time with distribution $g_0^{(\varepsilon)}(n)$. Now, with some small probability $p^{(\varepsilon)}$ the process is absorbed in state -1 or with probability $1 - p^{(\varepsilon)}$ the process starts over in state 1.

Such a process can be interpreted as the state of a machine which is successively repaired after break-downs. The states 0 and 1 then represents that the machine is broken or working while -1 is the absorption state of fatal non-repairable failure.

Respectively, $g_1^{(\varepsilon)}(n)$ is the distribution of the time between repair and failure and $g_0^{(\varepsilon)}(n)$ is the distribution of the time to locate the error after a break-down. The absorption probability $p^{(\varepsilon)}$ corresponds to a fatal error such that the machine can not be repaired. The first hitting time to the state -1 is the lifetime of the system.

We assume the following condition, preventing instant jumps:

E: $g_0^{(\varepsilon)}(0) = g_1^{(\varepsilon)}(0) = 0$ for all $\varepsilon \geq 0$.

Mathematically, this is described by a discrete time semi-Markov process.

Let $\left(\eta_k^{(\varepsilon)}, \kappa_k^{(\varepsilon)}\right)$ be a Markov renewal chain with phase space $X \times \{1, 2, \ldots\}$, where $X = \{-1, 0, 1\}$, and with transition probabilities

$$q_{ij}^{(\varepsilon)}(n) = \mathsf{P}\left\{\eta_{k+1}^{(\varepsilon)} = j, \kappa_{k+1}^{(\varepsilon)} = n/\eta_k^{(\varepsilon)} = i\right\}, \qquad i, j \in X,\ n = 1, 2, \ldots,$$

given by

$$q_{ij}^{(\varepsilon)}(n) = \begin{cases} g_1^{(\varepsilon)}(n) & i=1, j=0, \\ (1-p^{(\varepsilon)})g_0^{(\varepsilon)}(n) & i=0, j=1, \\ p^{(\varepsilon)} g_0^{(\varepsilon)}(n) & i=0, j=-1, \\ \chi(n=1) & i=j=-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\nu^{(\varepsilon)}(n) = \max\{k\colon \gamma^{(\varepsilon)}(k) \leq n\}$, where $\gamma^{(\varepsilon)}(0) = 0$ and $\gamma^{(\varepsilon)}(k) = \kappa_1^{(\varepsilon)} + \cdots + \kappa_k^{(\varepsilon)}$ for $k \geq 1$.

The discrete time semi-Markov process $\eta^{(\varepsilon)}(n)$ can be defined by the following relation,

$$\eta^{(\varepsilon)}(n) = \eta^{(\varepsilon)}_{\nu^{(\varepsilon)}(n)}, \qquad n = 0, 1, \dots .$$

Let

$$\nu_j^{(\varepsilon)} = \min\left\{k \geq 1\colon \eta_k^{(\varepsilon)} = j\right\}.$$

Then the absorption time is given by $\mu^{(\varepsilon)} = \gamma^{(\varepsilon)}\big(\nu_{-1}^{(\varepsilon)}\big)$ and the first regeneration time is given by $\tau_1^{(\varepsilon)} = \gamma^{(\varepsilon)}\big(\nu_1^{(\varepsilon)}\big)$.

The process described above is illustrated in Figure 1.

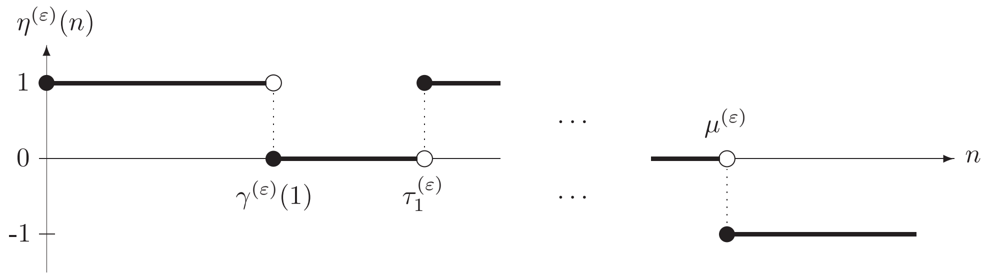

FIGURE 1. Realization of the process $\eta^{(\varepsilon)}(n)$.

In the definition of a regenerative process with regenerating stopping time it is assumed that the regeneration times are proper random variables. In the process defined above this is not the case. However, the transition probabilities from the absorbing state can be modified in such a way that the return times to state 1 are proper random variables, and that the probabilities $\mathsf{P}\big\{\eta^{(\varepsilon)}(n) = i, \mu^{(\varepsilon)} > n\big\}$ are the same for the modified process. We can then apply the results from Sections 2 and 3 to the modified process and interpret the results for the original process.

The weak continuity conditions at $\varepsilon = 0$ are formulated in terms of the local characteristics of the alternating regenerative process as follows.

F: **(a)** $g_i^{(\varepsilon)}(n) \to g_i^{(0)}(n)$ as $\varepsilon \to 0$, $n = 0, 1, \dots$, $i = 0, 1$.
(b) $p^{(\varepsilon)} \to p^{(0)} \in [0, 1)$ as $\varepsilon \to 0$.

We also need the following non-periodicity condition.

G: At least one of the distributions $g_0^{(0)}(n)$ and $g_1^{(0)}(n)$ is non-periodic.

We introduce the following mixed power-exponential moment generating functions for distributions of sojourn times,

$$\psi_i^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} g_i^{(\varepsilon)}(n), \qquad \rho \geq 0,\ r = 0, 1, \dots,\ i = 0, 1. \tag{18}$$

Also, consider the following mixed power-exponential moment generating functions,

$$\phi^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} f^{(\varepsilon)}(n), \qquad \rho \geq 0, \ r = 0, 1, \dots, \tag{19}$$

where

$$f^{(\varepsilon)}(n) = \mathsf{P}\left\{\tau_1^{(\varepsilon)} = n, \mu^{(\varepsilon)} > \tau_1^{(\varepsilon)}\right\}, \qquad n = 0, 1, \dots$$

For the exponential moment generating functions, the following relation is obtained,

$$\begin{aligned} \phi^{(\varepsilon)}(\rho, 0) &= \left(1 - p^{(\varepsilon)}\right) \sum_{n=0}^{\infty} e^{\rho n} \, \mathsf{P}\left\{\kappa_1^{(\varepsilon)} + \kappa_2^{(\varepsilon)} = n\right\} \\ &= \left(1 - p^{(\varepsilon)}\right) \psi_0^{(\varepsilon)}(\rho, 0) \psi_1^{(\varepsilon)}(\rho, 0), \qquad \rho \geq 0. \end{aligned} \tag{20}$$

From this it follows that existence of (18) and (19) for ε small enough is guaranteed by the following Cramér type condition:

H: There exists $\delta > 0$ such that
- **(a)** $\overline{\lim}_{0 \leq \varepsilon \to 0} \psi_i^{(\varepsilon)}(\delta, 0) < \infty$, $i = 0, 1$.
- **(b)** $(1 - p^{(0)}) \psi_0^{(0)}(\delta, 0) \psi_1^{(0)}(\delta, 0) > 1$.

We will also use the following mixed power-exponential moment generating functions,

$$\omega_i^{(\varepsilon)}(\rho, r) = \sum_{n=0}^{\infty} n^r e^{\rho n} q_i^{(\varepsilon)}(n), \qquad \rho \geq 0, \ r = 0, 1, \dots, \ i = 0, 1, \tag{21}$$

where

$$q_i^{(\varepsilon)}(n) = \mathsf{P}\left\{\eta^{(\varepsilon)}(n) = i, \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\right\}, \qquad n = 0, 1, \dots, \ i = 0, 1.$$

If condition **E**–**H** hold, then condition $\mathbf{A} - \mathbf{D}$ hold, so the results in Section 2 can be applied. Lemma 2.1 implies that for ε small enough there exists a unique root $\rho^{(\varepsilon)}$ of the characteristic equation

$$\phi^{(\varepsilon)}(\rho, 0) = 1. \tag{22}$$

It is worth noticing that the solution to equation (22) satisfies $\rho^{(\varepsilon)} = 0$ if and only if $p^{(\varepsilon)} = 0$, and $\rho^{(\varepsilon)} > 0$ if and only if $p^{(\varepsilon)} > 0$.

It follows from Theorem 2.2 that that for ε sufficiently small,

$$\lim_{n \to \infty} \mathsf{P}\left\{\eta^{(\varepsilon)}(n) = j / \mu^{(\varepsilon)} > n\right\} = \pi_j^{(\varepsilon)}\left(\rho^{(\varepsilon)}\right), \qquad j = 0, 1,$$

where

$$\pi_j^{(\varepsilon)}\left(\rho^{(\varepsilon)}\right) = \frac{\omega_j^{(\varepsilon)}(\rho^{(\varepsilon)}, 0)}{\omega_0^{(\varepsilon)}(\rho^{(\varepsilon)}, 0) + \omega_1^{(\varepsilon)}\left(\rho^{(\varepsilon)}, 0\right)}, \qquad j = 0, 1. \tag{23}$$

If conditions $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$ hold for the generating functions (19) and (21), it follows from Theorem 3.1 that we can build an asymptotic expansion for the quasi-stationary distribution (23). However, it is more convenient to use perturbation conditions for local characteristics of the process $\eta^{(\varepsilon)}(n)$. Therefore, we formulate perturbation conditions on the generating functions of the distributions of sojourn times and the absorption probabilities, and then show how these conditions are related to $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$.

We assume the following:

$\mathbf{P_3^{(k)}}$: $p^{(\varepsilon)} = p^{(0)} + p[1]\varepsilon + \cdots + p[k]\varepsilon^k + o(\varepsilon^k)$, where $|p[n]| < \infty$, $n = 1, \dots, k$.

$\mathbf{P_4^{(k)}}$: $\psi_i^{(\varepsilon)}(\rho^{(0)}, r) = \psi_i^{(0)}(\rho^{(0)}, r) + \psi_i[1, r]\varepsilon + \cdots + \psi_i[k - r, r]\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \dots, k$, $i = 0, 1$, where $|\psi_i[n, r]| < \infty$, $n = 1, \dots, k - r$, $r = 0, \dots, k$, $i = 0, 1$.

Observe that for $n = 0, 1, \dots,$

$$\mathsf{P}\{\eta^{(\varepsilon)}(n) = i, \tau_1^{(\varepsilon)} \wedge \mu^{(\varepsilon)} > n\} = \begin{cases} \mathsf{P}\{\kappa_1^{(\varepsilon)} \le n, \kappa_1^{(\varepsilon)} + \kappa_2^{(\varepsilon)} > n\} & i = 0, \\ \mathsf{P}\{\kappa_1^{(\varepsilon)} > n\} & i = 1. \end{cases}$$

Using this relation, we obtain for $\rho \ge 0$,

$$\omega_i^{(\varepsilon)}(\rho, 0) = \begin{cases} \psi_1^{(\varepsilon)}(\rho, 0)\varphi_0^{(\varepsilon)}(\rho, 0) & i = 0, \\ \varphi_1^{(\varepsilon)}(\rho, 0) & i = 1, \end{cases} \tag{24}$$

where, for $i = 0, 1$,

$$\varphi_i^{(\varepsilon)}(\rho, 0) = \begin{cases} (\psi_i^{(\varepsilon)}(\rho, 0) - 1)/(e^\rho - 1) & \rho > 0, \\ \psi_i^{(\varepsilon)}(0, 1) & \rho = 0. \end{cases} \tag{25}$$

Under condition **H**, the derivative of any order of the function $\varphi_i^{(\varepsilon)}(\rho, 0)$ exists for $0 \le \rho \le \beta < \delta$ and sufficiently small ε. Denote the derivative of order r of this function by $\varphi_i^{(\varepsilon)}(\rho, r)$. It follows directly from (25) that

$$\psi_i^{(\varepsilon)}(\rho, 0) = \varphi_i^{(\varepsilon)}(\rho, 0)(e^\rho - 1) + 1, \qquad \rho \ge 0. \tag{26}$$

By differentiating equation (26) r times and rearranging, it follows that the derivative of order $r = 1, 2, \dots,$ of the function $\varphi_i^{(\varepsilon)}(\rho, 0)$ is given by the recursive relation

$$\varphi_i^{(\varepsilon)}(\rho, r) = \begin{cases} (\psi_i^{(\varepsilon)}(\rho, r) - e^\rho \sum_{j=0}^{r-1} \binom{r}{j} \varphi_i^{(\varepsilon)}(\rho, j))/(e^\rho - 1) & \rho > 0, \\ (\psi_i^{(\varepsilon)}(0, r+1) - \sum_{j=0}^{r-1} \binom{r+1}{j} \varphi_i^{(\varepsilon)}(0, j))/(r+1) & \rho = 0. \end{cases}$$

In the following, suppose that condition $\mathbf{P_3^{(k)}}$ holds, together with condition $\mathbf{P_4^{(k)}}$ if $\rho^{(0)} > 0$, or together with condition $\mathbf{P_4^{(k+1)}}$ if $\rho^{(0)} = 0$. Then the following asymptotic expansion holds,

$$\varphi_i^{(\varepsilon)}\left(\rho^{(0)}, r\right) = \varphi_i^{(0)}\left(\rho^{(0)}, r\right) + \varphi_i[1, r]\varepsilon + \cdots + \varphi_i[k-r, r]\varepsilon^{k-r} + o(\varepsilon^{k-r}). \tag{27}$$

Denote $\varphi_i[0, r] = \varphi_i^{(0)}(\rho^{(0)}, r)$.

In the case $\rho^{(0)} > 0$, the coefficients for $i = 0, 1$, are given by

$$\begin{aligned} &\varphi_i[n, r]\left(e^{\rho^{(0)}} - 1\right) \\ &\quad = \begin{cases} \psi_i[n, 0] - \delta(n, 0) & n = 0, \dots, k,\ r = 0, \\ \psi_i[n, r] - e^{\rho^{(0)}} \sum_{j=0}^{r-1} \binom{r}{j} \varphi_i[n, j] & n = 0, \dots, k-r,\ r = 1, \dots, k. \end{cases} \end{aligned} \tag{28}$$

In the case $\rho^{(0)} = 0$, the coefficients for $i = 0, 1$, are given by

$$\begin{aligned} &\varphi_i[n, r](r+1) \\ &\quad = \begin{cases} \psi_i[n, 1] & n = 0, \dots, k,\ r = 0, \\ \psi_i[n, r+1] - \sum_{j=0}^{r-1} \binom{r+1}{j} \varphi_i[n, j] & n = 0, \dots, k-r,\ r = 1, \dots, k. \end{cases} \end{aligned} \tag{29}$$

Differentiating equation (20) and (24) r times with respect to ρ and evaluating at $\rho = \rho^{(0)}$ yields for any $r = 0, 1, \dots,$

$$\phi^{(\varepsilon)}\left(\rho^{(0)}, r\right) = \left(1 - p^{(\varepsilon)}\right) \sum_{j=0}^{r} \binom{r}{j} \psi_0^{(\varepsilon)}\left(\rho^{(0)}, j\right) \psi_1^{(\varepsilon)}\left(\rho^{(0)}, r-j\right), \tag{30}$$

$$\omega_0^{(\varepsilon)}\left(\rho^{(0)}, r\right) = \sum_{j=0}^{r} \binom{r}{j} \psi_1^{(\varepsilon)}\left(\rho^{(0)}, j\right) \varphi_0^{(\varepsilon)}\left(\rho^{(0)}, r-j\right), \tag{31}$$

$$\omega_1^{(\varepsilon)}\left(\rho^{(0)}, r\right) = \varphi_1^{(\varepsilon)}\left(\rho^{(0)}, r\right). \tag{32}$$

It follows from equations (27)–(32) that conditions $\mathbf{P_1^{(k)}}$ and $\mathbf{P_2^{(k)}}$ are implied by conditions $\mathbf{P_3^{(k)}}$ and $\mathbf{P_4^{(k)}}$ in the case $\rho^{(0)} > 0$, and by conditions $\mathbf{P_3^{(k)}}$ and $\mathbf{P_4^{(k+1)}}$ in the case $\rho^{(0)} = 0$. We can find the relations between the coefficients by using arithmetic rules of asymptotic expansions.

The coefficients in condition $\mathbf{P_1^{(k)}}$ are for any $n = 0, \ldots, k-r$ and $r = 0, \ldots, k$ given by

$$a_{0,r} = \left(1 - p^{(0)}\right) h_{0,r}, \qquad a_{n,r} = \left(1 - p^{(0)}\right) h_{n,r} - \sum_{i=1}^{n} p[i] h_{n-i,r},$$
$$h_{n,r} = \sum_{i=0}^{n} \sum_{j=0}^{r} \binom{r}{j} \psi_0[i,j] \psi_1[n-i, r-j]. \tag{33}$$

The coefficients in condition $\mathbf{P_2^{(k)}}$ are for any $n = 0, \ldots, k-r$ and $r = 0, \ldots, k$ given by

$$b_{n,r}(\{0\}) = \sum_{i=0}^{n} \sum_{j=0}^{r} \binom{r}{j} \psi_1[i,j] \varphi_0[n-i, r-j],$$
$$b_{n,r}(\{1\}) = \varphi_1[n,r], \qquad b_{n,r}(X) = b_{n,r}(\{0\}) + b_{n,r}(\{1\}). \tag{34}$$

It follows from Theorem 3.1 that an asymptotic expansion of order k exists for the quasi-stationary distribution (23). We can build the expansion using equations (7), (8), (9), (28), (29), (33) and (34).

5. Perturbed Risk Processes

This section shows how the results of the present paper can be used to obtain approximations for the ruin probability in a perturbed discrete time risk process.

For each $\varepsilon \geq 0$, let $X_1^{(\varepsilon)}, X_2^{(\varepsilon)}, \ldots$ be a sequence of non-negative, independent and identically distributed random variables and set

$$Z_u^{(\varepsilon)}(n) = u + n - \sum_{k=1}^{n} X_k^{(\varepsilon)}, \qquad n = 0, 1, \ldots,$$

where u is a non-negative integer.

We can interpret $Z_u^{(\varepsilon)}(n)$ as the capital of an insurance company (in units equivalent to expected premium per time unit) and $X_n^{(\varepsilon)}$ as the claims at moment n.

Let us denote $p^{(\varepsilon)} = \mathsf{P}\{X_1^{(\varepsilon)} > 0\}$ and $\mu^{(\varepsilon)} = \sum_{u=0}^{\infty} u g^{(\varepsilon)}(u)$ where

$$g^{(\varepsilon)}(u) = \mathsf{P}\left\{X_1^{(\varepsilon)} = u / X_1^{(\varepsilon)} > 0\right\}, \qquad u = 0, 1, \ldots$$

An object of interest is the infinite time horizon ruin probability which is defined as

$$\Psi^{(\varepsilon)}(u) = \mathsf{P}\left\{\min_{n \geq 0} Z_u^{(\varepsilon)}(n) < 0\right\}, \qquad u = 0, 1, \ldots$$

Define $\alpha^{(\varepsilon)} := \mathsf{E} X_1^{(\varepsilon)} = p^{(\varepsilon)} \mu^{(\varepsilon)}$. It can be shown that if $\alpha^{(\varepsilon)} \geq 1$, then $\Psi^{(\varepsilon)}(u) = 1$ for all $u \geq 0$. In the case $\alpha^{(\varepsilon)} \leq 1$, the ruin probability $\Psi^{(\varepsilon)}(u)$ satisfies the following discrete time renewal equation,

$$\Psi^{(\varepsilon)}(u) = q^{(\varepsilon)}(u) + \sum_{k=0}^{u} \Psi^{(\varepsilon)}(u-k) f^{(\varepsilon)}(k), \qquad u = 0, 1, \ldots, \tag{35}$$

where, for $u = 0, 1, \ldots$,

$$G^{(\varepsilon)}(u) = \sum_{k=0}^{u} g^{(\varepsilon)}(k), \qquad f^{(\varepsilon)}(u) = \alpha^{(\varepsilon)} \frac{1 - G^{(\varepsilon)}(u)}{\mu^{(\varepsilon)}}, \qquad q^{(\varepsilon)}(u) = \sum_{k=u+1}^{\infty} f^{(\varepsilon)}(k).$$

A derivation of this equation can be found, for example, in Petersson and Silvestrov (2012, 2013). It is similar with the well-known technique for deriving the corresponding renewal equation for a continuous time risk process, given, for example, in Feller (1966) and Grandell (1991).

We now introduce the following mixed power-exponential moment generating functions,

$$\varphi^{(\varepsilon)}(\rho, r) = \sum_{u=0}^{\infty} u^r e^{\rho u} \left(1 - G^{(\varepsilon)}(u)\right), \qquad \rho \geq 0,\ r = 0, 1, \ldots$$

Let us assume the following conditions:

I: **(a)** $g^{(\varepsilon)}(u) \to g^{(0)}(u)$ as $\varepsilon \to 0$, $u = 0, 1, \ldots$
(b) $p^{(\varepsilon)} \to p^{(0)}$ as $\varepsilon \to 0$.

J: There exists $\delta > 0$ such that
(a) $\overline{\lim}_{0 \leq \varepsilon \to 0} \varphi^{(\varepsilon)}(\delta, 0) < \infty$.
(b) $(\alpha^{(0)}/\mu^{(0)})\varphi^{(0)}(\delta, 0) > 1$.

Under conditions **I** and **J** there exists a unique non-negative root $\rho^{(\varepsilon)}$ for sufficiently small ε to the characteristic equation

$$\sum_{u=0}^{\infty} e^{\rho u} f^{(\varepsilon)}(u) = 1. \tag{36}$$

Using this, we can transform the renewal equation (35) into the following form,

$$\widetilde{\Psi}^{(\varepsilon)}(u) = \tilde{q}^{(\varepsilon)}(u) + \sum_{k=0}^{u} \widetilde{\Psi}^{(\varepsilon)}(u-k)\tilde{f}^{(\varepsilon)}(k), \qquad u = 0, 1, \ldots, \tag{37}$$

where

$$\widetilde{\Psi}^{(\varepsilon)}(u) = e^{\rho^{(\varepsilon)}u}\Psi^{(\varepsilon)}(u), \qquad \tilde{q}^{(\varepsilon)}(u) = e^{\rho^{(\varepsilon)}u} q^{(\varepsilon)}(u), \qquad \tilde{f}^{(\varepsilon)}(u) = e^{\rho^{(\varepsilon)}u} f^{(\varepsilon)}(u).$$

There is a close connection between renewal equations and regenerative processes. In fact, the solution $x(n)$ of a discrete time renewal equation where the distribution $f(n)$ and the forcing function $q(n)$ satisfies $0 \leq q(n) \leq 1 - \sum_{k=0}^{n} f(k)$ for all $n \geq 0$, can be related to the one-dimensional distributions of some discrete time regenerative process.

In our case, there exists a discrete time regenerative process $\xi^{(\varepsilon)}(n)$, $n = 0, 1, \ldots$, with regeneration times $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \ldots$, and phase space $\{0, 1\}$ such that for $u = 0, 1, \ldots$, we have

$$\mathsf{P}\left\{\xi^{(\varepsilon)}(u) = 1\right\} = \widetilde{\Psi}^{(\varepsilon)}(u) \tag{38}$$

and

$$\mathsf{P}\left\{\xi^{(\varepsilon)}(u) = 1, \tau_1^{(\varepsilon)} > u\right\} = \tilde{q}^{(\varepsilon)}(u), \qquad \mathsf{P}\left\{\tau_1^{(\varepsilon)} = u\right\} = \tilde{f}^{(\varepsilon)}(u). \tag{39}$$

We next show how this process can be constructed. It is similar with the construction in the corresponding continuous time model, given in Ekheden and Silvestrov (2011).

Let $\kappa_1^{(\varepsilon)}, \kappa_2^{(\varepsilon)}, \ldots,$ be a sequence of independent random variables, each with distribution $\tilde{f}^{(\varepsilon)}(n)$, and let $U_1^{(\varepsilon)}, U_2^{(\varepsilon)}, \ldots,$ be a sequence of independent random variables uniformly distributed on the interval $[0, 1]$. Furthermore, we assume that the two sequences are independent.

For $k = 0, 1, \ldots$, let

$$v^{(\varepsilon)}(k) = \begin{cases} \tilde{q}^{(\varepsilon)}(k)/(1 - \tilde{F}^{(\varepsilon)}(k)) & \text{if } 1 - \tilde{F}^{(\varepsilon)}(k) > 0, \\ 0 & \text{if } 1 - \tilde{F}^{(\varepsilon)}(k) = 0, \end{cases}$$

where $\tilde{F}^{(\varepsilon)}(k) = \sum_{u=0}^{k} \tilde{f}^{(\varepsilon)}(u)$, $k = 0, 1, \ldots$.

Let us now, for every $n = 1, 2, \ldots$, define a random process by

$$\eta_n^{(\varepsilon)}(k) = \chi\left(U_n^{(\varepsilon)} \leq v^{(\varepsilon)}(k)\right), \qquad k = 0, 1, \ldots .$$

Using this, we can define a regenerative process $\xi^{(\varepsilon)}(n)$ with regeneration times $\tau_k^{(\varepsilon)} = \kappa_1^{(\varepsilon)} + \cdots + \kappa_k^{(\varepsilon)}$, $k = 1, 2, \ldots$, by

$$\xi^{(\varepsilon)}(n) = \eta_{\nu^{(\varepsilon)}(n)+1}^{(\varepsilon)}\left(\zeta^{(\varepsilon)}(n)\right), \qquad n = 0, 1, \ldots,$$

where $\nu^{(\varepsilon)}(n) = \max\{k \colon \tau_k^{(\varepsilon)} \leq n\}$ is the number of regenerations up to and including time n, and $\zeta^{(\varepsilon)}(n) = n - \tau_{\nu^{(\varepsilon)}(n)}^{(\varepsilon)}$ is the time since the last regeneration.

By definition, $\xi^{(\varepsilon)}(n)$ is a regenerative process with phase space $\{0, 1\}$ and regeneration times $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \ldots$. It can be checked that for this process, relations (38) and (39) hold.

If conditions **I**$-$**J** hold, then conditions **A**–**D** hold for the functions $\tilde{f}^{(\varepsilon)}(u)$ and $\tilde{q}^{(\varepsilon)}(u)$. It follows from Theorem 2.2 that

$$\mathsf{P}\left\{\xi^{(\varepsilon)}(u) = 1\right\} \to \pi^{(\varepsilon)}, \qquad u \to \infty, \tag{40}$$

where

$$\pi^{(\varepsilon)} = \frac{\sum_{u=0}^{\infty} \mathsf{P}\left\{\xi^{(\varepsilon)}(u) = 1, \tau_1^{(\varepsilon)} > u\right\}}{\sum_{u=0}^{\infty} \mathsf{P}\left\{\tau_1^{(\varepsilon)} > u\right\}}. \tag{41}$$

Rewriting in terms of the claim distributions, relations (40) and (41) yield

$$e^{\rho^{(\varepsilon)} u} \Psi^{(\varepsilon)}(u) \to \frac{\sum_{n=0}^{\infty} e^{\rho^{(\varepsilon)} n} \sum_{k=n+1}^{\infty} (1 - G^{(\varepsilon)}(k))}{\sum_{n=0}^{\infty} n e^{\rho^{(\varepsilon)} n} (1 - G^{(\varepsilon)}(n))} \quad \text{as } u \to \infty. \tag{42}$$

Relation (42) can be seen as a discrete time analogue of the classical Cramér–Lundberg approximation.

Let us introduce the following mixed power-exponential moment generating functions,

$$\psi^{(\varepsilon)}(\rho, r) = \sum_{u=0}^{\infty} u^r e^{\rho u} g^{(\varepsilon)}(u), \qquad \rho \geq 0,\ r = 0, 1, \ldots,$$

$$\omega^{(\varepsilon)}(\rho, r) = \sum_{u=0}^{\infty} u^r e^{\rho u} \sum_{k=u+1}^{\infty} (1 - G^{(\varepsilon)}(k)), \qquad \rho \geq 0,\ r = 0, 1, \ldots .$$

In order to build an asymptotic expansion for the stationary distribution $\pi^{(\varepsilon)}$, we impose the following perturbation conditions:

$\mathbf{P_5^{(k)}}$: $p^{(\varepsilon)} = p^{(0)} + p[1]\varepsilon + \cdots + p[k]\varepsilon^k + o(\varepsilon^k)$, where $|p[n]| < \infty$, $n = 1, \ldots, k$.

$\mathbf{P_6^{(k)}}$: $\psi^{(\varepsilon)}(\rho^{(0)}, r) = \psi^{(0)}(\rho^{(0)}, r) + \psi[1, r]\varepsilon + \cdots + \psi[k - r, r]\varepsilon^{k-r} + o(\varepsilon^{k-r})$, for $r = 0, \ldots, k$, where $|\psi[n, r]| < \infty$, $n = 1, \ldots, k - r$, $r = 0, \ldots, k$.

The moment generating functions $\varphi^{(\varepsilon)}(\rho, 0)$, $\psi^{(\varepsilon)}(\rho, 0)$ and $\omega^{(\varepsilon)}(\rho, 0)$ are linked by the relations

$$\omega^{(\varepsilon)}(\rho, 0) = \begin{cases} (\varphi^{(\varepsilon)}(\rho, 0) - \varphi^{(\varepsilon)}(0, 0))/(e^{\rho} - 1) & \rho > 0, \\ \varphi^{(\varepsilon)}(0, 1) & \rho = 0, \end{cases} \tag{43}$$

and

$$\varphi^{(\varepsilon)}(\rho, 0) = \begin{cases} (\psi^{(\varepsilon)}(\rho, 0) - 1)/(e^{\rho} - 1) & \rho > 0, \\ \psi^{(\varepsilon)}(0, 1) & \rho = 0. \end{cases} \tag{44}$$

Using relations (43) and (44) we can build asymptotic expansions for $\varphi^{(\varepsilon)}(\rho^{(0)}, r)$ and $\omega^{(\varepsilon)}(\rho^{(0)}, r)$ using the same techniques as in Section 4. From this one can continue and obtain asymptotic expansions for the stationary distribution $\pi^{(\varepsilon)}$ and the root $\rho^{(\varepsilon)}$ of the characteristic equation (36) as follows,

$$\begin{aligned} \pi_l^{(\varepsilon)} &= \pi^{(0)} + \pi_1\varepsilon + \cdots + \pi_l\varepsilon^l + o(\varepsilon^l), \\ \rho_r^{(\varepsilon)} &= \rho^{(0)} + a_1\varepsilon + \cdots + a_r\varepsilon^r + o(\varepsilon^r). \end{aligned} \tag{45}$$

Using (42) and (45) we obtain approximations of the ruin probability of the form

$$\widehat{\Psi}_{r,l}^{(\varepsilon)}(u) = e^{-\rho_r^{(\varepsilon)}u}\pi_l^{(\varepsilon)}. \tag{46}$$

By different choices of the parameters r and l one can control the highest order of moments of claim distributions involved in the approximation.

For any non-negative integer-valued function $u^{(\varepsilon)} \to \infty$ in such a way that $\varepsilon^r u^{(\varepsilon)} \to \lambda_r \in [0,\infty)$, as $\varepsilon \to \infty$, this approximation has asymptotic relative error zero, meaning that

$$\frac{\Psi^{(\varepsilon)}(u^{(\varepsilon)})}{\widehat{\Psi}_{r,l}^{(\varepsilon)}(u^{(\varepsilon)})} \to 1 \quad \text{as } \varepsilon \to 0.$$

In the case $\rho^{(0)} > 0$, the approximation in equation (46) generalises the Cramér–Lundberg approximation for discrete time risk processes while the case $\rho^{(0)} = 0$ corresponds to a generalisation of the diffusion approximation.

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