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STORAGE IMPULSIVE PROCESSES ON INCREASING TIME INTERVALS

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Abstract. The Storage Impulsive Process (SIP) $S(t)$ is a sum of (jointly independent) random variables defined on the embedded Markov chain of a homogeneous Markov process.

The SIP is considered in the series scheme on increasing time intervals t/ε, with a small parameter $\varepsilon \to 0$, $\varepsilon > 0$. The SIP is investigated in the average and diffusion approximation scheme. The large deviation problem is considered under corresponding scaling with an asymptotically small diffusion.

Анотація. Імпульсні процеси накопичення (ІПН) задаються сумами (незалежними в сукупності) випадкових величин, визначених на вкладеному ланцюгу Маркова однорідного марковського процесу.

ІПН розглядаються у схемі серій на зростаючих інтервалах часу t/ε, з малим параметром серії $\varepsilon \to 0$, $\varepsilon > 0$. ІПН досліджуються у схемах усереднення та дифузійної апроксимації. Проблема великих відхилень розглядається при відповідному нормуванні з асимптотично малою дифузією.

Аннотация. Импульсные процессы накопления (ИПН) задаются суммами (независимыми в совокупности) случайных величин, определенных на вложенной цепи Маркова однородного марковского процесса.

ИПН рассматриваются в схеме серий на возрастающих интервалах времени t/ε, с малым параметром серии $\varepsilon \to 0$, $\varepsilon > 0$. ИПН исследуются в схемах укрупнения и диффузионной аппроксимации. Проблема больших уклонений рассматривается при соответствующей нормировке с асимптотически малой диффузией.

1. Introduction

The Storage Impulsive Process (SIP) $S(t)$ is a sum of (jointly independent) random variables defined on the embedded Markov chain of a homogeneous Markov process

$$S(t) = u + \sum_{n=1}^{\nu(t)} \alpha_n(x_n), \qquad t \ge 0,\ u \in \mathbb{R}^d. \tag{1}$$

The time homogeneous Markov process $x(t)$, $t \ge 0$, is defined on a standard phase space $(E, \mathcal{E})$ by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \qquad x \in E,$$

for a real valued test function $\varphi(x)$, $x \in E$, with a bounded sup-norm:

$$\|\varphi(x)\| := \sup_{x\in E} |\varphi(x)|.$$

The embedded Markov chain x_n, $n \ge 0$, is defined by

$$x_n := x(\tau_n), \qquad n \ge 0,$$

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where the renewal moments of jumps are given by

$$\tau_{n+1} = \tau_n + \theta_{n+1}, \qquad n \geq 0,\ \tau_0 = 0,$$

and the sojourn times θ_{n+1}, $n \geq 0$, are such that

$$\mathsf{P}(\theta_{n+1} \geq t \mid x_n = x) = e^{-q(x)t} =: \mathsf{P}(\theta_x \geq t).$$

The stochastic kernel $P(x, B)$, $x \in E$, $B \in \mathcal{E}$, defines the transition probabilities of the embedded Markov chain

$$P(x, B) = \mathcal{P}\{x_{n+1} \in B \mid x_n = x\}.$$

The counting process is defined by

$$\nu(t) := \max\{n > 0 \colon \tau_n \leq t\}, \qquad t \geq 0.$$

The random variables in (1) have the distribution functions

$$\Phi_x(dv) = P\{\alpha_n(x) \in dv\} := \mathsf{P}\{\alpha_n(x_n) \in dv \mid x_n = x\}, \qquad x \in E.$$

The SIP may be considered as a random evolution process [1, Ch.2]. The switching Markov process $x(t)$, $t \geq 0$, describes a random environment.

A1: The main assumption is the uniform ergodicity of the Markov process $x(t)$, $t \geq 0$, with the stationary distribution $\pi(B)$, $B \in \mathcal{E}$, satisfying the equation:

$$\pi(dx)q(x) = q\rho(dx), \qquad q = \int_E \pi(dx)q(x).$$

The stationary distribution $\rho(B)$, $B \in \mathcal{E}$, of the embedded Markov chain x_n, $n \geq 0$, satisfies the equation

$$\rho(B) = \int_E \rho(dx)P(x, B), \qquad B \in \mathcal{E},\ \rho(E) = 1.$$

Provided that the main assumption **A1** takes place the potential operator R_0 may be given by a solution of the equation [1, Ch. 2]

$$QR_0 = R_0Q = \Pi - I, \qquad \Pi\varphi(x) := \int_E \pi(dx)\varphi(x).$$

2. SIP on increasing time intervals in average scheme.

The SIP on increasing time intervals in average scheme is considered in the series scheme with the small parameter $\varepsilon \to 0$, $\varepsilon > 0$, in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon)} \alpha_n(x_n), \qquad t \geq 0,\ \varepsilon > 0,\ u \in \mathbb{R}^d. \tag{2}$$

The random evolution approach [1, Ch. 3, 5] is an effective method of asymptotic analysis (2) when $\varepsilon \to 0$.

Proposition 2.1. *The SIP* (2) *in the average scheme convergences weakly*

$$S^\varepsilon(t) \Rightarrow S^0(t) = u + \widehat{a}_0 t, \qquad \varepsilon \to 0, \tag{3}$$

where the average velocity is such that

$$\widehat{a}_0 = q\widehat{a}, \qquad \widehat{a} = \int_E \rho(dx)a(x), \qquad a(x) = \int_{\mathbb{R}^d} v\,\Phi_x(dv). \tag{4}$$

Proof of Proposition 2.1 is based on the random evolution approach [1, Ch. 3] by using a solution of the singular perturbation problem [1, Ch. 5].

Remark 2.1. For simplcity without loss of generality the proof is realized for the SIP given on real line $\mathbb{R}$, $d = 1$.

According to the definition of a random evolution [1, Ch. 2] we consider the two component Markov process

$$S^\varepsilon(t),\ x^\varepsilon(t) := x(t/\varepsilon), \qquad t \geq 0. \tag{5}$$

Lemma 2.1. *The Markov process* (5) *is characterized by the generator*

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-1}q(x)\int_E P(x,dy)\int_{\mathbb{R}^d}\Phi_y(dv)[\varphi(u+\varepsilon v,y) - \varphi(u,x)]. \tag{6}$$

The proof of Lemma 2.1 is a direct consequence of the definition of the generator [1, Ch. 3].

Remark 2.2. The generator (6) may be rewritten as follows

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-1}\left[Q + Q_0\Phi_x^\varepsilon\right]\varphi(u,x), \tag{7}$$

where, by definition,

$$Q_0\varphi(x) := q(x)\int_E P(x,dy)\varphi(y),$$

$$\Phi_x^\varepsilon\varphi(u) := \int_{\mathbb{R}^d}\Phi_x(dv)[\varphi(u+\varepsilon v) - \varphi(u)].$$

On a test function $\varphi(u)$ being smooth enough,

$$\Phi_x^\varepsilon\varphi(u) = \varepsilon[a(x)\varphi'(u) + \delta^\varepsilon(x)\varphi(u)]$$

with the negligible term:

$$\|\delta^\varepsilon(x)\varphi(u)\| \to 0, \qquad \varepsilon \to 0,\ \varphi(u) \in C^2(\mathbb{R}).$$

Lemma 2.2. *The generator* (7) *admits the following asymptotic expansion:*

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-1}Q + Q_0\mathbb{A}(x) + \delta^\varepsilon(x)\right]\varphi(u,x),$$

where

$$\mathbb{A}(x)\varphi(u) := a(x)\varphi'(u),$$

and the negligible term is such that

$$\sup_{x\in E}\|\delta^\varepsilon(x)\varphi(u,x)\| \to 0, \qquad \varepsilon \to 0,\ \varphi(u,\cdot) \in C^2(\mathbb{R}).$$

Then a solution of the singular perturbation problem [1, Ch. 5] *may be used for the truncated operator*

$$L_0^\varepsilon\varphi(u,x) := \left[\varepsilon^{-1}Q + Q_0\mathbb{A}(x)\right]\varphi(u,x). \tag{8}$$

Lemma 2.3. *The truncated operator* (8) *on a perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x),$$

admits the following asymptotic representation [1, Proposition 5.1]*:*

$$L_0^\varepsilon\varphi^\varepsilon(u,x) = \widehat{a}_0\varphi'(u) + \delta^\varepsilon(x)\varphi(u).$$

The negligible term may be written in explicit form:

$$\delta^\varepsilon(x)\varphi(u) = \varepsilon Q_0\mathbb{A}(x)R_0\widehat{\mathbb{A}}(x)\varphi(u).$$

$$\widehat{\mathbb{A}}(x) := \widehat{\mathbb{A}}_0 - Q_0\mathbb{A}(x), \qquad \widehat{\mathbb{A}}_0 := \Pi Q_0\mathbb{A}(x)\Pi.$$

Conclusion 2.1. *The generator* (6) *of the random evolution* (5) *admits the asymptotic representation*

$$L^\varepsilon \varphi^\varepsilon(u,x) = \widehat{a}_0 \varphi'(u) + \delta^\varepsilon(x)\varphi(u) \tag{9}$$

with the negligible term $\delta^\varepsilon(x)\varphi(u)$.

The representation (9) *implies the weak convergence* (3)–(4) [1, Ch. 6] *because the limit operator*

$$L^0\varphi(u) := \widehat{a}_0\varphi'(u), \qquad \varphi(u) \in C^1(\mathbb{R}), \tag{10}$$

defines the evolution

$$S^0(t) = u + \widehat{a}_0 t, \qquad t \geq 0,\ S^0(0) = u.$$

Remark 2.3. The limit operator (10) in the Euclidean space $\mathbb{R}^d$ has the following representation:

$$\widehat{a}_0\varphi'(u) := \sum_{k=1}^{d} \widehat{a}_k^0 \varphi_k'(u), \qquad \varphi_k'(u) := \partial\varphi(u)/\partial u_k,$$

$$\widehat{a}_k^0 = q\widehat{a}_k, \qquad \widehat{a}_k = \int_E \rho(dx) a_k(x), \qquad a_k(x) = \int_{\mathbb{R}} v_k \Phi_x(dv).$$

3. SIP in diffusion approximation scheme.

It is well known that the diffusion approximation of stochastic systems may be realized under some additional *Balance Condition (BC)*.

We consider two different BC for SIP, namely the *total* and *local* ones.

3.1. SIP under total balance condition.

The SIP in the series scheme with the parameter $\varepsilon \to 0$, $\varepsilon > 0$, in the diffusion approximation scheme under the *Total Balance Condition (TBC)*:

$$a(x) = \int_{R^d} v\, \Phi_x(dv) \equiv 0, \tag{11}$$

is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon^2)} \alpha_n(x_n), \qquad t \geq 0,\ \varepsilon > 0.$$

Proposition 3.1. *Under the TBC* (11)*, the weak convergence*

$$S^\varepsilon(t) \Rightarrow W_\sigma(t), \qquad \varepsilon \to 0,$$

takes place.

The limit Brownian motion process $W_\sigma(t)$, $t \geq 0$, *is defined by the variance matrix*

$$\widehat{C} = \sigma^*\sigma = q\widehat{B},$$

$$\widehat{B} = \int_E \rho(dx) B(x), \qquad B(x) = \int_{\mathbb{R}^d} v^* v\, \Phi_x(dv).$$

Proof of Proposition 3.1. As in Section 2, we start by characterizing the coupled Markov process.

Lemma 3.1. *The Markov process*

$$S^\varepsilon(t),\ x^\varepsilon(t) := x\left(t/\varepsilon^2\right), \qquad t \geq 0,$$

is characterized by the generator

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-2} q(x) \int_E P(x,dy) \int_{\mathbb{R}^d} \Phi_x(dv)[\varphi(u+\varepsilon v, y) - \varphi(u,x)]. \tag{12}$$

The generator (12) *may be rewritten as follows*

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-2}[Q + Q_0\mathit{\Phi}^\varepsilon_x]\varphi(u,x), \tag{13}$$

where

$$\mathit{\Phi}^\varepsilon_x\varphi(u) := \int_{\mathbb{R}^d}\Phi_x(dv)[\varphi(u+\varepsilon v) - \varphi(u)] = \varepsilon^2\left[\frac{1}{2}B(x)\varphi''(u) + \delta^\varepsilon(x)\varphi(u)\right], \tag{14}$$

with the negligible term $\delta^\varepsilon(x)\varphi(u)$.

Lemma 3.2. *The generator* (13)–(14) *admits the asymptotic expansion*

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}Q + Q_0\mathbb{B}(x)\right]\varphi(u,x) + \delta^\varepsilon(x)\varphi(u)$$

with negligible term $\delta^\varepsilon(x)\varphi(u)$. *Here by definition*

$$\mathbb{B}(x)\varphi(u) = \frac{1}{2}B(x)\varphi''(u). \tag{15}$$

Then the solution of singular perturbation problem [1, Ch. 5] *can be used for the truncated operator*

$$\mathbb{L}^\varepsilon_0\varphi(u,x) = \left[\varepsilon^{-2}Q + Q_0\mathbb{B}(x)\right]\varphi(u,x). \tag{16}$$

Lemma 3.3. *The truncated operator (16) on a perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon^2\varphi_2(u,x), \tag{17}$$

admits the asymptotic representation

$$L^\varepsilon_0\varphi^\varepsilon(u,x) = \frac{1}{2}\widehat{C}\varphi''(u) + \delta^\varepsilon(x)\varphi(u).$$

Proof. Considering (16) and (17),

$$\begin{aligned} L^\varepsilon_0\varphi^\varepsilon &= \left[\varepsilon^{-2}Q + Q_0\mathbb{B}(x)\right]\left[\varphi(u) + \varepsilon^2\varphi_2(u,x)\right] \\ &= \varepsilon^{-2}Q\varphi(u) + [Q\varphi_2(u,x) + Q_0\mathbb{B}(x)\varphi(u)] + \delta^\varepsilon(x)\varphi(u). \end{aligned}$$

It is obvious

$$Q\varphi(u) = 0.$$

The equation

$$Q\varphi_2(u,x) + Q_0\mathbb{B}(x)\varphi(u) = \widehat{L}_0\varphi(u)$$

can be solved under the solvability condition [1, Ch.5]:

$$\widehat{L}_0\Pi = \Pi Q_0\mathbb{B}(x)\Pi.$$

Transforming (15) gives us

$$\widehat{L}_0\varphi(u) = \frac{1}{2}\widehat{C}\varphi''(u).$$

Indeed

$$\begin{aligned} \widehat{L}_0\varphi(u) &= \int_E \pi(dx)q(x)\int_E P(x,dy)\frac{1}{2}B(y)\varphi''(u) \\ &= \frac{1}{2}q\int_E \rho(dx)B(x)\varphi''(u) = \frac{1}{2}q\widehat{B}\varphi''(u). \end{aligned}$$

□

Remark 3.1. The limit generator $\widehat{L}_0$ in the Euclidean space $\mathbb{R}^d$ is represented as follows:

$$\widehat{L}_0\varphi(u) = \frac{q}{2}\sum_{k,r=1}^{d} B_{kr}\varphi''_{kr}(u),$$

$$\widehat{B} = [B_{kr}; 1 \le k,r \le d], \qquad \varphi''_{kr}(u) := \partial^2\varphi(u)/\partial u_k\partial u_r,$$

$$B_{kr} = \int_E \rho(dx)B_{kr}(x), \qquad B_{kr}(x) = \int_{\mathbb{R}} v_k v_r\,\Phi_x(dv).$$

The proof of Proposition 3.1 is finished by using the asymptotic representation

$$L^\varepsilon \varphi^\varepsilon(u,x) = \widehat{L}_0\varphi(u) + \delta^\varepsilon(x)\varphi(u), \tag{18}$$

and convergence Theorem 6.3 [1, Ch.6]. The negligible term in (18) may be written in the explicit form. □

3.2. **SIP under the Local Balance Condition (LBC).** The LBC means that the average value of jumps is such that

$$\widehat{a} := \int_E \rho(dx)a(x) \neq 0. \tag{19}$$

The SIP in the series scheme under the LBC (19) with the parameter $\varepsilon \to 0$, $\varepsilon > 0$, is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon^2)} \alpha_n(x_n) - q\widehat{a}t/\varepsilon, \qquad t \geq 0. \tag{20}$$

Proposition 3.2. *Under the LBC* (19)*, the weak convergence*

$$S^\varepsilon(t) \Rightarrow W_\sigma(t), \qquad \varepsilon \to 0,$$

takes place.

The limit Brownian motion $W_\sigma(t)$, $t \geq 0$, is defined by the variance matrix

$$\begin{aligned}
\widehat{C} = \sigma^*\sigma = q\widehat{B}, \qquad & \widehat{B} = \widehat{B}_0 + \widehat{B}_1,\\
\widehat{B}_0 = \int_E \rho(dx)B_0(x), \qquad & B_0(x) = \int_{\mathbb{R}^d} v^*v\Phi_x(dv),\\
\widehat{B}_1 = \int_E \rho(dx)B_1(x), \qquad & B_1(x) = 2\widehat{a}^*(x)R_0\widehat{a}(x),\\
\widehat{a}(x) &:= a_0(x) - q\widehat{a},\\
a_0(x) &:= q(x)\int_E P(x,dy)a(y).
\end{aligned} \tag{21}$$

Here the potential operator R_0 is defined as the solution of the equation

$$QR_0 = R_0Q = \Pi - I$$

[1, Ch. 3].

Proof of Proposition 3.2. As in the previous section we start using the generator of the two component Markov process.

Lemma 3.4. *The two component Markov process $S^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon^2)$, $t \geq 0$, is characterized by the generator*

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-2}q(x)\int_E P(x,dy)\int_{\mathbb{R}^d}\Phi(dv)[\varphi(u+\varepsilon v,y) - \varphi(u,x)] - \varepsilon^{-1}\widehat{a}_0\varphi'(u,x). \tag{22}$$

This generator can be written as follows

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}[Q + Q_0\Phi_x^\varepsilon] - \varepsilon^{-1}\widehat{\mathbb{A}}_0\right]\varphi(u,x) \tag{23}$$

with $\widehat{\mathbb{A}}_0\varphi(u) := \widehat{a}_0\varphi'(u)$

$$\begin{aligned}
\Phi_x^\varepsilon\varphi(u) &= \int_{\mathbb{R}^d}\Phi_x(dv)[\varphi(u+\varepsilon v) - \varphi(u)]\\
&= \varepsilon a(x)\varphi'(u) + \varepsilon^2\frac{1}{2}B(x)\varphi''(u) + \varepsilon^2\delta^\varepsilon(x)\varphi(u).
\end{aligned}$$

Lemma 3.5. *The generator* (22) *admits the asymptotic expansion*

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}\widehat{\mathbb{A}}(x) + Q_0\mathbb{B}(x)\right]\varphi(u,x) + \delta^\varepsilon(x)\varphi(u,x).$$

Here

$$\begin{aligned} \widehat{\mathbb{A}}(x)\varphi(u) &= \widehat{a}(x)\varphi'(u), \\ \widehat{a}(x) &:= a_0(x) - \widehat{a}_0, \\ a_0(x) &:= q(x)\int_E P(x,dy)a(y). \end{aligned} \tag{24}$$

Note that the following balance condition

$$\Pi\widehat{a}(x) = 0 \tag{25}$$

takes place.

Now a solution of singular perturbation problem [1, Ch.5] can be used for the truncated operator

$$L_0^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}\widehat{\mathbb{A}}(x) + Q_0\mathbb{B}(x)\right]\varphi(u,x). \tag{26}$$

Lemma 3.6. *The truncated operator* (26) *on a perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)$$

admits the asymptotic representation

$$L_0^\varepsilon\varphi^\varepsilon(u,x) = \frac{1}{2}\widehat{C}\varphi''(u) + \delta^\varepsilon(x)\varphi(u).$$

Proof. Let us consider

$$\begin{aligned} L_0^\varepsilon\varphi^\varepsilon(u,x) &= [\varepsilon^{-2}Q + \varepsilon^{-1}\widehat{\mathbb{A}}(x) + Q_0\mathbb{B}(x)][\varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)] \\ &= \varepsilon^{-2}Q\varphi(u) + \varepsilon^{-1}[Q\varphi_1 + \widehat{\mathbb{A}}(x)\varphi] + [Q\varphi_2 + \widehat{\mathbb{A}}(x)\varphi_1 + Q_0\mathbb{B}(x)\varphi] \\ &\quad + \delta^\varepsilon(x)\varphi(u). \end{aligned}$$

We get the equations

$$\begin{aligned} Q\varphi(u) &= 0, \\ Q\varphi_1(u,x) + \widehat{\mathbb{A}}(x)\varphi(u) &= 0, \\ Q\varphi_2(u,x) + \widehat{\mathbb{A}}(x)\varphi_1(u,x) + Q_0\mathbb{B}(x)\varphi(u) &= \widehat{L}_0\varphi(u). \end{aligned}$$

The first equation is obvious. The second equation satisfies the solvability condition (25). Hence

$$\varphi_1(u,x) = R_0\widehat{\mathbb{A}}(x)\varphi(u).$$

Now the third equation is

$$Q\varphi_2 + \left[\widehat{\mathbb{A}}_0(x) + Q_0\mathbb{B}(x)\right]\varphi(u) = \widehat{L}_0\varphi(u), \tag{27}$$

where

$$\widehat{\mathbb{A}}_0(x)\varphi(u) := \widehat{\mathbb{A}}(x)R_0\widehat{\mathbb{A}}(x)\varphi(u). \tag{28}$$

The solvability condition for (27) gives

$$\widehat{L}_0\Pi = \Pi\left[\widehat{\mathbb{A}}_0(x) + Q_0\mathbb{B}(x)\right]\Pi.$$

Using (28), (24), and (15) we calculate the limit generator

$$\widehat{L}_0\varphi(u) = \frac{1}{2}\widehat{C}\varphi''(u),$$

where the variance matrix $\widehat{C}$ is represented in (21).

Note that (see (24))

$$\widehat{\mathbb{A}}_0(x)\varphi(u) = \widehat{\mathbb{A}}(x)R_0\widehat{\mathbb{A}}(x)\varphi(u) = \widehat{\mathbb{A}}(x)R_0\widehat{a}(x)\varphi'(u) = \widehat{a}(x)R_0\widehat{a}(x)\varphi''(u) = \frac{1}{2}B_1(x)\varphi''(u).$$

Here

$$\widehat{a}(x) = a_0(x) - \widehat{a}_0. \qquad \square \quad \square$$

4. Large deviation in the scheme of asymptotically small diffusion

The SIP in the scheme of asymptotically small diffusion is considered under two different balance conditions, namely total and local ones.

4.1. The SIP under the total balance condition. The total balance condition means that the mean values of jumps of SIP equal totaly zero:

$$a(x) = \int_{\mathbb{R}^d} v\,\Phi_x(dv) \equiv 0. \tag{29}$$

The SIP in the scheme of asymptotically small diffusion is considered in the following scaling [3]:

$$S^\varepsilon(t) = u + \varepsilon^2 \sum_{n=1}^{\nu(t/\varepsilon^3)} \alpha_n(x_n), \qquad t \ge 0,\ \varepsilon > 0,\ u \in \mathbb{R}^d. \tag{30}$$

The coupled Markov process

$$S^\varepsilon(t),\ x^\varepsilon(t) := x\left(t/\varepsilon^3\right), \qquad t \ge 0,$$

is defined by the generator

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-3}q(x)\int_E P(x,dy)\int_{\mathbb{R}^d}\Phi_y(dv)\left[\varphi(u+\varepsilon^2 v, y) - \varphi(u,x)\right],$$

which can be rewritten as follows

$$L^\varepsilon\varphi(u,x) = \varepsilon^{-3}[Q + Q_0\Phi_x^\varepsilon]\varphi(u,x), \tag{31}$$

where, by definition,

$$\Phi_x^\varepsilon\varphi(u) := \int_{\mathbb{R}^d}\Phi_x(dv)\left[\varphi\left(u+\varepsilon^2 v\right) - \varphi(u)\right] = \varepsilon^4[\mathbb{B}(x)\varphi(u) + \delta^\varepsilon(x)\varphi(u)].$$

Here

$$\mathbb{B}(x)\varphi(u) := \frac{1}{2}B(x)\varphi''(u).$$

Hence the generator (31) admits the asymptotic expansion

$$L^\varepsilon\varphi(u,x) = L_0^\varepsilon\varphi(u,x) + \delta^\varepsilon(x)\varphi(u,x), \tag{32}$$

$$L_0^\varepsilon\varphi(u,x) = \left[\varepsilon^{-3}Q + \varepsilon Q_0\mathbb{B}(x)\right]\varphi(u,x).$$

The truncated operator (32) on a perturbed test function

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon^4\varphi_1(u,x),$$

admits the asymptotic representation

$$L_0^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon[Q\varphi_1 + Q_0\mathbb{B}(x)\varphi(u)] + \delta^\varepsilon(x)\varphi(u). \tag{33}$$

The representations (32) and (33) give

$$L^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon\left[\widehat{\mathbb{C}}\varphi(u) + \delta^\varepsilon(x)\varphi(u,x)\right],$$

where the main part

$$\varepsilon\widehat{\mathbb{C}}\varphi(u) = \varepsilon\frac{1}{2}\widehat{C}\varphi''(u)$$

is the generator of a small diffusion.

4.2. **Large deviation for SIP under the total balance condition.** We investigate the large deviation problem for SIP by using the asymptotic analysis of the exponential generator of large deviation

$$H^\varepsilon \varphi(u,x) = e^{-\varphi/\varepsilon} \varepsilon L^\varepsilon e^{\varphi/\varepsilon} \tag{34}$$

[2, Part I].

Proposition 4.1. *The large deviation for SIP* (30) *under the total balance condition* (29) *is realized by the exponential generator of small diffusion*

$$H\varphi(u) = \frac{1}{2}\widehat{C}[\varphi'(u)]^2, \tag{35}$$

$$\widehat{C} = q\int_E \rho(dx)B(x), \qquad B(x) = \int_{\mathbb{R}^d} v^* v\, \Phi_x(dv).$$

Proof of Proposition 4.1.

Lemma 4.1. *The exponential generator* (34) *on a perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln\left[1+\varepsilon^2\varphi_1(u,x)\right]$$

admits the asymptotic representation

$$H^\varepsilon\varphi^\varepsilon(u,x) = Q\varphi_1 + \frac{1}{2}Q_0 B(x)[\varphi'(u)]^2 + h^\varepsilon(x)\varphi(u)$$

with the negligible term

$$\|h^\varepsilon(x)\varphi(u)\| \to 0, \qquad \varepsilon \to 0,\ \varphi(u) \in C^3(\mathbb{R}).$$

Proof of Lemma 4.1. Let us calculate

$$\begin{aligned}
H^\varepsilon\varphi^\varepsilon &= e^{-\varphi/\varepsilon}\left[1+\varepsilon^2\varphi_1\right]^{-1}\varepsilon L^\varepsilon[1+\varepsilon^2\varphi_1]e^{\varphi/\varepsilon}\\
&= e^{-\varphi/\varepsilon}\left[1-\varepsilon^2\varphi_1\right]\varepsilon L_0^\varepsilon[1+\varepsilon^2\varphi_1]e^{\varphi/\varepsilon} + h^\varepsilon(x)\varphi(u)\\
&= e^{-\varphi/\varepsilon}\left[1-\varepsilon^2\varphi_1\right]\varepsilon^{-2}Q\left[1+\varepsilon^2\varphi_1\right]e^{\varphi/\varepsilon} + e^{-\varphi/\varepsilon}\varepsilon^{-2}Q_0\Phi_x^\varepsilon e^{\varphi/\varepsilon} + h^\varepsilon(x)\varphi(u)\\
&= Q\varphi_1 + \frac{1}{2}Q_0 B(x)[\varphi'(u)]^2 + h^\varepsilon(x)\varphi(u). \qquad \square
\end{aligned}$$

Now the solution of the singular perturbation problem [1, Ch.5] gives

$$H^\varepsilon\varphi^\varepsilon(u,x) = H\varphi(u) + h^\varepsilon(x)\varphi(u). \tag{36}$$

The asymptotic representation (36) completes the proof of Proposition 4.1. □

Remark 4.1. The exponential generator of small diffusion (35) in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, is represented as follows:

$$H\varphi(u) = \frac{1}{2}\varphi'^*(u)\widehat{C}\varphi'(u),$$

where $\varphi'^*(u) = (\varphi'_k(u), 1 \leq k \leq d)$ is a vector-row, $\varphi'(u) = (\varphi'_k(u), 1 \leq k \leq d)$ is a vector-column, $\widehat{C} = [\widehat{C}_{kr;1\leq k,r\leq d}]$ is the variance matrix.

4.3. **Large deviation for SIP under the local balance condition.** *The Local Balance Condition (LBC)* means that the average value of jumps is not equal to zero:

$$\widehat{a} := \int_E \rho(dx) a(x) \neq 0. \tag{37}$$

The SIP under LBC (37) is considered in the following scaling:

$$S^\varepsilon(t) = u + \varepsilon^2 \sum_{n=1}^{\nu(t/\varepsilon^3)} \alpha_n(x_n) - \widehat{a}_0 t/\varepsilon. \tag{38}$$

Lemma 4.2. *The coupled Markov process $S^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon^3)$, $t \geq 0$, is determined by the generator (compare* (22)*)*

$$L^\varepsilon \varphi(u,x) = \varepsilon^{-3} q(x) \int_E P(x,dy) \int_{\mathbb{R}^d} \Phi_y(dv) \left[\varphi(u+\varepsilon^2 v, y) - \varphi(u,x)\right] - \varepsilon^{-1} \widehat{a}_0 \varphi'_u(u,x).$$

Or, in a different form,

$$L^\varepsilon \varphi(u,x) = \left[\varepsilon^{-3}[Q + Q_0 \Phi_x^\varepsilon] - \varepsilon^{-1} \widehat{\mathbb{A}}_0\right] \varphi(u,x),$$
$$\Phi_x^\varepsilon \varphi(u) = \int_{\mathbb{R}^d} \Phi_x(dv) \left[\varphi(u+\varepsilon^2 v) - \varphi(u)\right].$$

Proposition 4.2. *The large deviation for SIP* (38) *under the LBC* (37) *is realized by the exponential generator of small diffusion*

$$H\varphi(u) = \frac{1}{2} \widehat{C} [\varphi'(u)]^2, \tag{39}$$

$$\widehat{C} = q[\widehat{B}_1 + \widehat{B}_2],$$
$$\widehat{B}_k = \int_E \rho(dx) B_k(x), \qquad k = 1,2, \tag{40}$$
$$B_1(x) = \int_{\mathbb{R}^d} v^* v \Phi_x(dv), \qquad B_2(x) = 2\widehat{a}(x) R_0 \widehat{a}(x),$$
$$\widehat{a}(x) = a_0(x) - \widehat{a}_0, \qquad a_0(x) := q(x) \int_E P(x,dy) a(x).$$

The exponential generator of large deviation (39)–(40) *contains two components. One of them is the variance matrix of the second moment of jumps. The second component $\widehat{B}_2$ is defined by the fluctuation of the first moment of jumps.*

Proof of Proposition 4.2. To prove the proposition we need the following lemma:

Lemma 4.3. *The exponential generator* (34) *under the local balance condition* (37) *on the perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln\left[1 + \varepsilon \varphi_1(u,x) + \varepsilon^2 \varphi_2(u,x)\right]$$

admits the asymptotic representation

$$\begin{aligned} H^\varepsilon \varphi^\varepsilon(u,x) &= \varepsilon^{-1} \left[Q\varphi_1 + \widetilde{\mathbb{A}}(x)\varphi(u)\right] \\ &\quad + \left[Q\varphi_2 - \varphi_1 Q \varphi_1 + \frac{1}{2} Q_0 B(x)[\varphi'(u)]^2\right] + h^\varepsilon(x)\varphi(u) \end{aligned} \tag{41}$$

with the negligible term

$$\|h^\varepsilon(x)\varphi(u)\| \to 0, \qquad \varepsilon \to 0, \; \varphi(u) \in C^3(\mathbb{R}).$$

Proof. Proof of Lemma 4.3 is based on the following asymptotic representations:

$$H_Q^\varepsilon \varphi^\varepsilon(u,x) := e^{-\varphi^\varepsilon/\varepsilon}\varepsilon^{-2}Qe^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1}Q\varphi_1 + [Q\varphi_2 - \varphi_1 Q\varphi_1] + h_q^\varepsilon(x)\varphi(u),$$

$$H_\varphi^\varepsilon \varphi^\varepsilon(u,x) := e^{-\varphi^\varepsilon/\varepsilon}\varepsilon^{-2}Q_0\Phi_x^\varepsilon e^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1}Q_0\mathbb{A}(x)\varphi(u) + Q_0\mathbb{A}(x)\varphi_1(u,x) + h_\varphi^\varepsilon(x)\varphi(u),$$

$$H_a^\varepsilon \varphi^\varepsilon(u,x) := e^{-\varphi^\varepsilon/\varepsilon}\widehat{\mathbb{A}}_0 e^{\varphi^\varepsilon/\varepsilon} = \varepsilon^{-1}\widehat{a}_0\varphi'(u) + h_a^\varepsilon(x)\varphi(u).$$

Thus, the relation

$$H^\varepsilon \varphi^\varepsilon(u,x) = [H_Q^\varepsilon + H_\varphi^\varepsilon - H_a^\varepsilon]\varphi^\varepsilon(u,x)$$

gives (41) with (see (24)–(25))

$$\widetilde{\mathbb{A}}(x)\varphi(u) := \widetilde{a}(x)\varphi'(u),$$
$$\widetilde{a}(x) := Q_0 a(x) - \widehat{a}_0. \qquad \square$$

Now the solution of the singular perturbation problem [1, Ch. 5] may be used for the equations

$$\begin{aligned} &Q\varphi_1 + \widetilde{A}(x)\varphi(u) = 0, \Pi\widetilde{A}(x) = 0; \\ &Q\varphi_2 - \varphi_1 Q\varphi_1 + \frac{1}{2}B_1(x)[\varphi'(u)]^2 = \widehat{H}\varphi(u). \end{aligned} \tag{42}$$

The first equation in (42) has the solution

$$\varphi_1(u,x) = R_0\widetilde{a}(x)\varphi'(u), \qquad Q\varphi_1 = \widetilde{a}(x)\varphi'(u).$$

Hence, the second equation in (42) may be rewritten as follows

$$Q\varphi_2 + \frac{1}{2}[B_1(x) + B_2(x)][\varphi'(u)]^2 = \widehat{H}\varphi(u)$$

with $B_2(x)$ given in (40).

The solvability condition [1, Ch. 5] for the last equation gives Proposition 4.2. $\square$

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