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THE LOGISTIC S.D.E.

We consider the logistic S.D.E which is obtained by addition of a diffusion coefficient of the type $\beta\sqrt{x}$ to the usual and deterministic Verhust-Volterra differential equation. We show that this S.D.E is the limit of a sequence of birth and death Markov chains. This permits to interpret the solution V_t as the size at time t of a self-controlled tumor which is submitted to a radiotherapy treatment. We mainly focus on the family of stopping times T_ε, where T_ε is the first hitting of level $\varepsilon > 0$ by (V_t). We calculate their Laplace transforms and also the first moment of T_ε. Finally we determine the asymptotic behavior of T_ε, as $\varepsilon \to 0$.

1. INTRODUCTION

1) The (deterministic) logistic equation:

$$(1.1) \qquad \begin{cases} \frac{dV(t)}{dt} = r\,V(t)\left(1 - \frac{V(t)}{\kappa}\right) - c\,V(t) \\ \\ V(0) = v > 0 \end{cases}$$

where the parameters r, c, v and κ are positive, was first introduced by Verhulst [40], then studied by Volterra [41] and more recently by a lot of authors, see for instance [33],[27],[41],[34],[15]. At the beginning, equation (1.1) has been introduced to model the time-evolution of the size $V(t)$ of a population. The goal was to add a limit to the exponential growth described by Malthus (1798) introducing both external constraints of the environment and the natural self limitation of the phenomenon.

The solution of (1.1) can be easily calculated, see Proposition 3.1 below. The simple and explicit formula for $V(t)$ permits to determine its limit behavior when time goes to infinity.

Afterwards, an extention to the Verhulst equation was considered by many authors, among them we can cite: [32],[39],[30],[23],[42],[19],[21],[18],[20],[29],[3],[2]. The stochastic generalization of (1.1) is called the *stochastic logistic equation* and takes the form:

$$(1.2) \qquad \begin{cases} dV_t = V_t\,(a - b\,V_t)\,dt + \beta\,V_t\,dB_t \\ \\ V_0 = v > 0 \end{cases}$$

where $(B_t)_{t \geq 0}$ a standard Brownian motion starting from zero and a, b, β are real parameters. Note that if $\beta = 0$, $a = r - c$ and $b = \dfrac{r}{\kappa}$ then (1.2) and (1.1) coincide. In other words, (1.2) is actually an extension of (1.1). Introducing noise to the initial deterministic model is reasonable to take into account uncertainty related to either the internal growth of the population or external influences as interaction with the other species and environmental factors. See for instance [28],[26], [24] for applications in the field of dynamics of population. We can also mention other extensions of (1.2), cf. [36],[37],[35].

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2) The Brownian motion (B_t) in (1.2) is a random perturbation which is arbitrary added to the O.D.E. (1.1). In particular the form of the diffusion coefficient is unexplained. One of our aims is to prove that the form of (1.2) appears naturally when we model the behavior of a tumor submitted to a treatment of either radiotherapy or phototherapy. Such treatment is applied to a population of cancer cells which evolve independently from each other. This aggregate of cells constitutes the tumor and its size corresponds to the number of all the elements. We have to take into account two antagonist forces. The first one is the natural duplication of cells. The second has two parts, the first one is the effect of the treatment and the second results from the self-limitation of the tumor (for instance a maximal possible volume).

We denote by $X_n(t)$ the size of the tumor at time t. Let $X_n(0)$ be the integer part of vn where $v > 0$. We suppose that $\big(X_n(t)\big)$ is a birth and death process, i.e. a Markov chain in continuous time valued in $\mathbb{N}$ and with jumps equal to ± 1. More precisely we propose (see Section 2.1 for details) the following infinitesimal behavior:

$$\mathbb{P}\big(X_n(t + \Delta t) = i + 1 \,|\, X_n(t) = i\big) \;\;=\;\; \Big[ri \; + \frac{\beta^2}{2} i^2\Big]\Delta t + o(\Delta t), \quad \forall\, i \in \mathbb{N}$$

$$\mathbb{P}\big(X_n(t + \Delta t) = i - 1 \,|\, X_n(t) = i\big) \;\;=\;\; \Big[ci \; + \big(\frac{\beta^2}{2} + \frac{r}{\kappa}\frac{1}{n}\big)i^2\Big]\Delta t + o(\Delta t), \quad \forall\, i \in \mathbb{N}^*.$$

One interesting feature of the above model is its small number of parameters and r, c and κ have biological significance: r is the intrinsic growth rate of the tumor, c is its rate of decay due to cancer treatment, κ is the carrying capacity of the environment. It can be proved, see Proposition 2.2, that the proportion of cancer cells $\big(\dfrac{X_n(t)}{n}\big)$ converges in distribution as $n \to \infty$ to the diffusion process V_t solution of (1.2) where $a = r - c$, $b = \dfrac{r}{\kappa}$. Therefore this approximation scheme based on biological considerations allows to consider V_t as the "limit size" at time t of a tumor submitted to a treatment. This approach is developed in detail in Section 2. The process V_t is a solution of the S.D.E. (1.2) and can be approximated by the Euler scheme. In [36], [13] the authors defined different possible approximation of V_t, leading to either an Itô integral or a Stratonovich one.

3) S.D.E. (1.2) which is valued in $[0, \infty[$ admits a unique and explicit solution, see [19], Theorem 2.2, page 167. We also study under which conditions V_t is either recurrent or transient. We prove that in some cases either V_t admits a unique and invariant probability measure (p.m.) or goes to 0 as $t \to \infty$. All these results are given in Propositions 3.3 and 3.6. We also study the moments of V_t, see Proposition 3.8. Then we interpret the above results when V_t models the size of a tumor at time t. This allows to have a better understanding of the role of the parameters. We partially recover the asymptotic dynamics of the Verhulst's function, cf (3.2), however new interesting effects appear due to the presence of the stochastic perturbation. Randomness introduces more flexibility than the deterministic setting.

4) In the rest of the paper, i.e. Section 4, we study the first hitting times of fixed levels by the logistic process (V_t). These random times have a clear biological interpretation, since they represent the first time when the size of the tumor reaches a given level $\epsilon > 0$. We can calculate the Laplace transform of the hitting time $T_\varepsilon := \inf\{t \geq 0, V_t = \varepsilon\}$.

Theorem 1.1. *The Laplace transform of the hitting time T_ε is given by:*

$$(1.3) \qquad\qquad \mathbb{E}_v\left[\exp(-\lambda\, T_\varepsilon)\right] = \Big(\frac{v}{\varepsilon}\Big)^{\sqrt{\frac{2\lambda}{\beta^2} + q^2} + q} \; \frac{\varphi(\lambda, v)}{\varphi(\lambda, \epsilon)}, \quad \lambda \geq 0$$

where

$$\varphi(\lambda, x) = \begin{cases} U\left(\sqrt{\dfrac{2\lambda}{\beta^2} + q^2} + q, 1 + 2\sqrt{\dfrac{2\lambda}{\beta^2} + q^2}; \dfrac{2bx}{\beta^2}\right) & \text{if } \varepsilon \leq v \\[3mm] M\left(\sqrt{\dfrac{2\lambda}{\beta^2} + q^2} + q, 1 + 2\sqrt{\dfrac{2\lambda}{\beta^2} + q^2}; \dfrac{2bx}{\beta^2}\right) & \text{if } \varepsilon \geq v \end{cases}$$

U is the Tricomi hypergeometric function and M is the Kummer function.

Definitions of functions U and M are given by (4.2), resp. (4.1). Theoretically, identity (1.3) gives the law of the random variable T_ε, but it is impossible to inverse this Laplace transform. However the explicit form of the Laplace transform of T_ε permits to get interesting consequences. We can prove that T_ε admits exponential moments, see Theorem 4.1. Taking the λ-derivative at 0 in (4.2) gives the value of the first order moment of T_ε, see Propositions 4.6 and 4.8. In particular if $a = \beta^2$ and $\varepsilon < V_0$, then:

$$\mathbb{E}[T_\varepsilon | V_0 = v] = \frac{1}{b}\left(\frac{1}{v} - \frac{1}{\varepsilon}\right).$$

If we go back to the interpretation of V_t as the size of a tumor at time t, it is crucial to know how fast the tumor can be reduced to a very small level ϵ. In other words, we would like to determine the asymptotic behavior of T_ε, as $\varepsilon \to 0$. The complete and explicit results are given in Theorem 4.3 and the three different regimes are governed by the values of the parameter $q := \dfrac{1}{2} - \dfrac{a}{\beta^2}$.

We have postponed in Section 5 all the technical proofs of results stated in Sections 2, 3 and 4.

2. An approximation of the logistic diffusion by continuous Markov chains

2.1. **Biological considerations.** Markov chains have been already associated with the logistic equation, see [39],[23], [30] and [38] but the convergence to the limit diffusion process has never been proven. Our setting is the one of modeling evolution of a tumor exposed to a treatment, for instance radiotherapy. Random models of tumors have been already considered, see for example [4] and [5]. We propose a very simple model in which a tumor is a collection of independent and non-interacting cells. The process $X_n(t)$ stands for the number of malignant cells at time t and we assume that $(X_n(t))_{t \geq 0}$ is a Markov chain in continuous time such that $X_n(0) = \lfloor vn \rfloor$ where $\lfloor vn \rfloor$ is the integer part of vn. We prove, see Proposition 2.2, the convergence of $X_n(t)/n$ as n tends to infinity to the solution of the logistic equation (1.2). This shows that the logistic S.D.E. admits a biological interpretation and can be considered itself as a pertinent model of tumor size.

Since $X_n(t)$ is a Markov chain, its behavior is determined by its infinitesimal dynamics. We suppose that it depends on three real functions α, γ and δ, defined on $\mathbb{R}_+$. The function α expresses the natural (intrinsic) growth of the tumor. The functions γ and δ will take into account the response of cancer cells to the treatment and the environment respectively. We detail how these two antagonist forces act.

- Each cell can duplicate: it dies and gives birth to two new cells. More precisely, a cell gives rise to two new cells between t and $t + \Delta t$ with probability: $\alpha(X_t)\,\Delta t$. However, from a mathematical point of view it is more convenient to consider this phenomenon as the addition of one cell to the population. Since all the cells evolve independently from each other, then the offspring distribution is the binomial law $\mathcal{B}(i; \alpha(i)\Delta t)$ where i stands for X_t, i.e.

$$(2.1) \qquad \mathbb{P}\big(X_n(t + \Delta t) = i + 1 \,|\, X_n(t) = i\big) = i\,\alpha(i)\,\Delta t + o(\Delta t), \quad \forall\, i \in \mathbb{N}.$$

- The death of a cancer cell is due to two factors. The first one comes from the limited size of the underlying environment, i.e. when the tumor reaches a "fixed" size, its growing is stopped. We suppose that each cell will die with probability $\delta(\frac{i}{n})$ between t and $t + \Delta t$. The second cause of death comes from the treatment applied to the tumor, each cell can die with probability $\gamma(i)$ in the time interval $[t, t + \Delta t]$. Finally, we can summarize death of malignant cells as

$$(2.2) \qquad \mathbb{P}\big(X_n(t + \Delta t) = i - 1 \,|\, X_n(t) = i\big) = i\left[\gamma(i) + \delta(\tfrac{i}{n})\right]\Delta t + o(\Delta t), \quad \forall\, i \in \mathbb{N}^*.$$

The size evolution of the tumor is then well captured by the process $\big(X_n(t)\big)$. However, to deal with the case $n \to \infty$, the right quantity to consider is the proportion $V_n(t)$ of cancer cells among the whole population, i.e.

$$(2.3) \qquad V_n(t) := \frac{1}{n}X_n(t), \quad t \geq 0.$$

It is clear that $\big(V_n(t)\big)$ is a continuous time Markov chain which takes its values in $\dfrac{\mathbb{N}}{n}$ and $V_n(0) = \dfrac{\lfloor vn \rfloor}{n}$.

2.2. Study of the Markov chain $V_n(t)$.

Recall that α, γ and δ are three non negative real functions defined on $\mathbb{R}_+$.

For all $x \in \mathbb{R}_+$, we set:

$$(2.4) \qquad q_n(x, y) := \begin{cases} nx\alpha(nx) & \text{if } y = x + \frac{1}{n} \\ nx\big(\gamma(nx) + \delta(x)\big) & \text{if } y = x - \frac{1}{n} \text{ and } x > 0 \\ -q_n(x, x + \frac{1}{n}) - q_n(x, x - \frac{1}{n}) & \text{if } y = x \text{ and } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q_n(x) := -q_n(x, x) = nx\alpha(nx) + nx\big(\gamma(nx) + \delta(x)\big).$$

It is well-known that a Markov chain in continuous time is characterized by its generator, i.e. its Q-matrix, see for instance in [31]. According to (2.1), (2.2) and (2.3), we adopt the following definition.

Definition 2.1. $\big(V_n(t), t \geq 0\big)$ *is the continuous time Markov chain which starts from* $\dfrac{\lfloor vn \rfloor}{n}$, *takes its values in* $\dfrac{\mathbb{N}}{n}$ *and with Q-matrix:* $Q_n := \big(q_n(x, y)\big)_{x,y \in \frac{\mathbb{N}}{n}}$.

We briefly recall the dynamics of the Markov chain $\big(V_n(t), t \geq 0\big)$. Let $(Y_n(k))_{k \in \mathbb{N}}$ be the skeleton associated with $V_n(t)$: $\big(Y_n(k)\big)$ is a Markov chain with values in the set $\left\{\frac{i}{n}; i \in \mathbb{N}\right\}$. Its transition matrix Π_n verifies

$$(2.5) \qquad \Pi_n(0, 0) + \Pi_n\left(0, \tfrac{1}{n}\right) = 1, \qquad \Pi_n\left(\tfrac{i}{n}, \tfrac{i+1}{n}\right) + \Pi_n\left(\tfrac{i}{n}, \tfrac{i-1}{n}\right) = 1 \qquad i \geq 1$$

and the above coefficients of matrix Π_n can be expressed in terms of the ones of matrix Q_n :

$$(2.6) \qquad \begin{cases} \mathbb{P}\big(Y_n(k+1) = x + \tfrac{1}{n} | Y_n(k) = x\big) = \Pi_n\left(x, x + \tfrac{1}{n}\right) = \dfrac{q_n\left(x, x + \tfrac{1}{n}\right)}{q_n(x)} \\[2mm] \mathbb{P}\big(Y_n(k+1) = x - \tfrac{1}{n} | Y_n(k) = x\big) = \Pi_n\left(x, x - \tfrac{1}{n}\right) = \dfrac{q_n\left(x, x - \tfrac{1}{n}\right)}{q_n(x)}. \end{cases}$$

Let $\xi_1, \xi_2, \cdots$ be a collection of i.i.d. random variables with exponential distribution, independent from the Markov chain $\big(Y_n(k)\big)$. Suppose that $V_n(0) = Y_n(0) = \frac{\lfloor vn \rfloor}{n} =: v_0$.

Then, $V_n(t)$ remains at level v_0 up to $T_1 := \dfrac{\xi_1}{q_n(v_0)}$. At that first jump time, V_n moves to $Y_n(1)$:

$$V_n(t) = v_0, \quad \forall t \in [0, T_1[, \quad V_n(T_1) = Y_n(1).$$

Set $Y_n(1) = v_1$. The next jump time for $(V_n(t))$ occurs at $T_2 := T_1 + \dfrac{\xi_2}{q_n(v_1)}$ and so on.

The process $(V_n(t))$ is a Markov process.

Proposition 2.2. *Let us consider α, β, γ functions defined by*

$$(2.7) \qquad \alpha(t) = \frac{\beta^2}{2} t + r, \quad \gamma(t) = \frac{\beta^2}{2} t + c, \quad \delta(t) = \frac{r}{\kappa} t$$

Then the sequence of Markov chains $((V_n(t))_{t \geq 0})_{n>0}$ converges in law to the process $(V_t)_{t \geq 0}$, solution of

$$(2.8) \qquad dV_t = V_t \left(r - c - \frac{r}{\kappa} V_t \right) dt + \beta\, V_t\, dB_t$$

with $V_0 = v$.

Remark 2.3. *The infinitesimal generator of $(V_n(t)\, ; t \geq 0)$, cf. [31], is for any $x \in \mathbb{R}_+$, and $f : \mathbb{R}_+ \to \mathbb{R}$ with support in $(0; \infty)$:*

$$\mathcal{L}_n f(x) = q_n(x) \left[\Pi_n \left(x, x - \tfrac{1}{n} \right) f \left(\left(x - \tfrac{1}{n} \right)_+ \right) + \Pi_n \left(x, x + \tfrac{1}{n} \right) f \left(x + \tfrac{1}{n} \right) - f(x) \right]$$

where the coefficients $\Pi_n \left(x, x - \tfrac{1}{n} \right)$ and $\Pi_n \left(x, x + \tfrac{1}{n} \right)$ are given by (2.6).

Under more general conditions than the ones of Proposition 2.2:

$$(2.9) \qquad \lim_{t \to \infty} \frac{\alpha(t)}{t} = \lim_{t \to \infty} \frac{\gamma(t)}{t} = \frac{\beta^2}{2} > 0$$

and

$$(2.10) \qquad \lim_{t \to \infty} (\alpha(t) - \gamma(t)) = c_0 \in \mathbb{R},$$

we can prove that

$$\lim_{n \to \infty} \mathcal{L}_n f(x) = \mathcal{L} f(x), \quad \forall\, x \geq 0.$$

Indeed, we can write

$$\mathcal{L}_n(f)(x) = \tau_n^1(x)\, f'(x) + \tau_n^2(x)\, \frac{f''(x)}{2}$$

with

$$(2.11) \qquad \tau_1(x) = x \left[\alpha(nx) - \gamma(nx) - \delta(x) \right],$$

$$(2.12) \qquad \tau_2(x) = x \left[x\, \frac{\alpha(nx)}{nx} + x\, \frac{\gamma(nx)}{nx} + \frac{\delta(x)}{n} \right].$$

To assure the convergence of $\mathcal{L}_n f(x)$ when n goes to infinity, it seems natural to ask that both $\tau_n^1(x)$ and $\tau_n^2(x)$ converge when n goes to infinity. According to (2.12), the convergence of $\tau_n^2(x)$ is established if

$$\lim_{t \to \infty} \frac{\alpha(t)}{t} = \frac{\beta_1^2}{2} \quad \text{and} \quad \lim_{t \to \infty} \frac{\gamma(t)}{t} = \frac{\beta_2^2}{2}, \quad \text{with } \beta_1, \beta_2 \geq 0.$$

However, if $\beta_1 \neq \beta_2$, then $\alpha(nx) - \gamma(nx) \sim \frac{\beta_1^2 - \beta_2^2}{2} n\, x$ and consequently $\tau_n^1(x)$ diverges as $n \to \infty$. Therefore we impose that $\beta_1 = \beta_2 = \beta$. Note that $\tau_n^1(x)$ converges if and only if (2.10) holds.

Under (2.7), conditions (2.9) and (2.10) are satisfied with $c_0 := r - c$. However the proof of Proposition 2.2 that is postponed in Section 5.1, does not use this result.

Remark 2.4. (1) The coefficient β of the diffusion (2.8) comes from the common rate of growth of $\alpha(i)$ and $\gamma(i)$ which is of the type $(\beta^2/2)i$.

(2) Proposition 2.2 gives a limit procedure for approximating the law of $(V_t)_{t\geq 0}$.

(3) It is convenient to introduce :

$$(2.13) \qquad a = r - c, \; b = \frac{r}{\kappa}.$$

Then, it is clear that (1.2) and (2.8) are equivalent.

(4) We can study SDE (1.2) for itself. However, we will keep here in memory that the diffusion V_t can model the size of a cancer tumor. In that case the parameters r, c, κ and v are biologically significant and permit to interpret the results.

Four parameters influence the evolution of the tumor size:
- the initial size $v > 0$;
- the intrinsic growth rate r of the tumor;
- the rate of decrease c of the tumor due to the treatment;
- the carrying capacity of the environment κ which is the maximal size the environment can support without negative impact. This capacity could vary according to time and context. It is nevertheless difficult to take into account these fluctuations and we restrict ourselves to κ being a constant. This parameter is actually a fictive upper bound since we will see in Section 3 below that the diffusion can exceed κ but with a "small" probability.

3. Study of the logistic diffusion

We now focus on equation (1.2). Relations (2.13) imply that equations (1.1) and (3.1) are equivalent.

The case without noise corresponds to $\beta = 0$. Then equation (1.2) reduces to the Verhulst equation:

$$(3.1) \qquad \begin{cases} dV_t = V_t \, (a - b\,V_t)\, dt, \\[2mm] V_0 = v > 0. \end{cases}$$

(3.1) is a classical ordinary differential equation which can be easily solved.

Proposition 3.1. Let a, b and v be real numbers such that $b, v > 0$.

1. Equation (3.1) has a unique solution given by:

$$(3.2) \qquad V(t) = \begin{cases} \dfrac{a\,v\,e^{at}}{a + bv\,(e^{at} - 1)} & \text{if } a \neq 0, \\[3mm] \dfrac{v}{1 + bvt} & \text{if } a = 0. \end{cases}$$

2. The function V takes its values in $(0, \infty)$. If $a > bv$ (resp. $a < bv$, $a = bv$) then V is an increasing (resp. decreasing, constant) function. Moreover $\lim\limits_{t\to\infty} V(t)$ equals a/b (resp 0) when $a > 0$ (resp. $a \leq 0$).

Remark 3.2. It is interesting to interpret item 2 of the above proposition when $V(t)$ represents the size of tumor at time t. This actually means that a and b are expressed in terms of r, c and κ via (2.13) whose interpretation is given by Remark 2.4. First, observe that Proposition 2.2 holds when $\beta = 0$. In that case the sequence of Markov chains $(V_n)_{n\in\mathbb{N}}$ converges in distribution to the continuous function V given by (3.2). Moreover, relations (2.7) reduce to:

$$\alpha(i) = r, \quad \gamma(i) = c, \quad \delta(i) = \frac{r}{\kappa}i.$$

Note that $a > bv$ is equivalent to $c < r\left(1 - \dfrac{V_0}{\kappa}\right)$. This condition means that the level of the treatment is low and cannot compensate the natural duplication of tumor cells. This is coherent since in that case it is expected that the tumor grows.

Note that, in any case, the limit size of the tumor is less than κ since:

$$(3.3) \qquad \lim_{t \to \infty} V(t) = \frac{a}{b} = \left(1 - \frac{c}{r}\right)\kappa < \kappa.$$

We introduce a parameter q which governs the behavior of the diffusion (see Proposition 3.3 below):

$$(3.4) \qquad q = \frac{1}{2} - \frac{a}{\beta^2}.$$

Note that:

$$q \leq 0 \qquad \Leftrightarrow \qquad a \geq \frac{\beta^2}{2}.$$

$\gamma(u, v)$ denotes the Gamma-distribution with shape parameter $u > 0$ and scale parameter $v > 0$, i.e. the law with density :

$$\gamma(u, v)(x) := \frac{1}{\Gamma(u)\, v^u}\, x^{u-1}\, e^{-\frac{x}{v}}\, \mathbf{1}_{(0;\infty)}(x).$$

As usual, $\gamma(u)$ stands for $\gamma(u, 1)$.

Proposition 3.3. *(1) Equation (1.2) admits a unique positive solution V and*

$$(3.5) \qquad V_t = \frac{\exp\left\{\beta B_t + (a - \frac{\beta^2}{2})t\right\}}{\dfrac{1}{v} + b \displaystyle\int_0^t \exp\left\{\beta B_s + (a - \frac{\beta^2}{2})s\right\}ds}, \quad t \geq 0.$$

(2) The diffusion V is recurrent if and only if $q \leq 0$.

(3) If $q < 0$, the diffusion V converges in law towards the unique stationary probability distribution $\gamma(-2q, \frac{\beta^2}{2b})$.

(4) If $q > 0$, the diffusion goes a.s. to zero when time goes to infinity.

Remark 3.4. *The comparison theorem related to one dimensional diffusions (see for instance [22] Prop 2.18 p. 293) can be applied in our context. Let us consider diffusions V_t and V'_t being the respective solutions of S.D.E. (1.2) with respective drift coefficients a, a':*

$$dV_t = (aV_t - bV_t^2)dt + \beta V_t dB_t\,;\, V_0 = v > 0$$
$$dV'_t = (a'V'_t - bV_t'^2)dt + \beta V'_t dB_t\,;\, V'_0 = v > 0$$

such that $a \leq a'$. Then $\mathbb{P}(V_t \leq V'_t\,;\, \forall\, t \geq 0) = 1$.

Remark 3.5. *Let us show that, in the setting of tumor, the behavior of $V(t)$ is more complex and therefore reacher than the one of the solution of the Verhulst equation. In particular, we have a better understanding of the role of the carrying capacity κ.*

(1) The condition $q < 0$ is equivalent to $r > c + \dfrac{\beta^2}{2}$ and means that the self-replication force is greater than the joint effect of noise and the effect of the treatment. In that case, the size of the tumor evolves after a long period of time to a random state which is gamma distributed. Moreover, by (2.13), the limit size V_∞ can be written in the following form: $V_\infty = \left(\dfrac{\beta^2}{2r}\, Z\right)\kappa$ with $Z \sim \gamma(-2q)$ and the law of Z does not depend on κ. Using moreover (2.13), we get:

$$\mathbb{E}(V_\infty) = \frac{\beta^2}{2r}\, (-2q)\kappa = \left(1 - \frac{c + \frac{\beta^2}{2}}{r}\right)\kappa < \kappa.$$

Note that we recover (3.3) *if* $\beta = 0$.

Then, in average, the limit size of the tumor is less that κ. *However it can exceed this threshold but the probability that the tumor size is beyong* κ *is equal to* $\mathbb{P}\left(Z > \frac{2r}{\beta^2}\right) > 0$ *and does not depend on* κ. *However, this quantity is really small if the intrinsic growth rate* r *is large.*

(2) *Otherwise, if* $q > 0$, *there is resorption of the tumor, as in the case when* $\beta = 0$, *see Proposition 3.1.*

(3) *It is clear from* (1.2) *that* $V_0(t) := E\big(V(t)\big)$ *does not solve the Verhulst equation. However, we will show, see Remark 3.9 below that the solution of* (1.2) *converges to the deterministic logistic equation, when the intensity* β *of noise goes to* 0.

Since the coefficients of equation (1.2) are locally Lipschitz, strong uniqueness holds (see for instance page 287 of [22]).

The stochastic logistic equation can be solved explicitly in a more general context than ours. Indeed, replacing the constants a, b and β in (1.2) by functions leads to:

$$(3.6) \qquad \begin{cases} dV_t = V_t \left(a(t) - b(t) V_t\right) dt + \beta(t) V_t \, dB_t \\ \\ V_0 = v > 0, \ t \in [0, T]. \end{cases}$$

Under additional assumptions an explicit solution of (3.6) has been given in [19], Theorem 2.2, page 167 which takes the form (3.5) in our setting.

The existence of the invariant probability measure has been proved in [42] *or* in [3]. As for the asymptotic behavior when $q < 0$, it follows from Proposition 3.6 below, see Subsection 5.2 for details.

We complete the above proposition by giving the behavior of the diffusion near the two boundary points 0 and ∞. The classification of the boundary points of a diffusion can be found for instance either in [17] p. 108 or in [7] or in [22]. Let us introduce:

$$(3.7) \qquad s(x) = \begin{cases} \displaystyle\int_1^x y^{2q-1} \exp\left\{\frac{2}{\beta^2} b(y-1)\right\} dy & \text{if } x > 1, \\ \\ -\displaystyle\int_x^1 y^{2q-1} \exp\left\{\frac{2}{\beta^2} b(y-1)\right\} dy & \text{if } 0 < x < 1, \end{cases}$$

and

$$(3.8) \qquad m(dx) = \frac{2}{\beta^2} x^{-2q-1} e^{-\frac{2}{\beta^2} b \, (x-1)} 1_{\{x>0\}} dx.$$

Proposition 3.6. (1) *The point* 0 *is neither an exit nor a starting point, i.e. the process* V *cannot start from* 0 *and neither visits* 0 *in finite time.*

(2) *The scale function and speed measure of* V *are* s *and* m *respectively.*

(3) *Let* T_ε *be the hitting time of level* ε, $T_\varepsilon := \inf\{t \geq 0, V_t = \varepsilon\}$.

  (a) *For any* $\varepsilon \leq v$, $\mathbb{P}(T_\varepsilon < +\infty | V_0 = v) = 1$.

  (b) *When* $\varepsilon \geq v$,

$$\mathbb{P}(T_\varepsilon < +\infty | V_0 = v) = \begin{cases} 1 & \text{if } q \leq 0 \\ \dfrac{s(v) - s(0^+)}{s(\varepsilon) - s(0^+)} & \text{if } q > 0. \end{cases}$$

  Note that in the case $q > 0$,

$$(3.9) \qquad s(v) - s(0^+) = e^{-\frac{2b}{\beta^2}} \int_0^v y^{2q-1} \exp\left\{\frac{2b}{\beta^2} y\right\} dy = \frac{e^{-\rho}}{2q} v^{2q} M(2q, 1+2q, \rho v)$$

  where $\rho := \dfrac{2b}{\beta^2}$ *and* M *is the Tricomi function defined by* (4.1).

The proof of Proposition 3.6 is based on the explicit calculation of the scale function and speed measure of V. Using standard analysis we can verify the given criterium related to the boundary points, see Section 5.2 for details.

Remark 3.7. *Suppose that $q > 0$ and $\varepsilon \geq v$. Let $(\overline{V}_t)$ be the one-sided maximum of (V_t), i.e. $\overline{V}_t := \max\limits_{0 \leq u \leq t} V_u$. Taking the limit $t \to \infty$ in the identity $\mathbb{P}(T_\varepsilon > t | V_0 = v) = \mathbb{P}(\overline{V}_t < \varepsilon | V_0 = v)$ and using item 3 (b) of Proposition 3.6 lead to:*

$$\mathbb{P}(\overline{V}_\infty < \varepsilon | V_0 = v) = \mathbb{P}(T_\varepsilon = \infty | V_0 = v)) = \frac{s(\varepsilon) - s(v)}{s(\varepsilon) - s(0^+)}.$$

Therefore relation (3.7) implies that the distribution function of $\overline{V}_\infty$ is determined.

We now study the integrability of V_t.

Proposition 3.8. *Let V be the solution of equation (1.2).*
1. *For any $t > 0$ and $k \geq 2$, the random variable V_t^k is integrable.*
2. *Set $m_k(t) := \mathbb{E}\left(V_t^k\right)$. Then, the sequence of functions $\left(m_k(t)\right)_{k \geq 1}$ satisfies the following recursive equations:*

$$(3.10) \quad m_k(t) = v^k + k\left(a + \frac{k-1}{2}\beta^2\right) \int_0^t m_k(s)ds - kb \int_0^t m_{k+1}(s)ds, \quad k \geq 1, t \geq 0.$$

3. *For any $k \geq 1$ and $t \geq 0$,*

$$(3.11) \quad m_k(t) \leq \max\left\{v^k, \left(\frac{2a + (k-1)\beta^2}{2b}\right)^k\right\}.$$

4. *Let V^0 be the solution of the Verhulst equation (3.1) with initial condition $V^0(0) = V_0$. Then,*

$$(3.12) \quad \mathbb{E}\left(V_t\right) < V^0(t), \quad \forall\, t > 0.$$

The proof is a consequence of the Itô formula, see Section 5.3.

Remark 3.9. (1) *Let $(V_t^\beta)_{t \geq 0}$ be the process defined by (3.5). It is clear that V_t^β converges a.s. and uniformly on compact sets as $\beta \to 0$ to the Verhulst solution V_t given by (3.2). Consequently any solution of (1.2) converges as β goes to 0 to the deterministic logistic equation.*

(2) *Suppose that $q < 0$. Thanks to the explicit form of the solution (3.5), we can recover the fact that V_t converges in distribution towards $\gamma\left(-2q, \frac{\beta^2}{2b}\right)$, as $t \to \infty$. Indeed, we modify (3.5) as:*

$$\begin{aligned}
V_t &= \left[\frac{e^{-(a-\frac{\beta^2}{2})\,t - \beta\,B_t}}{v} + b\int_0^t e^{-(a-\frac{\beta^2}{2})\,(t-s) - \beta\,(B_t - B_s)}ds\right]^{-1} \\
&= \left[\frac{e^{-(a-\frac{\beta^2}{2})\,t - \beta\,B_t}}{v} + b\int_0^t e^{-(a-\frac{\beta^2}{2})\,u - \beta\,(B_t - B_{t-u})}du\right]^{-1}.
\end{aligned}$$

For any fixed $t > 0$, the process $(B_t, (B_t - B_{t-u})_{0 \leq u \leq t})$ is distributed as the process $(B_t, (B_u)_{0 \leq u \leq t})$. Consequently:

$$(3.13) \quad V_t \stackrel{(d)}{=} \left[\frac{e^{-(a-\frac{\beta^2}{2})\,t - \beta\,B_t}}{v} + b\int_0^t e^{-(a-\frac{\beta^2}{2})\,u - \beta\,B_u}du\right]^{-1}.$$

Note that $q < 0$ implies that $a > \frac{\beta^2}{2}$. Then, we deduce :

$$\lim_{t \to \infty} e^{-(a-\frac{\beta^2}{2})\,t - \beta\,B_t} = 0, \quad a.s.$$

From [9], we know that we have the following identity in law:

$$\int_0^\infty e^{\mu_1 B_s - \mu_2 s}\, ds \stackrel{(d)}{=} \frac{2}{\mu_1^2\, Z}, \qquad \mu_2 > 0$$

where Z is a $\gamma\left(\frac{2\mu_2}{\mu_1^2}\right)$ distributed random variable.

Finally, taking the limit $t \to \infty$ in (3.13), we deduce that V_t converges in distribution to $V_\infty \sim \gamma\left(-2q, \frac{\beta^2}{2b}\right)$.

4. Hitting times

Recall $(V_t)_{t\geq 0}$ is the process solving (1.2). We adopt the classical notation: under the p.m. $\mathbb{P}_v$ the logistic process starts at level v and $\mathbb{E}_v$ stands for the related expectation. For any $\varepsilon > 0$, we denote by T_ε the first passage time at level ε of $(V_t)_{t\geq 0}$:

$$T_\varepsilon := \inf\{t \geq 0,\ V_t = \varepsilon\}$$

with the convention $\inf \emptyset = +\infty$.

4.1. Calculation of the Laplace transform of T_ε and asymptotic distributions.
We introduce the hypergeometric confluent functions of the first and second type M and U which will play a central role in our study. We begin with

$$(4.1) \qquad M(a,b,t) = {}_1F_1(a;b;t) = \sum_{n=0}^\infty \frac{(a)_n t^n}{(b)_n n!}, \quad a, t \in \mathbb{C}, b \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ when $n \geq 1$.

${}_1F_1(a;b;t)$ is known as the hypergeometric confluent function (see [1] page 504). The second function of interest is the Tricomi function which is given as
$$(4.2)$$
$$U(a,b,t) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a,b,t) + \frac{\Gamma(b-1)}{\Gamma(a)} t^{1-b} M(a-b+1, 2-b, t), \quad a, t \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}.$$

Function U can be extended by continuity when b is an integer, see Step 1 of Section 5.4.

Functions M and U are also denoted respectively Φ and Ψ in the literature, see Sections 9.9 and 9.10 in [25].

Recall that, from (3.4), we have $q = \frac{1}{2} - \frac{a}{\beta^2}$. It is convenient to introduce:

$$(4.3) \qquad\qquad\qquad \rho = \frac{2b}{\beta^2}$$

and

$$(4.4) \qquad\qquad\qquad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} \text{ for all } \lambda \geq -\frac{\beta^2 q^2}{2}.$$

In the calculus of the Laplace transform of T_ε, we discriminate $\varepsilon \leq v$ and $\varepsilon \geq v$.

Theorem 4.1. *(1) Let $v, \varepsilon > 0$. The Laplace transform of T_ε is given by:*
$$(4.5)$$
$$\mathbb{E}_v\left[\exp(-\lambda T_\varepsilon)\, 1_{\{T_\varepsilon < \infty\}}\right] = \left(\frac{v}{\varepsilon}\right)^{\sqrt{\frac{2\lambda}{\beta^2}+q^2}+q} \frac{F\left(\sqrt{\frac{2\lambda}{\beta^2}+q^2}+q, 1+2\sqrt{\frac{2\lambda}{\beta^2}+q^2}, \rho v\right)}{F\left(\sqrt{\frac{2\lambda}{\beta^2}+q^2}+q, 1+2\sqrt{\frac{2\lambda}{\beta^2}+q^2}, \rho \varepsilon\right)}$$

where $\lambda \geq 0$ and

$$F := \begin{cases} U & \text{if } v \geq \varepsilon \\ M & \text{if } v \leq \varepsilon. \end{cases}$$

(2) *There exists a negative real number $\sigma_c(\varepsilon, v)$ such that:*

$$\mathbb{E}_v \left[\exp(\lambda\, T_\varepsilon)\, 1_{\{T_\varepsilon < \infty\}} \right] < \infty, \quad \forall\, \lambda < -\sigma_c(\varepsilon, v).$$

Remark 4.2.   *(1) Recall that according to Proposition 3.6, $\mathbb{P}_v(T_\varepsilon < \infty) = 1$ (resp. $\mathbb{P}_v(T_\varepsilon < \infty) < 1$) if $\varepsilon \leq v$ (resp. $\varepsilon > v$). Taking the limit $\lambda \to 0$ in (4.5) we recover the results given in Proposition 3.6.*
(2) *We discuss in Remarks 5.8 and 5.9 the maximal value of $\lambda > 0$ so that $\mathbb{E}_v \left[\exp(\lambda\, T_\varepsilon)\, 1_{\{T_\varepsilon < \infty\}} \right]$ is finite.*
(3) *By the strong Markov property:*

$$u \leq v \leq w, \quad \mathbb{E}_w \left[e^{-\lambda T_u} \right] = \mathbb{E}_w \left[e^{-\lambda T_v} \right]\, \mathbb{E}_v \left[e^{-\lambda T_u} \right].$$

As a consequence, if the left-hand side is finite for $\lambda < 0$, the two factors of the right-hand side are finite for this same λ. It implies that the abscissa of convergence $\sigma_c(\varepsilon, v)$ (introduced in [43]) of the Laplace transform $\lambda \mapsto \mathbb{E}_v \left[\exp(-\lambda\, T_\varepsilon) \right]$ verifies:

$$0 < v \leq w < \varepsilon \Rightarrow \sigma_c(\varepsilon, v) \geq \sigma_c(\varepsilon, w), \quad 0 < \varepsilon < v \leq w \Rightarrow \sigma_c(\varepsilon, v) \leq \sigma_c(\varepsilon, w).$$

Theoretically, the law of T_ε is completely determined by the knowledge of its Laplace transform given in Theorem 4.1. However we cannot go further, i.e. calculate the density function of this random variable. This leads naturally to investigate the asymptotic behavior of T_ε as $\varepsilon \to 0$. From Proposition 3.6 the random variable T_ε is finite if $\varepsilon \leq v$ and it is clear that $\varepsilon \mapsto T_\varepsilon$ is decreasing and goes to infinity as $\varepsilon \to 0$. We can actually determine the rate of convergence.

Theorem 4.3. *Recall that the parameters ρ and q have been defined by (4.3) and (3.4) respectively.*

(1) *When $q > 0$, then $\sqrt{-\ln \varepsilon} \left(\frac{T_\varepsilon}{-\ln \varepsilon} - \frac{1}{\beta^2 q} \right)$ converges in law to $\frac{Z}{\beta^2 \sqrt{q^3}}$ where Z is a standard normal r.v., as $\varepsilon \to 0$.*

(2) *When $q < 0$, $\varepsilon^{2|q|}\, T_\varepsilon$ converges in law, when ε goes to 0, to $\frac{\rho^{2q}\Gamma(2|q|)}{\beta^2 |q|}\xi$ where ξ is an exponential random variable.*

(3) *When $q = 0$, then $\frac{T_\varepsilon}{(\ln \varepsilon)^2}$ converges in law, when ε goes to 0, to the hitting time of $\frac{1}{\sqrt{2a}}$ by a reflected Brownian motion starting at 0.*

Remark 4.4. *The asymptotic behavior of T_ε as $\varepsilon \to 0$ also holds in mean. Indeed, we deduce directly from Proposition 4.8 below:*

$$\mathbb{E}_v(T_\varepsilon) \sim \begin{cases} \dfrac{1}{\beta^2 q} \ln(1/\varepsilon) & \text{when } q > 0 \\ \dfrac{\rho^{2q}\Gamma(2|q|)}{\beta^2 |q|}\varepsilon^{2q} & \text{when } q < 0 \\ \dfrac{1}{2a}(\ln \varepsilon)^2 & \text{when } q = 0. \end{cases}$$

Remark 4.5.   *(1) According to Proposition 3.3, we know that in the case where $q > 0$, then V_t goes to 0 as $t \to \infty$. One way to measure how fast V_t goes to 0 is to determine the rate of convergence of T_ε as $\varepsilon \to 0$. Indeed, we can easily prove that item 1 of Theorem 4.3 implies that $\dfrac{T_\varepsilon}{-\ln \varepsilon} \to \dfrac{1}{\beta^2 q}$ in probability. This*

is equivalent to say $\lim\limits_{t\to\infty}\dfrac{\ln \underline{V}_t}{t} = -\beta^2 q$ in the probability sense, where $\underline{V}_t :=$ $\min\limits_{0\le u\le t} V_u$.

(2) When $q < 0$, it is easy to deduce from item 2 of Theorem 4.3 that $t\big(\underline{V}_t^{-2q}\big)$ converges in distribution to an exponential random variable with rate parameter $\frac{\beta^2 |q|}{\Gamma(2|q|)}\big(\frac{\beta^2}{2b}\big)^{2q}$, as $t \to \infty$.

(3) When $q = 0$, then item 3 of Theorem 4.3 implies that $\dfrac{1}{\sqrt{t}} \ln\big(1/\underline{V}_t\big)$ converges in distribution to $\dfrac{1}{\sqrt{T^*}}$, where $T^* := \inf\big\{t \ge 0, |B_t| = \dfrac{1}{\sqrt{2a}}\big\}$. Recall that its Laplace transform equals $\dfrac{1}{\cosh\left(\sqrt{\frac{\lambda}{a}}\right)}$ and its expectation is $\dfrac{1}{2a}$.

4.2. **First moment of the passage time.** We distinguish two cases according to the respective positions of the starting point v and the level ε. We begin with $v < \varepsilon$. Recall that according to (4.3), $\rho = \dfrac{2b}{\beta^2}$. We introduce two functions involved in the following development of function M,

(4.6) $\quad M(\alpha + x, 1 + \alpha + 2\,x\ ;\ y) = \hat{b}_\alpha(y) + b_\alpha(y)\,x + \mathcal{O}(x^2)$ where $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \dots\}$

$$\hat{b}_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n}\frac{x^n}{n!} = M(\alpha, \alpha + 1, x)$$

$$b_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + n}\left(\psi(1+\alpha) - \psi(1+\alpha+n) + \frac{1}{\alpha} - \frac{1}{\alpha+n}\right)\frac{x^n}{n!}$$

where $x \in \mathbb{R}$, $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$ and $\psi = \frac{\Gamma'}{\Gamma}$ is the digamma function.

Proposition 4.6. *Suppose $v < \varepsilon$.*

(1) *If $q > 0$,*

(4.7) $$\mathbb{P}_v(T_\varepsilon < \infty) = \left(\frac{v}{\varepsilon}\right)^{2q}\frac{\hat{b}_{2q}(\rho\,v)}{\hat{b}_{2q}(\rho\,\varepsilon)}.$$

(2) *As for the expectation of T_ε, we have in the case $q \ge 0$:*

(4.8) $$\mathbb{E}_v[T_\varepsilon \mathbf{1}_{\{T_\varepsilon < \infty\}}] = \frac{1}{\beta^2\,q}\left(\ln\left(\frac{\varepsilon}{v}\right) + \frac{b_{2q}(\rho\,\varepsilon)}{\hat{b}_{2q}(\rho\,\varepsilon)} - \frac{b_{2q}(\rho\,v)}{\hat{b}_{2q}(\rho\,v)}\right)$$

and otherwise ($q < 0$):

(4.9) $$\mathbb{E}_v[T_\varepsilon] = \frac{1}{\beta^2\,|q|}\left[\ln\left(\frac{\varepsilon}{v}\right) + \sum_{n=1}^{\infty}\frac{1}{(1-2q)_n}\frac{(\rho\varepsilon)^n}{n} - \sum_{n=1}^{\infty}\frac{1}{(1-2q)_n}\frac{(\rho v)^n}{n}\right].$$

Remark 4.7. (1) *When $v < \varepsilon$ and $q > 0$, we have already given, in item 3 b) of Proposition 3.6, the value of $\mathbb{P}_v(T_\varepsilon < \infty)$. Using identity (3.9), we can easily prove that probability equals the right hand-side of (4.7).*

(2) *Recall that according to Proposition 3.6, $\mathbb{P}_v(T_\varepsilon < \infty) = 1$ for any $q \le 0$.*

Proposition 4.8. *When $v > \epsilon$, we obtain the first moment of the hitting time to ε starting from v, according to the value of the parameter q:*

(4.10) $$\mathbb{E}_v[T_\varepsilon] = f_q(v) - f_q(\varepsilon)$$

where
(i) If $2q \in]0, \infty[\backslash \{1, 2, \dots\}$ then

$$(4.11) \quad f_q(x) = \frac{1}{\beta^2 \, q} \left[\ln x + \sum_{n=1}^{\infty} \frac{1}{n \, (1 - 2q)_n} (\rho x)^n + \Gamma(-2q) \sum_{n=0}^{\infty} \frac{2q}{n! \, (2q + n)} (\rho x)^{2q+n} \right].$$

(ii) If $2q = m \in \{1, 2, \dots\}$ then

$$\begin{aligned} f_q(x) \quad = \quad & \frac{2}{\beta^2 \, m} \left[\ln x + \frac{1}{(m-1)!} \sum_{n=1}^{m-1} \frac{(-1)^n (m - n - 1)!}{n} (\rho \, x)^n \right. \\ (4.12) \qquad & + \frac{(-1)^m}{(m-1)!} \sum_{n \geq 0} \frac{1}{(n+m)n!} \left(\psi(n+1) - \ln(\rho \, x) + \frac{1}{n+m} \right) (\rho \, x)^{n+m} \right]. \end{aligned}$$

with ψ the digamma function.
(iii) If $2q = 0$ then

$$\begin{aligned} f_q(x) \quad = \quad & \frac{2}{\beta^2} \left[-(\gamma + \ln \rho) \ln x - \frac{1}{2} (\ln x)^2 + \sum_{n=1}^{\infty} \left(-H_n - \frac{1}{n} + \gamma + \ln \rho \right) \frac{(\rho x)^n}{n! \, n} \right] \\ (4.13) \qquad & - \frac{2}{\beta^2} \ln x \sum_{n=1}^{\infty} \frac{(\rho \, x)^n}{n! \, n}. \end{aligned}$$

with γ is the Euler constant and $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is the harmonic series.
(iv) If $2q \in]-\infty, 0[\backslash \{-1, -2, \dots\}$ then

$$(4.14) \qquad f_q(x) = \frac{1}{\beta^2 \, |q|} \left[\ln x + \sum_{n=1}^{\infty} \frac{1}{n \, (1 - 2q)_n} (\rho x)^n + \Gamma(-2q)(\rho x)^{2q} \hat{b}_{2q}(\rho x) \right].$$

(v) If $2q = -m \in \{-1, -2, \dots\}$ then

$$(4.15) \qquad\qquad f_q(x) = \frac{2}{\beta^2} (m - 1)! \sum_{l=1}^{m} \frac{1}{l} \frac{(\rho x)^{-l}}{(m - l)!}.$$

Remark 4.9.   *(1) It is actually possible to prove that for any $x > 0$, the map $q \to f_q(x)$ is continuous, see Section 5.7, step (ii).*
  (2) The function f_q takes a simple form when $2q \in \{1, 2, \dots\}$; more specifically when $q = 1/2$, i.e. $m = 1$, we have

$$f_q(x) = \frac{2}{\beta^2} \left[\ln x - \sum_{n=1}^{\infty} \frac{(\rho \, x)^n}{n!} \left(\psi(n) - \ln(\rho \, x) + \frac{1}{n} \right) \right].$$

Using $\psi(n) + \frac{1}{n} = H_n - \gamma$ in the above identity and (4.10) leads to

$$\begin{aligned} \mathbb{E}_v[T_\varepsilon] \quad = \quad & \frac{2}{\beta^2} \left[\ln \left(\frac{v}{\varepsilon} \right) \right. \\ & - \sum_{n=1}^{\infty} \frac{(\rho \, v)^n}{n!} (H_n - \gamma - \ln(\rho \, v)) \\ & + \left. \sum_{n=1}^{\infty} \frac{(\rho \, \varepsilon)^n}{n!} (H_n - \gamma - \ln(\rho \, \varepsilon)) \right]. \end{aligned}$$

But $H_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}$ implies $\sum_{n=1}^{\infty} \frac{z^n}{n!} H_n = \exp(z) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k! \, k} z^k$, and thus

$$\mathbb{E}_v[T_\varepsilon] = \frac{2\exp(\rho\,v)}{\beta^2}\Big[\gamma + \ln(v) - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!\,k}(\rho\,v)^k\Big]$$
$$+\frac{2\exp(\rho\,\varepsilon)}{\beta^2}\Big[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!\,k}(\rho\,\varepsilon)^k - \gamma - \ln(\varepsilon)\Big],$$

$$\mathbb{E}_v[T_\varepsilon] = \frac{2\exp(\rho\,v)}{\beta^2}\Big[\gamma + \ln(v) + \int_0^{\rho\,v}\frac{e^{-t}-1}{t}\,dt\Big]$$
$$+\frac{2\exp(\rho\,\varepsilon)}{\beta^2}\Big[\int_0^{\rho\,\varepsilon}\frac{1-e^{-t}}{t}\,dt - \gamma - \ln(\varepsilon)\Big].$$

5. Proofs

5.1. **Proof of Proposition 2.2.**

Proof. We use Theorem 4.1, chapter 7 in [11] to prove the convergence in law of the processes $(V_n(t))_{t\geq 0}$ to the process $(V_t)_{t\geq 0}$, solution of the SDE (2.8).

We cannot apply directly Theorem 4.1 because the coefficient of the diffusion of V, $\frac{\beta^2}{2}x^2$, vanishes at 0. We actually prove the convergence of the processes $Z_n(t) := \ln\left(\frac{1}{n} + V_n(t)\right)$ to the process $(\ln(V_t))_{t\geq 0}$.

The process $Z_n(t)$ takes its values in the set $\{\ln k - \ln n; k \in \mathbb{N}^*\}$ and its infinitesimal generator satisfies for all $x \in \{\ln k - \ln n; k \in \mathbb{N}^*\}$:

$$\mathcal{L}^{Z_n} f(x) = q_n\left(e^x - \tfrac{1}{n}\right)\Big[\Pi_n\left(e^x - \tfrac{1}{n}, e^x - \tfrac{2}{n}\right) f\left(\ln\left(e^x - \tfrac{1}{n}\right)\right)$$
$$+\Pi_n\left(e^x - \tfrac{1}{n}, e^x\right) f\left(\ln\left(e^x + \tfrac{1}{n}\right)\right) - f(x)\Big].$$

Under (2.7), we obtain:

$$\mathcal{L}^{Z_n} f(x) = (ne^x - 1)\left(\tfrac{\beta^2}{2}(ne^x - 1) + c + \tfrac{r}{\kappa}\left(e^x - \tfrac{1}{n}\right)\right)\left(f\left(\ln\left(e^x - \tfrac{1}{n}\right)\right) - f(x)\right)$$
(5.1)
$$+ (ne^x - 1)\left(\tfrac{\beta^2}{2}(ne^x - 1) + r\right)\left(f\left(\ln\left(e^x + \tfrac{1}{n}\right)\right) - f(x)\right).$$

Using Itô's rule and considering SDE (2.8), we introduce the diffusion $X_t := lnV_t$, with $(V_t)_{t\geq 0}$ solution of (1.2) with $a = r - c$ and $b = \frac{r}{\kappa}$; $(X_t)_{t\geq 0}$ is solution of the following SDE

(5.2)
$$dX_t = \left(r - c - \frac{\beta^2}{2} - \frac{r}{\kappa}e^{X_t}\right) + \beta dB_t,$$

with infinitesimal generator

$$\mathcal{L}^X f(x) = \frac{\beta^2}{2}f''(x) + \left(r - c - \frac{\beta^2}{2} - \frac{r}{\kappa}e^x\right)f'(x).$$

As in the Theorem 4.1 in [11], we denote

$$a(x) = \beta^2,\ b(x) = r - c - \frac{\beta^2}{2} - \frac{r}{\kappa}e^x.$$

Let $r > 0$ and $|x| \le r$. Using a Taylor expansion of f at order 2, we get:

$$\mathcal{L}^{Z_n} f(x) = n^2 \beta^2 (e^x - \tfrac{1}{n})^2 \left[f'(x) \ln(1 - \tfrac{e^{-2x}}{n^2}) \right.$$
$$+ \tfrac{1}{2} f''(x) (\ln^2(1 - \tfrac{e^{-x}}{n}) + \ln^2(1 + \tfrac{e^{-x}}{n})) + o(\tfrac{1}{n^2}) \Big]$$
$$+ n(e^x - \tfrac{1}{n}) \left[\left(c + \tfrac{r}{\kappa}(e^x - \tfrac{1}{n}) \right) \left(f'(x) \ln(1 - \tfrac{e^{-x}}{n}) + o(\tfrac{1}{n}) \right) \right.$$
$$\left. + r \left(f'(x) \ln(1 + \tfrac{e^{-x}}{n}) + o(\tfrac{1}{n}) \right) \right]$$

We can conclude

$$(5.3) \qquad \lim_n \sup_{|x| \le r} \left| \mathcal{L}^{Z_n} f(x) - \mathcal{L}^X f(x) \right| = 0$$

We now check all the assumptions of Theorem 4.1 in [11].

We denote by h_1 and h_2 the two functions defined on $(0, \infty)$ by $h_1(x) = x$ and $h_2(x) = x^2$.

Following the notations of Theorem 4.1 in [11], we define

$$(5.4) \qquad B_n(t) := \int_0^t (\mathcal{L}^{Z_n} h_1)(Z_n(s)) \, ds,$$

$$M_n(t) := Z_n(t) - B_n(t)$$

and

$$A_n(t) := \int_0^t \left[(\mathcal{L}^{Z_n} h_2)(Z_n(s)) - 2Z_n(s)(\mathcal{L}^{Z_n} h_1)(Z_n(s)) \right] ds.$$

By definition of the infinitesimal generator, the process

$M_n(t) = h_1(Z_n(t)) - \int_0^t (\mathcal{L}^{Z_n} h_1)(Z_n(s)) \, ds$ is a local martingale.

Similarly the process

$$(5.5) \qquad h_2(Z_n(t)) - \int_0^t (\mathcal{L}^{Z_n} h_2)(Z_n(s)) \text{ is a local martingale.}$$

For any process Y, the quantity $\Delta Y(t) := Y(t) - Y(t-)$ stands for its jump at time t. By Itô's rule applied to $M_n(t)^2$ and $Z_n(t)^2$, we deduce that the processes

$$(5.6) \qquad M_n^2(t) - \sum_{0 < s \le t} (\Delta M_n(s))^2 = M_n^2(t) - \sum_{0 < s \le t} (\Delta Z_n(s))^2$$

and

$$(5.7) \qquad Z_n^2(t) - 2 \int_0^t Z_n(s)(\mathcal{L}^{Z_n} h_1)(Z_n(s)) \, ds - \sum_{0 < s \le t} (\Delta Z_n(s))^2$$

are local martingales.

Combining (5.5) and (5.6), we obtain that $M_n^2(t) - \int_0^t A_n(s) \, ds$ is a local martingale.

We now check the assumptions of convergence of Theorem 4.1 in [11].

T denotes any positive time.

We define $\tau_n^r := \inf\{t > 0; |Z_n(t)| \ge r \text{ or } |Z_n(t-)| \ge r\}$.

The jumps of the process V_n are $\tfrac{1}{n}$ or $-\tfrac{1}{n}$, therefore with $\varepsilon \in \{-1, 0, 1\}$ and $t \le T \wedge \tau_n^r$,

$$|\Delta Z_n(t)| = |\ln(\tfrac{1}{n} + V_n(t-) + \tfrac{\varepsilon}{n}) - \ln(\tfrac{1}{n} + V_n(t-))| = |\ln(1 + \tfrac{\varepsilon}{n} \exp(-Z_n(t-)))|.$$

Assumption (4.3) of Theorem 4.1 in [11] follows:

$$\lim_n \mathbb{E}\left[\sup_{t \leq T \wedge \tau_n^r} |\Delta Z_n(t)|^2\right] = 0$$

Since the processes A_n and B_n are continuous, then assumptions (4.4) and (4.5) hold. Note that $\mathcal{L}^X h_1 = b$, then by (5.4):

$$B_n(t) - \int_0^t b(Z_n(s))\,ds = \int_0^t (\mathcal{L}^{Z_n}h_1 - \mathcal{L}^X h_1)(Z_n(s))\,ds.$$

Using (5.3), we deduce:

$$\lim_n \sup_{t \leq T \wedge \tau_n^r}\left|B_n(t) - \int_0^t b(Z_n(s))\,ds\right| \leq \lim_n \sup_{t \leq T \wedge \tau_n^r}\int_0^t \sup_{|x| \leq r}\left|(\mathcal{L}^{Z_n} - \mathcal{L}^X)h_1(x)\right|\,ds = 0$$

and assumption (4.6) of Theorem 4.1 in [11] is satisfied.

$$
\begin{aligned}
A_n(t) - \int_0^t a(Z_n(s))\,ds &= \int_0^t \left[\mathcal{L}^{Z_n}h_2(Z_n(s)) - 2Z_n(s)\mathcal{L}^{Z_n}h_1(Z_n(s)) - a(Z_n(s))\right]\,ds \\
&= \int_0^t \left[\mathcal{L}^{Z_n}h_2 - (a + 2h_1 b) + 2h_1 b - 2h_1 \mathcal{L}^{Z_n}h_1\right](Z_n(s))\,ds \\
&= \int_0^t \left[\mathcal{L}^{Z_n}h_2 - \mathcal{L}^X h_2 - 2h_1\left(\mathcal{L}^{Z_n}h_1 - \mathcal{L}^X h_1\right)\right](Z_n(s))\,ds.
\end{aligned}
$$

It is clear that (5.3) implies:

$$\lim_n \sup_{t \leq T \wedge \tau_n^r}\left|A_n(t) - \int_0^t a(Z_n(s))\,ds\right|$$

$$\leq \lim_n \sup_{t \leq T \wedge \tau_n^r}\int_0^t \sup_{|x| \leq r}\left|(\mathcal{L}^{Z_n} - \mathcal{L}^X)h_2(x)\right| + 2r \sup_{|x| \leq r}\left|(\mathcal{L}^{Z_n} - \mathcal{L}^X)h_1(x)\right|\,ds = 0$$

Assumption (4.7) of Theorem 4.1 in [11] is satisfied.

The martingale problem associated with the operator $\mathcal{L}^X$ is well posed since by Theorem 5.15 in [22], SDE (5.2) admits a unique solution up to an explosion time; but according to Proposition 3.3,

$$X_t = ln(V_t) = \beta B_t + (r - c - \tfrac{\beta^2}{2})t - \ln\left(\frac{1}{v} - \frac{r}{\kappa}\int_0^t e^{\beta B_s + (r - c - \frac{\beta^2}{2})s}\,ds\right)$$

is this solution and it is defined on $[0, \infty)$. Proposition 4.11 in [22] establishes the equivalence of martingale problem and weak solution to SDE.

We can apply Theorem 4.1 in [11]: the process $(Z_n(t))_{t \geq O}$ converges in distribution to $(V_t)_{t \geq 0}$. By the continuous mapping theorem, $(V_n(t))_{t \geq 0} = (e^{Z_n(t)} - \frac{1}{n})_{t \geq 0}$ converges in distribution to $(V(t))_{t \geq 0} = (e^{X(t)})_{t \geq 0}$.

$\square$

5.2. **Proof of Propositions 3.3 and 3.6.**

Proof. For a real valued diffusion with generator:

$$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x),$$

the associated scale function s and speed measure m can be calculated easily. Recall, see for instance [7], Chap. II that we have:

$$s'(x) = \exp\left\{-\int^x \frac{2b(y)}{\sigma^2(y)}\,dy\right\}, \quad m(dx) = m'(x)dx = \frac{2}{\sigma^2(x)\,s'(x)}\,dx.$$

Since $\sigma(x) = \beta x$ and $b(x) = x(a - bx)$ then (3.7) and (3.8) follow directly.

1) We claim that 0 is not an exit time. It is clear that (3.7) and (3.8) imply:

$$(5.8) \qquad s'(x) \backsim k\, x^{2q-1}, \quad m'(x) \backsim k'\, x^{-2q-1} \text{ when } x \to 0.$$

Since $m(x, 1) = \displaystyle\int_x^1 m'(y)\, dy$, we deduce that $m(0, 1) < \infty$ iff $q < 0$ and in that case:

$$\int_0^1 m(x, 1)\, s'(x)\, dx \backsim m(0, 1) \int_0^1 s'(x)\, dx = +\infty.$$

When $q > 0$, then $m(x, 1) \backsim k\, x^{-2q}$ as $x \to 0+$ and consequently

$$\int_0^1 m(x, 1)\, s'(x)\, dx = +\infty.$$

The case $q = 0$ can be treated similarly.

2) We prove that 0 is not a starting point for the diffusion (V_t).
   • If $q > 0$, then

$$\int_0^1 (s(1) - s(x))\, m'(x)\, dx \backsim (s(1) - s(0_+)) \int_0^1 m'(x)\, dx = +\infty.$$

   • If $q = 0$, then $s(1) - s(x) \sim -k \ln x$ when $x \to 0$. Therefore:

$$\int_x^1 (s(1) - s(y))\, m'(y)\, dy \backsim k \int_x^1 \frac{\ln y}{y}\, dy \ \to \infty, \quad x \to 0.$$

   • The case $q < 0$ can be studied similarly.

3) Since 0 is neither an exit nor a starting boundary, then 0 is natural. Consequently, it is attractive iff $s(0^+) > -\infty$. Relation (5.8) obviously implies that $s(0^+) > -\infty \ \Leftrightarrow \ q > 0$.

4) We now prove that V is recurrent iff $q \le 0$. Recall that if $0 < u < v < w$, then

$$\mathbb{P}_v(T_w < T_u) = \frac{s(v) - s(u)}{s(w) - s(u)}$$

where $\mathbb{P}_v$ is the conditional probability $\mathbb{P}(\cdot | V_0 = v)$, s is the scale function and

$$T_x := \inf\{t \ge 0,\, V_t = x\}$$

with the convention: $\inf \emptyset = \infty$.

a) We consider $\varepsilon < v < z$. Then,

$$\mathbb{P}_v(T_\varepsilon < +\infty) = \lim_{z \to +\infty} \mathbb{P}_v(T_\varepsilon < T_z) = \lim_{z \to +\infty} \frac{s(z) - s(v)}{s(z) - s(\varepsilon)} = 1$$

since identity (3.7) implies $s(+\infty) = \infty$.

b) Let us now consider that $\varepsilon > v > z$. Since 0 is not an exit boundary for the diffusion:

$$\mathbb{P}_v(T_\varepsilon < +\infty) = \lim_{z \to 0} \mathbb{P}_v(T_\varepsilon < T_z) = \lim_{z \to 0} \frac{s(v) - s(z)}{s(\varepsilon) - s(z)} = \begin{cases} 1 & \text{if } q \le 0 \\[2mm] \dfrac{s(v) - s(0^+)}{s(\varepsilon) - s(0^+)} < 1 & \text{if } q > 0. \end{cases}$$

As a result, the process (V_t) is recurrent iff $q \le 0$.

In the case $q > 0$ the numerator and the denominator can be simplified. Indeed, from (3.7), we have

$$s(x) - s(0^+) = e^{-\frac{2b}{\beta^2}} \int_0^x y^{2q-1} \exp\left\{\frac{2b}{\beta^2} y\right\} dy.$$

Expending the exponential gives:

$$s(x) - s(0^+) = x^{2q} e^{-\rho} \sum_{k \geq 0} \frac{\rho^k}{(k+2q)k!} x^k = \frac{e^{-\rho}}{2q} M(2q, 1+2q, \rho x)$$

where $\rho := \dfrac{2b}{\beta^2}$ and M is the Tricomi function defined by (4.1).

5) Suppose that $q > 0$. We have to prove that $\lim\limits_{t \to \infty} V_t = 0$ almost surely. Note that 0 being an attractive boundary we know that V_t converges to 0 at infinity with positive probability. Let $0 < \varepsilon < v/2$ and let us introduce the following sequence of stopping times: $\sigma_0 := 0$, $\sigma_1 := \inf\{t \geq 0, V_t = \varepsilon\}$ and inductively

$$\sigma_{2k} := \inf\{t \geq \sigma_{2k-1}, V_t = 2\varepsilon\}, \ \sigma_{2k+1} := \inf\{t \geq \sigma_{2k}, V_t = \varepsilon\}, \ k \geq 1.$$

According to the above steps 4) a) and b), for any $k \geq 1$ we have:

$$\mathbb{P}(\sigma_{2k-1} - \sigma_{2k-2} < \infty | \sigma_{2k-2} < \infty) = 1, \quad \rho := \mathbb{P}(\sigma_{2k} - \sigma_{2k-1} < \infty | \sigma_{2k-1} < \infty) < 1.$$

Since ρ does not depend on k, these relations imply the existence of a finite random number K such that $\sigma_{2K-1} < \infty$ and $\sigma_{2K} = \infty$ a.s. This means that $V_t \leq 2\varepsilon$ for any $t \geq \sigma_{2K-1}$. In other words $\lim\limits_{t \to \infty} V_t = 0$ a.s.

$\square$

5.3. Proof of Proposition 3.8.

Proof. **1)** Let $t > 0$ and $k \geq 2$. Using (1.2) and Itô formula we get:

$$(5.9) \qquad V_t^k = v^k + k \int_0^t V_s^{k-1} \big[V_s(a - bV_s)ds + \beta V_s dB_s \big] + \frac{k(k-1)}{2} \beta^2 \int_0^t V_s^k ds.$$

Define : $T_n := \inf\{t \geq 0, \ V_t = n\}$ and

$$\varphi_n(t) := \mathbb{E}\big(V_{t \wedge T_n}^k\big).$$

Replacing t by $t \wedge T_n$ in (5.9), using the facts that $V_t \geq 0$ and $b > 0$ and taking the expectation we get :

$$\varphi_n(t) \leq v^k + \Big(ka + \frac{k(k-1)}{2}\beta^2\Big) \int_0^t \varphi_n(s)ds. \quad t \geq 0.$$

Gronwall's lemma implies that:

$$\varphi_n(t) = \mathbb{E}\big(V_{t \wedge T_n}^k\big) \leq v^k \exp\Big\{t\Big(ka + \frac{k(k-1)}{2}\beta^2\Big)\Big\}.$$

Taking $n \to \infty$ in the above inequality and using Fatou lemma give:

$$\mathbb{E}\big(V_t^k\big) \leq v^k \exp\Big\{t\Big(ka + \frac{k(k-1)}{2}\beta^2\Big)\Big\}.$$

Consequently, $\max\limits_{0 \leq u \leq t} E(V_u^k) < \infty$ for any $k \geq 1$ and $t \geq 0$. Therefore we can take the expectation in (5.9). This gives item 2.

2) Using Hölder inequality and the fact that V_t is a non-constant random variable for $t > 0$, we have:

$$m_k(t) = \mathbb{E}(V_t^k) < \big(\mathbb{E}(V_t^{k+1})\big)^{\frac{k}{k+1}} = m_{k+1}(t)^{\frac{k}{k+1}}.$$

Consequently, relation (3.10) implies

$$(5.10) \qquad m_k'(t) < km_k(t)\Big(a + \frac{k-1}{2}\beta^2 - bm_k(t)^{1/k}\Big), \quad t > 0.$$

Since $m_k'(t) < 0$ if $m_k(t) > \Big(\dfrac{2a + (k-1)\beta^2}{2b}\Big)^k$ we deduce (3.11).

Inequality (3.12) can be proved similarly using identity (5.10) with $k = 1$, i.e. $m_1'(t) < am_1(t) - bm_1(t)^2$ and $V_0'(t) = aV_0(t) - bV_0(t)^2$.

$\square$

5.4. **Proof of Theorem 4.1.** We begin with two preliminary sections devoted to notations. The proof of Theorem 4.1 actually starts at step 3. *In* [3] the authors proposed another method to determine the eigenfunctions.

Step 1: definition of function U

When $b \neq 0, \pm 1, \pm 2, \cdots$, then $U(a, b; t)$ has been defined by (4.2). Function U can be extended to any $a, b \in \mathbb{R}$ as follows:

(1) When $-a \notin \mathbb{N}$, $b = n + 1$ where $n \in \mathbb{N}$ (cf formula (9.10.6) page 264 in [25]):

$$U(a, n+1; t) = \frac{(-1)^{n+1}}{\Gamma(a-n)} \sum_{k=0}^{\infty} \frac{(a)_k}{(n+k)!k!} \big[\psi(a+k) - \psi(1+k) - \psi(n+1+k) + \ln t\big] t^k$$

$$+ \frac{1}{\Gamma(a)} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!(a-n)_k}{k!} t^{k-n}, \quad t \in \mathbb{R}, a \neq n.$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the the digamma function.

We get the particular case $a = n, b = n + 1$ using the recurrence relation (9.10.16) page 266 in [25]:

$$U(n, n+1; t) = \frac{t}{t+n} U(n, n+2; t).$$

(2) When $a = -m$ ($m \in \mathbb{N}$) and $b = n + 1$, according to formula (9.10.7) page 264 in [25]:

$$U(-m, n+1; t) = (-1)^m \frac{(m+n)!}{n!} M(-m, n+1; t), \quad n \in \mathbb{N}.$$

(3) Using identity formula (9.10.8) page 265 in [25], we get:

$$U(a, -n; t) = t^{1+n} U(1+a+n, 1+n; t), \quad a, t \in \mathbb{R}, n \in \mathbb{N}.$$

This identity permits to come down to the two above cases.

Then $U(a, b; t)$ is meaningful for arbitrary values of the parameters a and b. Moreover $U(a, b; t)$ is an analytic function of $t > 0$ and an entire function of a and b.

Step 2: notations

It is convenient to recall the values of q, ρ and z_λ given by (3.4), (4.3) and (4.4) resp.:

$$\rho = \frac{2b}{\beta^2}, \quad q = \frac{1}{2} - \frac{a}{\beta^2}, \quad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} \text{ for all } \lambda \geq -\frac{\beta^2 q^2}{2}.$$

Let $z \mapsto \sqrt{z}$ be the analytic continuation of the square root function to the set $\mathbb{C} \backslash]-\infty, 0[$ and define:

$$(5.11) \qquad r_1(z) := \sqrt{\frac{2z}{\beta^2} + q}, \quad z \in \mathbb{C} \backslash]-\infty, -\frac{q^2 \beta^2}{2}],$$

$$(5.12) \qquad r_2(\lambda) := i \sqrt{-\frac{2\lambda}{\beta^2} - q^2}, \quad \lambda \in]-\infty, -\frac{q^2 \beta^2}{2}].$$

Step 3: proof of item 1

We recall the classical method which permits to calculate its Laplace transform. Let $\mathcal{L}$ be the generator of process (V_t), i.e.

$$(5.13) \qquad \mathcal{L}f(x) := \frac{\beta^2}{2} x^2 f''(x) + x(a - bx)f'(x), \quad x > 0.$$

Consider an eigenfunction f_λ associated with the eigenvalue $\lambda > 0$ of $\mathcal{L}$, i.e. a solution of

$$(5.14) \qquad \mathcal{L}f = \lambda\,f.$$

Lemma 5.1. *Let $v, \varepsilon > 0$ and f_λ be a solution of* (5.14) *and such that it is bounded on any interval of the type $[\varepsilon, \infty[$ (resp. $[0, \varepsilon])$ when $\varepsilon \leq v$ (resp. $\varepsilon \geq v$), then*

$$(5.15) \qquad f_\lambda(\varepsilon)\mathbb{E}_v\big(e^{-\lambda T_\varepsilon}1_{\{T_\varepsilon < \infty\}}\big) = f_\lambda(v).$$

If moreover $f_\lambda(\varepsilon) \neq 0$, we have:

$$(5.16) \qquad \mathbb{E}_v\big(e^{-\lambda T_\varepsilon}1_{\{T_\varepsilon < \infty\}}\big) = \frac{f_\lambda(v)}{f_\lambda(\varepsilon)}.$$

Proof. Since f_λ solves (5.14), the process $\big(f_\lambda(V_t)e^{-\lambda t},\ t \geq 0\big)$ is a local martingale. If we choose f_λ such that it is bounded on any interval of the type $[\varepsilon, \infty[$ (resp. $[0, \varepsilon])$ when $\varepsilon \leq v$ (resp. $\varepsilon \geq v$) then $\big(f_\lambda(V_{t \wedge T_\varepsilon})e^{-\lambda t \wedge T_\varepsilon},\ t \geq 0\big)$ is a bounded process and is therefore a martingale. The stopping theorem gives (5.15) $\qquad\qquad\qquad\square$

This approach leads us to first determine the eigenfunctions of $\mathcal{L}$ and their behaviors in the vicinity of 0 and $+\infty$ (see Lemma 5.2). Secondly we prove, cf Lemma 5.3, that these functions do not vanish.

Lemma 5.2. *Let $\lambda \geq -\frac{\beta^2 q^2}{2}$. We introduce the two following functions:*

$$\begin{aligned} h_\lambda(x) &= x^{z_\lambda + q}U\left(z_\lambda + q, 1 + 2z_\lambda; \rho x\right), & x > 0 \\ \tilde{h}_\lambda(x) &= x^{z_\lambda + q}M\left(z_\lambda + q, 1 + 2z_\lambda; \rho x\right), & x > 0 \end{aligned}$$

where M and U have been defined by (4.1) *and in the above step 1* (4.2)*.*

(1) For any solution f_λ of (5.14) *there exists two constants C_1 and C_2 such that:*

$$(5.17) \qquad f_\lambda(x) = C_1 h_\lambda(x) + C_2 \tilde{h}_\lambda(x), \quad x > 0.$$

(2) The function h_λ (resp. $\tilde{h}_\lambda$) is the unique solution, up to a multiplicative constant, to the equation (5.14) *being bounded on every interval $[\varepsilon, +\infty[$ with $\varepsilon > 0$ (resp. on $]0, v]$, with $v > 0$).*

Proof. a) According to (5.13), the function f_λ is a solution of the following equation:

$$(5.18) \qquad \frac{\beta^2}{2}\,x^2 f''(x) + \big(a\,x - b\,x^2\big)f'(x) = \lambda\,f(x), \quad x > 0.$$

We consider the function u associated with the function f by the relation :

$$(5.19) \qquad f(x) = x^{-\frac{a}{\beta^2}}\,e^{\frac{b}{\beta^2}x}\,u(x).$$

By consecutive derivations we get:

$$(5.20) \qquad \frac{f'(x)}{f(x)} + \frac{a - b\,x}{\beta^2\,x} = \frac{u'(x)}{u(x)}$$

and

$$(5.21) \qquad \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 - \frac{a}{\beta^2\,x^2} = \frac{u''(x)}{u(x)} - \left(\frac{u'(x)}{u(x)}\right)^2.$$

Squaring the two sides of (5.20) gives:

$$(5.22) \qquad \left(\frac{u'(x)}{u(x)}\right)^2 - 2\left(\frac{a - bx}{\beta^2 x}\right)\left(\frac{f'(x)}{f(x)}\right) - \left(\frac{f'(x)}{f(x)}\right)^2 = \left(\frac{a - bx}{\beta^2 x}\right)^2.$$

Substituting (5.22) to (5.21) leads to:

$$\frac{f''(x)}{f(x)} = \frac{u''(x)}{u(x)} - \left(\frac{a-bx}{\beta^2 x}\right)^2 - 2\left(\frac{a-bx}{\beta^2 x}\right)\left(\frac{f'(x)}{f(x)}\right) + \frac{a}{\beta^2 x^2}.$$

Using (5.20), we obtain:

$$\frac{f''(x)}{f(x)} = \frac{u''(x)}{u(x)} + \left(\frac{a-bx}{\beta^2 x}\right)^2 - 2\left(\frac{a-bx}{\beta^2 x}\right)\left(\frac{u'(x)}{u(x)}\right) + \frac{a}{\beta^2 x^2}.$$

We divide (5.18) by f and then use (5.20) to get:

$$u''(x) = \left(\frac{2\lambda - a}{\beta^2 x^2} + \left(\frac{a-bx}{\beta^2 x}\right)^2\right) u(x).$$

The change of variables $x = \frac{\beta^2 t}{2b}$ (i.e. $t = \frac{2b}{\beta^2} x$) leads to a Whittaker's equation (page 505 of [1] or [16]), with $w(t) = u(x)$.

$$w''(t) + \left[-\frac{1}{4} + \frac{a}{\beta^2}\frac{1}{t} + \left(\frac{1}{4} - z_\lambda^2\right)\frac{1}{t^2}\right] w(t) = 0.$$

This Whittaker's equation admits two linearly independent solutions (see [6], Section 6.9) given by:

$$M_{a/\beta^2, z_\lambda}(t) = \exp\left(-\frac{t}{2}\right) t^{z_\lambda + \frac{1}{2}} M\left(z_\lambda + q, 1 + 2z_\lambda; t\right)$$

and

$$W_{a/\beta^2, z_\lambda}(t) = \exp\left(-\frac{t}{2}\right) t^{z_\lambda + \frac{1}{2}} U\left(z_\lambda + q, 1 + 2z_\lambda; t\right)$$

where M and U are the confluent hypergeometric functions of the first and second kind, defined by (4.1) and (4.2). Then $u(x) = w(t)$ is a linear combination of these functions M and U. Using (5.19) we get (5.17).

b) We now determine the asymptotic behaviors of $h_\lambda(x)$ and $\widetilde{h}_\lambda(x)$ as $x \to 0$ and $x \to \infty$.

Since $\lim_{x\to 0} M(a,b;x) = 1$ we deduce

$$\lim_{x\to 0} \widetilde{h}_\lambda(x) = \lim_{x\to 0}\left(x^{z_\lambda + q} M\left(z_\lambda + q, 1 + 2z_\lambda; \rho x\right)\right) = 0.$$

Using (4.2) we have: $h_\lambda(x) \sim Cte\, x^{q - z_\lambda}$ when $x \to 0$. Since $z_\lambda > q$, $\lim_{x\to 0} h_\lambda(x) = \infty$.

Recall that from [1] p. 504 (13.1.8) and (13.1.4) respectively, we have:

$$U(a,b;t) \sim t^{-a}, \quad M(a,b;t) \sim \frac{\Gamma(b)}{\Gamma(a)} e^t t^{a-b}, \quad t \to \infty.$$

Consequently:

$$(5.23) \qquad\qquad \lim_{x\to\infty} \widetilde{h}_\lambda(x) = \infty, \quad \lim_{x\to\infty} h_\lambda(x) = 1.$$

$\square$

Lemma 5.3. *For any $x > 0$, $h_\lambda(x)$ and $\widetilde{h}_\lambda(x)$ are positive.*

Proof. The definition (4.1) of function M implies that $\widetilde{h}_\lambda(x) > 0$ for any $x > 0$. According to (5.15) and Lemma 5.2 , we have

$$h_\lambda(\varepsilon)\mathbb{E}_v\left(e^{-\lambda T_\varepsilon}\right) = h_\lambda(v), \quad \varepsilon \leq v.$$

Therefore, if $h_\lambda(\varepsilon) = 0$, then $h_\lambda(v) = 0$ for any $v \in [\varepsilon, \infty[$. This generates a contradiction since h_λ is analytic on $]0, \infty[$. Consequently, $h_\lambda(x) \neq 0$ for any $x > 0$. The second identity in (5.23) implies that $h_\lambda(x)$ is positive. $\square$

It is clear that Lemmas 5.1-5.3 and (5.16) imply item 1 of Theorem 4.1.

Step 4: the method to prove item 2

Our strategy is based on the holomorphic extension of the Laplace transform of a non-negative random variable X. Recall (see [43], chapter 2, page 37) that the convergence abscissa of its Laplace transform:

$$L_X(\lambda) := \mathbb{E}(e^{-\lambda X}), \quad \lambda \geq 0.$$

is the unique real number, possibly infinite, $\sigma_c \in [-\infty, 0]$ such that:

$$\mathbb{E}(e^{-sX}) < \infty \quad \text{for all } s > \sigma_c \quad \text{and } \mathbb{E}(e^{-sX}) = \infty \quad \text{for all } s < \sigma_c.$$

Then, $\widehat{L}_X(z) := \mathbb{E}(e^{-zX})$ is the unique holomorphic extension of L to the domain $\{z \in \mathbb{C}, Re(z) > \sigma_c\}$.

Our approach is based on the following general result.

Lemma 5.4. *(Landau Theorem)*

Let F be an holomorphic function in a domain $\mathcal{D}_F$ which contains the real half-line $(\sigma_F; +\infty)$ where $\sigma_F \leq 0$ and such that

$$F(\lambda) = L_X(\lambda), \quad \forall \lambda \geq 0.$$

(1) Then, $\sigma_c \leq \sigma_F$.
(2) If F is not bounded in the neighborhood of σ_F, then $\sigma_c = \sigma_F$.

Proof. The proof is inspired by the one of Theorem 5b, page 58 in [43].

By analytic continuation,

$$(5.24) \qquad F(\lambda) = \widehat{L}_X(\lambda), \quad \lambda > \max\{\sigma_c, \sigma_F\}.$$

Suppose that $\sigma_c > \sigma_F$ and let us show that this inequality generates a contradiction, i.e.

$$(5.25) \qquad \mathbb{E}(e^{(-\sigma_c + \varepsilon)X}) < \infty, \quad \text{for some } \varepsilon > 0.$$

Since F is analytic at σ_c, there exists $0 < \varepsilon < \sigma_c - \sigma_F$ such that:

$$F(\sigma_c - \varepsilon) = \sum_{k \geq 0} \frac{(-2\varepsilon)^k}{k!} F^{(k)}(\sigma_c + \varepsilon).$$

Relation (5.24) implies

$$F^{(k)}(\sigma_c + \varepsilon) = \widehat{L}_X^{(k)}(\sigma_c + \varepsilon) = (-1)^k \mathbb{E}(X^k e^{-(\sigma_c + \varepsilon)X})$$

and therefore

$$\begin{aligned}
F(\sigma_c - \varepsilon) &= \sum_{k \geq 0} \frac{(2\varepsilon)^k}{k!} \mathbb{E}(X^k e^{-(\sigma_c + \varepsilon)X}) \\
&= \mathbb{E}\Big(\sum_{k \geq 0} \frac{(2\varepsilon X)^k}{k!} e^{-(\sigma_c + \varepsilon)X}\Big) \\
&= \mathbb{E}(e^{(-\sigma_c + \varepsilon)X}).
\end{aligned}$$

This proves(5.25). Item 2 is direct consequence of the fact that $\widehat{L}_X$ is holomorphic in $\{z \in \mathbb{C}, Re(z) > \sigma_c\}$. $\qquad \square$

Step 5: proof of item 2, when $v \geq \varepsilon$

According to Lemmas 5.1 and 5.2, the function

$$g_1(z) = \Big(\frac{v}{\varepsilon}\Big)^{r_1(z)+q} \frac{U(r_1(z) + q, 1 + 2r_1(z); \rho v)}{U(r_1(z) + q, 1 + 2r_1(z); \rho \varepsilon)}$$

seems to be the good candidate to give an holomorphic extension of the Laplace transform of T_ε under $\mathbb{P}_v$. We prove a technical result (see Lemma 5.5 below) which permits to

prove that g_1 is holomorphic on a suitable set which contains any real number λ such that $\lambda > -\dfrac{q^2\beta^2}{2}$, see Lemma 5.5. Then, the proof of item 2 of Theorem 4.1 follows immediately.

We begin with notations. We introduce

$$\mathcal{U}_* = \{z \in \mathbb{C}; U(z + q, 1 + 2z; \rho\varepsilon) \neq 0\}$$

and function g as:

(5.26) $$g(z) = \left(\frac{v}{\varepsilon}\right)^{z+q} \frac{U(z+q, 1+2z; \rho v)}{U(z+q, 1+2z; \rho\varepsilon)}, \quad z \in \mathcal{U}_*.$$

We introduce two new sets $\mathcal{U}_1$ and $\mathcal{U}_2$:

$$\mathcal{U}_1 = \left\{z \in \mathbb{C}\backslash] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_1(z) = \sqrt{\tfrac{2z}{\beta^2} + q^2} \in \mathcal{U}_*\right\},$$

$$\mathcal{U}_2 = \left\{\lambda \in] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_2(\lambda) = i\sqrt{-\tfrac{2\lambda}{\beta^2} - q^2} \in \mathcal{U}_*\right\}.$$

The following result is a preliminary step to prove that function g_1 is holomorphic on $\mathcal{U}_1 \cup \mathcal{U}_2$ (cf Lemma 5.5 below).

Lemma 5.5.   (1) Set $\mathcal{U}_*$ and function g are symmetric, i.e. $z \in \mathcal{U}_* \Rightarrow -z \in \mathcal{U}_*$ and $g(-z)=g(z)$.
   (2) $\mathcal{U}_*$ is an open set.
   (3) g is a meromorphic function over $\mathbb{C}$ and its restriction to $\mathcal{U}_*$ is holomorphic.
   (4) The set $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$ is discrete and therefore the set $\mathcal{U}_1 \cup \mathcal{U}_2$ is open in $\mathbb{C}$.

Proof. a) It is easy to prove that relation (4.2) implies:

$$U(-z + q, 1 - 2z; x) = x^{2z}U(z + q, 1 + 2z; x), \quad x \in \mathbb{R}.$$

We deduce that $\mathcal{U}_*$ is symmetric and

$$g(-z) = \left(\frac{v}{\varepsilon}\right)^{-z+q} \frac{(\rho v)^{2z} \, U(z+q, 1+2z; \rho v)}{(\rho\varepsilon)^{2z} \, U(z+q, 1+2z; \rho\varepsilon)} = g(z), \; z \in \mathcal{U}_*.$$

From [25] page 261 section 9.9, we have:

(5.27) $$\frac{M(a, b; z)}{\Gamma(b)} = \sum_{k \geq 0} \frac{(a)_k}{\Gamma(b+k)} \frac{z^k}{k!}.$$

Recall Gamma function admits an holomorphic extension to $\{z \in \mathbb{C}, -z \notin \mathbb{N}\}$. It follows that, for any real t, the maps $z \mapsto M(z + q, 1 + 2z; t)$ and $z \mapsto M(q - z, 1 - 2z; t)$ are holomorphic when $z \notin \{0, \pm\tfrac{1}{2}, \pm\tfrac{2}{2}, \pm\tfrac{3}{2}, \cdots\}$. It is easy to deduce items 2 and 3 from (4.2).

c) Since function r_1 (resp. r_2) is continuous, the set $\mathcal{U}_1 = r_1^{-1}(\mathcal{U}_*)$ is open both in $\mathbb{C}\backslash] - \infty, -\tfrac{q^2\beta^2}{2}]$ and $\mathbb{C}$. However $\mathcal{U}_2 = r_2^{-1}(\mathcal{U}_*)$ is only open in $] - \infty, -\tfrac{q^2\beta^2}{2}]$.

d) Item 3 implies that

$$\begin{aligned} \mathcal{U}_{1,c} &:= \left\{z \in \mathbb{C}\backslash] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_1(z) \in \mathcal{U}_*^c\right\} \\ &= \left\{z \in \mathbb{C}\backslash] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; U(r_1(z) + q, 1 + 2r_1(z); \rho\varepsilon) = 0\right\} \end{aligned}$$

is discrete.

Similarly, by analyticity,

$$\begin{aligned} \mathcal{U}_{2,c} &:= \left\{\lambda \in] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_2(\lambda) \in \mathcal{U}_*^c\right\} \\ &= \left\{\lambda \in] - \infty, -\tfrac{q^2\beta^2}{2}] \; ; \; U(r_2(\lambda) + q, 1 + 2r_2(\lambda); \rho\varepsilon) = 0\right\} \end{aligned}$$

is also discrete.

Note that $\mathcal{U}_{1,c}$ (resp. $\mathcal{U}_{2,c}$) is the complement of $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) in $\mathbb{C}\backslash] - \infty, -\frac{q^2\beta^2}{2}]$ (resp. $] - \infty, -\frac{q^2\beta^2}{2}])$ and $(\mathcal{U}_1 \cup \mathcal{U}_2)^c = \mathcal{U}_{1,c} \cup \mathcal{U}_{2,c}$.

In order to prove that the union of $\mathcal{U}_{1,c}$ and $\mathcal{U}_{2,c}$ is discrete, it is sufficient to prove that every sequence of complex numbers in $\mathcal{U}_{1,c}$ cannot have an accumulation point in $\mathcal{U}_{2,c}$. Suppose, contrary to our claim, there exists a sequence $(\lambda_n)_{n\geq 1}$ of points in $\mathcal{U}_{1,c}$ converging to $\lambda \in \mathcal{U}_{2,c}$. Using the definition of $\mathcal{U}_{1,c}$ and $\mathcal{U}_{2,c}$, we have:

$$U(r_1(\lambda_n) + q, 1 + 2r_1(\lambda_n); \rho\varepsilon) = 0, \quad \forall\, n \geq 1.$$

and

$$U(r_2(\lambda) + q, 1 + 2r_2(\lambda); \rho\varepsilon) = U(-r_2(\lambda) + q, 1 - 2r_2(\lambda); \rho\varepsilon) = 0.$$

Since λ is a negative real number, the sequence

$$(r_1(\lambda_n))_{n\geq 1} = (\sqrt{2\lambda_n/\beta^2 + q^2})_{n\geq 1}$$

presents at least one of the two accumulation points

$$i\sqrt{-2\lambda/\beta^2 - q^2} = r_2(\lambda) \text{ and } -i\sqrt{-2\lambda/\beta^2 - q^2} = -r_2(\lambda).$$

As a consequence, one of these two points is an accumulation point of zeros of the meromorphic function $z \mapsto U(z + q, 1 + 2z, \frac{2b\varepsilon}{\beta^2})$. This generates a contradiction, since a meromorphic function has isolated zeros. $\qquad\square$

Lemma 5.6. *The function g_1 defined on the open set $\mathcal{U}_1 \cup \mathcal{U}_2$ by*

$$g_1(z) = \begin{cases} g\big(r_1(z)\big) & \text{if } z \in \mathcal{U}_1 \\ g\big(r_2(z)\big) & \text{if } z \in \mathcal{U}_2 \end{cases}$$

is holomorphic on $\mathcal{U}_1 \cup \mathcal{U}_2$, meromorphic on $\mathbb{C}$ and the set of poles of g_1 is included in the discrete set $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$.

Proof. 1) The restriction of function g_1 to $\mathcal{U}_1$ could be written as the composition of two holomorphic functions: $g_{1|\mathcal{U}_1} = g \circ r_{1|\mathcal{U}_1}$; function g_1 is therefore holomorphic on $\mathcal{U}_1$.

If g_1 is continuous in $\mathcal{U}_1 \cup \mathcal{U}_2$, according to ([14] exercise 11, pages 100 and 463) g_1 is holomorphic on $\mathcal{U}_1 \cup \mathcal{U}_2$. Function g_1 is clearly continuous on $\mathcal{U}_1$. We now proceed to show the continuity at any point $\lambda \in \mathcal{U}_2$. Let $(\lambda_n)_{n\geq 1}$ be a sequence of points in $\mathcal{U}_1 \cup \mathcal{U}_2$, converging to λ. The two accumulation points of $(\lambda_n)_{n\geq 1}$ are $\pm r_2(\lambda)$. However, since g is an even function, the sequence $(g_1(\lambda_n))_{n\geq 1}$ converges to $g\big(r_2(\lambda)\big) = g_1(\lambda)$.

2) It is clear that the points of $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$ are the zero of the denominator of (5.26), they are poles or removable singularities of the function g_1. The function g_1 is meromorphic and the set of its non-essential singularities is the discrete set $(\mathcal{U}_1 \cup \mathcal{U}_2)^c$.

$\qquad\square$

Lemma 5.7. *Function g_1 is analytic at any real point $\lambda \geq -\dfrac{\beta^2 q^2}{2}$ (resp. $\lambda \geq 0$) if $q \geq 0$ (resp. $q < 0$).*

Proof. As function g_1 is analytic on the set $\mathcal{U}_1 \cup \mathcal{U}_2$, we will prove that $\mathcal{U}_1 \cup \mathcal{U}_2$ contains the interval $[-\frac{\beta^2 q^2}{2}; +\infty[$ if $q \geq 0$ and the interval $[0; +\infty[$ if $q < 0$.

$$\begin{aligned} \mathbb{R} \cap \mathcal{U}_1 &= \{\lambda \in] - \tfrac{\beta^2 q^2}{2}; +\infty[; r_1(\lambda) \in \mathcal{U}_*\} \\ &= \{\lambda \in] - \tfrac{\beta^2 q^2}{2}; +\infty[; U(r_1(\lambda) + q, 1 + 2r_1(\lambda); \rho\varepsilon) \neq 0\}. \end{aligned}$$

If $q \geq 0$ then the three numbers $r_1(\lambda) + q$, $1 + 2r_1(\lambda)$ and $\rho\varepsilon$ are positive and so does the number $U(r_1(\lambda) + q, 1 + 2r_1(\lambda); \rho\varepsilon)$, according to [12] page 290 section 6.16. Therefore, if $q \geq 0$ then $\mathbb{R} \cap \mathcal{U}_1 =] - \frac{\beta^2 q^2}{2}; +\infty[$.

Similarly, since $U\big(q + r_2(-\frac{\beta^2 q^2}{2}), 1 + 2r_2(-\frac{\beta^2 q^2}{2}); \rho\varepsilon\big) = U(q, 1; \frac{2b\varepsilon}{\beta^2}) \neq 0$ if $q \geq 0$, the number $-\frac{\beta^2 q^2}{2}$ is in $\mathcal{U}_2$.

If $q < 0$ then $U\big(q + r_1(0), 1 + 2r_1(0); \rho\varepsilon\big) = U\big(0, 1 + 2|q|; \rho\varepsilon\big) = 1 \neq 0$; as a consequence, $0 \in \mathcal{U}_1$ and $[0; +\infty[\subset \mathcal{U}_1$.

$\square$

Remark 5.8. *Set*

$$\sigma_1(q) := \max \mathcal{U}_{2,c} = \max\left\{\lambda \in]-\infty, -\tfrac{q^2\beta^2}{2}[\; ; \; U\big(q + r_2(\lambda), 1 + 2r_2(\lambda); \rho\varepsilon\big) = 0\right\},$$

where function r_2 has been defined by (5.12) and by convention, $\max \emptyset = -\infty$.

(1) *Suppose $q \geq 0$. We deduce from above that $\sigma_c(\varepsilon, v) \leq \sigma_1(q)$, the real number $\sigma_c(\varepsilon, v)$ equals $\sigma_1(q)$ if it is a singularity of g_1, i.e. either $U\big(q + r_2(\sigma_1(q)), 1 + 2r_2(\sigma_1(q)); \rho v\big) \neq 0$ or $\sigma_1(q)$ is a zero of $U\big(q + r_2(\cdot), 1 + 2r_2(\cdot); \rho v\big)$ with order less than the order of $U\big(q + r_2(\cdot), 1 + 2r_2(\cdot); \rho\varepsilon\big)$.*

(2) *Similarly, when $q < 0$, $\sigma_c(\varepsilon, v)$ is lower than $\max\{\sigma_1(q), \sigma_2(q)\}$, where*

$$\sigma_2(q) := \max\left\{\lambda \in [-\tfrac{q^2\beta^2}{2}, 0[, U\big(q + r_1(\lambda), 1 + 2r_1(\lambda); \rho\varepsilon\big) = 0\right\}$$

and function r_1 has been defined by (5.11).

Step 6: proof of item 2, when $v < \varepsilon$
We proceed similarly to the above step 5. We introduce

$$\widetilde{\mathcal{U}}_* = \{z \in \mathbb{C}; \; M(z + q, 1 + 2z; \rho\varepsilon) \neq 0\}$$

where M is the hypergeometric confluent function defined by (4.1) and $\widetilde{g}$ is the function:

$$\widetilde{g}(z) := \left(\frac{v}{\varepsilon}\right)^{z+q} \frac{M(z + q, 1 + 2z; \rho v)}{M(z + q, 1 + 2z; \rho\varepsilon)}, \quad z \in \widetilde{\mathcal{U}}_*.$$

We define

$$\widetilde{\mathcal{U}}_1 := \Big\{z \in \mathbb{C}\backslash]-\infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_1(z) = \sqrt{\tfrac{2z}{\beta^2} + q^2} \in \widetilde{\mathcal{U}}_*\Big\}$$

and

$$\widetilde{g}_1(\lambda) := \widetilde{g}\big(r_1(z)\big), \quad z \in \widetilde{\mathcal{U}}_1.$$

There is a main difference with the case $v > \varepsilon$: the function $\widetilde{g}$ is not even. This implies that $\widetilde{g}_1$ cannot be continuously extended to $\big\{\lambda \in]-\infty, -\tfrac{q^2\beta^2}{2}] \; ; \; r_2(\lambda) = i\sqrt{-\tfrac{2\lambda}{\beta^2} - q^2} \in \widetilde{\mathcal{U}}_*\big\}$. However, we can prove similarly to Lemmas 5.5-5.7:

(1) The set $\widetilde{\mathcal{U}}_*$ is open and the function $\widetilde{g}$ is holomorphic on $\widetilde{\mathcal{U}}_*$.
(2) The function $\widetilde{g}_1$ is holomorphic on $\widetilde{\mathcal{U}}_1$ and meromorphic on $\mathbb{C}\backslash]-\infty, -\tfrac{q^2\beta^2}{2}]$.
(3) The function $\widetilde{g}_1$ is analytic at any real point $\lambda > -\dfrac{\beta^2 q^2}{2}$ (resp. $\lambda \geq 0$) if $q \geq 0$ (resp. $q < 0$).

Remark 5.9. *Set*

$$\widetilde{\sigma}_2(q) := \max\left\{\lambda \in [-\tfrac{q^2\beta^2}{2}, 0[, M\big(q + r_1(\lambda), 1 + 2r_1(\lambda); \rho\varepsilon\big) = 0\right\}.$$

(1) *Suppose $q \geq 0$. We deduce from above that $\sigma_c(\varepsilon, v) = -\tfrac{q^2\beta^2}{2}$.*
(2) *When $q < 0$, $\sigma_c(\varepsilon, v)$ is in the interval $[-\tfrac{q^2\beta^2}{2}; \widetilde{\sigma}_2(q)] \subset]-\infty; 0[$.*

5.5. Proof of Theorem 4.3.

Proof. Since we study the asymptotic behavior of T_ε as $\varepsilon \to 0$, we can suppose that $\varepsilon < v$.

1) In the case $q > 0$, we will prove that the r.v. $\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)$ converges in law to $\frac{1}{\beta^2\sqrt{q^3}}Z$ where Z is a standard normal r.v.

By (4.5) and (4.4), the Laplace transform of this r.v. satisfies

(5.28)
$$
\mathbb{E}_v\left[\exp\left(-\lambda\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = \exp\left(\lambda\frac{\sqrt{-\ln\varepsilon}}{\beta^2 q}\right)
$$
$$
\times\frac{v^{q+z_{\lambda(\varepsilon)}}U(q+z_{\lambda(\varepsilon)}, 1+2z_{\lambda(\varepsilon)}; \rho v)}{\varepsilon^{q+z_{\lambda(\varepsilon)}}U(q+z_{\lambda(\varepsilon)}, 1+2z_{\lambda(\varepsilon)}; \rho\varepsilon)}
$$

where $\lambda(\varepsilon) := \frac{\lambda}{\sqrt{-\ln\varepsilon}}$ and

$$
z_{\lambda(\varepsilon)} = \sqrt{\frac{2\lambda(\varepsilon)}{\beta^2} + q^2} = q + \frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} - \frac{\lambda^2}{2q^3\beta^4(-\ln\varepsilon)} + \mathcal{O}\left(\frac{1}{(-\ln\varepsilon)^{\frac{3}{2}}}\right).
$$

Using this expression or direct consequences of it in (5.28), we obtain

(5.29)
$$
\mathbb{E}_v\left[\exp\left(-\lambda\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = A_1(\varepsilon)\exp\left(\frac{\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q}\right)
$$

where,

$$
A_1(\varepsilon) := v^{2q+\mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)}\exp\left(\frac{\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q} - \frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) A_2(\varepsilon),
$$

$$
A_2(\varepsilon) := \frac{U\left(2q + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right), 1 + 2q + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right); \rho v\right)}{\varepsilon^{2q}U\left(2q + \frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right); \rho\varepsilon\right)}.
$$

Using the following identity ([25] page 505, formula (13.1.29))

(5.30)
$$
U(a,b,t) = t^{1-b}U(1+a-b, 2-b, t), \quad t \in \mathbb{R}, b \notin \mathbb{N},
$$

we obtain the convergence of the function U of the numerator $A_2(\varepsilon)$:

(5.31)
$$
U\left(2q + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right), 1 + 2q + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right); \rho v\right) = (\rho v)^{-2q} + o(1).
$$

The function U at the denominator requires a more precise study; Definition (4.2) leads to the following:

$$
U\left(2q + \frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right); \rho\varepsilon\right)
$$
$$
= \frac{\Gamma(-2q - \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right))}{\Gamma(-\frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right))}M(2q + o(1), 1 + 2q + o(1); \rho\varepsilon)
$$
$$
+ \frac{\Gamma(2q + o(1))}{\Gamma(2q + o(1))}\rho^{-2q+o(1)}\varepsilon^{-2q}\exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right)
$$
$$
M(o(1), 1 - 2q + o(1); \rho\varepsilon).
$$

When $2q$ is not an integer, the first quotient of Gamma functions converges to 0 since $\dfrac{\Gamma(-2q+2x)}{\Gamma(x)} \sim \Gamma(-2q)x$, when $x \to 0$, whereas when $2q$ is an integer, according to

$$(5.32) \qquad \frac{1}{\Gamma(-n+x)} = (-1)^n\, n!\, \left(x - \psi(n+1)\,x^2 + o(x^2)\right), \quad x \to 0, \ n \in \{1, 2, \cdots\}$$

this ratio goes to $\dfrac{(-1)^{2q}}{2(2q)!}$ as ε goes to 0. In these two cases, the quotient remains bounded when ε goes to 0.

By these considerations, the definition (4.1) of M and properties of Gamma function imply:

$$U\left(2q + \frac{\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right), 1 + 2q + \frac{2\lambda}{q\beta^2\sqrt{-\ln\varepsilon}} + \mathcal{O}\left(\frac{1}{\ln\varepsilon}\right); \rho\varepsilon\right)$$

$$(5.33) \quad = \mathcal{O}(1) + (1+o(1))\rho^{-2q+o(1)}\varepsilon^{-2q}\exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right).$$

By using (5.31) and (5.33) in (5.29), we obtain successively:

$$\mathbb{E}_v\left[\exp\left(-\lambda\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q}\right)$$

$$\times \frac{v^{2q+\mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)}\exp\left(-\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right)(\rho v)^{-2q}}{\varepsilon^{2q}\left[(1+o(1))\rho^{-2q+o(1)}\varepsilon^{-2q}\exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) + \mathcal{O}(1)\right]},$$

$$\mathbb{E}_v\left[\exp\left(-\lambda\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = \exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q}\right)$$

$$\times \frac{\exp\left(-\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right)(1+o(1))}{\exp\left(\frac{2\lambda\sqrt{-\ln\varepsilon}}{\beta^2 q} - \frac{\lambda^2}{q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right) + \mathcal{O}(1)},$$

$$\mathbb{E}_v\left[\exp\left(-\lambda\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = \exp\left(\frac{\lambda^2}{2q^3\beta^4} + \mathcal{O}\left(\frac{1}{\sqrt{-\ln\varepsilon}}\right)\right).$$

The Laplace transform of T_ε being analytic in a neighbourhood of zero, we can conclude that the characteristic function of $\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)$ satisfies:

$$\lim_{\varepsilon\to 0}\mathbb{E}_v\left[\exp\left(i\xi\sqrt{-\ln\varepsilon}\left(\frac{T_\varepsilon}{-\ln\varepsilon} - \frac{1}{\beta^2 q}\right)\right)\right] = \exp\left(\frac{-\xi^2}{2q^3\beta^4}\right).$$

The Lévy continuity theorem establishes the result.

2) We now consider the case $q < 0$.

For every $\varepsilon \in]0, v]$, set $\lambda(\varepsilon) := \lambda\varepsilon^{2|q|}$. Using standard analysis, we get:

$$z_{\lambda(\varepsilon)} = |q| + \frac{\varepsilon^{2|q|}}{\beta^2|q|} + \mathcal{O}(\varepsilon^{4|q|}), \quad \varepsilon \to 0.$$

a) Recall that the Laplace transform of T_ε is given by (4.5). We study the limits of the numerator and the denominator. The first one is straightforward since

$$v^{z_{\lambda(\varepsilon)}+q} = v^{\frac{\lambda\varepsilon^{2|q|}}{\beta^2|q|}+\mathcal{O}(\varepsilon^{4|q|})} \xrightarrow[\varepsilon\to 0]{} 1$$

implies that

$$\lim_{\varepsilon \to 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} U\left(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho v\right) = 1.$$

b) Next, we determine the limit of the denominator of (4.5). According to the definition (4.2) of the function U, we have:

$$U\left(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} M\left(q + z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right)$$

$$+ \frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} M\left(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right).$$

Note that:

$$\lim_{\varepsilon \to 0} \Gamma(2z_{\lambda(\varepsilon)}) = \Gamma(2|q|), \quad \lim_{\varepsilon \to 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} = 1, \quad \lim_{\varepsilon \to 0} M\left(q + z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = 1.$$

Using moreover

$$(5.34) \qquad \Gamma(x) = \frac{1}{x} - \gamma + o(1), \quad x \to 0,$$

we have:

$$\frac{1}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} = \rho^{2q+\mathcal{O}(\varepsilon^{2|q|})} \varepsilon^{2q+\mathcal{O}(\varepsilon^{2|q|})} \frac{1}{\Gamma} \left(\frac{\lambda\varepsilon^{2|q|}}{\beta^2|q|} + \mathcal{O}(\varepsilon^{4|q|}) \right) \xrightarrow[\varepsilon \to 0]{} \rho^{2q} \frac{\lambda}{\beta^2|q|}.$$

• We begin with studying the case where $2q$ is not an integer, i.e. $2|q| \notin \mathbb{N}$. We have successively:

$$\lim_{\varepsilon \to 0} \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} = \frac{\Gamma(2q)}{\Gamma(2q)} = 1, \quad \lim_{\varepsilon \to 0} M\left(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = 1.$$

Finally,

$$\lim_{\varepsilon \to 0} \varepsilon^{z_{\lambda(\varepsilon)}+q} U\left(z_{\lambda(\varepsilon)} + q, 1 + 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = 1 + \rho^{2q}\Gamma(2|q|)\frac{\lambda}{\beta^2|q|},$$

and

$$(5.35) \qquad \lim_{\varepsilon \to 0} \mathbb{E}_v \left[e^{-\lambda \varepsilon^{2|q|} T_\varepsilon} \right] = \frac{1}{1 + \rho^{2q}\Gamma(2|q|)\frac{\lambda}{\beta^2|q|}}.$$

• The case $-2q \in \mathbb{N}$ is more complicated.

According to the asymptotic expansion (5.32) of the function $\frac{1}{\Gamma}$ in the vicinity of $2q = -2|q|$, we deduce:

$$\frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(q - z_{\lambda(\varepsilon)})} = \frac{\frac{1}{\Gamma}(q - z_{\lambda(\varepsilon)})}{\frac{1}{\Gamma}(-2z_{\lambda(\varepsilon)})} = \frac{\frac{1}{\Gamma}(-2|q| - \frac{\lambda}{\beta^2|q|}\varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|}))}{\frac{1}{\Gamma}(-2|q| - 2\frac{\lambda}{\beta^2|q|}\varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|}))} \xrightarrow[\varepsilon \to 0]{} \frac{1}{2}, \quad 2|q| \in \mathbb{N},$$

$$\frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(q + z_{\lambda(\varepsilon)})} (\rho\varepsilon)^{-2z_{\lambda(\varepsilon)}} \xrightarrow[\varepsilon \to 0]{} \Gamma(2|q|)\rho^{2q}\frac{\lambda}{\beta^2|q|}.$$

It remains to study $M\left(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right)$. From (5.27), it can be written as:

$$M\left(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)} \left(q - z_{\lambda(\varepsilon)}\right)_k \rho^k \frac{\varepsilon^k}{k!}.$$

▷ The terms of the series corresponding to $k \in \{1, \ldots, 2|q| - 1\}$ converge to 0 since relation (5.32) implies that the ratio $\dfrac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)}$ is bounded.

▷ The terms of the series corresponding to $k \in \{2|q|+1, \dots\}$ converge to 0 when ε goes to 0, uniformly in k : the product $\left(q - z_{\lambda(\varepsilon)}\right)_k$ contains $k-1$ bounded factors and one factor containing $\varepsilon^{2|q|}$; more precisely,

$$
\begin{aligned}
\left(q - z_{\lambda(\varepsilon)}\right)_k &= \left(-2|q| - \tfrac{\lambda}{\beta^2|q|}\varepsilon^{2|q|} + \mathcal{O}(\varepsilon^{4|q|})\right)_k \\
&\sim (-1)^{2|q|+1}(2|q|)!(-2|q|+k-1)!\frac{\lambda}{\beta^2|q|}\varepsilon^{2|q|}, \quad \varepsilon \to 0.
\end{aligned}
$$

$$
\frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + k)} \sim \frac{1}{(-1)^{2|q|-1}(2|q|-1)!\left(-2\frac{\lambda}{\beta^2|q|}\varepsilon^{2|q|}\right)((-2|q|+k)!)}.
$$

▷ The only remaining term corresponds to $k = 2|q|$:

$$
\left(q - z_{\lambda(\varepsilon)}\right)_{2|q|} \sim (-1)^{2|q|}(2|q|)!
$$

and

$$
\frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + 2|q|)} \sim \frac{1}{(-1)^{2|q|-1}(2|q|-1)!\left(-2\frac{\lambda}{\beta^2|q|}\varepsilon^{2|q|}\right)}.
$$

Consequently

$$
\lim_{\varepsilon \to 0} \left(q - z_{\lambda(\varepsilon)}\right)_{2|q|} \frac{\Gamma(1 - 2z_{\lambda(\varepsilon)})}{\Gamma(1 - 2z_{\lambda(\varepsilon)} + 2|q|)} \rho^{2|q|} \frac{\varepsilon^{2|q|}}{(2|q|)!} = \frac{\beta^2|q|}{2\lambda} \frac{\rho^{2|q|}}{\Gamma(2|q|)},
$$

$$
\lim_{\varepsilon \to 0} M\left(q - z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \rho\varepsilon\right) = 1 + \frac{\beta^2|q|}{2\lambda} \frac{\rho^{2|q|}}{\Gamma(2|q|)}.
$$

Finally we get (5.35).

3) Let us now consider the case $q = 0$, then $\frac{2}{\beta^2} = \frac{1}{a}$ and $\rho = \frac{b}{a}$.

For every $\varepsilon \in]0, v]$, let $\lambda(\varepsilon) = \frac{\lambda}{(\ln \varepsilon)^2}$. Then, according to (4.4) we have:

$$
z_{\lambda(\varepsilon)} = \sqrt{\frac{\lambda}{a}} \frac{1}{\ln(1/\varepsilon)}.
$$

As previously, we determine the limits of the numerator and the denominator of (4.5). It is clear that we have:

$$
\lim_{\varepsilon \to 0} v^{z_{\lambda(\varepsilon)}} = 1, \quad \lim_{\varepsilon \to 0} \varepsilon^{z_{\lambda(\varepsilon)}} = e^{-\sqrt{\frac{\lambda}{a}}}.
$$

We deduce the limit of the numerator of (4.5):

$$
\lim_{\varepsilon \to 0} v^{z_{\lambda(\varepsilon)}} U\left(z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{bv}{a}\right) = U\left(0, 1; \frac{bv}{a}\right) = 1.
$$

We now determine the limit of the denominator of (4.5) using (4.2), (5.34) and

$$
\lim_{\varepsilon \to 0} \frac{\Gamma(-2z_{\lambda(\varepsilon)})}{\Gamma(-z_{\lambda(\varepsilon)})} = \lim_{\varepsilon \to 0} \frac{\Gamma(2z_{\lambda(\varepsilon)})}{\Gamma(z_{\lambda(\varepsilon)})} = \frac{1}{2},
$$

$$
\lim_{\varepsilon \to 0} M\left(z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a}\right) = \lim_{\varepsilon \to 0} M\left(-z_{\lambda(\varepsilon)}, 1 - 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a}\right) = 1.
$$

Combining the previous limits, we get:

$$
\lim_{\varepsilon \to 0} \left\{\varepsilon^{z_{\lambda(\varepsilon)}} U\left(z_{\lambda(\varepsilon)}, 1 + 2z_{\lambda(\varepsilon)}; \frac{b\varepsilon}{a}\right)\right\} = \frac{1}{2}(1 + e^{2\sqrt{\frac{\lambda}{a}}})e^{-\sqrt{\frac{\lambda}{a}}} = \cosh\left(\sqrt{\frac{\lambda}{a}}\right)
$$

and finally,

$$\lim_{\varepsilon \to 0} \mathbb{E}_v \left[e^{-\lambda(\varepsilon)\,T_\varepsilon} \right] = \frac{1}{\cosh\left(\sqrt{\frac{\lambda}{a}} \right)}.$$

This is the Laplace transform of the hitting time of the level $1/\sqrt{2a}$ by a reflected Brownian motion starting at 0, see for instance ([22] p. 100). $\qquad\square$

5.6. Proof of Proposition 4.6.

Proof. Let $v < \varepsilon$.

(i) We first prove the identities (4.7) and (4.8) concerning the case $q > 0$. Recall that from Theorem 4.1 we have

$$(5.36) \qquad \mathbb{E}_v \left[\exp(-\lambda\,T_\varepsilon) 1_{\{T_\varepsilon < \infty\}} \right] = \frac{v^{z_\lambda + q} M\left(z_\lambda + q, 1 + 2z_\lambda, \rho v\right)}{\varepsilon^{z_\lambda + q} M\left(z_\lambda + q, 1 + 2z_\lambda, \rho\varepsilon\right)} = \frac{N_1(v)}{N_1(\varepsilon)}$$

with $\rho = 2b/\beta^2$ and

$$(5.37) \qquad z_\lambda = \sqrt{\frac{2\lambda}{\beta^2} + q^2} = q + \frac{\lambda}{\beta^2 q} + o(\lambda).$$

By the identity (4.6) with $\alpha = 2q$ and $x := \frac{\lambda}{\beta^2 q} + o(\lambda) = o(1)$, we obtain

$$\begin{aligned} N_1(v) &= v^{2q}\left(1 + \ln v \frac{\lambda}{\beta^2 q} + o(\lambda)\right)\left(\hat{b}_{2q}(\rho v) + b_{2q}(\rho v)\frac{\lambda}{\beta^2 q} + o(\lambda)\right) \\ &= v^{2q}\left[\hat{b}_{2q}(\rho v) + \left(\hat{b}_{2q}(\rho v)\ln v + b_{2q}(\rho v)\right)\frac{\lambda}{\beta^2 q} + o(\lambda)\right]. \end{aligned}$$

Then, we deduce (4.7) and (4.8).

(ii) In the case $q < 0$, $z_\lambda = -q - \frac{\lambda}{\beta^2 q} + o(\lambda)$ and therefore

$$x := z_\lambda + q = -\frac{\lambda}{\beta^2 q} + o(\lambda) = o(1), \quad \lambda \to 0.$$

According to the definition of M, i.e. formula (4.1), we have:

$$(5.38) \qquad M\left(x, 1 - 2q + 2x, \rho v\right) = 1 + \left(\sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k}\right) x + o(x).$$

Consequently,

$$\begin{aligned} N_1(v) &= \left(1 - \ln v \frac{\lambda}{\beta^2 q} + o(\lambda)\right)\left(1 - \left[\sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k}\right]\frac{\lambda}{\beta^2 q} + o(\lambda)\right) \\ &= 1 - \left(\ln v + \left[\sum_{k \geq 1} \frac{1}{(1 - 2q)_k} \frac{(\rho v)^k}{k}\right]\right)\frac{\lambda}{\beta^2 q} + o(\lambda). \end{aligned}$$

This gives (4.9). $\qquad\square$

5.7. Proof of Proposition 4.8.

Proof. We suppose here that $v > \varepsilon$. We proceed similarly to the proof of Proposition 4.6. We begin with the case $q > 0$. Recall that from Theorem 4.1 we have

$$(5.39) \qquad \mathbb{E}_v \left[\exp(-\lambda\,T_\varepsilon)\right] = \frac{v^{z_\lambda + q} U\left(z_\lambda + q, 1 + 2z_\lambda, \rho v\right)}{\varepsilon^{z_\lambda + q} U\left(z_\lambda + q, 1 + 2z_\lambda, \rho\varepsilon\right)} = \frac{N_2(v)}{N_2(\varepsilon)}$$

with $\rho = 2b/\beta^2$, z_λ which has been defined by (5.37) and:

$$N_2(v) := v^{z_\lambda + q} U\left(z_\lambda + q, 1 + 2z_\lambda, \rho v\right).$$

(i) Case $2q \in [0, \infty[\backslash\{1, 2, \dots\}$

We write $z_\lambda = q + x$ where $x := \dfrac{\lambda}{\beta^2 q} + o(\lambda) = o(1)$. We set:

$$N_2(x, v) := N_2(v) = v^{2q+x} U\left(2q + x, 1 + 2q + 2x, \rho\, v\right).$$

By the definition (4.2) of the function U, we have:

(5.40)

$$
\begin{aligned}
U\left(2q + x, 1 + 2q + 2x, \rho\, v\right) &= \frac{\Gamma(-2q - 2x)}{\Gamma(-x)} M\left(2q + x, 1 + 2q + 2x, \rho\, v\right) \\
&\quad + \frac{\Gamma(2q + 2x)}{\Gamma(2q + x)} (\rho\, v)^{-2q - 2x} M\left(-x, 1 - 2q - 2x; \rho\, v\right).
\end{aligned}
$$

Using (5.32), (5.38) and

(5.41) $\qquad \Gamma(x_0 + x) = \Gamma(x_0)\big(1 + \psi(x_0)\, x + o(x)\big), \quad x \to 0, \ x_0 \in \mathbb{R}\backslash\{0, -1, \dots\},$

we get

$$
\begin{aligned}
U(x + 2q, 1 + 2x + 2q, \rho\, v) &= (\rho\, v)^{-2q} \Big\{ 1 + \Big[\psi(2q) - 2\ln(\rho\, v) \\
&\quad - \sum_{n \geq 1} \frac{1}{(1 - 2q)_n} \frac{(\rho\, v)^n}{n} - \Gamma(-2q)\hat{b}_{2q}(\rho\, v)(\rho\, v)^{2q} \Big] x + o(x) \Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
N_2(x, v) &= \rho^{-2q} \Big\{ 1 + \Big[\psi(2q) - 2\ln\rho - \ln v \\
&\quad - \sum_{n \geq 1} \frac{1}{(1 - 2q)_n} \frac{(\rho\, v)^n}{n} - \Gamma(-2q)\hat{b}_{2q}(\rho\, v)(\rho\, v)^{2q} \Big] x + o(x) \Big\}.
\end{aligned}
$$

We easily deduce (4.11).

(ii) In the case where $2q = m$, where m is a positive integer, we can proceed as in the above step. However to perform this method needs tedious calculations. We propose another approach which is based on the fact this case can be obtained by a limit procedure. Indeed let f_q be the function defined by (4.11) where $q > 0$ and $2q$ does not belong to $\{1, 2, \cdots\}$. We set:

$$2q = m + y, \quad -1 < y < 1.$$

We will take later $y \to 0$, i.e. $q \to m/2$.

We modify f_q as follows:

$$f_q(x) = A_1(q) + A_2(q)$$

where $x > 0$ is fixed and:

$$
\begin{aligned}
A_1(q) &:= \frac{1}{\beta^2 q} \Big(\ln x + \sum_{n=1}^{m-1} \frac{1}{n(1 - 2q)_n} (\rho x)^n \Big), \\
A_2(q) &:= \frac{1}{\beta^2 q} \Big(\sum_{n \geq 0} \lambda_n(y)(\rho x)^{n+m} \Big), \\
\lambda_n(y) &:= \frac{(m + y)\Gamma(-m - y)}{(n + m)\Gamma(1 + n - y)} \Big(\frac{(\rho x)^y \Gamma(1 + n - y)}{\left(1 + \frac{y}{n+m}\right) n!} - 1 \Big).
\end{aligned}
$$

By (5.41) and (5.32), we have:

$$\lim_{y \to 0} \lambda_n(y) = \frac{(-1)^m}{(n + m)n!\,(m - 1)!} \Big(\psi(n + 1) - \ln(\rho x) + \frac{1}{n + m} \Big).$$

and

(5.42) $\qquad\qquad\qquad f_{\frac{m}{2}}(x) = \lim_{q \to \frac{m}{2}} f_q(x) \text{ exists}$

where

$$f_{\frac{m}{2}}(x) = \frac{2}{\beta^2\,m}\left[\ln x + \frac{1}{(m-1)!}\sum_{n=1}^{m-1}\frac{(-1)^n(m-n-1)!}{n}(\rho\,x)^n\right.$$

$$\left. + \frac{(-1)^m}{(m-1)!}\sum_{n\geq 0}\frac{1}{(n+m)n!}\left(\psi(n+1) - \ln(\rho\,x) + \frac{1}{n+m}\right)(\rho\,x)^{n+m}\right].$$

We claim that this result permits to prove (4.12). Indeed suppose that $2q = m$ where $m \in \{1, 2, \cdots\}$. Recall that $2q = 1 - \frac{2a}{\beta^2}$. Let $(a_n)_{n\geq 1}$ and $(a'_n)_{n\geq 1}$ such that:

$$a_n < a < a'_n, \quad 2q_n := 1 - \frac{2a_n}{\beta^2} \notin \{1, 2, \cdots\}, \quad 2q'_n := 1 - \frac{2a'_n}{\beta^2} \notin \{1, 2, \cdots\}$$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n = a$.

Denote $(V_t^{(a)})_{t\geq 0}$ the diffusion solution of (1.2) with initial value v, parameters a, b and β, where a, v and β are fixed. Then, Remark 3.4 tells us

$$V_t^{(a_n)} \leq V_t^{(a)} \leq V_t^{(a'_n)}, \forall\, t \geq 0.$$

Since $\varepsilon < v$, that relation implies: $T_\varepsilon^{(a_n)} \leq T_\varepsilon^{(a)} \leq T_\varepsilon^{(a'_n)}$. Taking the expectation and using the previous step we obtain:

$$f_{q_n}(v) - f_{q_n}(\varepsilon) = \mathbb{E}_v(T_\varepsilon^{(a_n)}) \leq \mathbb{E}_v(T_\varepsilon^{(a)}) \leq f_{q'_n}(v) - f_{q'_n}(\varepsilon) = \mathbb{E}_v(T_\varepsilon^{(a'_n)}).$$

Finally (4.12) follows taking the limit $n \to \infty$ in the above inequality and using (5.42).

(iii) Case $q = 0$. We have:

$$x := z_\lambda = \sqrt{\frac{2\lambda}{\beta^2}} = o(1), \quad \lambda \to 0.$$

and in (5.39), the numerator $N_2(v)$ equals $N_2(x, v) = (\rho v)^x U(x, 1 + 2x, \rho v)$. Note that we add an extra term ρ^x which cancels when we consider the ratio $N_2(x, v)/N_2(x, \varepsilon)$. With this choice, $x \mapsto N_2(x, v)$ is symmetric, see below.

According to the definition (4.2) of the function U and (5.40), we get:

$$N_2(x, v) = \frac{\Gamma(-2x)}{\Gamma(-x)}(\rho v)^x M(x, 1 + 2x; \rho\,v) + \frac{\Gamma(2x)}{\Gamma(x)}(\rho v)^{-x} M(-x, 1 - 2x; \rho\,v).$$

By

(5.43) $$\frac{1}{\Gamma}(x) = x + \gamma\,x^2 + o(x^2), \quad x \to 0$$

and

$$M(x, 1 + 2x; y) = 1 + x\left(\sum_{n=1}^{\infty}\frac{y^n}{n!\,n}\right) + x^2\left(\sum_{n=1}^{\infty}\left(-H_n - \frac{1}{n}\right)\frac{y^n}{n!\,n}\right) + o(x^2), \quad x \to 0,$$

we obtain

$$\frac{\Gamma(-2x)}{\Gamma(-x)}(\rho\,v)^x M(x, 1 + 2x; \rho\,v) = \frac{1}{2} + x\left[\frac{\gamma}{2} + \frac{\ln(\rho\,v)}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n!\,n}\right]$$

$$+ x^2\left[\frac{\pi^2}{8} + \frac{\gamma^2}{4} + \frac{\ln^2(\rho\,v)}{4} - \frac{1}{2}\sum_{n=1}^{\infty}\left(H_n + \frac{1}{n}\right)\frac{(\rho\,v)^n}{n!\,n} + \frac{\gamma}{2}\ln(\rho\,v) + \frac{\gamma}{2}\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n!\,n}\right.$$

$$+\frac{1}{2}\ln(\rho\,v)\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n!\,n}\Bigg] + o(x^2).$$

By adding the same function in $-x$, we get:

$$N_2(x,v) = 1 + x^2\left[\frac{\pi^2}{4}+\frac{\gamma^2}{2}+\frac{\ln^2(\rho\,v)}{2}-\sum_{n=1}^{\infty}\left(H_n+\frac{1}{n}\right)\frac{(\rho\,v)^n}{n!\,n}+\gamma\,\ln(\rho\,v)\right.$$

$$\left.+\gamma\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n!\,n}+\ln(\rho\,v)\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n!\,n}+o(x^2)\right].$$

Making the quotient $\dfrac{N_2(x,v)}{N_2(x,\varepsilon)}$ and replacing x^2 by $\frac{2\lambda}{\beta^2}$ give (4.13).

(iv) and (v) We now consider $q<0$. Then,

$$x := z_\lambda + q = -\frac{\lambda}{\beta^2 q}+o(\lambda)=o(1),\quad \lambda\to 0.$$

and the denominator $N_2(v)$ in (5.39) is equal to $N_2(x,v)=v^x U(x,1-2q+2x,\rho v)$.
By (4.2), we have:

$$U\left(x,1-2q+2x;\rho\,v\right) = \frac{\Gamma(2q-2x)}{\Gamma(2q-x)}M\left(x,1-2q+2x,\rho\,v\right)$$

$$+\frac{\Gamma(-2q+2x)}{\Gamma(x)}\left(\rho\,v\right)^{2q-2x}M\left(2q-x,1+2q-2x,\rho\,v\right).$$

a) We first deal with the case $2q \in]-\infty,0[\backslash\{-1,-2,\dots\}$. By formula (4.2) defining the function U, (5.32), (5.41), (5.43) and (5.38) we get:

$$U\left(x,1+2x-2q;\rho\,v\right) = 1 + \left[\sum_{n=1}^{\infty}\frac{(\rho\,v)^n}{n(1-2q)_n}-\psi(2q)+\Gamma(-2q)\left(\rho\,v\right)^{2q}\hat{b}_{2q}(\rho\,v)\right]x+o(x).$$

and we deduce (4.14).

b) The case $-2q=m\in\mathbb{N}^*$ can be treated either as in the previous step or a limit procedure, see the above item *(ii)*. The second approach is based on the following modification of the function f_q defined by (4.14) (when $q<0$ and $q\notin\{-1,-2,\cdots\}$):

$$f_q(x) = \frac{1}{\beta^2\,|q|}\left[\ln x+\sum_{n=1}^{\infty}\frac{\Gamma(1+2|q|)}{\Gamma(1+2|q|+n)}\frac{(\rho x)^n}{n}+\Gamma(2|q|)\sum_{n=0}^{m-1}\frac{2|q|}{n-2|q|}\frac{(\rho x)^{n-2|q|}}{n!}\right.$$

$$\left.+\Gamma(2|q|)\frac{2|q|}{m-2|q|}\frac{(\rho x)^{m-2|q|}}{m!}+\Gamma(2|q|)\sum_{n=m+1}^{\infty}\frac{2|q|}{n-2|q|}\frac{(\rho x)^{n-2|q|}}{n!}\right].$$

Taking the limit $q\to -m/2$ in the above identity implies (4.15). $\qquad\square$

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.
2. L.H.R. Alvarez, *On the option interpretation of rational harvesting planning*, J. Math. Biol. **40** (2000), 383–405.
3. L.H.R. Alvarez and L. A. Shepp, *Optimal harvesting of stochastically fluctuating populations*, J. Math. Biol. **37** (1998), 155–177.
4. T. Bastogne, R. Keinj and P. Vallois, *Multinomial model-based formulations of TCP and NTCP for radiotherapy treatment planning*, Journal of Theoretical Biology **279** (2011), 55–62.

5. T. Bastogne, R. Keinj and P. Vallois, *Multinomial model-based formulations of TCP and NTCP for radiotherapy treatment planning*, Journal of Theoretical Biology **312** (2012), 76–86.

6. H. Bateman, *Higher Transcendental Functions 1*, Mac Graw-Hill, 1953.

7. A.N. Borodin and P. Salminen, *Handbook of Brownian motion - facts and formulae, Probability and its applications*, Birkhauser Verlag, Basal, 1996.

8. A. B. O. Daalhuis, *Confluent hypergeometric function*. NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

9. C. Donati-Martin, R. Ghomrasni and M. Yor, *On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options*, Rev. Mat. Iberoam. **17** (2001), 179–193.

10. H. Doss, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B (N.S.) **13** (1977), no. 2, 99–125.

11. S. N. Ethier and T. G. Kurtz, *Markov processes. Characterization and convergence*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986.

12. A. E. Erdélyi, *Higher transcendental functions. Vol. I. Based, in part, on notes left by Harry Bateman and compiled by the Staff of the Bateman Manuscript Project. (Repr. of the orig. 1953 publ. by McGraw-Hill Book Company, Inc., New York).* Bateman Manuscript Project, California Institute of Technology. Malabar, Florida: Robert E. Krieger Publishing Company. XXVI, 302 p. \$22.50 , 1981.

13. M.W. Feldman and J. Roughgarden, *A population stationnary distribution and chance of extinction in a stochastic environment with remarks on the theory of species packing*, Theor. Pop. Biol. **7** (1975), no. 2, 197–207.

14. E. Freitag and R. Busam, *Complex analysis*, Universitext. Springer-Verlag, Berlin, second edition, 2009.

15. G.F. Gause, *The struggle for existence*, Hafner, New York, 1964 (first published in 1934).

16. I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, series and products*, Academic Press, 1980.

17. K. Itô and Jr. McKean, *Diffusion processes and their sample paths*, Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125, Springer-Verlag, Berlin.

18. C. Ji, D. Jiang, N. Shi and D. O'Regan, *Existence, uniqueness, stochastic persistence and global stability of positive solutions of the logistic equation with random perturbation*, Math. Methods Appl. Sci. **30** (2007), no. 1, 77–89

19. D. Jiang and N. Shi, *A note on nonautonomous logistic equation with random perturbation,* J. Math. Anal. Appl. **303** (2005), no. 1, 164–172.

20. D. Jiang, N. Shi and X. Li, *Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation*, J. Math. Anal. Appl. **340** (2008), 588–597.

21. D. Jiang, B.X. Zhang, D.H. Wang and N.Z. Shi, *Existence, uniqueness and global attractivity of positive solutions and MLE of the parameters to the logistic equation with random perturbation*, Science in China Series A: Mathematics **50** (2007), no. 7, 977–986.

22. I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

23. R.J. Kryscio and C. Lefèvre, *On the extinction of the S-I-S stochastic logistic epidemic*, Journal of Applied Probability **27** (1989), 685–694.

24. G.S. Laddle and D.D. Siljak, *Stability of multispecies communities in randomly varying environment*, J. Math. Biol. **2** (1975), 165–178.

25. N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

26. R. Levins, *The effect of random variations of different types on population growth*, Proc. Nat. Acad. Sci. USA **62** (1969), 1061–1065.

27. A.J. Lotka, *Elements of physical biology*, Williams and Wilkins, Baltimore, 1925.

28. X. Mao, G. Marion and E. Renshaw, *Environmental Brownian noise suppresses explosions in population dynamics*, Stoch. Proc. Appl. **97** (2002), no. 1, 95–110.

29. R.M. May, *Stability in Randomly Fluctuating Versus Deterministic Environments*, Amer. Naturalist **107** (1973), no. 957, 621–650.

30. R.H. Norden, *On the distribution of the time of extinction in the stochastic logistic population model*, Adv. Appl. Prob. **14** (1982), no. 4, 687–708.

31. J.R. Norris, *Markov chains*, Cambridge University Press, 1997.

32. S. Pasquali, *The stochastic logistic equation: stationary solutions and their stability*, Rend. Sem. Mat. Univ. Padova **106** (2001), 165–183.

33. R. Pearl, L.J. Reed, *On the rate of growth of the Population of the United States since 1790 and its Mathematical Representation*, Proceedings of the National Academy of Sciences of the United States of America **6** (1920), no. 6, 275–288.
34. E.C. Pielou, *An introductory to mathematical ecology*, Wiley-Interscience, New York, 1975.
35. H. Schurz, *Modeling, analysis and discretization of stochastic logistic equations*, Intern. Journal of Numerical Analysis and Modeling **4** (2007), 178–197.
36. M. Turelli, *Random Environments and Stochastic Calculus*, Theoretical Population Biol. **12** (1977), 140–178.
37. M. Turelli, *A Reexamination of Stability in Randomly Varying versus Deterministic Environments with Comments on the Stochastic Theory of Limiting Similarity*, Theoretical Population Biol. **13** (1978), 244–267.
38. A. Singh and J.P. Hespanha, *A derivative matching approach to moment closure for the stochastic logistic model*, Bulletin of Mathematical Biology **69** (2007), 1909–1925.
39. W.Y. Tan and S. Piantadosi, *On stochastic growth processes with application to stochastic logistic growth*, Statistica Sinica **1** (1991), 527–540.
40. P.F. Verhulst, *Notice sur la loi que la population suit dans son accroissement*, Correspondance mathématique et physique **10** (1838), 113–121.
41. V. Voltera, *Variations and fluctuations of the number of individuals in animal species living together*, Journal du Conseil **3** (1928), 1–51.
42. G.C. Wake and S.D. Watt, *The relaxation of May's conjecture for the logistic equation*, Appl. Math. Lett. **9** (1996), no. 5, 59–62.
43. D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

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