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I. Bondarenko (Kyiv. Nat. Taras Shevchenko Univ.),
R. Kravchenko (Univ. Chicago, USA)

SCHREIER GRAPHS FOR A SELF-SIMILAR ACTION OF THE HEISENBERG GROUP

ГРАФИ ШРАЙЄРА САМОПОДІБНОЇ ДІЇ ГРУПИ ГЕЙЗЕНБЕРГА

We construct a faithful self-similar action of the discrete Heisenberg group with the following properties: This action is self-replicating, finite-state, level-transitive, and noncontracting. Moreover, there exist orbital Schreier graphs of action on the boundary of the tree with different degrees of growth.

Побудовано точну самоподібну дію дискретної групи Гейзенберга з наступними властивостями. Дія є рекурентною, скінченностановою, сферично транзитивною, нестискуючою та існують графи Шрайєра дії групи на межі дерева з різними степенями зростання.

1. Introduction. The celebrated theorem of Gromov shows that the Cayley graphs of a finitely generated group have polynomial growth if and only if the group is virtually nilpotent. There are groups of exponential and intermediate growth between polynomial and exponential, but it is still not clear what are the possible growth rates of Cayley graphs.

In this paper we consider Schreier graphs of groups, which are generalization of Cayley graphs. Schreier graphs model the action of a group on a set. There is no sense to ask the question about possible growth rates of Schreier graphs: every connected regular graph of even degree is a Schreier graph of a free group and one can realize any growth rate with natural restrictions. However this question may be interesting in certain classes of groups and for some natural actions. We want to ask the following question. Given a finitely generated group G and a nested sequence $\{H_n\}_{n\geq 1}$ of subgroups of finite index in G with trivial intersection $\cap_{n\geq 1}H_n = \{e\}$, consider the action of G on the coset tree of $\{H_n\}_{n\geq 1}$.

Question. In what cases the orbital Schreier graphs of the action of G on the boundary of the coset tree have polynomial growth?

It is hard to believe that there is an algebraic characterization of such groups, but rather certain geometric characterization of the action on a tree.

One natural class of groups for the above question are self-similar groups. Every self-similar group is given by its action on a regular rooted tree and we may consider the orbital Schreier graphs of the action on the boundary of the tree. There are examples of self-similar groups with Schreier graphs of polynomial growth with irrational degree, exponential growth, and even intermediate growth (see [1, 2]). It is known that all orbital Schreier graphs have polynomial growth for every contracting self-similar action (see [3], Proposition 2.13.8), where contracting property of the action corresponds to the expanding property of the associated dynamical system. One may consider how far contracting groups are from self-similar groups with polynomial Schreier graphs. Since nilpotent groups have polynomial growth, all their Schreier graphs also have polynomial growth. Therefore it is interesting to understand self-similar actions of nilpotent groups. If we do not add additional restrictions, then one can construct a non-contracting action even of the infinite cyclic group with orbital Schreier graphs of linear growth. But the question happens to be more interesting under additional assumption that the action is finite-state, i.e., can be described by a finite automaton. The result of Nekrashevych and Sidki [4] says that a faithful self-replicating finite-state action of the free abelian group $\mathbb{Z}^n$ is necessary contracting. This result does not hold for all nilpotent groups and the complete picture for finitely generated torsion-free nilpotent groups was described by the authors in [5]. The following problem seems to be interesting: characterize finitely generated finite-state self-similar groups (i. e., groups generated by finite automata), whose all orbital Schreier graphs have polynomial growth.

In this paper we consider the self-similar action of the discrete Heisenberg group constructed by the authors in [5] and prove that this action provides an example of a self-replicating finite-state and non-contracting self-similar action with orbital Schreier graphs of polynomial growth. We also compute the degree of growth for every orbital Schreier graph of this action. It happens that while the action is level-transitive, there are orbital Schreier graphs with different growth degrees. This seems to be the first example of a self-similar group with such properties.

2. Self-similar actions and Schreier graphs. Let X be a finite set with at least two elements. Let $X^* = \{x_1x_2\ldots x_n \colon x_i \in X, n \geq 0\}$ be the free monoid freely generated by X with empty word denoted by $\varnothing$. The set X^n of words of length n is called the *n-th level*. We will also consider the set $X^{\mathbb{N}}$ of all infinite words $x_1x_2\ldots$, $x_i \in X$.

Self-similar actions. Self-similar group actions are specific actions of a group on the spaces X^* and $X^{\mathbb{N}}$. We are not going to give definition of a self-similar action, but rather define how one can construct every self-similar action of a group with transitive action on X (see [3] for more information about self-similar groups).

Let G be a group, H a subgroup of finite index in G, and $\phi\colon H \to G$ a homomorphism. Let D be a set of coset representatives for H in G and let X be in bijection with D so that $|X| = |D| = [G\colon H]$ and $D = \{d_x \colon x \in X\}$. The *self-similar action* of the group G on the spaces X^* and $X^{\mathbb{N}}$ *associated to the triple* (G, ϕ, D) is constructed as follows: for every $x \in X$ and $v \in X^* \cup X^{\mathbb{N}}$ define the action of an element $g \in G$ recursively by the rule

$$g(\varnothing) = \varnothing \qquad \text{and} \qquad g(xv) = yh(v) \quad \text{with} \quad h = \phi\big(d_y^{-1} g d_x\big), \tag{1}$$

where $y \in X$ is the unique element such that $d_y^{-1} g d_x \in H$. The action may be not faithful. The kernel of the action is equal to the maximal normal ϕ-invariant subgroup of G called the *ϕ-core* ([3], Proposition 2.7.5). Every self-similar action of a group G with transitive action on X corresponds to a certain triple (G, ϕ, D) as above (see [3], Chapter 2).

Note that each self-similar action preserves the length of words and one can restrict the action to every level X^n. The action is called *level-transitive* if it is transitive on each level. A self-similar action is called *self-replicating* if it is transitive on X and ϕ is surjective.

Finite-state actions. The element h from equation (1) is called the *state* of g at x and is denoted by $g|_x$; iteratively one can define the state of g at every finite word by the rule $g|_{x_1x_2\ldots x_n} = g|_{x_1}|_{x_2}\cdots|_{x_n}$. A self-similar action (G, X^*) is called *finite-state* if for every $g \in G$ the set $\{g|_v \colon v \in X^*\}$ is finite. A finite-state self-similar action of a finitely generated group can be given by a finite graph (automaton): the vertices are the generators of the group and all their states, and for every state s we have an arrow $s \to s|_x$ labeled by $x|s(x)$ for each $x \in X$.

A self-similar action is called *contracting* if there exists a finite set $N \subset G$ with the property that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in N$ for all words $v \in X^*$ of length $\geq n$. Every contracting action is finite-state.

Schreier graphs. Let G be a group with a finite generating set S and H a subgroup of G. The Schreier coset graph for H in G with respect to S is the graph whose vertices are the left cosets $G/H = \{gH : g \in G\}$ and the edges are (gH, sgH) for $s \in S \cup S^{-1}$. Let G be acting on a set M from the left. The (simplicial) *Schreier graph* $\Gamma(G,S,M)$ *of the action* is the graph with the set of vertices M and two points $u, v \in M$ are connected by an edge if $s(u) = v$ for some $s \in S \cup S^{-1}$. The connected component of $\Gamma(G,S,M)$ around a point $w \in M$ is called the *orbital Schreier graph* $\Gamma_w(G,S)$. The graph $\Gamma_w(G,S)$ is the Schreier coset graph of G with respect to the stabilizer $St_G(w)$. Now to every self-similar action of the group G it is associated an uncountable family of orbital Schreier graphs $\Gamma_w(G,S)$ for $w \in X^{\mathbb{N}}$. The graphs $\Gamma_w(G,S)$ bring important information about the group action and were studied in relation to such topics as spectrum, growth, amenability, etc. (see [1, 2] and the reference therein).

3. Our results. Let G be the discrete Heisenberg group, which consists of upper unitriangular matrices of dimension 3 with integer coefficients. We use notation

$$(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

so that $G = \big\{(x,y,z) : x,y,z \in \mathbb{Z}\big\}$. The multiplication of the group elements written in this form can be performed by the rule

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2).$$

It is easy to see that the elements $(1,0,0)$ and $(0,1,0)$ are generators of the group G, and the element $(0,0,1)$ is a generator of the center $Z(G) = \big\{(0,0,z) : z \in \mathbb{Z}\big\}$ of the group.

Consider the subgroup $H = \big\{(x,2y,2z) : x,y,z \in \mathbb{Z}\big\}$ of index four in G and the map

$$\phi : H \to G, \qquad \phi(x,y,z) = (x, y/2, z/2).$$

Proposition 1. *The map ϕ is an isomorphism with trivial ϕ-core.*

Proof. It is clear that the map ϕ is bijective. Let us show that it is a homomorphism

$$\phi(x_1,y_1,z_1)\phi(x_2,y_2,z_2) = (x_1, y_1/2, z_1/2)(x_2, y_2/2, z_2/2) =$$

$$= (x_1+x_2, y_1/2+y_2/2, z_1/2+z_2/2+x_1y_2/2) =$$

$$= \phi\big((x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2)\big) =$$

$$= \phi\big((x_1,y_1,z_1)(x_2,y_2,z_2)\big).$$

It is left to compute the ϕ-core K. If $(x,y,z) \in K$ then $(x, y/2^k, z/2^k) \in K$ for all $k \in \mathbb{N}$ and therefore $y = z = 0$. Conjugating by $(0,1,0)$ we get

$$(0,1,0)^{-1}(x,0,0)(0,1,0) = (x,0,x) \in K$$

and hence $x = 0$. We have proved that the ϕ-core is trivial.

Proposition 1 is proved.

***Corollary* 1.** *For every choice of coset representatives D for H in G, every self-similar action associated to the triple (G, ϕ, D) is faithful.*

We choose the set of coset representatives $D = \{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}$ for H in G and construct the self-similar action (G, X^*) of the group G over the alphabet $X = \{1,2,3,4\}$ associated to the triple (G, ϕ, D). The action of the generators $a = (1,0,0)$ and $b = (0,1,0)$ of the group satisfies the following recursions:

$$a(1v) = 1a(v), \qquad a(2v) = 4a(v), \qquad a(3v) = 3a(v), \qquad a(4v) = 2(b^{-1}ab)(v),$$

$$b(1v) = 2v, \qquad b(2v) = 1b(v), \qquad b(3v) = 4v, \qquad b(4v) = 3b(v),$$

for $v \in X^*$; or in wreath recursion notation

$$a = (a, a, a, b^{-1}ab)(2,4), \qquad b = (e, b, e, b)(1,2)(3,4).$$

Theorem 1. *The constructed above self-similar action (G, X^*) of the Heisenberg group is faithful, level-transitive, self-replicating, finite-state, and non-contracting.*

Proof. The action is self-replicating, because it is transitive on X and the virtual endomorphism ϕ is surjective. Every self-replicating action is level-transitive by [3] (Proposition 2.8.2).

The elements a and b are finite-state, namely the computations

$$b^{-1}ab = \big(b^{-1}ab, a, b^{-2}ab^2, a\big)(1,3),$$

$$b^{-2}ab^2 = \big(b^{-1}ab, b^{-1}ab, b^{-1}ab, b^{-2}ab^2\big)(2,4)$$

show that the states of a are $a, b^{-1}ab, b^{-2}ab^2$ and the states of b are e, b. Hence the action (G, X^*) is finite-state.

Since the action is faithful and the group G is torsion-free, all powers a^{2n} are different. Since

$$a^{2n} = \big(a^{2n}, (ab^{-1}ab)^n, a^{2n}, (b^{-1}aba)^n\big),$$

we get $a^{2n}|_1 = a^{2n}$ and therefore the action is not contracting.

Theorem 1 is proved.

Since nilpotent groups have polynomial growth, all their Schreier graphs also have polynomial growth. Hence the constructed above self-similar action of the Heisenberg group provides an example of a self-replicating finite-state and non-contracting self-similar action with orbital Schreier graphs Γ_w of polynomial growth.

For every faithful level-transitive action, the action on every orbit on $X^{\mathbb{N}}$ is faithful. In the next theorem we describe orbits, where the action of the group G is free, and compute the growth of every orbital Schreier graph.

Theorem 2. *Let Ω be the collection of all pre-periodic sequences from $X^{\mathbb{N}}$ together with all sequences obtained from those by arbitrary changes of letters $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. Then the stabilizer $St_G(w)$ for $w \in \Omega$ is the infinite cyclic group and the stabilizer $St_G(w)$ for $w \in X^{\mathbb{N}} \setminus \Omega$ is trivial.*

Proof. Let us describe which elements $(x, y, z) \in H$ have a fixed letter depending on the parity of x, y, z. We use notation $g \cdot x = y \cdot h$ instead of equality in equation (1). By direct computation we get

$$(2x+1, 2y, 2z+1) \cdot 2 = 2 \cdot (2x+1, y, z+x+1),$$

$$(2x+1,2y,2z+1)\cdot 4 = 4\cdot(2x+1,y,z+x+1),$$

$$(x,2y,2z)\cdot 1 = 1\cdot(x,y,z),$$

$$(x,2y,2z)\cdot 3 = 3\cdot(x,y,z),$$

$$(2x,2y,2z)\cdot 2 = 2\cdot(2x,y,z+x),$$

$$(2x,2y,2z)\cdot 4 = 4\cdot(2x,y,z+x),$$

while the other triples do not have a fixed letter. Notice that in the above equations the states at 2 and 4 are equal, and the same holds for 1 and 3. That explains why one can make arbitrary letter changes $2\leftrightarrow 4$ and $1\leftrightarrow 3$ considering fixed sequences.

Let $g=(x,y,z)\in G$ fix an infinite sequence $w=x_1x_2\ldots\in X^{\mathbb{N}}$. If $y\neq 0$ then the equations above show that for some n the y-component of $g|_{x_1x_2\ldots x_n}$ will be odd and this element doesn't have a fixed letter. Hence $y=0$. Also notice that the x-component of each $g|_{x_1x_2\ldots x_n}$ is equal to x. We will trace what happens with the component z_n of $g|_{x_1x_2\ldots x_n}=(x,0,z_n)$. The sequence $\{z_n\}_{n\geq 1}$ satisfies the recurrence

$$z_0=z,\qquad z_{n+1}=\begin{cases} z_n/2, & x_n=1,3,\\ (z_n+x)/2, & x_n=2,4.\end{cases} \tag{2}$$

One can express z_n in terms of the word $v=x_1x_2\ldots x_n$ as

$$z_n=\frac{z+\alpha_v x}{2^n},\quad \text{where}\quad \alpha_v=\alpha_1+\alpha_2 2+\ldots+\alpha_n 2^{n-1}\quad\text{and}\quad \alpha_i=\begin{cases}0, & x_n=1,3,\\ 1, & x_n=2,4.\end{cases} \tag{3}$$

Since the action is finite-state, the sequence $\{z_n\}_{n\geq 1}$ assumes a finite number of values. Let the value $t=z_{n_1}$ be assumed infinitely many times and present the sequence w in the form $w = v_0v_1v_2\ldots$ with the shortest possible nonempty words $v_i\in X^*$ so that

$$(x,0,z)\cdot v_0=v_0\cdot(x,0,t)\qquad\text{and}\qquad (x,0,t)\cdot v_i=v_i\cdot(x,0,t)$$

for all $i\in\mathbb{N}$. Applying expression (3) to the element $(x,0,t)$ and the word v_i we get

$$t=\frac{t+\alpha_{v_i}x}{2^{|v_i|}}\ \Rightarrow\ \left(2^{|v_i|}-1\right)t=\alpha_{v_i}x\ \Rightarrow\ \frac{t}{x}=\frac{\alpha_{v_i}}{2^{|v_i|}-1}, \tag{4}$$

where $|v|$ denotes the length of the word v.

If $x=0$ then $t=0$ and $(x,0,t)$ is the identity element.

Let $x\neq 0$. Let us prove that for $i\geq 1$ all words v_i have the same length, all numbers α_{v_i} are equal, and hence all words v_i coincide up to arbitrary changes $1\leftrightarrow 3$ and $2\leftrightarrow 4$. Consider two words v_i and v_j and let $n_i=|v_i|\leq|v_j|=n_j$. Then we have

$$\alpha_{v_i}(2^{n_j}-1)=\alpha_{v_j}(2^{n_i}-1)\ \Rightarrow\ 2^{n_i}(2^{n_j-n_i}\alpha_{v_i}-\alpha_{v_j})=\alpha_{v_i}-\alpha_{v_j}.$$

Hence $\alpha_{v_i}\equiv\alpha_{v_j} \bmod 2^{n_i}$. It follows that the word v_i is a prefix of v_j up to arbitrary changes $1\leftrightarrow 3$ and $2\leftrightarrow 4$. Then $(x,0,t)\cdot u=u\cdot(x,0,t)$ for the prefix u of v_j of length n_i. Hence $n_i=n_j$ by the minimality of v_j in the presentation of w. The claim is proved.

We have proved that if the sequence w has a non-trivial stabilizer then it is pre-periodic up to changes $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. Hence $w \in \Omega$. Let us compute the stabilizer of such a sequence w. We can assume that $w = uvv\ldots$. Take the triple $(x, 0, z) \in St_G(vv\ldots)$ with the smallest positive x. The proportion in (4) shows that every element in $St_G(vv\ldots)$ is of the form $(mx, 0, mz)$. Since $(mx, 0, mz) = (x, 0, z)^m$, the stabilizer $St_G(vv\ldots)$ is the infinite cyclic group generated by $(x, 0, z)$. The stabilizer $St_G(w)$ is the preimage of $St_G(vv\ldots)$ under the isomorphism $\phi^{|u|}$, and therefore it is an infinite cyclic group too.

Conversely, let w coincide with $uvv\ldots$ up to changes $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. There exist $x, z \in \mathbb{Z}$, $x \neq 0$, such that $\big(2^{|v|} - 1\big)z = \alpha_v x$ and z, x are divisible by $2^{|v|}$. Then $(x, 0, z) \cdot v = v \cdot (x, 0, z)$. Since the action is self-replicating, there exists $g \in \phi^{-|u|}(G)$ such that $\phi^{|u|}(g) = (x, 0, z)$. Then g fixes the sequence w. The statement is proved.

***Corollary* 2.** *The growth of orbital Schreier graph Γ_w for $w \in X^{\mathbb{N}} \setminus \Omega$ is polynomial of degree 4. The growth of orbital Schreier graph Γ_w for $w \in \Omega$ is polynomial of degree 3.*

Proof. The Heisenberg group G has polynomial growth of degree 4. Since $St_G(w)$ is trivial for $w \in X^{\mathbb{N}} \setminus \Omega$, the Schreier graph Γ_w is just the Cayley graph of the group and therefore it has polynomial growth of degree 4.

Let $w \in \Omega$. Then the stabilizer $H = St_G(w)$ is the infinite cyclic group generated by $h = (x, 0, z)$ with $x > 0$. We consider the orbital Schreier graph Γ_w as the Schreier coset graph for H in G with the generating set $S = \big\{(\pm1, 0, 0), (0, \pm1, 0)\big\}$. Let $g_1, g_2, \ldots, g_m$ be coset representatives of length $\leq n$ for each coset in the ball of radius n in Γ_w around the vertex H. Each product $g_i h^k$ for $k = 1, \ldots, n$ is a group element of length $\leq 2n$. Since the number of such products is mn and the group G has polynomial growth of degree 4, the growth degree of the graph Γ_w is not greater than 3.

For the converse, consider the products $g_i h^k$ for $|k| \leq 2n$. Let $f \in G$ be an element of length $\leq n$. There exists g_i such that $g_i H = fH$ and thus $f^{-1} g_i = h^l$ for some $l \in \mathbb{Z}$. Note that the element $h^l = (lx, 0, lz)$ expressed as a word in S has length $\geq |l|$. Therefore $|l| \leq 2n$. We have proved that the products $g_i h^k$ for $|k| \leq 2n$ cover the ball of radius n in G. Hence the Schreier graph Γ_w has growth degree not less than 3 and the statement is proved.

A few remarks about the recurrence from equation (2). If x is odd and we want the sequence $\{z_n\}_{n\geq1}$ to assume only integer values, then the recurrence can be written in the form

$$z_{n+1} = \begin{cases} z_n/2, & z_n \text{ is even,} \\ (z_n + x)/2, & z_n \text{ is odd,} \end{cases}$$

independently on the word $x_1 x_2 \ldots$, i. e., it is iteration of a single map. Interestingly, the maps of this form were studied in relation to $3x+1$ conjecture (see [6, 7]). For even x we have a freedom to choose between $z_n/2$ and $(z_n + x)/2$. Let us encode our choice in the sequence $w = x_1 x_2 \ldots \in \{0, 1\}^{\mathbb{N}}$ and write

$$z_{n+1} = \begin{cases} z_n/2, & x_n = 0, \\ (z_n + x)/2, & x_n = 1. \end{cases}$$

Then the problem that we solved in the proof on Theorem 2 is the classification of all $x, z \in \mathbb{Z}$ and $w \in \{0, 1\}^{\mathbb{N}}$ such that the corresponding sequence $\{z_n\}_{n\geq1}$ with $z_1 = z$ assumes only integers (even) values.

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