UDC 517.9

D. Mosić (Univ. Niš, Serbia)

REPRESENTATIONS FOR THE GENERALIZED INVERSES OF A MODIFIED OPERATOR*

ЗОБРАЖЕННЯ УЗАГАЛЬНЕНИХ ОБЕРНЕНИХ ОПЕРАТОРІВ ДЛЯ МОДИФІКОВАНОГО ОПЕРАТОРА

Some explicit representations for the generalized inverses of a modified operator $A + YGZ$ are derived under some conditions, where $A, Y, Z,$ and G are operators between Banach spaces. These results generalize the recent works about the Drazin inverse and the Moore – Penrose inverse of complex matrices and Hilbert-space operators.

За певних умов встановлено деякі явні зображення узагальнених обернених операторів для модифікованого оператора $A + YGZ$, де A, Y, Z та G — оператори між банаховими просторами. Ці результати узагальнюють останні роботи щодо матриць Дразіна та Мура – Пенроуза, обернених до комплексних матриць та операторів у гільбертовому просторі.

1. Introduction. Let H, K and L be arbitrary Banach spaces. We use $\mathcal{B}(H,K)$ to denote the set of all linear bounded operators from H to K. Set $\mathcal{B}(H) = \mathcal{B}(H,H)$. For an operator $A \in \mathcal{B}(H,K)$, the symbols $N(A)$ and $R(A)$ will denote the null space and the range of A, respectively. An operator $B \in \mathcal{B}(K,H)$, $B \neq 0$, is an *outer inverse* of A, if $BAB = B$ holds. The outer inverse B of A with the prescribed range T and the null space S is uniquely determined and it is denoted by $A^{(2)}_{T,S}$. Some results concerning the outer invertible operators on Banach spaces are presented in [5].

An element a of a unital Banach algebra $\mathcal{A}$ is said to be *hermitian*, if $\|\exp(ita)\| = 1$, for all $t \in \mathbb{R}$. If $\mathcal{A}$ is a C^*-algebra, then $a \in \mathcal{A}$ is Hermitian if and only if a is self-adjoint.

The *Moore – Penrose inverse* of $A \in \mathcal{B}(H,K)$ is the operator $B \in \mathcal{B}(K,H)$ which satisfies

$$(1)\ ABA = A, \quad (2)\ BAB = B, \quad (3)\ AB \text{ is Hermitian}, \quad (4)\ BA \text{ is Hermitian}.$$

Recall that according to [7] (Lemma 2.1), there is at most one Moore – Penrose inverse and it is denoted by $A^\dagger$. If A is a linear bounded operator between two Hilbert spaces, then $A^\dagger$ exists if and only if $R(A)$ is closed (see [1]).

Let $W \in \mathcal{B}(K,H)$ be a fixed nonzero operator. An operator $A \in \mathcal{B}(H,K)$ is called Wg-*Drazin invertible*, if there exists some $B \in \mathcal{B}(H,K)$ satisfying

$$(5)\ BWAWB = B, \quad (6)\ AWB = BWA, \quad (7)\ A - AWBWA \text{ is quasinilpotent}.$$

The Wg-Drazin inverse B of A is unique, if it exists, and denoted by $A^{d,W}$ [3]. If $H = K$, $A \in \mathcal{B}(H)$ and $W = I_H$, then $B = A^d$ is the generalized Drazin inverse, or Koliha – Drazin inverse of A [6]. The Drazin inverse is a special case of the generalized Drazin inverse for which $A - A^2B$ is nilpotent instead of $A - A^2B$ is quasinilpotent. Obviously, if A is Drazin invertible, then it is generalized Drazin invertible. The group inverse is the Drazin inverse for which the condition $A - A^2B$ is nilpotent is replaced with $A = ABA$. We use $A^{\#}$ to denote the group inverse of A. Denote by (5) the equality $AB = BA$ and by (6) the condition $A - ABA$ is quasinilpotent.

* The author is supported by the Ministry of Science, Republic of Serbia (grant no. 174007).

If $\delta \subset \{1, 2, 3, 4, 5, 6\}$ and B satisfies the equations (i) for all $i \in \delta$, then B is an δ-inverse of A. The set of all δ-inverse of A is denote by $A\{\delta\}$. Observe that $A\{1, 2, 3, 4\} = \{A^{\dagger}\}$, $A\{2, 5, 6\} = \{A^d\}$ and $A\{1, 2, 5\} = \{A^{\#}\}$.

The generalized inverses and its applications are very important in various applied mathematical fields like numerical analysis, singular differential equations, singular difference equations, Markov chains, etc.

The Sherman – Morrison – Woodbury formula (or SMW formula) [8, 10] related to the inverse of matrix, i.e., the formula $(A + YGZ^T)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^T A^{-1}Y)^{-1}Z^T A^{-1}$, is an useful computational tool in applications to statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations. Deng [4] considered the more generalized case of the SMW formula when A and $A + YGZ^*$ are not invertible operator. Precisely, some conditions under which the SMW formula can be represented in the Moore – Penrose inverse and the generalized Drazin inverse forms for Hilbert space operators are investigated.

Wei [9] and J. Chen, Z. Xu [2] have discussed the expression of the Drazin inverse and the weighted Drazin inverse of a modified square matrix $A - CB$. These results can be applied to update finite Markov chains.

In this paper, we will prove that Sherman – Morrison – Woodbury formula has the analogous result concerning the δ-inverse of a modified operator $A + YGZ$ between Banach spaces, where $\delta \subset \{1, 2, 3, 4, 5\}$. Under some conditions, we will show that the Wg-Drazin inverse of a modified operator $A + YGZ$ exists and can be represented in terms of generalized SMW forms if and only if the Wg-Drazin inverse of the initial operator A exists. As a consequence, some results of Deng [4], Wei [9] and J. Chen, Z. Xu [2] are obtained.

2. Results. In the first theorem of this section, we consider the generalized inverse of a modified operator $A + YGZ$ and, under certain circumstances, we get that $A' \in A\{\delta\}$ if and only if $A' - A'YS'ZA' \in (A + YGZ)\{\delta\}$ ($\delta \subset \{1, 2, 3, 4, 5\}$). Thus, the generalized inverse of operator $A + YGZ$ is expressed in terms of generalized SMW formula. Notice that, we do not assume that S' and G' are the generalized inverses of S and G, respectively, that is, S' and G' can be arbitrary operators.

Theorem 2.1. *Suppose that* $A \in \mathcal{B}(H, K)$, $A' \in \mathcal{B}(K, H)$, $Y \in \mathcal{B}(L, K)$, $Z \in \mathcal{B}(H, L)$ *and* $G, G', S' \in \mathcal{B}(L)$. *Let* $B = A + YGZ$ *and* $S = G' + ZA'Y$.

(a) *If* $\delta \subset \{1, 2, 4\}$ *and*

$$ZA'A = Z, \quad YS'SGZ = YGZ, \quad YS'G'GZ = YS'Z, \tag{2.1}$$

then $A' \in A\{\delta\}$ *if and only if* $A' - A'YS'ZA' \in (A + YGZ)\{\delta\}$.

(b) *If* $\delta \subset \{1, 2, 3\}$ *and*

$$AA'Y = Y, \quad YGZ = YGSS'Z, \quad YS'Z = YGG'S'Z, \tag{2.2}$$

then $A' \in A\{\delta\}$ *if and only if* $A' - A'YS'ZA' \in (A + YGZ)\{\delta\}$.

(c) *If the conditions* (2.1) *and* (2.2) *hold, then* $A' \in A\{5\}$ *if and only if* $A' - A'YS'ZA' \in (A + YGZ)\{5\}$.

Proof. (a) If $P = A' - A'YS'ZA'$, by (2.1), we obtain

$$PB = (A' - A'YS'ZA')(A + YGZ) =$$
$$= A'A + A'YGZ - A'YS'(ZA'A) - A'YS'(ZA'Y)GZ =$$
$$= A'A + A'YGZ - A'YS'Z - A'YS'(S - G')GZ =$$
$$= A'A + A'Y(I_L - S'S)GZ - A'YS'(I_L - G'G)Z = A'A.$$

So, $A' \in A\{4\}$ if and only if $P \in B\{4\}$. Since, from $AA'Y = Y$,

$$PBP = A'AP = A'AA' - A'(AA'Y)S'ZA' = A'AA' - A'YS'ZA',$$

we deduce that $A'AA' = A' \Leftrightarrow PBP = P$, i.e., $A' \in A\{2\} \Leftrightarrow P \in B\{2\}$. Further, by the equalities

$$BPB = (A + YGZ)A'A = AA'A + YG(ZA'A) = AA'A + YGZ,$$

$A' \in A\{1\}$ is equivalent to $P \in B\{1\}$.

(b) The assumptions (2.2) imply

$$BP = (A + YGZ)(A' - A'YS'ZA') =$$
$$= AA' - (AA'Y)S'ZA' + YGZA' - YG(ZA'Y)S'ZA' =$$
$$= AA' - YS'ZA' - YGZA' - YG(S - G')S'ZA' =$$
$$= AA' - Y(I_L - GG')S'ZA' - YG(I_L - SS')ZA' = AA'.$$

Hence, $A' \in A\{3\}$ if and only if $P \in B\{3\}$. The rest follows in the same way as in the proof of part (a).

(c) Since $PB = A'A$ and $BP = AA'$, we deduce that $A' \in A\{5\}$ if and only if $P \in B\{5\}$.

Theorem 2.1 is proved.

As an application of Theorem 2.1, we get the following result related to the ordinary inverse, the outer inverse and the Moore – Penrose inverse.

***Corollary* 2.1.** *Suppose that the conditions of Theorem* 2.1 *are satisfied.*

(a) *Let T and S be subspaces of H and K, respectively, and let the equalities* (2.1) *or* (2.2) *hold. Then there exists $A^{(2)}_{T,S}$ and $A' = A^{(2)}_{T,S}$ if and only if $(A + YGZ)^{(2)}_{T,S}$ exists and $(A + YGZ)^{(2)}_{T,S} = A' - A'YS'ZA'$. Furthermore, if $A^{(2)}_{T,S}$ exists, then $(A + YGZ)^{(2)}_{T,S} = A^{(2)}_{T,S} - A^{(2)}_{T,S}YS'ZA^{(2)}_{T,S}$. In addition, let the equalities* (2.1) *and* (2.2) *hold.*

(b) *Then there exists $A^\dagger$ and $A' = A^\dagger$ if and only if $(A + YGZ)^\dagger$ exists and $(A + YGZ)^\dagger = A' - A'YS'ZA'$. Furthermore, if $A^\dagger$ exists, then $(A + YGZ)^\dagger = A^\dagger - A^\dagger YS'ZA^\dagger$.*

(c) *Then A is invertible and $A' = A^{-1}$ if and only if $A+YGZ$ is invertible and $(A+YGZ)^{-1} = A' - A'YS'ZA'$. Furthermore, if A is invertible, then $(A + YGZ)^{-1} = A^{-1} - A^{-1}YS'ZA^{-1}$.*

Proof. (a) By Theorem 2.1, $A' \in A\{2\}$ is equivalent to $P = A' - A'YS'ZA' \in B\{2\}$. Observe that $R(P) = R(PB) = R(A'A) = R(A')$ and $N(P) = N(BP) = N(AA') = N(A')$. Thus, $A' = A^{(2)}_{T,S} \Leftrightarrow P = B^{(2)}_{T,S}$.

The statements (b) and (c) follows directly from Theorem 2.1.

In the following theorem, we consider the explicit expression for the Wg-Drazin inverse of a modified operator $A + YGZ$.

Theorem 2.2. *Suppose that* $A, A' \in \mathcal{B}(H,K)$, $W \in \mathcal{B}(K,H)$, $Y \in \mathcal{B}(L,K)$, $Z \in \mathcal{B}(H,L)$ *and* $G, G', S' \in \mathcal{B}(L)$. *Let* $B = A + YGZ$ *and* $S = G' + ZWA'WY$. *If*

$$ZWA'WA = Z, \quad YS'SGZ = YGZ = YGSS'Z,$$

$$AWA'WY = Y, \quad YS'G'GZ = YS'Z = YGG'S'Z,$$

then there exists $A^{d,W}$ *and* $A^{d,W} = A'$ *if and only if* $(A+YGZ)^{d,W}$ *exists and* $(A+YGZ)^{d,W} = A' - A'WYS'ZWA'$. *Furthermore, if* $A^{d,W}$ *exists, then* $(A+YGZ)^{d,W} = A^{d,W} - A^{d,W}WYS'ZWA^{d,W}$.

Proof. In the similar way as in the proof of Theorem 2.1, for $T = A' - A'WYS'ZWA'$, we obtain $TWB = A'WA$ and $BWT = AWA'$ implying $A'WA = AWA' \Leftrightarrow TWB = BWT$. From $TWBWT = A'WAWT$, we get $TWBWT = T$ is equivalent to $A'WAWA' = A'$. Also, we have

$$B - BWTWB = A + YGZ - AWA'W(A + YGZ) = A - AWA'WA,$$

which gives $B - BWTWB$ is quasinilpotent if and only if $A - AWA'WA$ is quasinilpotent. So, $A^{d,W} = A' \Leftrightarrow B^{d,W} = T$.

Theorem 2.2 is proved.

If $H = K$ and $W = I_H$ in Theorem 2.2, we obtain the next corollary concerning representations for the generalized Drazin inverse and the group inverse of a operator on Banach space.

***Corollary* 2.2.** *Let* $A, A' \in \mathcal{B}(H)$, $Y \in \mathcal{B}(L,H)$, $Z \in \mathcal{B}(H,L)$ *and* $G, G', S' \in \mathcal{B}(L)$. *Set* $B = A + YGZ$ *and* $S = G' + ZA'Y$. *Assume that the equalities* (2.1) *and* (2.2) *hold.*

(a) *There exists* A^d *and* $A' = A^d$ *if and only if* $(A + YGZ)^d$ *exists and* $(A + YGZ)^d = A' - A'YS'ZA'$. *Furthermore, if* A^d *exists, then* $(A + YGZ)^d = A^d - A^dYS'ZA^d$.

(b) *There exists* $A^\#$ *and* $A' = A^\#$ *if and only if* $(A + YGZ)^\#$ *exists and* $(A + YGZ)^\# = A' - A'YS'ZA'$. *Furthermore, if* $A^\#$ *exists, then* $(A + YGZ)^\# = A^\# - A^\#YS'ZA^\#$.

(c) *If* $\delta \subset \{1,2,3,4,5,6\}$, *then* $A' \in A\{\delta\}$ *if and only if* $A' - A'YS'ZA' \in (A + YGZ)\{\delta\}$.

Remark. In Theorem 2.1, Corollary 2.1, Theorem 2.2 and Corollary 2.2, if we assume that G' is θ-inverse of G and S' is ϑ-inverse of S, for $\theta, \vartheta \subset \{1 - 6\}$, then we obtain various formulas for corresponding inverse of $A + YGZ$. In particular, if G' and S' are the Moore – Penrose (or the generalized Drazin) inverses of G and S, respectively, in Corollary 2.1 (or Corollary 2.2), we get as a spacial case [4] (Theorem 2.2) (or [4], Theorem 2.4) for Hilbert space operators.

For $G = G' = I_L$ in Theorems 2.1 and 2.2, we have the next results.

***Corollary* 2.3.** *Suppose that* $\delta \subset \{1,2,3,4\}$, $A \in \mathcal{B}(H,K)$, $A' \in \mathcal{B}(K,H)$, $Y \in \mathcal{B}(L,K)$, $Z \in \mathcal{B}(H,L)$ *and* $S' \in \mathcal{B}(L)$. *Let* $B = A + YZ$ *and* $S = I_L + ZA'Y$. *If*

$$ZA'A = Z, \quad AA'Y = Y, \quad YS'SZ = YZ = YSS'Z,$$

then $A' \in A\{\delta\}$ *if and only if* $A' - A'YS'ZA' \in (A + YZ)\{\delta\}$.

Notice that [9] (Theorem 2.1) and [2] (Theorem 2.1) concerning the Drazin inverse and the weighted Drazin inverse of complex matrices are particular cases of the following corollary.

***Corollary* 2.4.** *Suppose that* $A, A' \in \mathcal{B}(H,K)$, $W \in \mathcal{B}(K,H)$, $Y \in \mathcal{B}(L,K)$, $Z \in \mathcal{B}(H,L)$ *and* $S' \in \mathcal{B}(L)$. *Let* $B = A + YZ$ *and* $S = I_L + ZWA'WY$. *If*

$$ZWA'WA = Z, \quad AWA'WY = Y, \quad YS'SZ = YZ = YSS'Z,$$

then there exists $A^{d,W}$ *and* $A^{d,W} = A'$ *if and only if* $(A+YZ)^{d,W}$ *exists and* $(A+YZ)^{d,W} = A' - A'WYS'ZWA'$. *Furthermore, if* $A^{d,W}$ *exists, then* $(A+YZ)^{d,W} = A^{d,W} - A^{d,W}WYS'ZWA^{d,W}$.

If S is invertible and $S' = S^{-1}$ in Corollaries 2.3 and 2.4, the following consequences recover [4] (Corollaries 2.3 and 2.6).

***Corollary* 2.5.** *Suppose that* $\delta \subset \{1,2,3,4\}$, $A \in \mathcal{B}(H,K)$, $A' \in \mathcal{B}(K,H)$, $Y \in \mathcal{B}(L,K)$, $Z \in \mathcal{B}(H,L)$, $S' \in \mathcal{B}(L)$ *and* $B = A + YZ$. *Let* $S = I_L + ZA'Y$ *be invertible and* $S' = S^{-1}$. *If*

$$ZA'A = Z, \quad and \quad AA'Y = Y,$$

then $A' \in A\{\delta\}$ *if and only if* $A' - A'Y(I + ZA'Y)^{-1}ZA' \in (A + YZ)\{\delta\}$.

***Corollary* 2.6.** *Suppose that* $A, A' \in \mathcal{B}(H,K)$, $W \in \mathcal{B}(K,H)$, $Y \in \mathcal{B}(L,K)$, $Z \in \mathcal{B}(H,L)$, $S' \in \mathcal{B}(L)$ *and* $B = A + YZ$. *Let* $S = I_L + ZWA'WY$ *be invertible and* $S' = S^{-1}$. *If*

$$ZWA'WA = Z, \quad AWA'WY = Y,$$

then there exists $A^{d,W}$ *and* $A^{d,W} = A'$ *if and only if* $(A + YZ)^{d,W}$ *exists and* $(A + YZ)^{d,W} = A' - A'WY(I_L + ZWA'WY)^{-1}ZWA'$. *Furthermore, if* $A^{d,W}$ *exists, then* $(A + YZ)^{d,W} = A^{d,W} - A^{d,W}WY(I_L + ZWA'WY)^{-1}ZWA^{d,W}$.

References

1. *Ben-Israel A., Greville T. N. E.* Generalized inverses: theory and applications. – 2nd ed. – New York: Springer, 2003.
2. *Chen J., Xu Z.* Representations for the weighted Drazin inverse of a modified matrix // Appl. Math. and Comput. – 2008. – **203**. – P. 202 – 209.
3. *Dajić A., Koliha J. J.* The weighted g-Drazin inverse for operators // J. Austral. Math. Soc. – 2007. – **82**. – P. 163 – 181.
4. *Deng C. Y.* A generalization of the Sherman – Morrison – Woodbury formula // Appl. Math. Lett. – 2011. – **24**. – P. 1561 – 1564.
5. *Djordjevic D. S., Stanimirovic P. S.* Splittings of operators and generalized inverses // Publ. Math. Debrecen. – 2001. – **59**, № 1-2. – P. 147 – 159.
6. *Koliha J. J.* A generalized Drazin inverse // Glasgow Math. J. – 1996. – **38**. – P. 367 – 381.
7. *Rakočević V.* Moore – Penrose inverse in Banach algebras // Proc. Roy. Irish Acad. A. – 1988. – **88**. – P. 57 – 60.
8. *Sherman J., Morrison W. J.* Adjustment of an inverse matrix corresponding to a change in one elements of a given matrix // Ann. Math. Statist. – 1950. – **21**. – P. 124 – 127.
9. *Wei Y.* The Drazin inverse of a modified matrix // Appl. Math. and Comput. – 2002. – **125**. – P. 295 – 301.
10. *Woodbury M. A.* Inverting modified matrices // Techn. Rept. Statist. Res. Group. – Princeton, NJ: Princeton Univ., 1950. – **42**.

Received 08.07.14