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POLYNOMIAL INEQUALITIES IN REGIONS WITH INTERIOR ZERO ANGLES IN THE BERGMAN SPACE *

ПОЛІНОМІАЛЬНІ НЕРІВНОСТІ В ОБЛАСТЯХ ІЗ ВНУТРІШНІМИ НУЛЬОВИМИ КУТАМИ У ПРОСТОРІ БЕРГМАНА

We investigate the order of growth of the moduli of arbitrary algebraic polynomials in the weighted Bergman space $A_p(G,h)$, $p>0$, in regions with interior zero angles at finitely many boundary points. We obtain estimations for algebraic polynomials in bounded regions with piecewise smooth boundary.

Вивчається порядок зростання модулів довільних алгебраїчних поліномів у ваговому просторі Бергмана $A_p(G,h)$, $p>0$, в областях із внутрішніми нульовими кутами у скінченній кількості точок. Отримано оцінки для алгебраїчних поліномів в обмежених областях з кусково-гладкою межею.

1. Introduction and main results. Let $G\subset\mathbb{C}$ be a finite region, with $0\in G$, bounded by a Jordan curve $L:=\partial G$, $\Omega:=\operatorname{ext} L:=\overline{\mathbb{C}}\backslash\overline{G}$, where $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$, $\Delta:=\{w:\ |w|>1\}$ and let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n\in\mathbb{N}$. Let $w=\Phi(z)$ be the univalent conformal mapping of Ω onto the Δ with usual normalization, and $\Psi:=\Phi^{-1}$. For $t\geq 1$, $z\in\mathbb{C}$, we us set

$$L_t:=\{z:\ |\Phi(z)|=t\}\quad (L_1\equiv L),\qquad G_t:=\operatorname{int} L_t,\qquad \Omega_t:=\operatorname{ext} L_t.$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L, located in the positive direction. For some fixed R_0, $1<R_0<\infty$, and $z\in G_{R_0}$, consider a so-called generalized Jacobi weight function $h\,(z)$ being defined as follows:

$$h(z):=h_0(z)\prod_{j=1}^{m}|z-z_j|^{\gamma_j}\,,\quad z\in G_{R_0},\tag{1.1}$$

where $\gamma_j>-2$, for all $j=1,2,\ldots,m$, and the function h_0 is uniformly separated from zero in G_{R_0}, i.e., there exists a constant $c_0:=c_0(G_{R_0})>0$ such that, for all $z\in G_{R_0}$

$$h_0(z)\geq c_0>0.$$

For any $p>0$ and for Jordan region G, lets define

$$\|P_n\|_p:=\|P_n\|_{A_p(h,G)}:=\left(\iint\limits_G h(z)\,|P_n(z)|^p\,d\sigma_z\right)^{1/p}<\infty,\quad 0<p<\infty,$$
$$\|P_n\|_\infty:=\|P_n\|_{A_\infty(1,G)}:=\|P_n\|_{C(\overline{G})}\,,\quad p=\infty,\tag{1.2}$$

where σ_z is the two-dimensional Lebesgue measure. Clearly, $\|\cdot\|_{A_p}$ is the quasinorm (i.e., a norm for $1\leq p\leq\infty$ and a p-norm for $0<p<1$).

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In this work, we study the following Nikol'skii-type inequality:

$$\|P_n\|_\infty \leq c_1\lambda_n(G,h,p)\,\|P_n\|_p\,, \tag{1.3}$$

where $c_1 = c_1(G,h,p) > 0$ is a constant independent of n and P_n, and $\lambda_n(G,h,p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region G, weight function h and of p. The estimate of (1.3)-type for some $(G,\ p,\ h)$ was investigated in [21, p. 122–133], [15], [20] (Sect. 5.3), [2–8, 14, 23] (see also references therein).

Further, analogous of (1.3) for some regions and the weight function $h(z)$ were obtained: in [8] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps; in [9] ($h(z) \equiv 1$) and [11] ($h(z) \neq 1$) for $p > 0$ and for regions bounded by quasiconformal curve; in [7] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [10] for $p > 0$ and for regions bounded by asymptotically conformal curve.

In this work, we investigate similar problems for $z \in \overline{G}$ in regions bounded by piecewise smooth curve having interior zero angles and for weight function $h\left(z\right)$, defined in (1.1) and for $p > 0$.

Let us give some definitions and notations that will be used later in the text.

Following [18, p. 97, 22], the Jordan curve (or arc) L is called K-quasiconformal ($K \geq 1$), if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let S be rectifiable Jordan curve or arc and let $z = z(s)$, $s \in [0,|S|]\,,\ |S| := \operatorname{mes} S$, be the natural parametrization of S.

Definition 1.1. *We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write a region $G \in C_\theta$, if $\partial G \in C_\theta$.*

According to [22], we have the following fact.

***Corollary* 1.1.** *If $S \in C_\theta$, then S is $(1+\varepsilon)$-quasiconformal for arbitrary small $\varepsilon > 0$.*

According to the „three-point" criterion [12, p. 100], every piecewise smooth curve (without any cusps) is quasiconformal.

Now we define a new class of regions with piecewise smooth boundary, where having exterior corners and interior cusps simultaneously.

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k,m}$ denotes $j = k, k+1, \ldots, m$.

Definition 1.2 [7]. *We say that a Jordan region $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1,m}$, if $L = \partial G$ consists of the union of finite smooth arcs $\{L_j\}_{j=1}^{m}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j\pi$, $0 < \lambda_j \leq 2$, at the corner points $\{z_j\}_{j=1}^{m} \in L$, where two arcs meet.*

Let m_1 be the number of exterior angles, which are not cusps, and thus $m - m_1$ is the number of cusps. It is clear from Definition 1.2, that each region $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j \leq 2$, $j = \overline{1,m}$, may have exterior nonzero $\lambda_j\pi$, $0 < \lambda_j < 2$, angles at the points $\{z_j\}_{j=1}^{m_1} \in L$, and interior zero angles ($\lambda_j = 2$) at the the points $\{z_j\}_{j=m_1+1}^{m} \in L$. If $m_1 = m = 0$, then the region G doesn't have such angles, and in this case we will write $G \in C_\theta$; if $m_1 = m \geq 1$, then G has only $\lambda_i\pi$, $0 < \lambda_i < 2$, $i = \overline{1,m_1}$, exterior nonzero angles; if $m_1 = 0$ and $m \geq 1$, then G has only interior zero angles, and in this case we will write $G \in C_\theta(2, \ldots, 2)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^m \in L$ defined in (1.1) and in Definition 1.2 are identical and $w_j := \Phi(z_j)$.

For the simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we will take $m_1 = 1,\ m = 2$. Then, after this assumption, in the future we will have region $G \in C_\theta(\lambda_1, 2),\ 0 < \lambda_1 < 2$, such that at the point $z_1 \in L$ the region G have exterior nonzero angle $\lambda_1\pi,\ 0 < \lambda_1 < 2$, and at the point $z_2 \in L$-interior zero angle.

Now we can state our new results.

Theorem 1.1. *Let $p > 0,\ G \in C_\theta(\lambda_1, \lambda_2)$ for some $0 < \lambda_1,\ \lambda_2 < 2,\ h(z)$ be defined as in* (1.1). *Then, for any $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2,\ j = 1, 2,$ and arbitrary small $\varepsilon > 0$, we have*

$$\|P_n\|_\infty \le c_1 \mu_{n,1} \|P_n\|_p\,, \tag{1.4}$$

where $c_1 = c_1(G, \gamma_1, \gamma_2, \lambda_1, \lambda_2, p, \varepsilon) > 0$ is the constant, independent of z and n, and

$$\mu_{n,1} := \begin{cases} n^{\frac{(2+\widetilde{\gamma})\cdot\widetilde{\lambda}}{p}}, & \textit{if} \quad (2+\gamma)\cdot\widetilde{\lambda} > 1, \\ (n \ln n)^{1/p}, & \textit{if} \quad (2+\gamma)\cdot\widetilde{\lambda} = 1, \\ n^{1/p}, & \textit{if} \quad (2+\gamma)\cdot\widetilde{\lambda} < 1, \end{cases} \tag{1.5}$$

$$\gamma := \max\{\gamma_1, \gamma_2\}, \qquad \widetilde{\gamma}_j := \max\{0, \gamma_j\}; \qquad \widetilde{\lambda} := \max\left\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\right\}, \qquad \widetilde{\lambda}_j := \max\{1; \lambda_j\} + \varepsilon.$$

Now, we assume that the curve L at both points have interior zero angles. In this case we obtain the following theorem.

Theorem 1.2. *Let $p > 0,\ G \in C_\theta(2, 2),\ h(z)$ be defined as in* (1.1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N},\ \gamma_j > -2,\ j = 1, 2$, we have*

$$\|P_n\|_\infty \le c_2 \mu_{n,2} \|P_n\|_p\,, \tag{1.6}$$

where $c_2 = c_2(G, \gamma_1, \gamma_2, p) > 0$ is the constant, independent of z and n, and $\widetilde{\gamma}$ is defined as in (1.5) *and*

$$\mu_{n,2} := \begin{cases} n^{\frac{2(2+\widetilde{\gamma})}{p}}, & \textit{if} \quad \gamma > -\dfrac{3}{2}, \\ (n \ln n)^{1/p}, & \textit{if} \quad \gamma = -\dfrac{3}{2}, \\ n^{1/p}, & \textit{if} \quad \gamma < -\dfrac{3}{2}. \end{cases} \tag{1.7}$$

Now we will estimate of $|P_n(z)|$ at the critical points $z_j,\ j = 1, 2$.

Theorem 1.3. *Let $p > 0,\ G \in C_\theta(\lambda_1, 2)$ for some $0 < \lambda_1 < 2,\ h(z)$ be defined as in* (1.1). *Then, for any $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2,\ j = 1, 2$, and arbitrary small $\varepsilon > 0$, we obtain*

$$|P_n(z_j)| \le c_3 \mu_{n,3} \|P_n\|_p\,, \tag{1.8}$$

where $c_3 = c_3(G, \gamma_1, \gamma_2, \lambda_1, p, \varepsilon) > 0$ is the constant, independent of z and n;

$$\mu_{n,3} := \begin{cases} n^{\frac{(2+\gamma_1)\cdot\widetilde{\lambda}_1}{p}}, & \text{if} \quad \gamma_1 > \dfrac{1}{\widetilde{\lambda}_1} - 2, \\ (n\ln n)^{1/p}, & \text{if} \quad \gamma_1 = \dfrac{1}{\widetilde{\lambda}_1} - 2, \\ n^{1/p}, & \text{if} \quad \gamma_1 < \dfrac{1}{\widetilde{\lambda}_1} - 2, \end{cases}$$

for $j = 1$, *and*

$$\mu_{n,3} := \begin{cases} n^{\frac{2(2+\gamma_2)}{p}}, & \text{if} \quad \gamma_2 > -\dfrac{3}{2}, \\ (n\ln n)^{1/p}, & \text{if} \quad \gamma_2 = -\dfrac{3}{2}, \\ n^{1/p}, & \text{if} \quad \gamma_2 < -\dfrac{3}{2}, \end{cases}$$

for $j = 2$.

Combining Theorems 1.1 and 1.2 with the estimate for $|P_n(z)|$, $z \in \Omega$, in [25] (Corollaries 1.2 and 1.3), we can obtain estimation for $|P_n(z)|$ in the whole complex plane.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set that $d(z, M) = \operatorname{dist}(z, M) := \inf\{|z - \zeta| : \zeta \in M\}$ and $R := 1 + \dfrac{\varepsilon_0}{n}$.

***Corollary* 1.2.** *Under the conditions of Theorem* 1.1, *the following is true*:

$$|P_n(z)| \le c_4 \|P_n\|_p \begin{cases} \mu_{n,1}, & z \in \overline{G}_R, \\ \dfrac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_R)} \mu_{n,4}, & z \in \Omega_R, \end{cases} \tag{1.9}$$

where $c_4 = c_4(G, \gamma_1, \gamma_2, \lambda_1, p, \varepsilon) > 0$ *is the constant, independent of* z *and* n; $\mu_{n,1}$ *is defined as in* (1.5) *and*

$$\mu_{n,4} := \begin{cases} n^{\frac{\widetilde{\gamma}\cdot\widetilde{\lambda}_1}{p}}, & \text{if} \quad \gamma \cdot \widetilde{\lambda}_1 > 1, \\ (n\ln n)^{1/p}, & \text{if} \quad \gamma \cdot \widetilde{\lambda}_1 = 1, \\ n^{1/p}, & \text{if} \quad \gamma \cdot \widetilde{\lambda}_1 < 1. \end{cases}$$

***Corollary* 1.3.** *Under the conditions of Theorem* 1.2, *the following is true*:

$$|P_n(z)| \le c_3 \|P_n\|_p \begin{cases} \mu_{n,2}, & z \in \overline{G}_R, \\ \dfrac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_R)} \mu_{n,5}, & z \in \Omega_R, \end{cases} \tag{1.10}$$

where $c_5 = c_5(G, \gamma_1, \gamma_2, p) > 0$ *is the constant, independent of* z *and* n; $\mu_{n,2}$ *is defined as in* (1.7) *and*

$$\mu_{n,5} := \begin{cases} n^{\frac{2\tilde{\gamma}}{p}}, & if \quad \gamma > \dfrac{1}{2}, \\ (n \ln n)^{1/p}, & if \quad \gamma = \dfrac{1}{2}, \\ n^{1/p}, & if \quad \gamma < \dfrac{1}{2}. \end{cases}$$

The sharpness of the estimations (1.4), (1.6), (1.8), (1.9) and (1.10), can be discussed by comparing them with the following result.

***Remark* 1.1** ([9] (Theorem 1.15), [2]). (a) For any $n \in \mathbb{N}$ there exist polynomials $Q_n^*, T_n^* \in \wp_n$ such that for unit disk B and weight function $h^*(z) = |z - z_1|^2$ the following is true:

$$|Q_n^*(z)| \geq c_6 n \, \|Q_n^*\|_{A_2(B)} \quad \text{for all} \quad z \in \overline{B},$$
$$|T_n^*(z_1)| \geq c_7 n^2 \, \|T_n^*\|_{A_2(h^*, B}.$$

(b) For any $n \in \mathbb{N}$ there exists a polynomial $P_n^* \in \wp_n$, region $G_1^* \subset \mathbb{C}$, compact $F^* \Subset \Omega \backslash \overline{G}_1^*$ and constant $c_8 = c_8(G_1^*, F^*) > 0$ such that

$$|P_n^*(z)| \geq c_8 \frac{\sqrt{n}}{d(z, L)} \, \|P_n^*\|_{A_2(G_1^*)} \, |\Phi(z)|^{n+1} \quad \text{for all} \quad z \in F^*.$$

2. Some auxiliary results. Throughout this work, for the nonnegative functions $a > 0$ and $b > 0$, we will use the notations $a \preceq b$ (order inequality), if $a \leq cb$ and $a \asymp b$ are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1 [1]. *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z \colon |z - z_1| \preceq \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*:

(a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So, the statements $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent.*

(b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2} \preceq \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \preceq \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^{-2}},$$

where $0 < r_0 < 1$ is constants, depending on G.

***Corollary* 2.1.** *Under the assumptions of Lemma 2.1, for $z_3 \in L_{r_0}$,*

$$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}.$$

***Corollary* 2.2.** *If $L \in C_\theta$, then, for all $\varepsilon > 0$,*

$$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon}.$$

For $0 < \delta_j < \delta_0 := \dfrac{1}{4} \min \{|z_1 - z_2|\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \{z \colon |z - z_j| \leq \delta_j\}$, $\delta := := \min_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^{2} \Omega(z_j, \ \delta)$, $\widehat{\Omega} := \Omega \backslash \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \ \delta))$, $\Delta(\delta) := \bigcup_{j=1}^{m} \Phi(\Omega(z_j, \ \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

The following lemma is a consequence of the results given in [17, 19, 27], and of estimate for the $|\Psi'|$ (see, for example, [13], Theorem 2.8) for $0 < \lambda_j < 2,\ j = \overline{1,m}$:

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{2.1}$$

Lemma 2.2 [27]. *Let $G \in C_\theta(\lambda_1, \ldots, \lambda_m), 0 < \lambda_j < 2,\ j = \overline{1,m}$. Then, for all $\varepsilon > 0$:*

i) *for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \preceq |\Psi(w) - \Psi(w_j)| \preceq |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \preceq$ $\preceq |\Psi'(w)| \preceq |w - w_j|^{\lambda_j - 1 - \varepsilon}$,*

ii) *for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \preceq d(\Psi(w), L)| \preceq (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \preceq |\Psi'(w)| \preceq$ $\preceq (|w| - 1)^{-\varepsilon}$.*

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.1).

Lemma 2.3 [5]. *Let L be a K-quasiconformal curve, $h(z)$ is defined in* (1.1). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n = 1, 2, \ldots$, we have*

$$\|P_n\|_{A_p(h,G_R)} \preceq \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \quad p > 0, \tag{2.2}$$

where $\widetilde{R} = 1 + c(R - 1)$ and c is independent of n and R.

Lemma 2.4. *Let $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j \le 2$, $j = \overline{1,m}$. Then, for arbitrary $P_n(z) \in \wp_n$ and any $p > 0$, we obtain*

$$\|P_n\|_{A_p(h,G_{1+c/n})} \preceq \|P_n\|_{A_p(h,G)}. \tag{2.3}$$

Proof. For $0 < \lambda_j < 2$, $j = \overline{1,m}$, this follows from Lemma 2.4 and Corollary 1.1 and from the fact, that according to the „three-point" criterion [18, p. 100], any piecewise smooth curve without cusps is a quasiconformal. If $\lambda_j = 2$, for all $j = \overline{1,m}$, then the region G have exterior 2π angles (i.e., interior cusps) at the every point z_j, $j = \overline{1,m}$. Then in the neighborhood of the this points the region G have a boundary with outside wedge. Therefore, as well known from theory of conformal mappings, the distance from the corner point to the level curve L_R is less than of such distance from the other points. Furthermore, the area between boundary L and level curve L_R in the neighborhood of the such corners will be smaller than such in the case of without angles.

3. Proof of theorems. ***3.1. Proof of Theorems* 1.1 *and* 1.2.** Suppose that $G \in C_\theta(\lambda_1; 2)$, for some $0 < \lambda_1 < 2$ and $h(z)$ be defined as in (1.1). Let $\{\xi_j\}$, $1 \le j \le m \le n$, be the zeros (if any exist) of $P_n(z)$ lying on Ω. Lets define the function Blaschke with respect to the zeros $\{\xi_j\}$ of the polynomial $P_n(z)$:

$$\widetilde{B}_j(z) := \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)}, \quad z \in \Omega, \tag{3.1}$$

and let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z), \quad z \in \Omega. \tag{3.2}$$

It is easy that the

$$B_m(\xi_j) = 0, \qquad |B_m(z)| \equiv 1, \quad z \in L; \qquad |B_m(z)| < 1, \quad z \in \Omega. \tag{3.3}$$

Then, for each ε_1, $0<\varepsilon_1<1$, there exists circle $\left\{w\colon\ |w|=R_1:=1+\varepsilon_2, 0<\varepsilon_2<\dfrac{\varepsilon_1}{n}\right\}$ such that for any $j=1,2$, the following is holds:

$$\left|\widetilde{B}_j(\zeta)\right|>1-\varepsilon_2,\quad \zeta\in L_{R_1}.$$

So, from (3.2), we get

$$|B_m(\zeta)|>(1-\varepsilon_2)^m\succeq 1,\quad \zeta\in L_{R_1}. \tag{3.4}$$

For any $p>0$ and $z\in\Omega$ let us set

$$Q_{n,p}(z):=\left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2}. \tag{3.5}$$

The function $Q_{n,p}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$, $Q_{n,p}(\infty)=0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $Q_{n,p}(z)$ and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region Ω, we have

$$Q_{n,p}(z)=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}Q_{n,p}(\zeta)\frac{d\zeta}{\zeta-z},\quad z\in\Omega_{R_1}. \tag{3.6}$$

According to (3.1)–(3.5), we get

$$|P_n(z)|^{p/2}=\frac{\left|B_m(z)\Phi^{n+1}(z)\right|^{p/2}}{2\pi d(z,L_{R_1})}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2}|d\zeta|\preceq$$

$$\preceq\left|\Phi^{n+1}(z)\right|^{p/2}\int\limits_{L_{R_1}}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z|}. \tag{3.7}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w=\Phi(z)$ and applying the Hölder inequality, we obtain

$$\left(\int\limits_{L_{R_1}}|P_n(\zeta)|^{\frac{p}{2}}|d\zeta|\right)^2\le\int\limits_{|t|=R_1}h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}\le$$

$$\le\int\limits_{|t|=R_1}h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}=$$

$$=\int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}=:A_nD_n(w), \tag{3.8}$$

where $f_{n,p}(t):=h^{1/p}(\Psi(t))P_n(\Psi(t))(\Psi'(t))^{2/p}$, $|t|=R_1$.

To estimate integral A_n, we separate the circle $|t|=R_1$ to n equal parts δ_n with $\operatorname{mes}\delta_n=\dfrac{2\pi R_1}{n}$ and by applying the mean value theorem, we get

$$A_n: \ = \int\limits_{|t|=R_1} |f_{n,p}(t)|^p \, |dt| =$$

$$= \sum_{k=1}^{n} \int\limits_{\delta_k} |f_{n,p}(t)|^p \, |dt| = \sum_{k=1}^{n} \left|f_{n,p}(t_k')\right|^p \operatorname{mes} \delta_k, \quad t_k' \in \delta_k.$$

On the other hand, by applying mean value estimation

$$\left|f_{n,p}(t_k')\right|^p \leq \frac{1}{\pi\left(\left|t_k'\right| - 1\right)^2} \iint\limits_{\left|\xi - t_k'\right| < \left|t_k'\right| - 1} |f_{n,p}(\xi)|^p \, d\sigma_\xi,$$

we obtain

$$(A_n)^2 \preceq \sum_{k=1}^{n} \frac{\operatorname{mes} \delta_k}{\pi\left(\left|t_k'\right| - 1\right)^2} \iint\limits_{\left|\xi - t_k'\right| < \left|t_k'\right| - 1} |f_{n,p}(\xi)|^p \, d\sigma_\xi, \quad t_k' \in \delta_k.$$

Taking into account that at most two of the discs with center t_k' are intersecting, we have

$$A_n \preceq \frac{\operatorname{mes} \delta_1}{(|t_1'| - 1)^2} \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi \preceq n \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi.$$

According to Lemma 2.4, for A_n we get

$$A_n \preceq n \iint\limits_{G_R \setminus G} h(\zeta) \, |P_n(\zeta)|^p \, d\sigma_\zeta \preceq n \, \|P_n\|_p^p. \tag{3.9}$$

To estimate the integral $D_n(w)$, denote by $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, for any fixed $\rho > 1$, we introduce

$$\Delta_1(\rho) := \left\{ t = re^{i\theta} : r > \rho, \ \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$

$$\Delta_2(\rho) := \left\{ t = re^{i\theta} : r > \rho, \ \frac{\varphi_1 + \varphi_2}{2} \leq \theta < \frac{\varphi_1 + \varphi_0}{2} \right\}, \tag{3.10}$$

$$\Delta_j := \Delta_j(1), \qquad \Omega^j := \Psi(\Delta_j), \qquad \Omega_\rho^j := \Psi(\Delta_j(\rho)),$$

$$L^j := L \cap \overline{\Omega}^j, \qquad L_\rho^j := L_\rho \cap \overline{\Omega}_\rho^j, \quad j = 1, 2; \qquad L = L^1 \cup L^1, \qquad L_\rho = L_\rho^1 \cup L_\rho^2.$$

Under these notations, from (3.8) for the $D_n(w)$, we get

$$D_n(w) = \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) \, |\Psi(t) - \Psi(w)|^2} \preceq$$

$$\preceq \sum_{j=1}^{2} \int\limits_{\Phi(L_{R_1}^j)} \frac{|dt|}{\prod_{j=1}^{2} |\Psi(t) - \Psi(w_j)|^{\gamma_j} \, |\Psi(t) - \Psi(w)|^2} \asymp$$

$$\asymp \sum_{j=1}^{2} \int\limits_{\Phi(L_{R_1}^j)} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{\gamma_j}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{j=1}^{2} D_{n,j}(w), \tag{3.11}$$

since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct. So, we need to evaluate the $D_{n,j}(w)$. For this, we take $z \in L_R$ and introduce the notations:

$$\Phi(L_{R_1}) = \Phi\left(\bigcup_{j=1}^{2} L_{R_1}^j\right) = \bigcup_{j=1}^{2} \Phi(L_{R_1}^j) = \bigcup_{j=1}^{2}\bigcup_{i=1}^{3} K_i^j(R_1), \tag{3.12}$$

where

$$K_1^j(R_1) := \left\{t \in \Phi(L_{R_1}^j) :\ |t-w_j| < \frac{c_1}{n}\right\},$$

$$K_2^j(R_1) := \left\{t \in \Phi(L_{R_1}^j) :\ \frac{c_1}{n} \le |t-w_j| < c_2\right\},$$

$$K_3^j(R_1) := \left\{t \in \Phi(L_{R_1}^j) : c_2 \le |t-w_j| < c_3 < \operatorname{diam} \overline{G}\right\}, \quad j=1,2.$$

Analogously,

$$\Phi(L_R) = \Phi\left(\bigcup_{j=1}^{2} L_R^j\right) = \bigcup_{j=1}^{2} \Phi(L_R^j) = \bigcup_{j=1}^{2}\bigcup_{i=1}^{3} K_i^j(R),$$

where

$$K_1^j(R) := \left\{\tau \in \Phi(L_R^j) :\ |\tau-w_j| < \frac{2c_1}{n}\right\},$$

$$K_2^j(R) := \left\{\tau \in \Phi(L_R^j) :\ \frac{2c_1}{n} \le |\tau-w_j| < c_2\right\},$$

$$K_3^j(R) := \left\{\tau \in \Phi(L_R^j) : c_2 \le |\tau-w_j| < c_3 < \operatorname{diam} \overline{G}\right\}, \quad j=1,2.$$

Then, after these definitions, taking arbitrary fixed $w=\Phi(z) \in \Phi(L_R)$, the quantity $D_{n,j}(w)$ can be written as follows:

$$D_{n,j}(w) = \sum_{i=1}^{3} \int\limits_{K_i^j(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{\gamma_j}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{i=1}^{3} D_{n,j}^i(w). \tag{3.13}$$

The quantity $D_{n,j}^i(w)$ we will estimate for each $i=1,2,3$ and $j=1,2$ separately, depending of location of the $w \in \Phi(L_R)$. Let $\varepsilon > 0$ be an arbitrary small fixed number.

Case 1. Let $w \in \Phi(L_R^1)$.

According to the above notations, we will make evaluations for case $w \in K_i^1(R)$ for each $i=1,2,3$.

1.1. Let $w \in K_1^1(R)$. In this case, we will estimate the quantity

$$D_{n,1}(w) = \sum_{i=1}^{3} \int\limits_{K_i^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} =: \sum_{i=1}^{3} D_{n,1}^i(w) \tag{3.14}$$

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

For each $i = 1,2,3$ and $j = 1,2$ we put $K_{i,1}^j(R_1) := \left\{ t \in \Phi(L_{R_1}^j) \colon |t - w_j| \geq |t - w| \right\}$, $K_{i,2}^j(R_1) := K_i^j(R_1) \setminus K_{i,1}^j(R_1)$.

1.1.1. If $\gamma_1 \geq 0$, then

$$D_{n,1}^1(w) = \int\limits_{K_1^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{\gamma_1} |\Psi(t) - \Psi(w)|^2} =$$

$$= \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} =:$$

$$=: D_{n,1}^{1,1}(w) + D_{n,1}^{1,2}(w) \tag{3.15}$$

and so Lemma 2.2 yields

$$D_{n,1}^{1,1}(w) \preceq \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|t - w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1 - 1 + \varepsilon}, & \text{if} \quad (2+\gamma_1)\lambda_1 > 1 - \varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\lambda_1 > 1 - \varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\lambda_1 < 1 - \varepsilon, \end{cases} \tag{3.16}$$

and

$$D_{n,1}^{1,2}(w) \preceq \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|t - w_1|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1 - 1 + \varepsilon}, & \text{if} \quad (2+\gamma_1)\lambda_1 > 1 - \varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\lambda_1 = 1 - \varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\lambda_1 < 1 - \varepsilon. \end{cases} \tag{3.17}$$

If $\gamma_1 < 0$, then

$$D_{n,1}^1(w) = \int\limits_{K_1^1(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{(-\gamma_1)} |dt|}{|\Psi(t) - \Psi(w)|^2} \preceq$$

$$\preceq \int\limits_{K_1^1(R_1)} \frac{|t - w_1|^{(-\gamma_1)(\lambda_1 - \varepsilon)} |dt|}{|t - w|^{2(\lambda_1+\varepsilon)}} \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1 - \varepsilon)} \int\limits_{K_1^1(R_1)} \frac{|dt|}{|t - w|^{2(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1 - \varepsilon)} \begin{cases} n^{2(\lambda_1+\varepsilon) - 1}, & \text{if} \quad 2\lambda_1 > 1 - \varepsilon, \\ \ln n, & \text{if} \quad 2\lambda_1 > 1 - \varepsilon, \\ 1, & \text{if} \quad 2\lambda_1 < 1 - \varepsilon, \end{cases} \preceq$$

$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad \lambda_1 > \dfrac{1}{2}-\varepsilon, \\ 1, & \text{if} \quad \lambda_1 \leq \dfrac{1}{2}-\varepsilon. \end{cases} \tag{3.18}$$

1.1.2. If $\gamma_1 \geq 0$, then

$$D_{n,1}^2(w) = \int\limits_{K_2^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} =$$

$$= \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} =:$$

$$=: D_{n,1}^{2,1}(w) + D_{n,1}^{2,2}(w) \tag{3.19}$$

and, so from Lemma 2.2, we get

$$D_{n,1}^{2,1}(w) \preceq \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq n^{(2+\gamma_1)\lambda_1+\varepsilon} \operatorname{mes} K_{2,1}^1(R_1) \preceq n^{(2+\gamma_1)\lambda_1-1+\varepsilon} \tag{3.20}$$

and

$$D_{n,1}^{2,2}(w) \preceq \int\limits_{K_{2,2}^1(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 = 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon. \end{cases} \tag{3.21}$$

Therefore, from (3.19)–(3.21) for $\gamma_1 \geq 0$, we have

$$D_{n,1}^2(w) \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 = 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon. \end{cases} \tag{3.22}$$

According to well known inequality

$$(a+b)^{\varepsilon} \leq c(\varepsilon)(a^{\varepsilon}+b^{\varepsilon}), \qquad a,b>0, \quad \varepsilon>0, \tag{3.23}$$

and using estimations

$$|t-w_1| \leq |t-w| + |w-w_1| \preceq |t-w| + \frac{1}{n}$$

and consequently,

$$|t-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)} \preceq |t-w|^{(-\gamma_1)(\lambda_1-\varepsilon)} + \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1-\varepsilon)},$$

for $\gamma_1 < 0$, from (3.14), we get

$$D_{n,1}^2(w) = \int\limits_{K_2^1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}\,|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq$$

$$\preceq \int\limits_{K_2^1(R_1)} \frac{|t-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)}\,|dt|}{|t-w|^{2(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq n^{\gamma_1(\lambda_1-\varepsilon)} \int\limits_{K_2^1(R_1)} \frac{|dt|}{|t-w|^{2(\lambda_1+\varepsilon)}} + \int\limits_{K_2^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq n^{(2+\gamma_1)\lambda_1-1+\varepsilon}. \tag{3.24}$$

1.1.3. If $\gamma_1 \geq 0$, then Lemma 2.2 implies

$$D_{n,1}^3(w) = \int\limits_{K_3^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} \preceq$$

$$\preceq c_2^{-\gamma_1} \int\limits_{K_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1+\varepsilon}} \preceq n^{2\lambda_1-1+\varepsilon}, \tag{3.25}$$

and for $\gamma_1 < 0$, also Lemma 2.4 yields

$$D_{n,1}^3(w) \preceq c_3^{-\gamma_1} \int\limits_{K_3^1(R_1)} \frac{|dt|}{|t-w|^{2\lambda_1+\varepsilon}} \preceq n^{2\lambda_1-1+\varepsilon}. \tag{3.26}$$

1.2. Let $w \in K_2^1(R)$.

1.2.1. For any $\gamma_1 > -2$

$$D_{n,1}^1(w) = \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} =:$$

$$=: D_{n,1}^{1,1}(w) + D_{n,1}^{1,2}(w), \tag{3.27}$$

and so, according to Lemmas 2.1 and 2.2, we obtain

$$D_{n,1}^{1,1}(w) \preceq \int\limits_{K_{1,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq \int\limits_{1/n}^{c} \frac{ds}{s^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 = 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon, \end{cases} \tag{3.28}$$

and

$$D_{n,1}^{1,2}(w) \preceq \int\limits_{K_{1,2}^1(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq n^{(2+\gamma_1)(\lambda_1+\varepsilon)} \operatorname{mes} K_{1,2}^1(R_1) \preceq n^{(2+\gamma_1)\lambda_1-1+\varepsilon}. \tag{3.29}$$

1.2.2. For any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we have

$$D_{n,1}^2(w) \preceq \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K_{2,2}^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq$$

$$\preceq \int\limits_{K_{2,1}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} + \int\limits_{K_{2,2}^1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \int\limits_{1/n}^{c_1} \frac{ds}{s^{(2+\gamma_1)(\lambda_1+\varepsilon)}} + \int\limits_{1/n}^{c_2} \frac{ds}{s^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon. \end{cases} \tag{3.30}$$

1.2.3. For any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we get

$$D_{n,1}^3(w) \preceq \int\limits_{K_3^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K_3^1(R_1)} \frac{|dt|}{|t-w|^{2(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \int\limits_{1/n}^{c_3} \frac{ds}{s^{2(\lambda_1+\varepsilon)}} \preceq \begin{cases} n^{2\lambda_1-1+\varepsilon}, & \text{if} \quad 2\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad 2\lambda_1 = 1-\varepsilon, \\ 1, & \text{if} \quad 2\lambda_1 < 1-\varepsilon. \end{cases}$$

1.3. Let $w \in K_3^1(R)$.

1.3.1. If $\gamma_1 \geq 0$, from Lemmas 2.1 and 2.2, we have

$$D_{n,1}^1(w) \preceq \int\limits_{K_1^1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}} \preceq \int\limits_{K_1^1(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq n^{\gamma_1(\lambda_1+\varepsilon)} \operatorname{mes} K_1^1(R_1) \preceq n^{\gamma_1(\lambda_1+\varepsilon)-1} \tag{3.31}$$

and, for $\gamma_1 < 0$,

$$D_{n,1}^1(w) \preceq \int\limits_{K_1^1(R_1)} |t-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)}\,|dt| \preceq \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1-\varepsilon)} \operatorname{mes} K_1^1(R_1) \preceq$$

$$\preceq \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1-\varepsilon)+1} \preceq 1. \tag{3.32}$$

1.3.2. In this case for any $\gamma_1 > -2$, according to Lemmas 2.1 and 2.2, we obtain

$$D^2_{n,1}(w) \preceq \int\limits_{K^1_{2,1}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} + \int\limits_{K^1_{2,2}(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq$$

$$\preceq \int\limits_{K^1_{2,1}(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} + \int\limits_{K^1_{2,2}(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \int\limits_{1/n}^{c_1} \frac{ds}{s^{(2+\gamma_1)(\lambda_1+\varepsilon)}} + \int\limits_{1/n}^{c_2} \frac{ds}{s^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq$$

$$\preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon. \end{cases} \tag{3.33}$$

1.3.3. Analogously, for any $\gamma_1 > -2$,

$$D^3_{n,1}(w) \preceq \int\limits_{K^1_3(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K^1_3(R_1)} \frac{|dt|}{|t-w|^{2(\lambda_1+\varepsilon)}} \preceq n^{2\lambda_1-1+\varepsilon}. \tag{3.34}$$

Combining estimates (3.14)–(3.34), for $w \in \Phi(L_R)$, we have

$$D_{n,1} \preceq \begin{cases} n^{(2+\widetilde{\gamma}_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\widetilde{\lambda}_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\widetilde{\lambda}_1 > 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\widetilde{\lambda}_1 < 1-\varepsilon, \end{cases} \tag{3.35}$$

where $\widetilde{\gamma}_1 := \max\{0;\gamma_1\}$, $\widetilde{\lambda}_1 := \max\{1;\lambda_1\}$.

Case 2. Let $w \in \Phi(L^2_R)$.

Analogously to the case 1, in this case we will obtain estimates for $w \in K^2_1(R)$, $w \in K^2_2(R)$ and $w \in K^2_3(R)$.

2.1. Let $w \in K^2_1(R) \cup K^2_2(R)$. We will estimate the quantity

$$D_{n,2}(w) = \sum_{i=1}^{3} \int\limits_{K^2_i(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2} =: \sum_{i=1}^{3} D^i_{n,2}(w) \tag{3.36}$$

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

According to the estimation [24, p. 181] for arbitrary continuum with simple connected complementary, the following holds:

$$|\Psi(t)-\Psi(w_2)| \succeq |t-w_2|^2. \tag{3.37}$$

We will use this fact in evaluations in this section instead of Lemma 2.2.

2.1.1. For each $i=1,2$, we obtain

$$\sum_{i=1}^{2} D_{n,2}^{i}(w)=\sum_{i=1}^{2}\int\limits_{K_i^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2}\preceq$$

$$\preceq\left(\int\limits_{K_{1,1}^2(R_1)}+\int\limits_{K_{2,1}^2(R_1)}\right)\frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_2}}+\left(\int\limits_{K_{1,2}^2(R_1)}+\int\limits_{K_{2,2}^2(R_1)}\right)\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}}\preceq$$

$$\preceq\left(\int\limits_{K_{1,1}^2(R_1)}+\int\limits_{K_{2,1}^2(R_1)}\right)\frac{|dt|}{|t-w|^{2(2+\gamma_2)}}\preceq n^{2(2+\gamma_2)-1}, \tag{3.38}$$

if $\gamma_2\geq 0$, and

$$\sum_{i=1}^{2} D_{n,2}^{i}(w)=\sum_{i=1}^{2}\int\limits_{K_i^2(R_1)}\frac{|\Psi(t)-\Psi(w_2)|^{(-\gamma_2)}\,|dt|}{|\Psi(t)-\Psi(w)|^2}\preceq n^3, \tag{3.39}$$

if $\gamma_2<0$.

2.1.2. For $i=3$ we get

$$D_{n,2}^{3}(w)=\int\limits_{K_3^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2}\preceq$$

$$\preceq c_2^{-\gamma_2}\int\limits_{K_3^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w)|^2}\preceq\int\limits_{K_3^2(R_1)}\frac{|dt|}{|t-w|^{2+\varepsilon}}\preceq n^{1+\varepsilon}, \tag{3.40}$$

if $\gamma_2\geq 0$, and

$$D_{n,2}^{3}(w)\preceq n^{1+\varepsilon}, \tag{3.41}$$

if $\gamma_2<0$.

2.2. Let $w\in K_3^2(R)$. For each $\gamma_2>-2$, analogously to subcase 2.1.1, we obtain

$$\sum_{i=1}^{2} D_{n,2}^{i}(w)=\sum_{i=1}^{2}\int\limits_{K_i^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}\,|\Psi(t)-\Psi(w)|^2}\preceq$$

$$\preceq\left(\int\limits_{K_{1,1}^2(R_1)}+\int\limits_{K_{2,1}^2(R_1)}\right)\frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_2}}+\left(\int\limits_{K_{1,2}^2(R_1)}+\int\limits_{K_{2,2}^2(R_1)}\right)\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}}\preceq$$

$$\preceq\left(\int\limits_{K_{1,1}^2(R_1)}+\int\limits_{K_{2,1}^2(R_1)}\right)\frac{|dt|}{|t-w|^{2(2+\gamma_2)}}+\left(\int\limits_{K_{1,2}^2(R_1)}+\int\limits_{K_{2,2}^2(R_1)}\right)\frac{|dt|}{|t-w_2|^{2(2+\gamma_2)}}\preceq$$

$$\preceq \begin{cases} n^{2(2+\gamma_2)-1}, & \text{if} \quad 2(2+\gamma_2) > 1, \\ \ln n, & \text{if} \quad 2(2+\gamma_2) = 1, \\ 1, & \text{if} \quad 2(2+\gamma_2) < 1. \end{cases} \tag{3.42}$$

2.2.2. For each $\gamma_2 > -2$, we have

$$D_{n,2}^3(w) = \int\limits_{K_3^2(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_2)|^{\gamma_2} |\Psi(t) - \Psi(w)|^2} \preceq$$

$$\preceq \int\limits_{K_3^2(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^2} \preceq \int\limits_{K_3^2(R_1)} \frac{|dt|}{|t-w|^{2+\varepsilon}} \preceq n^{1+\varepsilon}. \tag{3.43}$$

Combining (3.36) – (3.43), we obtain

$$D_{n,2}(w) \preceq \begin{cases} n^{2(2+\widetilde{\gamma}_2)-1}, & \text{if} \quad 2(2+\gamma_2) > 1, \\ \ln n, & \text{if} \quad 2(2+\gamma_2) = 1, \\ 1, & \text{if} \quad 2(2+\gamma_2) < 1, \end{cases} \tag{3.44}$$

where $\widetilde{\gamma}_2 := \max\{0; \gamma_2\}$.

Therefore, comparing relations (3.11), (3.13), (3.35) and (3.44), we get

$$D_n(w) \preceq \begin{cases} n^{(2+\widetilde{\gamma}_1)\widetilde{\lambda}_1 - 1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\widetilde{\lambda}_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\widetilde{\lambda}_1 > 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\widetilde{\lambda}_1 < 1-\varepsilon, \end{cases} + \begin{cases} n^{2(2+\widetilde{\gamma}_2)-1}, & \text{if} \quad 2(2+\gamma_2) > 1, \\ \ln n, & \text{if} \quad 2(2+\gamma_2) = 1, \\ 1, & \text{if} \quad 2(2+\gamma_2) < 1, \end{cases}$$

and consequently, from (3.7), (3.8) and (3.9), we completed the proof of Theorems 1.1 and 1.2 for any $z \in L_R$. So, it also true for $z \in \overline{G}$, and we completed the proofs.

***3.2. Proof of Theorem* 1.3.** Suppose that $G \in C_\theta(\lambda_1; 2)$, for some $0 < \lambda_1 < 2$; $h(z)$ be defined as in (1.1). For each $R > 1$, let $w = \varphi_R(z)$ denotes be a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0) = 0$, $\varphi_R'(0) > 0$, and let $\{\zeta_j\}$, $1 \le j \le m \le n$, be a zeros of $P_n(z)$ (if any exist) lying on G_R. Let

$$b_{m,R}(z) := \prod_{j=1}^{m} \widetilde{b}_{j,R}(z) =: \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}, \tag{3.45}$$

denotes a Blaschke function with respect to zeros $\{\zeta_j\}$, $1 \le j \le m \le n$, of $P_n(z)$ [26]. Clearly,

$$|b_{m,R}(z)| \equiv 1, \; z \in L_R, \qquad \text{and} \qquad |b_{m,R}(z)| < 1, \quad z \in G_R. \tag{3.46}$$

For any $p > 0$ and $z \in G_R$, let us set

$$T_{n.p}(z) := \left[\frac{P_n(z)}{b_{m,R}(z)} \right]^{p/2}. \tag{3.47}$$

The function $T_{n,p}(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_{n,p}(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_{n,p}(z)$ at the $z = z_j,\ j = 1, 2$, gives

$$T_{n,p}(z_1) = \frac{1}{2\pi i} \int\limits_{L_R} T_{n,p}(\zeta) \frac{d\zeta}{\zeta - z_1}.$$

Then, according to (3.46), we obtain

$$|P_n(z_j)|^{p/2} \leq \frac{|b_{m,R}(z_1)|^{p/2}}{2\pi} \int\limits_{L_R} \left| \frac{P_n(\zeta)}{b_{m,R}(\zeta)} \right|^{p/2} \frac{|d\zeta|}{|\zeta - z_j|} \preceq$$

$$\preceq \int\limits_{L_R} |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z_j|}. \tag{3.48}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we get

$$\left(\int\limits_{L_R} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z_j|} \right)^2 \leq$$

$$\leq \int\limits_{|t|=R} h(\Psi(t)) \, |P_n(\Psi(t))|^p \left|\Psi'(t)\right|^2 |dt| \int\limits_{|t|=R} \frac{|dt|}{h(\Psi(t)) \, |\Psi(t) - \Psi(w_j)|^2} =$$

$$= \int\limits_{|t|=R} |f_{n,p}(t)|^p \, |dt| \int\limits_{|t|=R} \frac{|dt|}{h(\Psi(t)) \, |\Psi(t) - \Psi(w_j)|^2}, \tag{3.49}$$

where $f_{n,p}(t)$ has been defined as in (3.8). Since $R > 1$ is arbitrary, then (3.49) holds also for $R = R_1 := 1 + \dfrac{\varepsilon_1}{n}$, $0 < \varepsilon_1 < 1$. So, we have

$$\left(\int\limits_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} \frac{|d\zeta|}{|\zeta - z_j|} \right)^2 \leq$$

$$\leq \left(\int\limits_{|t|=R_1} |f_{n,p}(t)|^p \, |dt| \right) \left(\int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) \, |\Psi(t) - \Psi(w_j)|^2} \right) =:$$

$$=: A_n D_n(w_j), \tag{3.50}$$

and, A_n and $D_n(w_j)$ has been defined as in (3.8) for $R = R_1$. Therefore, from (3.48) and (3.50), we obtain

$$|P_n(z_1)| \preceq A_n D_n(w_j), \tag{3.51}$$

where, according to (3.9), the estimate

$$A_n \preceq n \left\| P_n \right\|_p^p$$

is satisfied. For the estimate of the quantity $D_n(w_j)$ we use the notations at the estimation of the $D_n(w)$ as in (3.11)–(3.13). Therefore, under these notations, for the $D_n(w_j)$, we get

$$D_n(w_j) \preceq \sum_{j=1}^{2} \int\limits_{\Phi(L_{R_1}^j)} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{2+\gamma_j}} \preceq$$

$$\preceq \sum_{j=1}^{2}\sum_{i=1}^{3} \int\limits_{K_i^j(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_j)|^{2+\gamma_j}} =: \sum_{j=1}^{2}\sum_{i=1}^{3} D_{n,j}^i(w_j), \tag{3.52}$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. So, we need to evaluate the $D_{n,j}^i(w_j)$ for each $j=1,2$ and $i=1,2,3$.

Case 1. $j=1$:

$$D_{n,1}^1(w_1)+D_{n,1}^2(w_1) = \int\limits_{K_1^1(L_{R_1})\cup K_1^2(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq$$

$$\preceq \int\limits_{K_1^1(L_{R_1})\cup K_1^2(L_{R_1})} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)(\lambda_1+\varepsilon)}} \preceq \begin{cases} n^{(2+\gamma_1)\lambda_1-1+\varepsilon}, & \text{if} \quad (2+\gamma_1)\,\lambda_1 > 1-\varepsilon, \\ \ln n, & \text{if} \quad (2+\gamma_1)\,\lambda_1 = 1-\varepsilon, \\ 1, & \text{if} \quad (2+\gamma_1)\,\lambda_1 < 1-\varepsilon, \end{cases} \tag{3.53}$$

and

$$D_{n,1}^3(w_1) = \int\limits_{K_1^3(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} \preceq \frac{1}{c_2^{2+\gamma_1}} \int\limits_{K_1^3(L_{R_1})} |dt| \preceq 1. \tag{3.54}$$

Case 2. $j=2$:

$$D_{n,2}^1(w_2)+D_{n,2}^2(w_2) = \int\limits_{K_2^1(L_{R_1})\cup K_2^2(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}} \preceq$$

$$\preceq \int\limits_{K_2^1(L_{R_1})\cup K_2^2(L_{R_1})} \frac{|dt|}{|t-w_2|^{2(2+\gamma_2)}} \preceq \begin{cases} n^{2(2+\gamma_2)-1}, & \text{if} \quad 2\,(2+\gamma_2) > 1, \\ \ln n, & \text{if} \quad 2\,(2+\gamma_2) = 1, \\ 1, & \text{if} \quad 2\,(2+\gamma_2) < 1, \end{cases} \tag{3.55}$$

and

$$D_{n,2}^3(w_2) = \int\limits_{K_2^3(L_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{2+\gamma_2}} \preceq \frac{1}{c_2^{2+\gamma_2}} \int\limits_{K_2^3(L_{R_1})} |dt| \preceq 1. \tag{3.56}$$

Combining relations (3.51)–(3.56), we complete the proof.

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