UDC 514.7

A. Sarkar, A. Sil, A. K. Paul (Univ. Kalyani, West Bengal, India)

SOME CHARACTERIZATIONS OF THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS ADMITTING η-RICCI SOLITONS AND TRANS-SASAKIAN MANIFOLDS AS KAGAN SUBPROJECTIVE SPACES

ДЕЯКІ ХАРАКТЕРИСТИКИ ТРИВИМІРНИХ ТРАНС-МНОГОВИДІВ САСАКЯНА, ЩО ДОПУСКАЮТЬ η-СОЛІТОНИ РІЧЧІ, ТА ТРАНС-МНОГОВИДИ САСАКЯНА ЯК СУБПРОЕКТИВНІ ПРОСТОРИ КАГАНА

The object of the present paper is to study three-dimensional trans-Sasakian manifolds admitting η-Ricci soliton. Actually, we study such manifolds whose Ricci tensor satisfy some special conditions like cyclic parallelity, Ricci semisymmetry, ϕ-Ricci semisymmetry, after reviewing the properties of second order parallel tensors on such manifolds. We determine the form of Riemann curvature tensor of trans-Sasakian manifolds of dimension greater than three as Kagan subprojective spaces. We also give some classification results of trans-Sasakian manifolds of dimension greater than three as Kagan subprojective spaces.

Вивчаються тривимірні транс-многовиди Сасакяна, які допускають η-солітони Річчі. Власне, після огляду властивостей паралельних тензорів другого порядку на таких многовидах ми вивчаємо многовиди, тензор Річчі яких задовольняє деякі спеціальні умови, такі як циклічна паралельність, напівсиметрія Річчі, ϕ-напівсиметрія Річчі. Визначено форму тензора кривини Рімана для транс-многовидів Сасакяна, розмірність яких більша ніж 3, як субпроективних просторів Кагана. Також наведено деякі класифікаційні результати для транс-многовидів Сасакяна, розмірність яких більша ніж 3, як субпроективних просторів Кагана.

1. Introduction. In [18], R. S. Hamilton introduced the revolutionary concept of Ricci flow on surfaces. The concepts of Ricci flow in physics was introduced by Friedan [14] almost around in the same time but with different motivations. Now a days such geometric flows have become popular, largely, because of Perelman's [22] work which lead to the proof of well known Poincaré conjecture. A Ricci soliton is a special solution of Ricci flow. This is considered as a natural generalization of Einstein metric and is defined on a Riemannian manifold (M, g) by

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where $\pounds_V$ denotes the Lie derivative operator along a complete vector field V. V is known as potential vector field. λ is a constant, called soliton constant. S is the Ricci tensor and g is the metric. X, Y are the arbitrary vector fields on M. The Ricci soliton is said to be shrinking, steady or expanding as λ is negative, zero or positive, respectively [7]. The study of Ricci solitons on contact manifolds was initiated by R. Sharma [24]. Later several authors have studied Ricci soliton on almost contact manifolds. For example, we may refer the papers [12, 15, 16, 27]. In [5], it has been proved that a real hypersurface in a non-flat complex space form does not admit a Ricci soliton with ξ as soliton vector field and then the author adopted the notion of η-Ricci soliton. The η-Ricci soliton (g, ξ, λ, μ) on a Riemannian manifold is defined by

$$\pounds_\xi g + 2S + 2\lambda - 2\mu n \otimes n = 0,$$

where ξ is the Reeb vector field, μ is a constant and the other objects are as described in equation (1.1). For details see also [2, 6, 8, 23]. Since 1923 [13], second order parallel tensors are studied by several authors [20]. Second order parallel tensors were studied in the frame work of contact manifolds by R. Sharma [25].

If geodesics are represented by $n-2$ homogeneous linear equations for a suitable coordinate system in an affine space A_n, then A_n is called a subprojective space by B. Kagan [19]. T. Adati [1] have studied such spaces intensively and proved that Kagan subprojective spaces are conformally flat [1, p. 160]. In this paper, we would like to find form of Riemann curvature tensors of trans-Sasakian manifolds of dimension greater than three as Kagan subprojective spaces. We give some classification results of such spaces.

2. Preliminaries. Let M be a differentiable manifold of dimension $2n+1$. M is said to have almost contact structure [3] if there is a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

where $X \in \chi(M)$, the set of all differentiable vector fields on M. On such manifolds it can be also proved that

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X)$$

for $X, Y \in \chi(M)$. An almost contact structure is called almost contact metric structure if there exists a Riemannian metric g on M satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The (0,2) tensor field Φ defined by $\Phi(X,Y) = g(X, \phi Y)$ is known as fundamental 2-form of the manifold. If Φ is closed, an almost contact metric structure reduces to contact metric structure [3]. An almost contact metric structure (ϕ, ξ, η, g) on a differentiable manifold M is called a trans-Sasakian structure [21] if $(M \times R, J, G)$ belongs to the class W_4 in the Gray – Hervella classification [17]. Here J is the almost complex structure on $M \times R$ defined by $J\left(X, f\dfrac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\dfrac{d}{dt}\right)$, for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This fact may be formulated by the following equation [4]:

$$(\nabla_X \phi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\phi X, Y)\xi - \eta(Y)\phi X\big),$$

where α and β are smooth functions on M. The above formula implies

$$\nabla_X \xi = \beta(X - \eta(X)\xi) - \alpha\phi X, \tag{2.1}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.2}$$

The Ricci tensor [11] of a three-dimensional trans-Sasakian manifold is given by

$$S(X,Y) = \left(\frac{r}{2} + \xi\beta - \big(\alpha^2 - \beta^2\big)\right) g(X,Y) - \left(\frac{r}{2} + \xi\beta - 3\big(\alpha^2 - \beta^2\big)\right)\eta(X)\eta(Y) -$$

$$-\big(Y\beta + (\phi Y)\alpha\big)\eta(X) - \big(X\beta + (\phi X)\alpha\big)\eta(Y), \tag{2.3}$$

where r is the scalar curvature of the manifold. Again from [9], we known that the Riemann curvature of a three-dimensional trans-Sasakian manifold is given by

$$R(X,Y)Z = \Big(\frac{r}{2} + 2\xi\beta - 2\big(\alpha^2 - \beta^2\big)\Big)\,\big(g(Y,Z)X - g(X,Z)Y\big) -$$
$$-g(Y,Z)\,\Big[\Big(\frac{r}{2} + \xi\beta - 3\big(\alpha^2 - \beta^2\big)\Big)\,\eta(X)\xi - \eta(X)(\phi\mathrm{grad}\,\alpha - \mathrm{grad}\,\beta) +$$
$$+\,(X\beta + (\phi X)\alpha)\xi\Big] + g(X,Z)\,\Big[\Big(\frac{r}{2} + \xi\beta - 3\big(\alpha^2 - \beta^2\big)\Big)\,\eta(Y)\xi - \eta(Y)(\phi\mathrm{grad}\,\alpha - \mathrm{grad}\,\beta) +$$
$$+\,(Y\beta + (\phi Y)\alpha)\xi\Big] - \Big[\big(Z\beta + (\phi Z)\alpha\big)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) +$$
$$+\Big(\frac{r}{2} + \xi\beta - 3\big(\alpha^2 - \beta^2\big)\Big)\,\eta(Y)\eta(Z)\Big]\,X + \Big[(Z\beta + (\phi Z)\alpha)\eta(X) +$$
$$+\big(X\beta + (\phi X)\alpha\big)\eta(Z) + \Big(\frac{r}{2} + \xi\beta - 3\big(\alpha^2 - \beta^2\big)\Big)\,\eta(X)\eta(Z)\Big]\,Y. \tag{2.4}$$

Again

$$2\alpha\beta + \xi\alpha = 0.$$

3. Existence criteria of η-Ricci soliton on three-dimensional trans-Sasakian manifolds.

Theorem 3.1. *A three-dimensional trans-Sasakian manifold with constant ξ-sectional curvature admits η-Ricci soliton if and only if $\pounds_\xi g + 2S + 2\mu n \otimes n$ is parallel.*

Theorem 3.2. *A three-dimensional proper trans-Sasakian manifold with cyclic parallel Ricci tensor does not admit η-Ricci soliton. It reduces to Einstein manifold.*

Proof. Let $(M, g, \xi, \lambda, \mu)$ be a three-dimensional trans-Sasakian η-Ricci soliton. Then we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Expressing the Lie derivative in terms of covariant derivative and using (2.1), we obtain

$$S(X,Y) = -\frac{2\lambda + \beta}{2} g(X,Y) + \frac{2\mu - \beta}{2}\eta(X)\eta(Y). \tag{3.1}$$

In [12], it was proved that if T is a symmetric parallel tensor on a trans-Sasakian manifold of dimension three with non-zero ξ-sectional curvature, then

$$T(X,Y) = T(\xi,\xi)g(X,Y).$$

We see that $(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y)$ is a symmetric $(0,2)$ tensor. Hence, by using its property, we obtain Theorem 3.1. Consider the manifold has cyclic parallel Ricci tensor [9]. By virtue of (3.1) and (2.3), after simplification we have $S = -\dfrac{2(\lambda + \beta)}{2} g$.

Theorem 3.2 is proved.

4. Ricci semisymmetric three-dimensional trans-Sasakian manifold admitting η-Ricci soliton.

Theorem 4.1. *If a three-dimensional trans-Sasakian manifold of type (α, β), where $\alpha \neq \pm\beta$ and $\beta = \text{constant}$, admitting η-Ricci soliton is Ricci semisymmetric, then $\mu = \dfrac{\beta}{2}$.*

***Corollary* 4.1.** *A Ricci semisymmetric three-dimensional α-Sasakian manifold does not admit proper η-Ricci soliton.*

Proof. It is well known that a Riemannian manifold is called Ricci semisymmetric if

$$(R(X,Y)S)(U,V) = 0.$$

The above condition implies

$$S\big(R(X,Y)U,V\big) + S\big(U,R(X,Y)V =\big) = 0.$$

Putting $Y = V = \xi$, we have

$$S\big(R(X,\xi)U,\xi\big) + S\big(U,R(X,\xi)\xi\big) = 0.$$

Using (2.4) in the above equation, after straight forward computation we have $\mu = \dfrac{\beta}{2}$, provided $\alpha \neq \pm\beta$. So, we have Theorem 4.1 for α-Sasakian case $\alpha = \text{constant}$ and $\beta = 0$. Thus we have Corollary 4.1.

5. ϕ-Ricci symmetric three-dimensional trans-Sasakian manifold admitting η-Ricci soliton. The notion of ϕ-Ricci symmetry was given by the first author in [10]. An almost contact manifold is called ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2(\nabla_W Q)X = 0.$$

The manifold is called locally ϕ-Ricci symmetric if X and W are orthogonal to ξ.

Theorem 5.1. *A three-dimensional trans-Sasakian manifold admitting η-Ricci soliton is ϕ-Ricci symmetric if and only if $\mu = \dfrac{\beta}{2}$.*

Proof. By virtue of (3.1) we obtain

$$QX = -\frac{2\lambda+\beta}{2}X + \frac{2\mu-\beta}{2}\eta(X)\xi.$$

Hence,

$$\phi^2(\nabla_W Q)X = \alpha\left(\frac{\beta}{2}-\mu\right)\eta(X)\phi^2(\phi W) + \beta\left(\mu-\frac{\beta}{2}\right)\phi^2 W.$$

The above equation proves Theorem 5.1.

6. Form of Riemann curvature tensors of trans-Sasakian manifolds of dimension greater than three as Kagan subprojective spaces. Riemann curvature tensors for three-dimensional trans-Sasakian manifolds have been deduced in the paper [11]. In this section, we like to deduce the form of Riemann curvature tensors of trans-Sasakian manifolds of dimension greater than three as Kagan subprojective spaces [19].

Theorem 6.1. *The form of Riemann curvature tensor of a trans-Sasakian manifold of dimension greater than three as Kagan subprojective space is given by*

$$R(X,Y)Z = \left(\frac{r}{2m(2m-1)} - \frac{2}{2m-1}\left(\frac{r}{2m}(\alpha^2-\beta^2)\right)\right)\big(g(Y,Z)X - g(X,Z)Y\big) +$$

$$+\frac{1}{2m-1}\left(\frac{r}{2m} + (2m+1)(\alpha^2-\beta^2)\right)\big(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big).$$

Theorem 6.2. *A trans-Sasakian manifold of dimension greater than three as Kagan subprojective space is Einstein manifold. Hence, it does not admit Ricci soliton and η-Ricci soliton.*

Proof. T. Adati [1] have proved that Kagan subprojective spaces are conformally flat. We know that the Weyl conformal curvature tensor C of a $(2m+1)$-dimensional $(m>1)$ manifold is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2m-1}\Big[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\Big] + \\ + \frac{r}{2m(2m-1)}\big[g(Y,Z)X - g(X,Z)Y\big],$$

where S is Ricci tensor and Q is Ricci operator. Since Kagan subprojective spaces are conformally flat, we get

$$R(X,Y)Z = \frac{1}{2m-1}\Big[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\Big] - \\ - \frac{r}{2m(2m-1)}\Big[g(Y,Z)X - g(X,Z)Y\Big].$$

From the above equation, we find S and Q and obtain the results.

7. Some classification results of a trans-Sasakian manifold of dimension greater than three as Kagan subprojective space.

Definition 7.1. *A Riemannian manifold is called locally ϕ-symmetric* [26] *if*

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

for X, Y, Z orthogonal to ξ.

Theorem 7.1. *A trans-Sasakian manifold of dimension greater than three as Kagan subprojective space is locally ϕ-symmetric if and only if $\dfrac{dr}{2m} = 4(\alpha d\alpha - \beta d\beta)$.*

Theorem 7.2. *If the structure functions α and β of a trans-Sasakian manifold of dimension greater than three as Kagan subprojective space are same, then the manifold is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant.*

Theorem 7.3. *If the structure functions α and β of a trans-Sasakian manifold of dimension greater than three as Kagan subprojective space are constants, then the manifold is locally ϕ-symmetric.*

Proof. By using Theorem 6.1, we get

$$\phi^2(\nabla_W R)(X,Y)Z = \\ = \left(\frac{dr}{2m(2m-1)} - \frac{2}{2m-1}\left(\frac{dr}{2m} + 2\big(\alpha d\alpha - \beta d\beta\big)\right)\right)\Big(g(X,Z)Y - g(Y,Z)X\Big), \qquad (7.1)$$

for X, Y, Z orthogonal to ξ. Let us consider the following cases:

Case 1. Consider α and β as arbitrary functions:

Subcase 1.1: Let $\alpha \neq \beta$. In that case we get Theorem 7.1 from (7.1).

Subcase 1.2: Let α and β are equal functions. In that case $\alpha d\alpha - \beta d\beta = 0$. So, we obtain Theorem 7.2 from (7.1).

Case 2. Let α and β are constants. In that case, deducing S from R, we have Theorem 7.3.

References

1. T. Adati, *On subprojective space I*, Tohoku Math. J., **3**, 159 – 173 (1951).
2. A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, **30**, 489 – 496 (2016).
3. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progr. Math., **203**, (2002).
4. D. E. Blair, J. A. Oubina, *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Math., **34**, 199 – 207 (1990).
5. J. T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61**, 205 – 212 (2009).
6. C. Calin, C. Crasmareanu, *η-Ricci solitons on Hopf hypersurfaces in complex space forms, arxiv math. DG*.
7. B. Chow, D. Knopf, *The Ricci flow, An introduction*, Math. Surveys and Monogr., **110** (2004).
8. M. Crasmareanu, *Parallel tensor and Ricci solitons on N(K)-quasi-Einstein manifolds*, Indian J. Pure and Appl. Math., **43**, 359 – 369 (2012).
9. U. C. De, A. Sarkar, *On three-dimensional trans-Sasakian manifolds*, Extracta Math., **23**, 247 – 255 (2008).
10. U. C. De, A. Sarkar, *On ϕ-Ricci symmetric Sasakian manifolds*, Proc. Jangjeon Math. Soc., **11**, 47 – 52 (2008).
11. U. C. De, M. M. Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J., **43**, 247 – 255 (2003).
12. S. Debnath, A. Bhattacharya, *Second order parallel tensor in trans-Sasakian manifolds and connections with Ricci solitons*, Lobachevskii J. Math., **33**, 312 – 316 (2012).
13. L. P. Eisenhart, *Symmetric tensor of the second order whose first covariant derivatives are zero*, Trans. Amer. Math. Soc., **25**, 297 – 306 (1923).
14. D. Friedan, *Nonlinear models in $2+\varepsilon$-dimensions*, Ann. Phys., **163**, 318 – 419 (1985).
15. A. Ghosh, *Kenmotsu 3-metric as Ricci solitons*, Chaos, Solitons and Fractals, **44**, 647 – 650 (2011).
16. A. Ghosh, *Sasakian metric as a Ricci soliton and related results*, J. Geom. and Phys., **75**, 1 – 6 (2014).
17. A. Gray, L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura ed Appl., **123**, 35 – 58 (1980).
18. R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity, Contemp. Math., **71**, 237 – 262 (1988).
19. B. Kagan, *Über eine Erweiterung des Begriffes vom projektiven Raume und dem zugehörigen Absolut*, Tensor Analysis, **1**, 12 – 101 (1933).
20. H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish*, Ann. Math., **27**, 91 – 98 (1925).
21. J. A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen, **32**, 187 – 193 (1985).
22. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arxiv math. DG. 1021111599.
23. D. G. Prakasha, B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, J. Geom. (2016); DOI 10.1007/s00022-016-0345-z.
24. R. Sharma, *Certain results on K-contact and (K,μ)-contact manifolds*, J. Geom., **89**, 138 – 147 (2008).
25. R. Sharma, *Second order parallel tensors on contact manifolds*, I, Algebra, Group, Geom., **7**, 145 – 152 (1990).
26. T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J., **29**, 91 – 113 (1977).
27. M. Turan, U. C. De, A. Yildiz, *Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds*, Filomat, **26**, 363 – 370 (2012).

Received 08.06.17