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UNIQUENESS THEOREM FOR HOLOMORPHIC MAPPINGS ON ANNULI SHARING FEW HYPERPLANES

ТЕОРЕМА ЄДИНОСТІ ДЛЯ ГОЛОМОРФНИХ ВІДОБРАЖЕНЬ НА КІЛЬЦЯХ З КІЛЬКОМА СПІЛЬНИМИ ГІПЕРПЛОЩИНАМИ

We prove a uniqueness theorem of linearly nondegenerate holomorphic mappings from annulus to complex projective space $\mathbb{P}^n(\mathbb{C})$ with different multiple values and a general condition on the intersections of the inverse images of these hyperplanes.

Доведено теорему єдиності для лінійно невироджених голоморфних відображень з кільця до комплексного проективного простору $\mathbb{P}^n(\mathbb{C})$ із різними множинами значень і загальною умовою щодо перетину прообразів гіперплощин.

1. Introduction. In 1975, H. Fujimoto [3] proved that if two linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which have the same inverse images of $3n+2$ hyperplanes in general position counted with multiplicities then they are identical.

In 1983, L. Smiley [9] obtained a uniqueness theorem for meromorphic mappings which share $3n+2$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position without counting multiplicities (i.e., they have the same inverse images of $3n+2$ hyperplanes and are identical on these inverse images) and satisfy an additional condition "codimension of the intersections of inverse images of two different hyperplanes are at least two".

Later on, the unicity problem of meromorphic mappings with truncated multiplicities has been extended and deepened by contribution of many authors. These authors have improved the result of L. Smiley in the case where the number of hyperplanes is replaced by a smaller one. We state here the recent result of Z. Chen and Q. Yan [2] which is one of the best results available at present.

Take a meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}^m$ such that for positive integers $k,\ d,\ 1 \le d \le n$, and q hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim f^{-1}\left(\bigcap_{j=1}^{k+1} H_{i_j}\right) \le m-2, \quad 1 \le i_1 < \ldots < i_{k+1} \le q.$$

Let $\mathcal{F}\big(f, \{H_i\}_{i=1}^q, k, d\big)$ be the set of all linearly nondegenerate over $\mathbb{C}^m$ meromorphic maps $g\colon \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\min(\nu_{(f,H_j)}, d) = \min(\nu_{(g,H_j)}, d),\ 1 \le j \le q,$

(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

Denote by $\sharp S$ the cardinality of the set S.

Theorem A [2]. $\sharp\mathcal{F}\big(f, \{H_i\}_{i=1}^{2n+3}, 1, 1\big) = 1.$

In 2012, H. H. Giang, L. L. Quynh, and S. S. Quang [4] introduced new techniques to treat the case $k \geq 1$. However, they only considered the case where the mappings f and g share all hyperplanes with the same multiple values. Thus, our purpose of this paper is to prove a uniqueness theorem for annulus similar to the results of Giang, Quynh, and Quang in the case where the mappings f and g share all hyperplanes with different multiple values as following.

Theorem 1.1. *Let $f_1, f_2 : \mathbb{A}(R_0) \to \mathbb{P}^n(\mathbb{C})$ be two admissible linearly nondegenerate holomorphic mappings, where $\mathbb{A}(R_0) = \left\{ z \;\middle|\; 0 < \dfrac{1}{R_0} < |z| < R_0 \right\}$. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position and k $(\leq n)$ be a positive integer. Let k_i, $1 \leq i \leq q$, be positive integers or $+\infty$. Assume that*:

(i) $\min\{1, \nu^0_{(f_1,H_i),\leq k_i}\} = \min\{1, \nu^0_{(f_2,H_i),\leq k_i}\}$ *for* $i = 1, \ldots, q$;

(ii) $f_1^{-1}(\bigcap_{j=1}^{k+1} H_{i_j}) = \varnothing$, $1 \leq i_1 < \ldots < i_{k+1} \leq q$;

(iii) $f_1 = f_2$ *on* $\bigcup_{j=1}^{q} f_1^{-1}(H_j)$.

Then we have $f_1 \equiv f_2$ if either $q \geq 2(n+1)k$ and

$$\sum_{i=1}^{q} \frac{1}{k_i + 1} < \frac{(q-n-1)(q-2k+2kn) - 2qnk}{(q-2k+2kn)n}$$

or $q < 2(n+1)k$ and

$$\sum_{i=1}^{q} \frac{1}{k_i + 1} < \frac{q-n-1-(n+1)k}{n}.$$

2. Some definitions and results from Nevanlinna theory on annuli. In this section, we will recall some basic notions of Nevanlinna theory for meromorphic functions on annuli from [7] (see also [1, 5, 6]).

For a divisor ν on $\mathbb{A}(R_0)$, which we may regard as a function on $\mathbb{A}(R_0)$ with values in $\mathbb{Z}$ whose support is discrete subset of $\mathbb{A}(R_0)$, and for a positive integer M (maybe $M = \infty$), we define the counting function of ν as follows:

$$n_0^{[M]}(t) = \begin{cases} \sum_{1 \leq |z| \leq t} \min\{M, \nu(z)\}, & \text{if} \quad 1 \leq t < R_0, \\ \sum_{t \leq |z| < 1} \min\{M, \nu(z)\}, & \text{if} \quad \dfrac{1}{R_0} < t < 1, \end{cases}$$

and

$$N_0^{[M]}(r, \nu) = \int_{\frac{1}{r}}^{1} \frac{n_0^{[M]}(t)}{t} dt + \int_{1}^{r} \frac{n_0^{[M]}(t)}{t} dt, \quad 1 < r < \infty.$$

For brevity we will omit the character $^{[M]}$ if $M = \infty$.

For a divisor ν and a positive integer k (maybe $k = +\infty$), we define

$$\nu_{\leq k}(z) = \begin{cases} \nu(z), & \text{if} \quad \nu(z) \leq k, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nu_{>k}(z) = \begin{cases} \nu(z), & \text{if} \quad \nu(z) > k, \\ 0 & \text{otherwise.} \end{cases}$$

For a meromorphic function φ, we define

ν^0_φ (resp., ν^∞_φ) the divisor of zeros (resp., divisor of poles) of φ,
$\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$,
$\nu^0_{\varphi,\le k} = (\nu^0_\varphi)_{\le k}$, $\nu^0_{\varphi,>k} = (\nu^0_\varphi)_{>k}$.
Similarly, we define $\nu^\infty_{\varphi,\le k}$, $\nu^\infty_{\varphi,>k}$, $\nu_{\varphi,\le k}$, $\nu_{\varphi,>k}$ and their counting functions.

For a discrete subset $S \subset \mathbb{A}(R_0)$, we consider it as a reduced divisor (denoted again by S) whose support is S, and denote by $N_0(r, S)$ its counting function. We also set $\chi_S(z) = 0$ if $z \notin S$ and $\chi_S(z) = 1$ if $z \in S$.

Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. The proximity function of f is defined by

$$m_0(r,f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \left| f\left(\frac{e^{i\theta}}{r}\right)\right| d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta - \frac{1}{\pi}\int\limits_0^{2\pi} \log^+ \left|f(e^{i\theta})\right| d\theta$$

and the characteristic function of f is defined by

$$T_0(r,f) = m_0(r,f) + N_0(r, \nu^\infty_f).$$

Throughout this paper, we denote by $S_f(r)$ quantities satisfying:

(i) in the case $R_0 = +\infty$,

$$S_f(r) = O\big(\log(rT_0(r,f))\big)$$

for $r \in (1, +\infty)$ except for a set Δ_r such that $\displaystyle\int_{\Delta_r} r^{\lambda-1} dr < +\infty$ for some $\lambda \ge 0$,

(ii) in the case $R_0 < +\infty$,

$$S_f(r) = O\left(\log\left(\frac{T_0(r,f)}{R_0 - r}\right)\right) \quad \text{as} \quad r \to R_0$$

for $r \in (1, R_0)$ except for a set Δ'_r such that $\displaystyle\int_{\Delta'_r} \frac{dr}{(R_0 - r)^{\lambda+1}} < +\infty$ for some $\lambda \ge 0$.
The function f is said to be admissible if it satisfies

$$\limsup_{r\to+\infty} \frac{T_0(r,f)}{\log r} = +\infty \quad \text{in the case} \quad R_0 = +\infty$$

or

$$\limsup_{r\to R_0^-} \frac{T_0(r,f)}{-\log(R_0 - r)} = +\infty \quad \text{in the case} \quad 1 < R_0 < +\infty.$$

Thus for an admissible meromorphic function f on the annulus $\mathbb{A}(R_0)$, we have $S_f(r) = o(T_0(r,f))$ as $r \to R_0$ for all $1 \le r < R_0$ except for the set Δ_r or the set Δ'_r mentioned above, respectively (cf. [1]).

A meromorphic function a on $\mathbb{A}(R_0)$ is said to be small with respect to f if

$$T_0(r,a) = S_f(r).$$

Through this paper, by notation "$\|\, P$", we mean that the assertion P holds for all $1 \le r < R_0$ except for the set Δ_r or the set Δ'_r mentioned above, respectively.

Lemma 2.1 (Lemma on logarithmic derivatives [1, 5–7]). *Let f be a nonzero meromorphic function on $\mathbb{A}(R_0)$. Then for each $k \in \mathbb{N}$ we have*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = S_f(r), \quad 1 \le r < R_0.$$

Theorem 2.1 (First main theorem for meromorphic functions and values [1, 5–7]). *Let f be a meromorphic function on $\mathbb{A}(R_0)$. Then for each $a \in \mathbb{C}$ we have*

$$T_0(r, f) = T_0\left(r, \frac{1}{f-a}\right) + S_f(r), \quad 1 \le r < R_0.$$

Then for every small (with respect to f) function a ($a \not\equiv \infty$) on $\mathbb{A}(R_0)$, we obtain

$$T_0(r, f) \le T_0(r, f-a) + T_0(r, a) = T_0\left(r, \frac{1}{f-a}\right) + S_{1/(f-a)}(r) + S_f(r).$$

Similarly, we get

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r, f-a) + S_{1/(f-a)}(r) \le$$

$$\le T_0(r, f) + T_0(r, -a) + S_{1/(f-a)}(r) + S_f(r).$$

Therefore, we have the first main theorem for meromorphic functions and small function as follows.

Theorem 2.2 (First main theorem for meromorphic functions and small functions). *Let f be a meromorphic function on $\mathbb{A}(R_0)$ and let a be a small function with respect to f. Then we have*

$$T_0(r, f) = T_0\left(r, \frac{1}{f-a}\right) + S_f(r), \quad 1 \le r < R_0.$$

3. Nevanlinna theory for holomorphic mappings from an annulus into a projective space. Let f be a holomorphic mapping from an annulus $\mathbb{A}(R_0)$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \ldots : f_n)$. The characteristic function of f is defined by

$$T_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \left\|f(re^{i\theta})\right\| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \left\|f\left(\frac{1}{r}e^{i\theta}\right)\right\| d\theta - \frac{1}{\pi}\int_0^{2\pi} \log \left\|f(e^{i\theta})\right\| d\theta,$$

where $\|f\| = \left(|f_0|^2 + \ldots + |f_n|^2\right)^{\frac{1}{2}}$.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ given by $H = \left\{(\omega_0 : \ldots : \omega_n) \mid a_0\omega_0 + \ldots + a_N\omega_n = 0\right\}$. We set $(f, H) = a_0 f_0 + \ldots + a_n f_n$. The proximity function of f with respect to H is defined by

$$m_0(r, f, H) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\left\|f(re^{i\theta})\right\| \|H\|}{|(f, H)(re^{i\theta})|} d\theta +$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\left\|f\left(\frac{1}{r}e^{i\theta}\right)\right\| \|H\|}{\left|(f, H)\left(\frac{1}{r}e^{i\theta}\right)\right|} d\theta - \frac{1}{\pi}\int_0^{2\pi} \log \frac{\left\|f(e^{i\theta})\right\| \|H\|}{\left|(f, H)(e^{i\theta})\right|} d\theta,$$

where $\|H\| = \big(|a_0|^2 + \ldots + |a_n|^2\big)^{\frac{1}{2}}$.

By Jensen formula, we obtain

$$T_0(r,f) = m_0(r,f,H) + N_0(r,f^*H),$$

where f^*H denotes the pull back divisor of H by f.

Lemma 3.1. *Let f be as above. Let H_1 and H_2 be two distinct hyperplanes of $\mathbb{P}^n(\mathbb{C})$, then we have*

$$T_0\bigg(r, \frac{(f,H_1)}{(f,H_2)}\bigg) \le T_0(r,f) + O(1).$$

Let $\{H_i\}_{i=1}^q$, $q \ge n+2$, be a set of q hyperplanes in $\mathbb{P}^n(\mathbb{C})$. We say that the family $\{H_i\}_{i=1}^q$ is in general position if $\bigcap_{j=1}^{n+1} H_{i_j} = \varnothing$ for any $1 \le i_1 < \ldots < i_{n+1} \le q$. Using the same argument as in the proof of the Second Main Theorem for holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ (see [8], Theorem 3.1), we have the following Second Main Theorem for holomorphic curves from an annulus into $\mathbb{P}^n(\mathbb{C})$.

Theorem 3.1. *Let $f\colon \mathbb{A}(R_0) \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic mapping. Let $\{H_i\}_{i=1}^q$, $q \ge n+2$, be a set of q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then*

$$(q-n-1)T_0(r,f) \le \sum_{i=1}^q N_0^{[n]}(r, f^*H_i) + S_f(r), \quad 1 \le r < R_0,$$

where f^*H_i denotes the pull back divisor of H_i by f.

4. Proof of Theorem 1.1. In order to prove Theorem 1.1, we need the following.

Lemma 4.1. *Let f be nonconstant holomorphic mappings of $\mathbb{A}(R_0)$ into $\mathbb{P}^n(\mathbb{C})$. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ in general position and k $(\ge n)$ be a positive integer. Then*

$$N_0^{[n]}\big(r, \nu^0_{(f,H)}\big) \le n\bigg(1 - \frac{n}{k+1}\bigg) N_0^{[1]}\big(r, \nu^0_{(f,H),\le k}\big) + \frac{n}{k+1} N_0\big(r, \nu^0_{(f,H)}\big)$$

and

$$N_0^{[n]}\big(r, \nu^0_{(f,H)}\big) \le n\bigg(1 - \frac{n}{k+1}\bigg) N_0^{[1]}\big(r, \nu^0_{(f,H),\le k}\big) + \frac{n}{k+1} T_0(r,f) + S_f(r).$$

Proof. From

$$N_0^{[n]}\big(r, \nu^0_{(f,H)}\big) = N_0^{[n]}\big(r, \nu^0_{(f,H),\le k}\big) + N_0^{[n]}\big(r, \nu^0_{(f,H),>k}\big)$$

and

$$N_0^{[n]}(r, \nu^0_{(f,H),>k}) \le \frac{n}{k+1} N_0(r, \nu^0_{(f,H),>k}) \le \frac{n}{k+1}\Big(N_0\big(r, \nu^0_{(f,H)}\big) - N_0^{[n]}\big(r, \nu^0_{(f,H),\le k}\big)\Big)$$

we deduce that

$$N_0^{[n]}\big(r, \nu^0_{(f,H)}\big) \le \bigg(1 - \frac{n}{k+1}\bigg) N_0^{[n]}\big(r, \nu^0_{(f,H),\le k}\big) + \frac{n}{k+1} N_0\big(r, \nu^0_{(f,H)}\big) \le$$

$$\le n\bigg(1 - \frac{n}{k+1}\bigg) N_0^{[1]}\big(r, \nu^0_{(f,H),\le k}\big) + \frac{n}{k+1} N_0\big(r, \nu^0_{(f,H)}\big).$$

This completes the proof of the first inequality of the lemma. The second inequality of the lemma follows immediately because of $N_0\big(r, \nu^0_{(f,H)}\big) \le T_0(r,f) + S_f(r)$.

Lemma 4.2. *Let $f_1,\ f_2$ be nonconstant admissible holomorphic mappings of $\mathbb{A}(R_0)$ into $\mathbb{P}^n(\mathbb{C})$. Let $\{H_i\}_{i=1}^q$, $q \geq n+2$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let k_i, $1 \leq i \leq q$, be positive integers or $+\infty$. Assume that*

$$\min\left\{\nu^0_{(f_1,H_i),\leq k_i}, 1\right\} = \min\left\{\nu^0_{(f_2,H_i),\leq k_i}, 1\right\} \quad \textit{for all} \quad 1 \leq i \leq q.$$

If $\sum_{i=1}^q \dfrac{1}{k_i+1} < \dfrac{q-n-1}{n}$, then $\|T_0(r,f_2) = O(T_0(r,f_1))$ and $\|T_0(r,f_1) = O(T_0(r,f_2))$.

Proof. By the Second Main Theorem, we have

$$\|(q-n-1)T_0(r,f_2) \leq \sum_{i=1}^q N_0^{[n]}\big(r,\nu^0_{(f_2,H_i)}\big) + S_{f_2}(r) \leq$$

$$\leq \sum_{i=1}^q \left(n\left(1-\frac{n}{k_i+1}\right)N_0^{[1]}\big(r,\nu^0_{(f_2,H),\leq k_i}\big) + \frac{n}{k_i+1}T_0(r,f_2)\right) + S_{f_2}(r) \leq$$

$$\leq \sum_{i=1}^q \left(nN_0^{[1]}\big(r,\nu^0_{(f_1,H_i),\leq k_i}\big) + \frac{n}{k_i+1}T_0(r,f_2)\right) + S_{f_2}(r) \leq$$

$$\leq qn\, T_0(r,f_1) + n\sum_{i=1}^q \frac{1}{k_i+1}T_0(r,f_2) + S_{f_2}(r).$$

Thus

$$\left\|\left(q-n-1-n\sum_{i=1}^q \frac{1}{k_i+1}\right)T_0(r,f_2) \leq qnT_0(r,f_1) + S_{f_2}(r).\right.$$

Case 1. If $R_0 = +\infty$, then $\|S_{f_2}(r) = O\big(\log(rT_0(r,f_2))\big)$. Therefore there exists a positive constant K such that

$$\liminf_{r\to+\infty} \frac{S_{f_2}(r)}{T_0(r,f_2)} = \liminf_{r\to+\infty} \frac{K\log(rT_0(r,f_2))}{T_0(r,f_2)} = 0.$$

Case 2. If $1 < R_0 < +\infty$ then $\|S_{f_2}(r) = O\left(\log\left(\dfrac{T_0(r,f_2)}{R_0-r}\right)\right)$ as $r \to R_0$. Therefore there exists a positive constant K such that

$$\liminf_{r\to R_0,\ r\notin\Delta'_r} \frac{S_{f_2}(r)}{T_0(r,f_2)} = \liminf_{r\to R_0,\ r\notin\Delta'_r} \frac{K\log\left(\dfrac{T_0(r,f_2)}{R_0-r}\right)}{T_0(r,f_2)} \leq$$

$$\leq \limsup_{r\to R_0} \frac{K\log\left(\dfrac{1}{R_0-r}\right)}{T_0(r,f_2)} = 0.$$

Hence $\|T_0(r,f_2) = O(T_0(r,f_1))$. Similarly, we get $\|T_0(r,f_1) = O(T_0(r,f_2))$.

Lemma 4.2 is proved.

***Proof of Theorem* 1.1.** Assuming that $f_1 \not\equiv f_2$. By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f_1,H_1)}{(f_2,H_1)} \equiv \frac{(f_1,H_2)}{(f_2,H_2)} \equiv \ldots \equiv \frac{(f_1,H_{k_1})}{(f_2,H_{k_1})}}_{\text{group } 1} \not\equiv \underbrace{\frac{(f_1,H_{k_1+1})}{(f_2,H_{k_1+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_2})}{(f_2,H_{k_2})}}_{\text{group } 2} \not\equiv$$

$$\not\equiv \underbrace{\frac{(f_1,H_{k_2+1})}{(f_2,H_{k_2+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_3})}{(f_2,H_{k_3})}}_{\text{group } 3} \not\equiv \ldots \not\equiv \underbrace{\frac{(f_1,H_{k_{s-1}+1})}{(f_2,H_{k_{s-1}+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_s})}{(f_2,H_{k_s})}}_{\text{group } s},$$

where $k_s = q$.

For each $1 \le i \le q$, we set

$$\sigma(i) = \begin{cases} i+n, & \text{if} \quad i+n \le q, \\ i+n-q, & \text{if} \quad i+n > q, \end{cases}$$

and

$$P_i = (f_1,H_i)(f_2,H_{\sigma(i)}) - (f_2,H_i)(f_1,H_{\sigma(i)}).$$

Since $f_1 \not\equiv f_2$, the number of elements of each group is at most n. Then $\dfrac{(f_1,H_i)}{(f_2,H_i)}$ and $\dfrac{(f_1,H_{\sigma(i)})}{(f_2,H_{\sigma(i)})}$ belong to distinct groups. Therefore $P_i \not\equiv 0,\ 1 \le i \le q$. We set

$$P = \prod_{i=1}^{q} P_i \not\equiv 0$$

and

$$S = \bigcup_{1 \le i_1 < \ldots < i_{k+1} \le q} f_1^{-1}\left(\bigcap_{j=1}^{k+1} H_{i_j}\right).$$

Then S is an analytic set of codimension at most 2. By Jensen formula and by the definition of the characteristic function, we have

$$N_P(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log\left\|P(re^{i\theta})\right\| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left\|P\left(\frac{1}{r}e^{i\theta}\right)\right\| d\theta - \frac{1}{\pi}\int_0^{2\pi} \log\left\|P(e^{i\theta})\right\| d\theta \le$$

$$\le \frac{1}{2\pi}\sum_{i=1}^{q}\int_0^{2\pi} \log\left(|(f_1,H_i)|^2 + |(f_1,H_{\sigma(i)}|^2\right)^{\frac{1}{2}} d\theta +$$

$$+\frac{1}{2\pi}\sum_{i=1}^{q}\int_0^{2\pi} \log\left(|(f_2,H_i)|^2 + |(f_2,H_{\sigma(i)}|^2\right)^{\frac{1}{2}} d\theta \le$$

$$\leq \frac{1}{2\pi}\sum_{i=1}^{q}\int\limits_{0}^{2\pi}\log\Big(\|f_1\|\big(\|H_i\|^2+\|H_{\sigma(i)}\|^2\big)^{\frac{1}{2}}\Big)d\theta+$$

$$+\frac{1}{2\pi}\sum_{i=1}^{q}\int\limits_{0}^{2\pi}\log\Big(\|f_2\|\big(\|H_i\|^2+\|H_{\sigma(i)}\|^2\big)^{\frac{1}{2}}\Big)d\theta=$$

$$=q\left(\frac{1}{2\pi}\int\limits_{0}^{2\pi}\log(\|f_1\|)d\theta+\int\limits_{0}^{2\pi}\log(\|f_2\|)d\theta\right)+O(1)=$$

$$=q\big(T_0(r,f_1)+T_0(r,f_2)\big)+O(1). \tag{4.1}$$

On the other hand, we let $\xi := \dfrac{1}{z}$, then $f_1(\xi),\ f_2(\xi)$ are holomorphic mappings on $\mathbb{A}(R_0)$. By applying the Second Main Theorem we get, for $i=1,2,$

$$(q-n-1)T_0(r,f_i)\leq\sum_{j=1}^{q}N_0^{[n]}\big(r,\nu^0_{(f_i,H_j)}\big)+S_{f_i}(r). \tag{4.2}$$

We put $S_f(r)=S_{f_1}(r)+S_{f_2}(r),\ 1\leq r<R_0$.

Fix a point $z\notin I(f_1)\cup I(f_2)\cup S$. We assume that z is a zero of functions $(f_1,H_{i_1}),\ldots,(f_1,H_{i_t})$ with multiplicities $m_1,\ldots,m_t$, respectively, where $1\leq i_1<\ldots<i_t\leq q,\ t\leq k$, and z is not zero of any (f_1,H_i) for $i\notin\{i_1,\ldots,i_t\}$. For an index $i\in\{1,\ldots,q\}$, we distinguish the following four cases:

Case 1: $i,\sigma(i)\notin\{i_1,\ldots,i_t\}$. Then z is a zero point of P_i with multiplicity at least 1, since $f_1(z)=f_2(z)$. We denote $v(z)$ the number of indices i in this case. It is easy to see that $v(z)\geq$ $\geq q-2t$.

Case 2: $i\in\{i_1,\ldots,i_t\}$ and $\sigma(i)\notin\{i_1,\ldots,i_t\}$. Then z is a zero point of P_i with multiplicity at least $\min\Big\{\nu^0_{(f_1,H_i),\leq k_i},\nu^0_{(f_2,H_i),\leq k_i}\Big\}$.

Case 3: $\sigma(i)\in\{i_1,\ldots,i_t\}$ and $i\notin\{i_1,\ldots,i_t\}$. Then z is a zero point of P_i with multiplicity at least $\min\Big\{\nu^0_{(f_1,H_{\sigma(i)}),\leq k_i},\nu^0_{(f_2,H_{\sigma(i)}),\leq k_i}\Big\}$.

Case 4: $i,\sigma(i)\in\{i_1,\ldots,i_t\}$. Then z is a zero point of P_i with multiplicity at least

$$\min\Big\{\nu^0_{(f_1,H_i),\leq k_i},\nu^0_{(f_2,H_i),\leq k_i}\Big\}+\min\Big\{\nu^0_{(f_1,H_{\sigma(i)}),\leq k_i},\nu^0_{(f_2,H_{\sigma(i)}),\leq k_i}\Big\}.$$

Therefore, from the above four cases, it follows that

$$\nu_P(z)\geq 2\sum_{j=1}^{t}\min\Big\{\nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}(z),\nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}(z)\Big\}+v(z)\geq$$

$$\geq 2\sum_{j=1}^{t}\min\Big\{\nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}(z),\nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}(z)\Big\}+q-2t.$$

We consider the following two cases.

Case 1. If $q \geq 2(n+1)k$, then by using the fact that, for any two positive integer a and b, $\min\{a,b\} \geq \min\{a,n\} + \min\{b,n\} - n$, we get

$$\nu_P^0(z) \geq 2\sum_{j=1}^{t}\left(\min\left\{n, \nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}\right\} + \min\left\{n, \nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\} - n\right) + q - 2t =$$

$$= 2\sum_{j=1}^{t}\left(\min\left\{n, \nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}\right\} + \min\left\{n, \nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\}\right) - 2nt + q - 2t \geq$$

$$\geq 2\sum_{j=1}^{t}\left(\min\left\{n, \nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}\right\} + \min\left\{n, \nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\}\right) + q - 2(n+1)k \geq$$

$$\geq \left(2 + \frac{q-2(n+1)k}{2nk}\right)\sum_{i=1}^{q}\left(\min\left\{n, \nu^0_{(f_1,H_i),\leq k_i}\right\} + \min\left\{n, \nu^0_{(f_2,H_i),\leq k_i}\right\}\right) =$$

$$= \frac{q-2k+2kn}{2nk}\sum_{i=1}^{q}\left(\min\left\{n, \nu^0_{(f_1,H_i),\leq k_i}\right\} + \min\left\{n, \nu^0_{(f_2,H_i),\leq k_i}\right\}\right)$$

for all z outside the analytic set $I(f_1) \cup I(f_2) \cup S$.

Integrating both sides of the above inequality, we get

$$N_P(r) \geq \frac{q-2k+2kn}{2nk}\sum_{i=1}^{q}\left(N_0^{[n]}\big(r, \nu^0_{(f_1,H_i),\leq k_i}\big) + N_0^{[n]}\big(r, \nu^0_{(f_2,H_i),\leq k_i}\big)\right) =$$

$$= \frac{q-2k+2kn}{2nk}\sum_{i=1}^{q}\left(N_0^{[n]}\big(r, \nu^0_{(f_1,H_i)}\big) - N_0^{[n]}\big(r, \nu^0_{(f_1,H_i),>k_i}\big) + \right.$$

$$\left. + N_0^{[n]}\big(r, \nu^0_{(f_2,H_i)}\big) - N_0^{[n]}\big(r, \nu^0_{(f_2,H_i),>k_i}\big)\right) \geq$$

$$\geq \frac{q-2k+2kn}{2nk}\sum_{i=1}^{q}\left(N_0^{[n]}\big(r, \nu^0_{(f_1,H_i)}\big) + N_0^{[n]}\big(r, \nu^0_{(f_2,H_i)}\big) - \frac{n}{k_i+1}\big(T_0(r,f_1) + T_0(r,f_2)\big)\right). \tag{4.3}$$

Combining (4.2) and (4.3), it shows that

$$N_P(r) \geq \frac{q-2k+2kn}{2nk}\left(\left(q - n - 1 - \sum_{i=1}^{q}\frac{n}{k_i+1}\right)\big(T_0(r,f_1) + T_0(r,f_2)\big) - S_f(r)\right). \tag{4.4}$$

Thus, by (4.1) and (4.4) we have

$$q\big(T_0(r,f_1) + T_0(r,f_2)\big) \geq$$

$$\geq \frac{q-2k+2kn}{2nk}\left(\left(q - n - 1 - \sum_{i=1}^{q}\frac{n}{k_i+1}\right)\big(T_0(r,f_1) + T_0(r,f_2)\big) - S_f(r)\right).$$

This implies that

$$\left(q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{2qnk}{q-2k+2kn}\right)\big(T_0(r,f_1)+T_0(r,f_2)\big)\leq S_f(r). \tag{4.5}$$

Case 2. If $q<2(n+1)k$, then we get

$$\nu_P^0(z)\geq\left(2-\frac{q}{(n+1)k}\right)\sum_{j=1}^{t}\min\left\{\nu^0_{(f_1,H_{i_j}),\leq k_{i_j}},\nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\}+$$

$$+\frac{q}{(n+1)k}\sum_{j=1}^{t}\left(\min\left\{n,\nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}\right\}+\min\left\{n,\nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\}-n\right)+q-2t\geq$$

$$\geq\left(2-\frac{q}{(n+1)k}\right)t-\frac{qnt}{(n+1)k}+q-2t+$$

$$+\frac{q}{(n+1)k}\sum_{j=1}^{t}\left(\min\left\{n,\nu^0_{(f_1,H_{i_j}),\leq k_{i_j}}\right\}+\min\left\{n,\nu^0_{(f_2,H_{i_j}),\leq k_{i_j}}\right\}\right)\geq$$

$$\geq\frac{q}{(n+1)k}\sum_{i=1}^{q}\left(\min\left\{n,\nu^0_{(f_1,H_i),\leq k_i}\right\}+\min\left\{n,\nu^0_{(f_2,H_{i_j}),\leq k_i}\right\}\right)$$

for all z outside the analytic set $I(f_1)\cup I(f_2)\cup S$.

Integrating both sides of the above inequality, we obtain

$$N_P(r)\geq\frac{q}{(n+1)k}\sum_{i=1}^{q}\left(N_0^{[n]}\big(r,\nu^0_{(f_1,H_i),\leq k_i}\big)+N_0^{[n]}\big(r,\nu^0_{(f_2,H_i),\leq k_i}\big)\right)=$$

$$=\frac{q}{(n+1)k}\sum_{i=1}^{q}(N_0^{[n]}\big(r,\nu^0_{(f_1,H_i)}\big)-N_0^{[n]}\big(r,\nu^0_{(f_1,H_i),>k_i}\big)+$$

$$+N_0^{[n]}\big(r,\nu^0_{(f_2,H_i)}\big)-N_0^{[n]}\big(r,\nu^0_{(f_2,H_i),>k_i}\big))\geq$$

$$\geq\frac{q}{(n+1)k}\sum_{i=1}^{q}\left(N_0^{[n]}\big(r,\nu^0_{(f_1,H_i)}\big)+N_0^{[n]}\big(r,\nu^0_{(f_2,H_i)}\big)-\frac{n}{k_i+1}\big(T_0(r,f_1)+T_0(r,f_2)\big)\right). \tag{4.6}$$

Combining (4.2) and (4.6), it shows that

$$N_P(r)\geq\frac{q}{(n+1)k}\left(\left(q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}\right)\big(T_0(r,f_1)+T_0(r,f_2)\big)-S_f(r)\right). \tag{4.7}$$

Thus, by (4.1) and (4.7) we have

$$q\big(T_0(r,f_1)+T_0(r,f_2)\big)\geq$$

$$\geq\frac{q}{(n+1)k}\left(\left(q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}\right)\big(T_0(r,f_1)+T_0(r,f_2)\big)-S_f(r)\right).$$

This implies that

$$\left(q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-(n+1)k\right)\big(T_0(r,f_1)+T_0(r,f_2)\big)\leq S_f(r). \tag{4.8}$$

We consider the following two cases.

Case 1. If $R_0=+\infty$, then $\|S_f(r)=O(\log(r\big(T_0(r,f_1)+T_0(r,f_2)\big))$. Therefore there exists a positive constant K such that

$$\liminf_{r\to+\infty}\frac{S_f(r)}{T_0(r,f_1)+T_0(r,f_2)}=\liminf_{r\to+\infty}\frac{K\log(r\big(T_0(r,f_1)+T_0(r,f_2)\big))}{T_0(r,f_1))+T_0(r,f_2)}=0.$$

Letting $r\to\infty$, we get two subcases.

Subcase 1.1. If $q\geq 2(n+1)k$, then by (4.5) we have

$$q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-\frac{2qnk}{q-2k+2kn}\leq 0,$$

i.e.,

$$\sum_{i=1}^{q}\frac{1}{k_i+1}\geq\frac{(q-n-1)(q-2k+2kn)-2qnk}{(q-2k+2kn)n}.$$

This is a contradiction.

Subcase 1.2. If $q<2(n+1)k$, then by (4.8) we get

$$q-n-1-\sum_{i=1}^{q}\frac{n}{k_i+1}-(n+1)k\leq 0,$$

i.e.,

$$\sum_{i=1}^{q}\frac{1}{k_i+1}\geq\frac{q-n-1-(n+1)k}{n}.$$

This is a contradiction.

Case 2. If $1<R_0<+\infty$, then

$$\|S_f(r)=O\left(\log\left(\frac{T_0(r,f_1)+T_0(r,f_2)}{R_0-r}\right)\right)\quad\text{as}\quad r\to R_0.$$

Therefore, there exists a positive constant K such that

$$\liminf_{r\to R_0,\ r\notin\Delta'_r}\frac{S_f(r)}{T_0(r,f_1)+T_0(r,f_2)}=\liminf_{r\to R_0,\ r\notin\Delta'_r}\frac{K\log\left(\dfrac{T_0(r,f_1)+T_0(r,f_2)}{R_0-r}\right)}{T_0(r,f_1)+T_0(r,f_2)}\leq$$

$$\leq\limsup_{r\to R_0}\frac{K\log\left(\dfrac{1}{R_0-r}\right)}{T_0(r,f_1)+T_0(r,f_2))}=0.$$

Letting $r\to R_0$, by repeating the same arguments of the Case 1, we get a contradiction.

Hence, from the above two cases, it follows that $f_1\equiv f_2$.

Theorem 1.1 is proved.

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