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FUGLEDE – PUTNAM TYPE THEOREMS FOR EXTENSION OF *M*-HYPONORMAL OPERATORS

ТЕОРЕМИ ТИПУ ФУГЛЕДЕ – ПУТНАМА ДЛЯ РОЗШИРЕНЬ *M*-ГІПОНОРМАЛЬНИХ ОПЕРАТОРІВ

We consider k-quasi-M-hyponormal operator $T \in B(\mathcal{H})$ such that $TX = XS$ for some $X \in B(\mathcal{K}, \mathcal{H})$ and prove the Fuglede – Putnam type theorem when adjoint of $S \in B(\mathcal{K})$ is k-quasi-M-hyponormal or dominant operators. We also show that two quasisimilar k-quasi-M-hyponormal operators have equal essential spectra.

Розглянуто k-квазі-M-гіпонормальний оператор $T \in B(\mathcal{H})$ такий, що $TX = XS$ для деякого $X \in B(\mathcal{K}, \mathcal{H})$, та доведено теорему типу Фугледе – Путнама, коли спряженим до $S \in B(\mathcal{K})$ є або k-квазі-M-гіпонормальний, або домінуючий оператор. Також показано, що два квазіподібні k-квазі-M-гіпонормальні оператори мають однакові суттєві спектри.

1. Introduction. Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, and let $B(\mathcal{H}, \mathcal{K})$ denote the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ (We also write $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$.) Throughout this paper, the range and the null space of an operator T will be denoted by $\operatorname{ran}(T)$ and $\ker(T)$, respectively. Let $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ be the norm closure and the orthogonal complement of the subspace $\mathcal{M}$ of $\mathcal{H}$. The classical *Fuglede – Putnam theorem* [4] (Problem 152) asserts that if $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are normal operators such that $TX = XS$ for some operators $X \in B(\mathcal{K}, \mathcal{H})$, then $T^*X = XS^*$. The references [2, 6, 9, 10, 17 – 19] are among the various extensions of this celebrated theorem for nonnormal operators. According to [17], an operator $T \in \mathcal{H}$ is *dominant* if

$$\operatorname{ran}(T - \lambda I) \subseteq \operatorname{ran}(T - \lambda I)^* \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

From [1], it is seen that this condition is equivalent to the existence of a positive constant M_λ such that

$$(T - \lambda I)(T - \lambda I)^* \leq M_\lambda^2 (T - \lambda I)^*(T - \lambda I)$$

for each $\lambda \in \mathbb{C}$. An operator T is called *M-hyponormal* if there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$. If $M = 1$, T is hyponormal. We have the following inclusion relations:

$$\{\text{hyponormal}\} \subseteq \{M\text{-hyponormal}\} \subseteq \{\text{dominant}\}.$$

Mecheri [5] introduced *k-quasi-M-hyponormal* operators as follows. An operator T is k-quasi-M-hyponormal if there exists a real positive number M such that

$$T^{*k}\big((T - \lambda I)(T - \lambda I)^*\big)T^k \leq T^{*k}\big(M^2\big(T - \lambda I)^*(T - \lambda I)\big)T^k$$

for all $\lambda \in \mathbb{C}$, where k is a natural number. Evidently,

$$\{M\text{-hyponormal}\} \subseteq \{k\text{-quasi-}M\text{-hyponormal}\}.$$

For $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, we say that FP-*theorem* holds for the pair (T, S) if $TX = XS$ implies $T^*X = XS^*$, $\overline{\operatorname{ran}(X)}$ reduces T, and $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{\operatorname{ran}(X)}}$ and $S|_{\ker(X)^\perp}$ are unitarily equivalent normal operators for all $X \in B(\mathcal{K}, \mathcal{H})$. We say that an operator S is *quasiaffine transform* of an operator T if there exists an injective operator X with dense range such that $TX = XS$. Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$ are *quasisimilar* if there exist quasiaffinities $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $YS = TY$. In general quasisimilarity need not preserve the spectrum and essential spectrum. However, in special classes of operators quasisimilarity preserves spectra. For instance, it is well-known that two quasisimilar hyponormal operators have equal spectrum and equal essential spectrum.

Recall that an operator $T \in B(\mathcal{H})$ is k-*quasihyponormal* if $T^{*k}(T^*T - TT^*)T^k \geq 0$, where k is a positive integer and an operator $T \in B(\mathcal{H})$ is said to be (p, k)-quasihyponormal operators if $T^{*k}\big((T^*T)^p - (TT^*)^p\big)T^k \geq 0$, where k is a positive integer and $0 < p \leq 1$ [3, 19]. Recently, Tanahashi, Patel and Uchiyama [19] found some extensions of Fuglede – Putnam theorems involving (p, k)-quasihyponormal, dominant, and spectral operators.

Recall [8] that an operator $T \in B(\mathcal{H})$ is said to have the *single-valued extension property* (SVEP) if for every open subset D of $\mathbb{C}$ and any analytic function $f : D \to \mathcal{H}$ such that $(T - \lambda)f(\lambda) \equiv 0$ on D, it results $f(\lambda) \equiv 0$ on D. We say that a Hilbert space operator satisfies *Bishop property* (β) if, for every open subset D of $\mathbb{C}$ and every sequence $f_n : D \longrightarrow \mathcal{H}$ of analytic functions with $(T - \lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of D, $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of D. It is well-known that

$$\text{Bishop property}(\beta) \Rightarrow \text{SVEP}$$

(see [8] for more information). Mecheri [5] proved that k-quasi-M-hyponormal operators satisfies Bishop property (β). Recently, some spectral properties of k-quasi-M-hyponormal operators has been studied by Zuo and Mecheri [22]. In the present note, we seek some extensions of Fuglede – Putnam type theorems involving k-quasi-M-hyponormal operator and dominant operators. Let U be an open set in $\mathbb{C}$. Stampfli [16] investigated the equation $(T - \lambda I)f(\lambda) \equiv x$ for some non-zero $x \in \mathcal{H}$ and $f : U \to \mathcal{H}$ in an effort to discover necessary and or sufficient condition for analyticity of f when T is a dominant operator. In this paper, we show that if $T \in B(\mathcal{H})$ be k-quasi-M-hyponormal, if $0 \notin \delta \subseteq \mathbb{C}$ be closed, and if there exists a bounded function $f : \mathbb{C}\backslash\delta \to \mathcal{H}$ such that $(T - \lambda I)f(\lambda) \equiv x$ for some non-zero $x \in H$, then f is analytic at every non zero point and hence f has analytic extension everywere on $\mathbb{C}\backslash\delta$. In Section 3, we show that if $T, S \in B(\mathcal{H})$ are quasisimilar k-quasi-M-hyponormal operators, then they have equal spectrum.

2. Fuglede – Putnam type theorem. Throughout this paper we would like to present some known results as propositions which will be used in the sequel.

Proposition 2.1 [5]. *Let T be k-quasi-M-hyponormal operator*, $\operatorname{ran}(T^k)$ *be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k}).$$

Then $T_1 = T|_{\overline{\operatorname{ran}(T^k)}}$ is M-hyponormal, $T_3^k = 0$ *and* $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proposition 2.2 [15]. *Let $T \in B(\mathcal{H})$ and let $S \in B(\mathcal{K})$. Then the following assertions are equivalent*:

(i) *If* $TX = XS$ *where* $X \in B(\mathcal{K}, \mathcal{H})$, *then* $T^*X = XS^*$.

(ii) *If* $TX = XS$ *where* $X \in B(\mathcal{K}, \mathcal{H})$, *then* $\overline{\operatorname{ran}(X)}$ *reduces* T, *and* $\ker(X)^{\perp}$ *reduces* S, *the restrictions* $T|_{\overline{\operatorname{ran}(X)}}$ *and* $S|_{\ker(X)^{\perp}}$ *are normal.*

Proposition 2.3 [10]**.** *Let* T *and* S *be* M*-hyponormal operators and* $TX = XS^*$. *Then*

(i) $\overline{\operatorname{ran}(X)}$ *reduces* T *and* $\ker(X)$ *reduces* S.

(ii) $T|_{\overline{\operatorname{ran}(X)}}$ *and* $S^*|_{\ker(X)^{\perp}}$ *are unitarily equivalent normal operators.*

It is well-known that a normal part of hyponormal is reducing. This result remains true for dominant operators.

Proposition 2.4 [14, 17, 21]**.** *Let* $T \in B(\mathcal{H})$ *be dominant and* $\mathcal{M}$ *be an invariant subspace of* T. *Then*:

(i) *The restriction* $T|_{\mathcal{M}}$ *is dominant.*

(ii) *If the restriction* $T|_{\mathcal{M}}$ *is normal*, *then* $\mathcal{M}$ *reduces* T.

In the following lemma we prove, a normal part of a k-quasi-M-hyponormal operator is reducing.

Lemma 2.1. *If the restriction* $T|_{\mathcal{M}}$ *of the* k*-quasi-*M*-hyponormal operator* $T \in \mathcal{B}(\mathcal{H})$ *to an invariant subspace* $\mathcal{M}$ *is injective and normal*, *then* $\mathcal{M}$ *reduces* T.

Proof. Let T be k-quasi-M-hyponormal and $T_1 = T|_{\mathcal{M}}$ is injective and normal. Decompose T on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

The following inclusion relation holds by the k-quasi-M-hyponormality of T and Theorem 1 of [1]:

$$\operatorname{ran}(T^{*k}(T - \lambda I)) \subset \operatorname{ran}(T^{*k}(T^* - \overline{\lambda} I)) \subset \operatorname{ran}(T^* - \overline{\lambda} I)$$

for $\lambda \in \mathbb{C}$. Then, for any arbitrary vector $y \in \mathcal{M}^{\perp}$, $T_1^{*k} T_2 y = (T_1^* - \overline{\lambda}) u_{\lambda}$ for some $u_{\lambda} \in \mathcal{M}$. Choose v_{λ} such that $(T_1^* - \overline{\lambda} I) u_{\lambda} = (T_1 - \lambda I) v_{\lambda}$. It follows that $T_1^{*k} T_2 y = (T_1 - \lambda) v_{\lambda}$ and so

$$T_1^{*k} T_2 y \in \bigcap_{\lambda \in \mathbb{C}} \operatorname{ran}(T_1 - \lambda I).$$

Then, by [11] (Theorem 1), $T_1^{*k} T_2 y = 0$ and hence $T_2 y = 0$. Therefore, $T_2 = 0$.

Remark **2.1.** The condition $T|_{\mathcal{M}}$ is injective in Lemma 2.1 is indispensable because $\ker(T)$ for k-quasi-M-hyponormal operator T is not always reducing.

In [19], the authors considered the situation S and T^* are (p, k)-quasihyponormal operators and proved Fuglede – Putnam theorem for (S, T) if either S or T is injective. Now we study Fuglede – Putnam theorem for the case that T and S^* are k-quasi-M-hyponormal operators with the condition that either T or S^* is injective.

Theorem 2.1. *Let* $T \in B(\mathcal{H})$ *and* $S^* \in B(\mathcal{K})$ *be* k*-quasi-*M*-hyponormal operators. If either* T *or* S^* *is injective*, *then Fuglede – Putnam theorem holds for* (T, S).

Proof. Suppose T and S^* are k-quasi-M-hyponormal operators and $TX = XS$ for any operator $X \in B(\mathcal{K}, \mathcal{H})$. Since $\overline{\operatorname{ran(X)}}$ is invariant under T and $\ker(X)^{\perp}$ is invariant under S^*, we decompose T, S and X into

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\operatorname{ran}(X)} \oplus \overline{\operatorname{ran}(X)}^{\perp},$$

$$S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \ker(X)^{\perp} \oplus \ker(X),$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \ker(X)^{\perp} \oplus \ker(X) \to \overline{\operatorname{ran}(X)} \oplus \overline{\operatorname{ran}(X)}^{\perp},$$

where T_1 and S_1^* are k-quasi-M-hyponormal operators by Proposition 2.1, and

$$X_1 : \ \ker(X)^{\perp} \to \overline{\operatorname{ran(X)}}$$

is injective with dense range.

From $TX = XS$, we have

$$T_1 X_1 = X_1 S_1. \tag{2.1}$$

First consider the case where T is injective. Clearly, T_1 is injective. It is not difficult to show from (2.1) that S_1 is injective or equivalently, $\operatorname{ran}(S_1^*)$ is dense. Incidently, S_1^* turns out to be a M-hyponormal operator. In particular, $\ker(S_1^*) \subset \ker(S_1)$ and hence $\ker(S_1^*) = 0$. From (2.1), it is easy to see that T_1^* is injective, thereby T_1 is M-hyponormal. Next consider the case that S^* is injective. Then S_1^* is injective and so T_1^* is injective by (2.1). Obviously, T_1 is an injective M-hyponormal operator, and, by (2.1), S_1 is injective. Therefore, S_1^* is M-hyponormal. Ultimately, if either T or S^* is injective, then T_1 and S_1^* are both M-hyponormal operators. Then, by Propositions 2.2 and 2.3 and (2.1), we obtain

$$T_1^* X_1 = X_1 S_1^*$$

and $T_1,\ S_1$ are normal operators. Since T_1 and S_1 are injective, $T_2 = S_2 = 0$ by Lemma 2.1. Hence,

$$T^* X = T_1^* X_1 = X_1 S_1^* = X S^*.$$

The rest of the proof follows from Proposition 2.2.

***Corollary* 2.1.** *Let $T \in B(\mathcal{H})$ and $S^* \in B(\mathcal{K})$ be k-quasi-M-hyponormal operators with reducing kernels. Then Fuglede – Putnam theorem holds for (T, S).*

Proof. By hypothesis, we can write $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $S = S_1^* \oplus S_2^*$ with respect to $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where T_1 and S_1^* are normal parts and T_2 and S_2 are pure parts. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad \text{on} \quad \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

From $TX = XS$, we have

$$\begin{pmatrix} T_1X_1 & T_1X_2 \\ T_2X_3 & T_2X_4 \end{pmatrix} = \begin{pmatrix} X_1S_1 & X_2S_2 \\ X_3S_1 & X_4S_2 \end{pmatrix}.$$

The underlying kernel conditions ensures of T_2 and S_2^* are injective. The operator T_2 is injective k-quasi-M-hyponormal and S_1 normal. From the above matrix relation, we obtain $T_2X_3 = X_3S_1$. Then by applying Theorem 2.1, we get $T_2^*X_3 = X_3S_1^*$, $\overline{\operatorname{ran}(X_3)}$ reduces T_2 and $T_2|_{\overline{\operatorname{ran}(X_3)}}$ is normal and so $X_3 = 0$. In a similar manner we have $X_2 = 0$ from $T_1X_2 = X_2S_2$ and $X_4 = 0$ from $T_2X_4 = X_4S_2$. Since T_1 and S_1 are normal and since $T_1X_1 = X_1S_1$, required result follows from classical Fuglede – Putnam theorem and Proposition 2.2.

Proposition 2.5 [21]. *If $T^* \in B(\mathcal{H})$ is M-hyponormal, $S \in B(\mathcal{K})$ is dominant, and $XT = SX$ for $X \in B(\mathcal{H}, \mathcal{K})$, then $XT^* = S^*X$.*

Now we consider the situation that where T is a k-quasi-M-hyponormal operator and S^* is a dominant operator.

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be k-quasi-M-hyponormal and $S^* \in B(\mathcal{K})$ be dominant. If either T or S^* is injective, then Fuglede – Putnam theorem holds for (T, S).*

Proof. Suppose that $T \in B(\mathcal{H})$ is k-quasi-M-hyponormal and $S^* \in B(\mathcal{K})$ is dominant such that $TX = XS$ for $X \in B(\mathcal{K}, \mathcal{H})$. Since $\overline{\mathrm{ran(X)}}$ is invariant under T and $\ker(X)^\perp$ is invariant under S^*, we can write T, S and X as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathrm{ran(X)}} \oplus \overline{\mathrm{ran(X)}}^\perp,$$

$$S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \ker(X)^\perp \oplus \ker(X)$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \ker(X)^\perp \oplus \ker(X) \to \overline{\mathrm{ran}(X)} \oplus \overline{\mathrm{ran}(X)}^\perp.$$

From $TX = XS$, we have

$$T_1 X_1 = X_1 S_1, \tag{2.2}$$

where T_1 is k-quasi-M-hyponormal by Proposition 2.1, S_1^* is dominant by Proposition 2.4 and

$$X_1 : \ \ker(X)^\perp \to \overline{\mathrm{ran}(X)}$$

is injective with dense range. First assume that T is injective. Then T_1 is injective. From (2.1), S_1 is injective. Since S_1^* is dominant, it turns out to be injective. By (2.2), we have that T_1^* is injective. Ultimately, T_1 is M-hyponormal. Applying Proposition 2.5 to (2.2), we obtain

$$T_1^* X_1 = X_1 S_1^*$$

and T_1, S_1 are normal operators. Since T_1 injective, $T_2 = 0$ by Lemma 2.1. Also $S_2 = 0$ by Proposition 2.4. Next assume S^* is injective. Then S_1^* is injective. Then by (2.2) T_1^* is injective. Ultimately, T_1 turns out to be M-hyponormal. Conclude as before that

$$T_1^* X_1 = X_1 S_1^*$$

and T_1, S_1 are injective normal operators and so $T_2 = S_2 = 0$. Hence,

$$T^* X = T_1^* X_1 = X_1 S_1^* = XS^*.$$

The rest of the proof follows from Proposition 2.2.

***Corollary* 2.2.** *Let $T \in B(\mathcal{H})$ be dominant and $S^* \in B(K)$ be k-quasi-M-hyponormal operator. If either T or S^* is injective, then Fuglede – Putnam theorem holds for (T, S).*

Proof. From $TX = XS$, we have $S^*X^* = X^*T^*$. Applying Theorem 2.2, it follows that $SX^* = X^*T$. The rest of the proof follows from Proposition 2.2.

***Corollary* 2.3.** *Let $T \in B(\mathcal{H})$ be k-quasi-M-hyponormal operator with reducing kernel and $S^* \in B(\mathcal{K})$ be dominant operator such that $TX = XS$ for $X \in B(\mathcal{K}, \mathcal{H})$. Then Fuglede–Putnam theorem holds for (T, S).*

Proof. Let $T \in B(\mathcal{H})$ be k-quasi-M-hyponormal with reducing kernel and $S^* \in B(\mathcal{K})$ be dominant. We decompose T, S and X as follows:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \ker(T)^\perp \oplus \ker(T)$$

and

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \ker(S)^\perp \oplus \ker(S).$$

Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad \text{on} \quad \ker(S)^\perp \oplus \ker(S) \to \ker(T)^\perp \oplus \ker(T).$$

From $TX = XS$, we have

$$\begin{pmatrix} T_1X_1 & T_1X_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1S_1 & 0 \\ X_3S_1 & 0 \end{pmatrix}.$$

The equations $T_1X_2 = 0$ and $X_3S_1 = 0$ yields $X_2 = X_3 = 0$ because T_1 and S_1^* are injective. Applying Theorem 2.2 to $T_1X_1 = X_1S_1$, it follows $T_1^*X_1 = X_1S_1^*$.

Stampfli and Wadhwa [17] proved if T be dominant and S be a normal operator and if $TX = XS$ where $X \in B(\mathcal{H})$ has dense range, then T is a normal operator (see [17], Theorem 1). This remarkable result for k-quasihyponormal operators has been studied by Gupta and Ramanujan [3]. Now we show this result remains true for k-quasi-M-hyponormal operators.

Theorem 2.3. *Let T be a k-quasi-M-hyponormal and S a normal operator. If S is quasiaffine transform of T, then T is a normal operator unitarily equivalent to S.*

Proof. Let T be k-quasi-M-hyponormal. By Proposition 2.1, decompose T on $\mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus$ $\oplus \ker(T^{*k})$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 = T|_{\overline{\operatorname{ran}(T^k)}}$ is M-hyponormal and $T_3^k = 0$. Let $S_1 = S|_{\overline{\operatorname{ran}(S^k)}}$. Decompose

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, S_1 is normal. Let $X_1 = X|_{\overline{\operatorname{ran}(S^k)}}$. Then

$$X_1 : \overline{\operatorname{ran}(S^k)} \to \overline{\operatorname{ran}(T^k)}$$

is injective and has dense range.

From $TX = XS$, we have

$$T_1 X_1 = X_1 S_1.$$

Since T_1 is M-hyponormal and S_1 is normal, it follows from [17] (Theorem 1) that T_1 is normal operator unitarily equivalent to S_1. Consequently, $\overline{\operatorname{ran}(T^k)}$ reduces T and so $T_2 = 0$ by Lemma 2.1. Since $X^*(\ker(T^{*k})) \subset \ker(S^{*k}) = \ker(S^*)$,

$$X^* T_3^* x = X^* T^* x = S^* X^* = 0$$

for each $x \in \ker(T^{*k})$. Since X has dense range, X^* is one-to-one. Therefore, $T_3^* x = 0$ for each $x \in \ker(T^{*k})$. Hence, $T_3 = 0$ and so $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ is normal.

Proposition 2.6 [16]. *Let $T \in B(\mathcal{H})$ be dominant and $\delta \subseteq \mathbb{C}$ be closed. If there exists a bounded function $f(z) : \mathbb{C}\backslash\delta \to \mathcal{H}$ such that $(T - zI)f(z) \equiv x$ for some non-zero $x \in \mathcal{H}$, then $f(z)$ is analytic on $\mathbb{C}\backslash\delta$.*

The above result proved for hyponormal operators by Radjabalipour [13]. This result for k-quasihyponormal with a condition $0 \notin \delta$ and its consequences has been studied by Gupta [2]. In the following theorem, we show this result hold true in the case of k-quasi-M-hyponormal operators.

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be k-quasi-M-hyponormal and $0 \notin \delta \subseteq \mathbb{C}$ be closed. If there exists a bounded function $f(\lambda) : \mathbb{C}\backslash\delta \to \mathcal{H}$ such that $(T - \lambda I)f(\lambda) \equiv x$ for some non-zero $x \in \mathcal{H}$, then f is analytic at every non-zero point and hence f has analytic extension everywhere on $\mathbb{C}\backslash\delta$.*

Proof. Suppose that T is k-quasi-M-hyponormal. By Proposition 2.1, decompose T on $\mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k})$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 = T|_{\overline{\operatorname{ran}(T^k)}}$ is M-hyponormal and $T_3^k = 0$.

Let $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$ and $x = x_1 \oplus x_2$ are the decomposition of f and x, respectively. Then

$$(T_1 - \lambda I)f_1(\lambda) + T_2 f_2(\lambda) \equiv x_1,$$

$$(T_3 - \lambda I)f_2(\lambda) \equiv x_2.$$

Since $T_3^k = 0$ and $0 \notin \delta$, $f_2(\lambda) = (T_3 - \lambda I)^{-1} x_2$ $(\lambda \neq 0)$ can be extended to a bounded entire function. Since k-quasi-M-hyponormal operators satisfies single valued extension property, we conclude $x_2 = 0$ (see [8], Proposition 1.2.16 9(f)). Hence $f_2(\lambda) = 0$. Therefore, for all $\lambda \notin \delta$,

$$(T_1 - \lambda I)f(\lambda) \equiv x_1.$$

M-hyponormality of T_1 ensures $f_1(\lambda)$ is analytic at every non zero point and has analytic extension every where on $\mathbb{C}\backslash\delta$ by Proposition 2.6.

If T and T^* are M-hyponormal, then T is normal [14]. Gupta [2] proved if T and T^* are k-quasihyponormal and T is injective, then T is normal. Now we establish a similar result for k-quasi-M-hyponormal operators.

***Corollary* 2.4.** *Let T be dominant or k-quasi-M-hyponormal and S^* be k-quasi-M-hyponormal. If either T or S is injective and S is a quasiaffine transform of T, then T and S are unitarily equivalent normal operators. In particular, if T, T^* are k-quasi-M-hyponormal and T is injective, then T is normal.*

Proof. Let T be dominant or k-quasi-M-hyponormal and S^* be k-quasi-M-hyponormal. Since S^* is k-quasi-M-hyponormal, there exists a real positive number M such that $\|(S-\lambda I)S^{*k}\| \leq \leq M\|(S-\lambda I)^* S^{*k}\|$. Therefore,

$$S^k(S-\lambda I)^*(S-\lambda I)S^{*k} \leq M(S-\lambda I)(S-\lambda I)^*.$$

Applying [14] (Theorem 2), it follows that

$$(S-\lambda)(S-\lambda)^* \geq c^2|(SS^*-S^*S)S^k|^2$$

for some $c>0$, where $|.|$ denote the positive part of operator. If $S^k(SS^*-S^*S)\neq 0$, then by [12] (Theorem 1) there exists a bounded function $f(\lambda)\colon \mathbb{C}\backslash\delta \to \mathcal{H}$ such that $(S-\lambda I)f(\lambda)\equiv x$ for some non-zero $x\in\mathcal{H}$ and so

$$(T-\lambda I)Xf(\lambda)\equiv Xx.$$

If T is k-quasi-M-hyponormal, then, by Theorem 2.4, we have $Xx=0$. If T is dominant, then we obtain $Xx=0$ by Proposition 2.6. Ultimately, $x=0$, a contradiction. Therefore,

$$S^k(SS^*-S^*S)=0.$$

Since S is a quasiaffine transform of T, $TX=XS$ for injective operator $X\in B(\mathcal{H})$. If T is injective, then S is injective, Since $S^k(SS^*-S^*S)=0$, S is normal. Then the required result follows by Theorem 2.3.

Spectral manifold (analytic), denoted by $X_T(\delta)$, of an operator $T\in B(\mathcal{H})$ is defined as follows:

$$X_T(\delta)=\big\{x\in H\colon (T-\lambda I)f(\lambda)\equiv x \text{ for some analytic function } f(\lambda)\colon \mathbb{C}\backslash\delta\to\mathcal{H}\big\}.$$

If a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be hyperinvariant of T if $\mathcal{M}$ is invariant under every operator which commutes with T.

From Theorem 2.4, $X_T(\delta)\neq\{0\}$ for k-quasi-M-hyponormal operators and it is known that k-quasi-M-hyponormal operators satisfies single valued extension property. The above results yields the following result by the method of [13] (Proposition 2).

***Corollary* 2.5.** *Let $T\in B(\mathcal{H})$ be k-quasi-M-hyponormal and $0\notin\delta\subseteq\mathbb{C}$ be closed. If there exists a bounded function $f\colon\mathbb{C}\backslash\delta\to\mathcal{H}$ such that $(T-\lambda I)f(\lambda)\equiv x$ for some non-zero $x\in H$, then T has non-zero hyperinvariant subspace $\mathcal{M}$ with $\sigma(T|_{\mathcal{M}})\subseteq\delta$. In particular, $\mathcal{M}$ is a nontrivial invariant subspace of T if δ is proper subset of $\sigma(T)$.*

3. Quasisimilarity. Equality of spectra of quasisimilar k-quasihyponormal operators has been proved in [3] by Gupta and Ramanujan. In Theorem 3.1, we show that spectrum of quasisimilar k-quasi-M-hyponormal operators are same. Recall, a subspace $\mathcal{M}$ of $\mathcal{H}$ is called *spectral maximal space* for T if $\mathcal{M}$ contains every invariant subspace $\mathcal{C}$ of T for which $\sigma(T|_{\mathcal{C}})\subset\sigma(T|_{\mathcal{M}})$. An operator $T\in B(\mathcal{H})$ is said to be decomposable if for any finite open covering $\{U_1,U_1,\ldots,U_1\}$ of spectrum of T, there exist spectral maximal subspaces $\mathcal{M}_1,\ \mathcal{M}_2,\ldots,\mathcal{M}_n$ of T such that

(a) $\mathcal{H}=\mathcal{M}_1+\mathcal{M}_2+\ldots+\mathcal{M}_n$

and

(b) $\sigma(T|_{\mathcal{M}_i}) \subset U_i$ for $i = 1, 2, \ldots, n$.

We say that an operator T is *subdecomposable operator* if it is the restriction of a decomposable operator to its invariant space (see [8]). It is well-known that T is decomposable if and only if T has Bishop property (β). The following result of Yang [20] is crucial to our purpose. It is known that two quasisimilar M-hyponormal operators have equal spectrum.

Proposition 3.1 ([20], Corollary 2.2)**.** *Let $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$ be two quasisimilar subdecomposible operators. Then $\sigma(T) = \sigma(S)$.*

Theorem 3.1. *If k-quasi-M-hyponormal operators $T, S \in B(\mathcal{H})$ are quasisimilar, then they have equal spectrum.*

Proof. Let $T, S \in B(\mathcal{H})$ be k-quasi-M-hyponormal operators. From [5], T and S satisfies Bishop property (β) and hence T and S are subdecomposible operators. Then, by Proposition 3.1, it follows that spectrum of T and S are equal.

Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are densely similar if there exist $X \in B(\mathcal{H}, \mathcal{K})$ and $Y \in B(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $YS = TY$, and are with dense ranges.

Theorem 3.2. *If k-quasi-M-hyponormal operators $T, S \in B(\mathcal{H})$ are densely similar, then they have equal essential spectrum.*

Proof. Since T and S are k-quasi-M-hyponormal operators, both T and S satisfies Bishop property (β). Then, by applying [8] (Theorem 3.7.13), it follows that essential spectrum of T and S are equal.

The following result is due to Yang [20].

Proposition 3.2 ([20], Theorem 2.10)**.** *Let $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$ be two quasisimilar M-hyponormal operators. Then $\sigma_e(T) = \sigma_e(S)$.*

Equality of essential spectrum of quasisimilar (p, k) quasihyponormal operators has been investigated by Kim and Kim [7]. Let $M_Q = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix}$ is an 2×2 upper-triangular operator matrix acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ and let $\sigma_e(T)$ denote the essential spectrum of T in $B(\mathcal{H})$.

Now we prove two quasisimilar k-quasi-M-hyponormal operators have equal essential spectra. The following result is due to Kim and Kim [7].

Proposition 3.3 [7]**.** *Let $\sigma_e(S) \cap \sigma_e(T)$ has no interior points. Then, for every $Q \in B(\mathcal{K}, \mathcal{H})$,*

$$\sigma_e(M_Q) = \sigma_e(S) \cup \sigma_e(T). \tag{3.1}$$

Theorem 3.3. *If k-quasi-M-hyponormal operators $T, S \in B(\mathcal{H})$ are quasisimilar, then they have equal essential spectrum.*

Proof. Let $T, S \in B(H)$ be quasisimilar k-quasi-M-hyponormal operators. Then there exist quasiaffinities X and Y such that $XT = SX$ and $YS = TY$. By Proposition 2.1, decompose T and S as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k})$$

and

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\operatorname{ran}(S^k)} \oplus \ker(S^{*k}),$$

where $T_1 = T|_{\overline{\mathrm{ran}(T^k)}}$, $S_1 = T|_{\overline{\mathrm{ran}(S^k)}}$ are M-hyponormal operators, $\sigma(T) = \sigma(T_1) \cup \{0\}$ and $\sigma(S) = \sigma(S_1) \cup \{0\}$. Since quasisimilar M-hyponormal operators, have same essential spectrum (see Proposition 3.2), in view of Propositions 2.1 and 3.3, it is enough to show that domain of T_3 is $\{0\}$ if and only if domain of S_3 is $\{0\}$. Since $XT = SX$, $XT^k = S^kX$. Let $0 \neq x \in H$ such that $T^{*k}x = 0$. Then, by the equality $XT^k = S^kX$, we have $S^{*k}Y^* = 0$. Since Y^* is one-to-one, we get that domain of S_3 is $\{0\}$ implies domain of T_3 is $\{0\}$. By a similar argument as above using the equality $YS = TY$ we obtain domain of T_3 is $\{0\}$ implies domain of S_3 is $\{0\}$.

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