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On recent advances in boundary value problems in the plane

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Abstract. The survey is devoted to recent advances in nonclassical solutions of the main boundary value problems such as the well–known Dirichlet, Hilbert, Neumann, Poincare and Riemann problems in the plane. Such solutions are essentially different from the variational solutions of the classical mathematical physics and based on the nonstandard point of view of the geometrical function theory with a clear visual sense. The traditional approach of the latter is the meaning of the boundary values of functions in the sense of the so-called angular limits or limits along certain classes of curves terminated at the boundary. This become necessary if we start to consider boundary data that are only measurable, and it is turned out to be useful under the study of problems in the field of mathematical physics, too. Thus, we essentially widen the notion of solutions and, furthermore, obtain spaces of solutions of the infinite dimension for all the given boundary value problems. The latter concerns to the Laplace equation as well as to its counterparts in the potential theory for inhomogeneous and anisotropic media.

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1. Introduction

The Dirichlet, Hilbert (Riemann–Hilbert), Neumann, Poincare and Riemann boundary value problems are basic in the theory of analytic functions and they are closely interconnected, see e.g. the monographs [24, 56] and [72] for the history, and also the recent papers [21, 34, 61–69] and [74].

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Recall that twice continuously differentiable solutions of the Laplace equation

$$\Delta u \ := \ \frac{\partial^2 u}{\partial x^2} \ + \ \frac{\partial^2 u}{\partial y^2} \ = \ 0 \qquad \forall\, z \in \mathbb{D} \tag{1.1}$$

are called **harmonic functions**. As well known, they are infinitely differentiable. The classic **Dirichlet problem** in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $z = x + iy$, is the problem on the existence of harmonic functions $u : \mathbb{D} \to \mathbb{R}$ such that

$$\lim_{z \to \zeta} u(z) \ = \ \varphi(\zeta) \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{1.2}$$

for a prescribed continuous function $\varphi : \partial\mathbb{D} \to \mathbb{R}$.

The request (1.2) is too strong and has no sense if the boundary function φ is only measurable. However, Luzin has shown in his dissertation that, for every measurable function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, there exists a harmonic function in $\mathbb{D}$ such that (1.2) holds for a.e. $\zeta \in \partial\mathbb{D}$ along any nontangential path, see, e.g., [51]. F. Gehring in [26] has rediscovered this fact in a similar way on the basis of another deep Luzin result, see Section 2. Furthermore, it was proved in [61] that the space of such solutions has the infinite dimension, see also [63].

Moreover, it was demonstrated in [34, 61–69] and [74] that all other boundary value problems mentioned above for harmonic and analytic functions as well as their generalizations in the extended sense are successively reduced to the first boundary value problem. In particular, it is well–known that the Neumann problem has no classical solutions generally speaking even for smooth boundary data, see e.g. [54]. The main goal of the short note [65] was to show that the problem has nonclassical solutions for arbitrary measurable data. The result was based on a reduction of this problem to the Hilbert boundary value problem recently solved for arbitrary measurable coefficients and for arbitrary measurable boundary data in [61].

Let us start from a more general **problem on directional derivatives**. The classic setting of the latter problem is to find a harmonic function $u : \mathbb{D} \to \mathbb{R}$ that admits a continuous extension to the boundary of $\mathbb{D}$ together with its first partial derivatives and satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} \ = \ \varphi(\zeta) \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{1.3}$$

with a prescribed continuous date $\varphi : \partial\mathbb{D} \to \mathbb{R}$ where $\frac{\partial u}{\partial \nu}$ denotes the derivative of u at ζ in a direction $\nu = \nu(\zeta)$, $|\nu(\zeta)| = 1$:

$$\frac{\partial u}{\partial \nu} \ := \ \lim_{t \to 0} \ \frac{u(\zeta + t \cdot \nu) - u(\zeta)}{t} \ . \tag{1.4}$$

The Neumann problem is a special case of the above problem on directional derivatives with the boundary condition

$$\frac{\partial u}{\partial n} \ = \ \varphi(\zeta) \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{1.5}$$

where n denotes the unit interior normal to $\partial\mathbb{D}$ at the point ζ.

In turn, the above problem on directional derivatives is a special case of **the Poincare problem** with the boundary condition

$$a \cdot u \ + \ b \cdot \frac{\partial u}{\partial \nu} \ = \ \varphi(\zeta) \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{1.6}$$

where $a = a(\zeta)$ and $b = b(\zeta)$ are real-valued functions given on $\partial\mathbb{D}$.

Recall that the classical setting of the **Riemann problem** in a smooth Jordan domain D of the complex plane $\mathbb{C}$ was on finding analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ that admit continuous extensions to ∂D and satisfy the boundary condition

$$f^+(\zeta) \ = \ A(\zeta) \cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1.7}$$

with prescribed Hölder continuous functions $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$.

Recall also that the **Riemann problem with shift** in D was on finding such functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ satisfying the condition

$$f^+(\alpha(\zeta)) \ = \ A(\zeta) \cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1.8}$$

where $\alpha : \partial D \to \partial D$ was a one-to-one sense preserving correspondence having the non-vanishing Hölder continuous derivative with respect to the natural parameter on ∂D. The function α is called a **shift function**. The special case $A \equiv 1$ gives the so-called **jump problem**.

The classical setting of the **Hilbert (Riemann–Hilbert) boundary value problem** was on finding analytic functions f in a domain $D \subset \mathbb{C}$ bounded by a rectifiable Jordan curve with the boundary condition

$$\lim_{z \to \zeta} \ \mathrm{Re}\, \{\overline{\lambda(\zeta)} \cdot f(z)\} \ = \ \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1.9}$$

with functions λ and φ that are continuously differentiable with respect to the natural parameter s on ∂D and, moreover, $|\lambda| \neq 0$ everywhere on ∂D. Hence without loss of generality one can assume that $|\lambda| \equiv 1$ on ∂D.

It is clear that if we start to consider the Hilbert and Riemann problems with measurable boundary data, the requests on the existence of the limits at all points $\zeta \in \partial D$ and along all paths terminating in ζ lose any sense (as well as the conception of the index). Thus, the notion of solutions of the Hilbert and Riemann problems should be widened in this case. The nontangential limits were a suitable tool from the function theory of one complex variable, see e.g. [61–69] and [74]. In [34] and [62], it was proposed an alternative approach based on the use of special families of curves terminating at the boundary, see [7], and admitting tangential curves.

Moreover, we extend the results on the boundary value problems to the case of quasiconformal functions (the Beltrami equations) as well as to A-harmonic functions that leads to problems of mathematical physics in inhomogeneous and anisotropic media, see [34,69] and [74]. The latter demands the application of the so-called logarithmic capacity zero that is invariant under quasiconformal mappings.

2. Angular limits in Dirichlet problem for Laplace equation

The following deep (non–trivial) result of Luzin was one the main theorems of his (1915) dissertation, see e.g. [51, p. 78].

Theorem A. *For any measurable function $\varphi : [0,1] \to \mathbb{R}$, there is a continuous function $\Phi : [0,1] \to \mathbb{R}$ such that $\Phi' = \varphi$ a.e.*

Just on the basis of Theorem A, Luzin proved the next significant result of his dissertation, see e.g. [51, p. 80], that is key for our goals.

Theorem B. *Let $\varphi(\vartheta)$ be real, measurable, almost everywhere finite and have the period 2π. Then there exists a harmonic function u in the unit disk $\mathbb{D}$ such that $u(z) \to \varphi(\vartheta)$ for a.e. ϑ as $z \to e^{i\vartheta}$ along any nontangential path.*

Here a path in $\mathbb{D}$ terminating at a point $\zeta = e^{i\vartheta} \in \partial\mathbb{D}$ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle in $\mathbb{D}$ with the vertex at ζ. Hence such limits are called also **angular limits**. The latter is a traditional tool of the geometric function theory, see e.g. monographs [19,44,51,59] and [60].

Note that the Luzin dissertation was published only in Russian in the book [51] prepared by his pupils Barri and Men'shov after his death but Theorem A was published with a complete proof in English in the book [70, p. 217], as Theorem VII(2.3). Hence Frederick Gehring in [26] has rediscovered Theorem B and his proof on the basis of Theorem A has in fact coincided with the original proof of Luzin. Since the proof is very short and nice and has a common interest, we give it for completeness here.

Proof. By Theorem A we can find a continuous function $\Phi(\vartheta)$ such that $\Phi'(\vartheta)=\varphi(\vartheta)$ for a.e. ϑ. Considering the Poisson integral

$$U(re^{i\vartheta}) \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\;\Phi(t)\;dt$$

for $0<r<1$, $U(0):=0$, we see by the Fatou result, see e.g. 3.441 in [76], p. 53, that $\frac{\partial}{\partial\vartheta}\,U(z)\to\Phi'(\vartheta)$ as $z\to e^{i\vartheta}$ along any nontangential path whenever $\Phi'(\vartheta)$ exists. Thus, the conclusion follows for the function $u(z)\;=\;\frac{\partial}{\partial\vartheta}\,U(z)$. □

Remark 2.1. Note that the given function u is harmonic in the punctured unit disk $\mathbb{D}\setminus\{0\}$ because the function U is harmonic in $\mathbb{D}$ and the differential operator $\frac{\partial}{\partial\vartheta}$ is commutative with the Laplace operator Δ. Setting $u(0)=0$, we see that

$$u(re^{i\vartheta}) \;=\; -\frac{r}{\pi}\int\limits_0^{2\pi}\frac{(1-r^2)\sin(\vartheta-t)}{(1-2r\cos(\vartheta-t)+r^2)^2}\;\Phi(t)\;dt\;\to\;0 \qquad \text{as } r\to 0\;,$$

i.e. $u(z)\to u(0)$ as $z\to 0$, and, moreover, the integral of u over each circle $|z|=r$, $0<r<1$, is equal to zero. Thus, by the criterion for a harmonic function on the averages over circles we have that u is harmonic in $\mathbb{D}$. The alternative argument for the latter is the removability of isolated singularities for harmonic functions, see e.g. [57].

Corollary 5.1 to Theorem 5.1 in [61] has strengthened Theorem B as the next.

Theorem C. *For each (Lebesgue) measurable function $\varphi:\partial\mathbb{D}\to\mathbb{R}$, the space of all harmonic functions $u:\mathbb{D}\to\mathbb{R}$ with the angular limits $\varphi(\zeta)$ for a.e. $\zeta\in\partial\mathbb{D}$ has the infinite dimension.*

Theorem C is the direct consequence of Theorem B and Theorem 5.1 in [61] :

Theorem 2.2. *The space of all harmonic functions in $\mathbb{D}$ with angular limit* 0 *at a.e. point of $\partial\mathbb{D}$ has the infinite dimension.*

We give its complete proof here in view of its importance because we will successively reduce all other boundary value problems to Theorem C.

Proof. Indeed, let $\Phi : [0,2\pi] \to \mathbb{R}$ be integrable and differentiable a.e. with $\Phi'(t) = 0$. Then the function

$$U(z) \; : = \; \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta - t) + r^2} \; \Phi(t) \; dt \; , \quad z = re^{i\vartheta}, \; r < 1 \; ,$$

is harmonic on $\mathbb{D}$ with $U(z) \to \Phi(\Theta)$ as $z \to e^{i\Theta}$, see e.g. Theorem 1.3 in [25] or Theorem IX.1.1 in [28], and $\frac{\partial}{\partial \vartheta} \; U(z) \to \Phi'(\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists, see e.g. 3.441 in [76], p. 53, or Theorem IX.1.2 in [28]. Thus, the harmonic function $u(z) = \frac{\partial}{\partial \vartheta} \; U(z)$ has nontangential limit 0 at a.e. point of $\partial\mathbb{D}$.

Let us give a subspace of such functions u with an infinite basis. Namely, let $\varphi : [0,1] \to [0,1]$ be the Cantor function, see e.g. 8.15 in [27], and let $\varphi_n : [0,2\pi] \to [0,1]$ be equal to $\varphi((t-a_{n-1})/(a_n - a_{n-1}))$ on $[a_{n-1}, a_n)$ where $a_0 = 0$ and $a_n = 2\pi(2^{-1} + \ldots + 2^{-n})$, $n = 1, 2, \ldots$ and 0 outside of $[a_{n-1}, a_n)$. Denote by U_n and u_n the harmonic functions corresponding to φ_n as in the first item.

By the construction the supports of the functions φ_n are mutually disjoint and, thus, the series $\sum\limits_{n=1}^{\infty} \gamma_n \varphi_n$ is well defined for every sequence $\gamma_n \in \mathbb{R}$, $n = 1, 2, \ldots$. If in addition we restrict ourselves to the sequences $\gamma = \{\gamma_n\}$ in the space l with the norm $\|\gamma\| = \sum\limits_{n=1}^{\infty} |\gamma_n|$, then the series is a suitable function Φ for the first item.

Denote by U and u the harmonic functions corresponding to the function Φ as in the first item and by $\mathbf{H}_0$ the class of all such u. Note that u_n, $n = 1, 2, \ldots$, form a basis in the space $\mathbf{H}_0$ with the locally uniform convergence in $\mathbb{D}$ which is metrizable.

Firstly, $\sum\limits_{n=1}^{\infty} \gamma_n \varphi_n \neq 0$ if $\gamma \neq 0$. Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \ldots$. Then $u \neq 0$ because the limits $\lim\limits_{z \to \zeta} U(z)$ exist for all $\zeta = e^{i\vartheta}$ with $\vartheta \in (a_{n-1}, a_n)$ and can be arbitrarily close to 0 as well as to γ_n.

Secondly, $u_m^* = \sum\limits_{n=1}^{m} \gamma_n \varphi_n \to u$ locally uniformly in $\mathbb{D}$ as $m \to \infty$. Indeed, elementary calculations give the following estimate of the remainder term

$$|u(z) - u_m^*(z)| \; \leq \; \frac{2r(1+r)}{(1-r)^3} \cdot \sum_{n=m+1}^{\infty} |\gamma_n| \; \to \; 0 \qquad \text{as} \qquad m \to \infty$$

in every disk $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \leq r\}$, $r < 1$. □

Remark 2.3. In Section 5, one can find our more refined results which are counterparts of Theorem A, B, C as well as 2.2 in terms of logarithmic capacity that makes possible to extend the theory of boundary value problems to the so-called $A-$harmonic functions corresponding to generalizations of the Laplace equation in inhomogeneous and anisotropic media.

By the well–known Lindelöf maximum principle, see e.g. Lemma 1.1 in [25], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions u on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under zero boundary data. In comparison with the examples in the proof of Theorem 2.2, here we give more elementary examples and constructions of solutions from [63].

Many such nontrivial solutions u for the Laplace equation can be given by the **Poisson–Stiltjes integral**

$$u(z) \ = \ \frac{1}{2\pi} \int\limits_{0}^{2\pi} P_r(\vartheta - t) \ d\Phi(t) \ , \quad z = re^{i\vartheta}, \ r < 1 \ , \tag{2.1}$$

with an arbitrary **singular function** $\Phi : [0, 2\pi] \to \mathbb{R}$, i.e., where Φ is of bounded variation and $\Phi' = 0$ a.e., and where we use the standard notation for the **Poisson kernel**

$$P_r(\Theta) \ = \ \frac{1 - r^2}{1 - 2r\cos\Theta + r^2} \ , \ r < 1 \ . \tag{2.2}$$

Indeed, u in (2.1) is harmonic for every function $\Phi : [0, 2\pi] \to \mathbb{R}$ of bounded variation and by the Fatou theorem, see e.g. Theorem I.D.3.1 in [44], $u(z) \to \Phi'(\Theta)$ as $z \to e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists. Thus, $u(z) \to 0$ as $z \to e^{i\Theta}$ for a.e. $\Theta \in [0, 2\pi]$ along any nontangential paths for every singular function Φ.

Example 2.4. The first natural example is given by the formula (2.1) with $\Phi(t) = \varphi(t/2\pi)$ where $\varphi : [0, 1] \to [0, 1]$ is the well–known **Cantor function**, see e.g. [18] and further references therein.

Example 2.5. However, the simplest example of such a kind is given by nondecreasing step-like data Φ_{ϑ_0} with values 0 and 2π and with the jump at $\vartheta_0 \in (0, 2\pi)$:

$$u(z) \ = \ P_r(\vartheta - \vartheta_0) \ = \frac{1 - r^2}{1 - 2r\cos(\vartheta - \vartheta_0) + r^2} \ , \quad z = re^{i\vartheta}, \ r < 1 \ .$$

We see that $u(z) \to 0$ as $z \to e^{i\Theta}$ for all $\Theta \in (0, 2\pi)$ except $\Theta = \vartheta_0$.

Note that the function u is harmonic in the unit disk $\mathbb{D}$ because

$$u(z) \;=\; \operatorname{Re}\frac{\zeta_0 + z}{\zeta_0 - z} \;=\; \frac{1 - |z|^2}{1 - 2\operatorname{Re} z\overline{\zeta_0} + |z|^2}\,, \qquad \zeta_0 = e^{i\vartheta_0}\,,\ z \in \mathbb{D}\,,$$

where the function $w = g(z) = g_{\zeta_0}(z) \colon = (\zeta_0 + z)/(\zeta_0 - z)$ is analytic (conformal) in $\mathbb{D}$ and maps $\mathbb{D}$ onto half-plane $\operatorname{Re} w > 0$, $g(0) = 1$, $g(\zeta_0) = \infty$.

Remark 2.6. On the basis of the latter examples, it was given an alternative proof of Theorem 2.2 and its strengthening for the class of all harmonic functions in $\mathbb{D}$ with the angular limit 0 at every point of $\partial\mathbb{D}$ except a countable collection of points, see Theorem 2.1 in [63]. In particular, that makes possible to formulate the corresponding more refined results in terms of the so-called logarithmic capacity.

Note also that the harmonic functions u given in the proof of Theorem 2.2 and in (2.1) cannot be represented in the form of the Poisson integral with any integrable function $\varphi : [0, 2\pi] \to \mathbb{R}$ because such integral would have nontangential limits φ a.e. in $\partial\mathbb{D}$, see e.g. Corollary IX.1.1 in [28]. Consequently, u do not belong to the classes h_p for any $p > 1$, see e.g. Theorem IX.2.3 in [28]. However, the functions u in (2.1) belong to the class h_1, see e.g. Theorem IX.2.2 in [28].

Recall that h^p, $p \in (0, \infty)$, denotes the class of harmonic functions u in $\mathbb{D}$ with

$$\sup_{r \in (0,1)} \left\{ \int\limits_0^{2\pi} |u(re^{i\vartheta})|^p \, d\vartheta \right\}^{\frac{1}{p}} \;<\; \infty$$

and functions in h^1 have angular limits a.e. in $\partial\mathbb{D}$, see e.g. Corollary IX.2.2 in [28].

3. Angular limits in Hilbert problem for analytic functions

Boundary value problems for analytic functions are due to the well-known Riemann dissertation (1851) contained a general setting of a problem on finding analytic functions with a connection between its real and imaginary parts on the boundary. However, it has contained no concrete boundary value problems.

The first concrete problem of such a type has been proposed by Hilbert (1904) and called by the Hilbert problem or the Riemann–Hilbert problem. Recall that this problem consists in finding an analytic function

f in a domain bounded by a rectifiable Jordan curve C with the linear boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \varphi(\zeta) \qquad\qquad \forall\, \zeta\in C \tag{3.1}$$

where it was assumed that the functions λ and φ are continuously differentiable with respect to the natural parameter s on C and, moreover, $|\lambda|\neq 0$ everywhere on C. Hence without loss of generality we may assume that $|\lambda|\equiv 1$ on C.

The first way for solving this problem based on the theory of singular integral equations was given by Hilbert (1904), see [35]. This attempt was not quite successful because of the theory of singular integral equations has been not yet enough developed at that time. However, just that way became the main approach in this research direction with important contributions of Georgian and Russian mathematicians and mechanicians, see e.g. [24, 56] and [72]. In particular, the existence of solutions to this problem was in that way proved for Hölder continuous λ and φ. But subsequent weakening conditions on λ and φ led to strengthening conditions on the contour C, say to the Lyapunov curves or the Radon condition of bounded rotation or even to smooth curves.

However, Hilbert (1905) has solved his problem with the above settings to (3.1) in the second way based on the reduction it to solving the corresponding two Dirichlet problems, see e.g. [36]. The main goal of the paper [61] was to show that this approach is more simple and leads to perfectly general results in the problem for the arbitrary rectifiable Jordan domains with coefficients λ and boundary data φ that are only measurable with respect to the natural parameter. The key was the Gehring–Luzin result, see Theorem B in Section 2, on the Dirichlet problem for harmonic functions. But the way of the reduction of the Hilbert problem to the corresponding 2 Dirichlet problems was original in [61].

First we have the result in the unit disk $\mathbb{D}$, see Theorems 2.1 and 5.2 in [61].

Theorem 3.1. *Let $\lambda:\partial\mathbb{D}\to\mathbb{C}$, $|\lambda(\zeta)|\equiv 1$, and $\varphi:\partial\mathbb{D}\to\mathbb{R}$ be measurable functions. Then there exist analytic functions $f:\mathbb{D}\to\mathbb{C}$ such that along any nontangential path*

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta\in\partial\mathbb{D} \tag{3.2}$$

and the space of such analytic functions has the infinite dimension.

The proof is short and, in view of its importance, we give it here.

Proof. First, consider the function $\alpha(\zeta) = \arg\lambda(\zeta)$ where $\arg\omega$ is the principal value of the argument of $\omega \in \mathbb{C}$ with $|\omega| = 1$, i.e., the unique number $\alpha \in (-\pi, \pi]$ such that $\omega = e^{i\alpha}$. Note that the function $\arg\omega$ is continuous on $\partial\mathbb{D} \setminus \{-1\}$ and the sets $\lambda^{-1}(\partial\mathbb{D} \setminus \{-1\})$ and $\lambda^{-1}(-1)$ are measurable because the function $\lambda(\zeta)$ is measurable. Thus, the function $\alpha(\zeta)$ is measurable on $\partial\mathbb{D}$. Furthermore, $\alpha \in L^\infty(\partial\mathbb{D})$ because $|\alpha(\zeta)| \le \pi$ for all $\zeta \in \partial\mathbb{D}$. Hence

$$g(z) \; : = \; \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \alpha(\zeta) \, \frac{z+\zeta}{z-\zeta} \, \frac{d\zeta}{\zeta} \, , \qquad z \in \mathbb{D} \, , \tag{3.3}$$

is an analytic function in $\mathbb{D}$ with $u(z) = \mathrm{Re}\; g(z) \to \alpha(\zeta)$ as $z \to \zeta$ along any nontangential path in $\mathbb{D}$ for a.e. $\zeta \in \partial\mathbb{D}$, see e.g. Corollary IX.1.1 in [28] and Theorem I.E.1 in [44]. Denote $\mathcal{A}(z) = \exp\{ig(z)\}$ that is an analytic function.

Since $\alpha \in L^\infty(\partial\mathbb{D})$, we have that $u \in h^p$ for all $p \ge 1$, see e.g. Theorem IX.2.3 in [28], and then also $v = \mathrm{Im}\; g \in h^p$ for all $p \ge 1$ by the theorem of M. Riesz (1927), see e.g. Theorem IX.2.4 in [28]. Hence there exists a function $\beta : \partial\mathbb{D} \to \mathbb{R}$, $\beta \in L^p$, for all $p \ge 1$ such that $v(z) \to \beta(\zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$ along any nontangential path, see e.g. Theorem IX.2.3 and Corollary IX.2.2 in [28]. Thus, by Theorem B there exists an analytic function $\mathcal{B} : \mathbb{D} \to \mathbb{C}$ such that $\mathrm{Re}\;\mathcal{B}(z) \to B(\zeta) \; : = \varphi(\zeta) \cdot \exp\{\beta(\zeta)\}$ as $z \to \zeta$ along any nontangential path for a.e. $\zeta \in \partial\mathbb{D}$. Finally, elementary calculations show that one of the desired analytic functions in (3.7) is $f = \mathcal{A} \cdot \mathcal{B}$.

Let $U : \mathbb{D} \to \mathbb{R}$ be a harmonic function with angular limit 0 at a.e. point of $\partial\mathbb{D}$ from Theorem 2.2. Then there is the unique harmonic function $V : \mathbb{D} \to \mathbb{R}$ with $V(0) = 0$ such that $\mathcal{C} = U + iV$ is an analytic function. Thus, setting in the last item $f = \mathcal{A}(\mathcal{B}+\mathcal{C})$ instead of $f = \mathcal{A}\cdot\mathcal{B}$, we obtain by Theorem 2.2 the space of analytic functions of the infinite dimension satisfying (3.7). □

Remark 3.2. As it follows from formula (3.3), the first analytic function $\mathcal{A}$ in the proof is calculated in the explicit form. The function $\beta : \partial\mathbb{D} \to \mathbb{R}$ in the proof can also explicitly be calculated by the following formula, see e.g. Theorem I.E.4.1 in [44], for a.e. $\zeta \in \partial\mathbb{D}$

$$\beta(\zeta) \; : = \; \lim_{\varepsilon \to +0} \; \frac{1}{\pi} \int\limits_{\varepsilon}^{\pi} \frac{\alpha(\zeta e^{-it}) - \alpha(\zeta e^{it})}{2 \;\mathrm{tg}\; \frac{t}{2}} \; dt \; . \tag{3.4}$$

The second analytic function $\mathcal{B}$ in the proof is equal to $\frac{\partial}{\partial\vartheta}\; G(z)$,

$z = re^{i\vartheta}$, with

$$G(z) \; : = \; \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}} \Phi(\zeta) \, \frac{z+\zeta}{z-\zeta} \, \frac{d\zeta}{\zeta} \, , \qquad z \in \mathbb{D} \, , \tag{3.5}$$

where $\Phi : \partial\mathbb{D} \to \mathbb{R}$ is a continuous function such that $\frac{\partial}{\partial\vartheta}\, \Phi(\zeta) = B(\zeta)$, $\zeta = e^{i\vartheta}$, for a.e. $\vartheta \in [0, 2\pi]$, see the nontrivial construction of Theorem VII(2.3) in [70].

The case of arbitrary rectifiable Jordan curves is reduced to the case of the unit circle as in Theorem 3.1 and Remark 5.2 in [61].

Theorem 3.3. *Let D be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter on ∂D. Then there exist analytic functions $f : D \to \mathbb{C}$ such that along any nontangential path*

$$\lim_{z\to\zeta} \; \mathrm{Re}\, \{\overline{\lambda(\zeta)} \cdot f(z)\} \; = \; \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{3.6}$$

with respect to the natural parameter on ∂D and the space of such analytic functions has the infinite dimension.

Proof. This case is reduced to the case of the unit disk $\mathbb{D}$ in the following way. First, by the Riemann theorem, see e.g. Theorem II.2.1 in [28], there exists a conformal mapping ω of any Jordan domain D onto $\mathbb{D}$. By the Caratheodory (1912) theorem ω can be extended to a homeomorphisms of $\overline{D}$ onto $\overline{\mathbb{D}}$ and, if ∂D is rectifiable, then by the theorem of F. and M. Riesz (1916) length $\omega^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}$ with $|E| = 0$, see e.g. Theorem II.C.1 and Theorems II.D.2 in [44]. Conversely, by the Lavrentiev (1936) theorem $|\omega(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ and length $\mathcal{E} = 0$, see [49], see also the point III.1.5 in [60].

Hence ω and ω^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [70], and continuous mappings transform compact sets into compact sets. Thus, a function $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\Phi = \varphi \circ \omega^{-1} : \partial\mathbb{D} \to \mathbb{R}$ is measurable with respect to the linear measure on $\partial\mathbb{D}$.

By the Lindelöf (1917) theorem, see e.g. Theorem II.C.2 in [44], if ∂D has a tangent at a point ζ, then $\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to \mathrm{const}$ as $z \to \zeta$. In other words, the conformal images of sectors in D with a vertex at ζ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $w = \omega(\zeta)$. Thus, nontangential paths in D are transformed under ω

into nontangential paths in $\mathbb{D}$. Finally, a rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and, thus, Theorem 3.3 follows from Theorem 3.1. $\Box$

Remark 3.4. The conceptions of a harmonic measure introduced by R. Nevanlinna in [57] and a principal asymptotic value based on one nice result of F. Bagemihl [6] make possible with a great simplicity and generality to formulate the existence theorems for the Dirichlet and Hilbert problems in arbitrary Jordan domains, see Theorem 4.1 and Remark 5.2 in [61].

In view of the theorems of Riemann and Caratheodory, this approach makes possible also to formulate the corresponding theorems for arbitrary simply connected domains D in $\mathbb{C}$ having at least 2 boundary points. The only difference is that the functions λ and φ should be given as functions of prime ends of D but not of points of ∂D and harmonic measures of sets of prime ends are given through the natural one-to-one correspondence between the prime ends of D and the boundary points of $\mathbb{D}$ under Riemann mappings $\omega : D \to \mathbb{D}$, see e.g. [17].

Moreover, in [64] it was proved the existence of multivalent solutions with the infinite number of branches for the Hilbert problem in the general settings of finitely connected domains bounded by mutually disjoint Jordan curves, measurable coefficients and measurable boundary data. The general theorem is formulated in terms of harmonic measure and principal asymptotic values. It is also given the corresponding reinforced criterion for domains with rectifiable boundaries stated in terms of the natural parameter and nontangential limits. Furthermore, it is shown that the dimension of the spaces of these solutions is infinite.

Let us start from the simplest kind of multiply connected domains. Recall that a domain D in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called **circular** if its boundary consists of finite number of mutually disjoint circles and points. We call such a domain **nondegenerate** if its boundary consists only of circles. The following statement was first proved as Theorem 2.1 in [64].

Theorem 3.5. *Let $\mathbb{D}_*$ be a bounded nondegenerate circular multiply connected domain and let $\lambda : \partial\mathbb{D}_* \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D}_* \to \mathbb{R}$ be measurable functions. Then there exist multivalent analytic functions $f : \mathbb{D}_* \to \mathbb{C}$ with the infinite number of branches such that*

$$\lim_{z\to\zeta} \mathrm{Re}\, \{\overline{\lambda(\zeta)} \cdot f(z)\} \;=\; \varphi(\zeta) \tag{3.7}$$

along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}_*$.

Proof. Indeed, by the Poincare theorem, see e.g. Theorem VI.1 in [28], there is a locally conformal mapping g of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\mathbb{D}_*$. Let $h : \mathbb{D}_* \to \mathbb{D}$ be the corresponding multivalent analytic function that is inverse to g. $\mathbb{D}_*$ without a finite number of cuts is simply connected and hence h has there only single-valued branches that are extended to the boundary by the Caratheodory theorem.

By Section VI.2 in [28], $\partial\mathbb{D}$ without a countable set of its points consists of a countable collection of arcs every of which is a one-to-one image of a circle in $\partial\mathbb{D}_*$ without its one point under every extended branch of h. Note that by the reflection principle g is conformally extended into a neighborhood of every such arc and, thus, nontangential paths to its points go into nontangential paths to the corresponding points of circles in $\partial\mathbb{D}_*$ and inversely.

Setting $\Lambda = \lambda \circ g$ and $\Phi = \varphi \circ g$ with the extended g on the given arcs of $\partial\mathbb{D}$ we obtain measurable functions on $\partial\mathbb{D}$. Thus, by Theorem 2.1 in [61], see Theorem 3.1 above, there exist analytic functions $F : \mathbb{D} \to \mathbb{C}$ such that

$$\lim_{w \to \eta} \ \mathrm{Re}\ \{\overline{\Lambda(\eta)} \cdot F(w)\} \ = \ \Phi(\eta) \tag{3.8}$$

along any nontangential path to a.e. $\eta \in \partial\mathbb{D}$. By the above arguments, we see that $f = F \circ h$ are desired multivalent analytic solutions of (3.7). □

To solve the Riemann–Hilbert problem in the case of domains bounded by a finite number of rectifiable Jordan curves we should extend to this case the known results of Caratheodory (1912), Lindelöf (1917), F. and M. Riesz (1916) and Lavrentiev (1936) for Jordan's domains. Namely, it was proved as Lemma 3.1 in [64] the following statements.

Lemma 3.6. *Let D be a bounded domain in $\mathbb{C}$ whose boundary components are Jordan curves, $\mathbb{D}_*$ be a bounded nondegenerate circular domain in $\mathbb{C}$ and let $\omega : D \to \mathbb{D}_*$ be a conformal mapping. Then*

(i) ω can be extended to a homeomorphism of $\overline{D}$ onto $\overline{\mathbb{D}_*}$;

(ii) arg $[\omega(\zeta) - \omega(z)]$ − arg $[\zeta - z] \to$ *const as $z \to \zeta$ whenever ∂D has a tangent at $\zeta \in \partial D$;*

(iii) for rectifiable ∂D, length $\omega^{-1}(E) = 0$ *whenever $|E| = 0$, $E \subset \partial\mathbb{D}_*$;*

(iv) for rectifiable ∂D, $|\omega(\mathcal{E})| = 0$ whenever length $\mathcal{E} = 0$, $\mathcal{E} \subset \partial D$.

Proof. (i) Indeed, we are able to transform $\mathbb{D}_*$ into a simply connected domain $\mathbb{D}^*$ through a finite sequence of cuts. Thus, we come to the desired conclusion applying the Caratheodory theorems to simply connected domains $\mathbb{D}^*$ and $D^* := \omega^{-1}(\mathbb{D}^*)$, see e.g. Theorem 9.4 in [17] and Theorem II.C.1 in [44].

(ii) In the construction from the previous item, we may assume that the point ζ is not the end of the cuts in D generated by the cuts in $\mathbb{D}_*$ under the extended mapping ω^{-1}. Thus, we come to the desired conclusion twice applying the Caratheodory theorems, the reflection principle for conformal mappings and the Lindelöf theorem for the Jordan domains, see e.g. Theorem II.C.2 in [44].

Points (iii) and (iv) are proved similarly to the last item on the basis of the corresponding results of F. and M. Riesz and Lavrentiev for Jordan domains with rectifiable boundaries, see e.g. Theorem II.D.2 in [44], and [49], see also the point III.1.5 in [60]. $\Box$

Lemma 3.6 makes possible to reduce the case of domains with rectifiable boundaries to the case of circular domains, see Theorem 3.1 in [64].

Theorem 3.7. *Let D be a bounded multiply connected domain in $\mathbb{C}$ whose boundary components are rectifiable Jordan curves and $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter on ∂D. Then there exist multivalent analytic functions $f : \mathbb{D} \to \mathbb{C}$ with the infinite number of branches such that along any nontangential path*

$$\lim_{z\to\zeta} \operatorname{Re}\,\{\overline{\lambda(\zeta)} \cdot f(z)\} \;=\; \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial D \tag{3.9}$$

with respect to the natural parameters of the boundary components of D.

Proof. This case is reduced to the case of a bounded nondegenerate circular domain $\mathbb{D}_*$ in the following way. First, there is a conformal mapping ω of D onto a circular domain $\mathbb{D}_*$, see e.g. Theorem V.6.2 in [28]. Note that $\mathbb{D}_*$ is not degenerate because isolated singularities of conformal mappings are removable that is due to the well-known Weierstrass theorem, see e.g. Theorem 1.2 in [17]. Without loss of generality, we may assume that $\mathbb{D}_*$ is bounded.

By point (i) in Lemma 3.6 ω can be extended to a homeomorphisms of $\overline{D}$ onto $\overline{\mathbb{D}_*}$. If ∂D is rectifiable, then by point (iii) in Lemma 3.6 length $\omega^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}_*$ with $|E| = 0$, and by (iv) in Lemma 3.6, conversely, $|\omega(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ with length $\mathcal{E} = 0$.

In the last case ω and ω^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [70], and continuous mappings transform compact sets into compact sets. Thus, a function $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\Phi = \varphi \circ \omega^{-1} : \partial\mathbb{D}_* \to \mathbb{R}$ is measurable with respect to the natural parameter on $\partial\mathbb{D}_*$.

By point (ii) in Lemma 3.6, if ∂D has a tangent at a point $\zeta \in \partial D$, then $\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to const$ as $z \to \zeta$. In other words, the conformal images of sectors in D with a vertex at ζ is asymptotically the same as sectors in $\mathbb{D}_*$ with a vertex at $w = \omega(\zeta)$. Thus, nontangential paths in D are transformed under ω into nontangential paths in $\mathbb{D}_*$ and inversely. Finally, a rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and, thus, Theorem 3.3 follows from Theorem 3.1. □

Theorem 4.1 in [64] is formulated for finitely connected domains bounded by arbitrary Jordan curves in terms of harmonic measure and the so-called principal asymptotic values. Finally, Theorems 5.1 in [64] says on the dimension of these spaces of solutions:

Theorem 3.8. *The spaces of solutions of the Hilbert boundary value problem in Theorems 3.5 and 3.7 have the infinite dimension.*

Proof. By Theorem 5.1 in [61], see Theorem 3.1 above, the space of solutions of the problem (3.8) has the infinite dimension. Thus, the conclusion follows by the construction of these solutions in the given theorems through the successive reduction to (3.8). □

Remark 3.9. Of course, results concerning to the infinite dimension of the space of solutions are not in some sense new and treated in terms of the infinite index of the Hilbert boundary value problem, see e.g. [29] and [55]. By the general theory of boundary value problems, each additional singularities, including singularities on the boundary contour, increases the index. Hence the above results can be also interpreted as the case of the infinite index.

Note that the considered situations admit the boundary functions with uncountable singularities. Such examples are given by Poisson-Stiltjes integral with the Cantor type functions under its differential, see e.g. [63] or Section 2 above, and the corresponding examples of analytic functions in the simplest case of the Hilbert problem under $\lambda = 1$ and $\varphi = 0$ a.e.

Indeed, a Cantor type set C is perfect, i.e. it is closed and without isolated points. Hence C is of the continuum cardinality by the well–known W.H. Young theorem, see [75]. The corresponding Cantor type function has the symmetric Lebesgue derivative $+\infty$ at every point in C except ends, see e.g. the survey [18]. Then by the Fatou theorem, see e.g. Theorem I.D.3.2 in [44], the corresponding harmonic function has the radial limit $+\infty$ on the set of the continuum (maximal possible) cardinality.

In this connection, it would be also interesting to study the problem on a maximal possible cardinality of the dimension of the spaces of solutions for the Hilbert problem.

4. Angular limits in Neumann problem for Laplace equation

It is well–known that the Neumann problem has no classical solutions generally speaking even for smooth boundary data, see e.g. [54]. On the basis of solving of Hilbert boundary value problem in [61], see the last section, it was proved in [65] the existence of nonclassical solutions of the Neumann problem for the harmonic functions in the Jordan rectifiable domains with arbitrary measurable boundary distributions of normal derivatives. The same is stated for a special case of the Poincare problem on directional derivatives. Moreover, it is shown that the spaces of the found solutions have the infinite dimension.

Let us start from the more general problem on directional derivatives, see Theorem 1 and Remark 1 in [65].

Theorem 4.1. *Let $\nu : \partial\mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable functions. Then there exist harmonic functions $u : \mathbb{D} \to \mathbb{R}$ such that*

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu}\,(z) \;=\; \varphi(\zeta) \tag{4.1}$$

along any nontangential paths to a.e. point $\zeta \in \partial\mathbb{D}$.

Remark 4.2. We are able to say more in the case of $\mathrm{Re}\; n(\zeta)\overline{\nu(\zeta)} > 0$. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν onto the inner normal n to $\partial\mathbb{D}$ at the point ζ. In view of (4.1), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in $\mathbb{D}$ along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$u(z) \;=\; u(z_0) \;-\; \int\limits_0^1 \frac{\partial u}{\partial \nu}\,(z_0 + \tau(z - z_0))\; d\tau\;. \tag{4.2}$$

Thus, at each point with condition (4.1), there is the directional derivative

$$\frac{\partial u}{\partial \nu}\,(\zeta) \;:=\; \lim_{t\to 0}\; \frac{u(\zeta + t\cdot\nu) - u(\zeta)}{t} \;=\; \varphi(\zeta)\;. \tag{4.3}$$

In particular, in the case of the Neumann problem, Re $n(\zeta)\overline{\nu(\zeta)} \equiv 1$ and we have by Theorem 4.1 and Remark 4.2 the following result, see Theorem 2 in [65].

Theorem 4.3. *For each measurable function $\varphi : \partial\mathbb{D} \to \mathbb{R}$, one can find harmonic functions $u : \mathbb{D} \to \mathbb{R}$ such that, at a.e. point $\zeta \in \partial\mathbb{D}$, there exist:*

1) the finite radial limit

$$u(\zeta) \ := \ \lim_{r\to 1} \ u(r\zeta) \tag{4.4}$$

2) the normal derivative

$$\frac{\partial u}{\partial n} \,(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\cdot n) - u(\zeta)}{t} \ = \ \varphi(\zeta) \tag{4.5}$$

3) the nontangential limit

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial n} \,(z) \ = \ \frac{\partial u}{\partial n} \,(\zeta) \tag{4.6}$$

where $n = n(\zeta)$ denotes the unit interior normal to $\partial\mathbb{D}$ at the point ζ.

Proof. To prove Theorem 4.1, let us show that the problem on directional derivatives is equivalent to the corresponding Riemann–Hilbert problem.

Indeed, let u be a harmonic function $u : \mathbb{D} \to \mathbb{R}$ satisfying the boundary condition (4.1). Then the functions $U = u_x$ and $V = -u_y$ satisfy the system of Cauchy-Riemann: $U_y = -V_x$ and $U_x = V_y$ in view of (1.1). Thus, the function $f = U + iV$ is analytic in $\mathbb{D}$ and along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}$

$$\lim_{z\to\zeta} \ \mathrm{Re}\ \nu(\zeta)\cdot f(z) \ = \ \varphi(\zeta) \tag{4.7}$$

that is equivalent to (4.1). Inversely, let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function satisfying the boundary condition (4.7). Then any indefinite integral F of f is also a single-valued analytic function in $\mathbb{D}$ and $u = \mathrm{Re}\, F$ is a harmonic function satisfying the boundary condition (4.1) because the directional derivative

$$\frac{\partial u}{\partial \nu} \ = \ \mathrm{Re}\ \overline{\nu}\cdot\nabla u \ = \ \mathrm{Re}\ \nu\cdot\overline{\nabla u} \ = \ (\nu, \nabla u) \tag{4.8}$$

is the scalar product of ν and the gradient ∇u interpreted as vectors in $\mathbb{R}^2$.

Thus, Theorem 4.1 is a direct consequence of Theorem 2.1 in [61], see Theorem 3.1 above on the Hilbert boundary value problem with $\lambda(\zeta) = \overline{\nu(\zeta)}$, $\zeta \in \partial\mathbb{D}$. □

The proof of the following result in domains bounded by rectifiable Jordan curves, Theorem 3 in [65], is perfectly similar to the proof of Theorem 4.1 above but it is based on more general Theorem 3.1 in [61], see Theorem 3.3 in Section 3.

Theorem 4.4. *Let D be a domain in $\mathbb{C}$ bounded by a rectifiable Jordan curve, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter. Then there exist harmonic functions $u : D \to \mathbb{R}$ such that along any nontangential paths*

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu}(z) \ = \ \varphi(\zeta) \tag{4.9}$$

for a.e. point $\zeta \in \partial D$ with respect to the natural parameter.

Remark 4.5. Again we are able to say more in the case with $\mathrm{Re}\, n\cdot\overline{\nu} > 0$ where $n = n(\zeta)$ is the unit inner normal at a point $\zeta \in \partial D$ with a tangent to ∂D. In view of (4.9), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in $\mathbb{D}$ along the straight line passing through the point ζ and being parallel to the vector ν because along this line, for z and z_0 that are close enough to ζ,

$$u(z) \ = \ u(z_0) \ - \int\limits_0^1 \frac{\partial u}{\partial \nu}\,(z_0 + \tau(z - z_0))\ d\tau\ . \tag{4.10}$$

Thus, at each point with the condition (4.9), there is the directional derivative

$$\frac{\partial u}{\partial \nu}(\zeta) \ := \ \lim_{t\to 0} \frac{u(\zeta + t\cdot\nu) - u(\zeta)}{t} \ = \ \varphi(\zeta)\ . \tag{4.11}$$

In particular, in the case of the Neumann problem, $\mathrm{Re}\, n(\zeta)\cdot\overline{\nu(\zeta)} \equiv 1 > 0$ and we have by Theorem 4.6 and Remark 4.5 the following significant result, Theorem 4 in [65]. Here we also apply the well-known fact that any rectifiable curve has a tangent a.e. with respect to the natural parameter.

Theorem 4.6. *Let D be a domain in $\mathbb{C}$ bounded by a rectifiable Jordan curve and $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to the natural parameter. Then one can find harmonic functions $u : D \to \mathbb{R}$ such that, at a.e. point $\zeta \in \partial D$ with respect to the natural parameter, there exist:*

1) the finite normal limit

$$u(\zeta) \ := \ \lim_{z\to\zeta} u(z) \tag{4.12}$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta)\ :=\ \lim_{t\to 0}\ \frac{u(\zeta+t\cdot n)-u(\zeta)}{t}\ =\ \varphi(\zeta) \tag{4.13}$$

3) the nontangential limit

$$\lim_{z\to\zeta}\ \frac{\partial u}{\partial n}\,(z)\ =\ \frac{\partial u}{\partial n}\,(\zeta) \tag{4.14}$$

where $n = n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ.

Note that here the tangent $\tau(s)$ to ∂D is measurable with respect to the natural parameter s as the derivative $d\zeta(s)/ds$ and, thus, the inner normal $n(s)$ to ∂D is also measurable with respect to the natural parameter.

Finally, we have the following significant result, Theorem 5 in [65].

Theorem 4.7. *The spaces of harmonic functions in Theorems 4.1, 4.3, 4.4 and 4.6, being nonclassical solutions of the problem on directional derivatives and the Neumann problem, correspondingly, have the infinite dimension for any prescribed measurable boundary data.*

Proof. In view of the equivalence of the problem on the directional derivatives to the corresponding Hilbert boundary value problem established under the proof of Theorem 4.1, the conclusion of Theorem 4.7 follows directly from Theorem 5.2 and Remark 5.2 in [61], see Theorems 3.1 and 3.3 in Section 3 above. □

5. Logarithmic capacity in Hilbert problem for analytic functions

Here we give more refined results for harmonic and analytic functions in terms of the so-called logarithmic capacity that makes possible to extend the theory of boundary value problems to quasiconformal functions (Beltrami equations) and to $A-$harmonic functions (generalizations of the Laplace equation in inhomogeneous and anisotropic media), see [21, 34, 69, 74] and Sections 6–7 and 9–10 further.

Recall some notions and facts which are relevant to logarithmic capacity, see e.g. [16, 57] and [58]. First of all, given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function of a set ν defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_{E} \log\left|\frac{1}{z-\zeta}\right|\, d\nu(\zeta) \tag{5.1}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the Borel set E is the quantity

$$C(E) = e^{-V} , \qquad V = \inf_{\nu} V_{\nu}(E) , \qquad V_{\nu}(E) = \sup_{z} U^{\nu}(z) . \quad (5.2)$$

Note that it is sufficient to take the supremum in (5.2) over the set E only. If $V = \infty$, then $C(E) = 0$. It is known that $0 \le C(E) < \infty$, $C(E_1) \le C(E_2)$ if $E_1 \subseteq E_2$, $C(E) = 0$ if $E = \bigcup\limits_{n=1}^{\infty} E_n$, with $C(E_n) = 0$, $n = 1, 2, \ldots$, see e.g. Lemma III.4 in [16].

It is well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [57]:

$$C(E) = \tau(E) := \lim_{n\to\infty} V_n^{\frac{2}{n(n-1)}} \qquad (5.3)$$

where V_n denotes the supremum (really, maximum) of the product

$$V(z_1, \ldots, z_n) = \prod_{k<l}^{l=1,\ldots,n} |z_k - z_l| \qquad (5.4)$$

taken over all collections of points $z_1, \ldots, z_n$ in the set E. Following Fékete, see [23], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E. By the geometric interpretation of the logarithmic capacity as the transfinite diameter we immediately see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is continuous by Hölder and, in particular, for conformal and quasiconformal mappings on the compact sets, see e.g. Theorem II.4.3 in [50].

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [16], **inner C_* and outer C^* capacities:**

$$C_*(E) := \sup_{F\subseteq E} C(E) \qquad (5.5)$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$, and

$$C^*(E) := \inf_{E\subseteq O} C(O) \qquad (5.6)$$

where infimum is taken over all open sets $O \subset \mathbb{C}$. Further, a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if

$$C^*(E) = C_*(E) , \qquad (5.7)$$

and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$. Note, see e.g. Lemma III.5 in [16], that the outer capacity is semiadditive, i.e.,

$$C^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} C^*(E_n)\ . \tag{5.8}$$

A function $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{z \in E : \varphi(z) \in O\} \tag{5.9}$$

are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [57]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [57]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [16].

Remark 5.1. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [16]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of the sigma-compactum (union of countable collection of compact sets) and the set of logarithmic capacity zero. It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. theorem II(7.4) in [70]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [16].

We are especially interested by functions $\varphi : \partial\mathbb{D} \to \mathbb{C}$ defined on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. However, in view of (5.3), it suffices to examine the corresponding problems on the segments of the real axis because any closed arc on $\partial\mathbb{D}$ admits a bi-Lipschitz (even infinitely smooth, the so-called stereographic) mapping g onto such a segment.

In this connection, recall that a mapping $g : X \to X'$ between metric spaces (X, d) and (X', d') is said to be **Lipschitz** if $d'(g(x_1), g(x_2)) \leqslant C \cdot d(x_1, x_2)$ for any $x_1, x_2 \in X$ and for a finite constant C. If, in addition, $d(x_1, x_2) \leqslant c \cdot d'(g(x_1), g(x_2))$ for any $x_1, x_2 \in X$ and for a finite constant c, then mapping g is called **bi-Lipschitz**.

First of all, it was proved the following analog of the Luzin theorem, see Theorem 3.1 in [69], cf. Theorem A in Section 2.

Theorem 5.2. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ such that $\Phi'(x) = \varphi(x)$ a.e. on (a, b) with respect to logarithmic capacity. Furthermore, the function Φ can be choosen such that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ for a prescribed $\varepsilon > 0$ and all $x \in [a, b]$.*

Then, on this basis, it was obtain the following analog of the Gehring–Luzin theorem, see Theorem 4.1 in [69], cf. Theorem B in Section 2.

Theorem 5.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, measurable and finite a.e. with respect to logarithmic capacity. Then there is a harmonic function $u(z)$, $z \in \mathbb{D}$, such that $u(z) \to \varphi(\vartheta)$ as $z \to e^{i\vartheta}$ along any nontangential path for all $\vartheta \in \mathbb{R}$ except a set of logarithmic capacity zero.*

We call $\lambda : \partial\mathbb{D} \to \mathbb{C}$ a **function of bounded variation**, write $\lambda \in \mathcal{BV}(\partial\mathbb{D})$, if

$$V_\lambda(\partial\mathbb{D}) \, := \, \sup \sum_{j=1}^{j=k} |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| \, < \, \infty \tag{5.10}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial\mathbb{D}$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_\lambda(\partial\mathbb{D})$ is called the **variation of the function** λ.

Remark 5.4. Note that by the definition $V_\lambda(\partial\mathbb{D}) = V_{\lambda\circ h}(\partial\mathbb{D})$, i.e., the variation is invariant under every homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}$ and, thus, the definition can be extended in a natural way to an arbitrary Jordan curve in $\mathbb{C}$ because a Jordan curve is a continuous one-to-one image of the unit circle in $\mathbb{C}$.

It was established in [69], see Theorem 5.1, the following interesting fact.

Theorem 5.5. *Let $\alpha : \partial\mathbb{D} \to \mathbb{R}$ be a function of bounded variation and let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function such that*

$$\lim_{z \to \zeta} \text{Re f(z)} \; = \; \alpha(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial\mathbb{D} \tag{5.11}$$

with respect to logarithmic capacity along any nontangential path. Then

$$\lim_{z\to\zeta} \text{Im f(z)} = \beta(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial\mathbb{D} \tag{5.12}$$

with respect to logarithmic capacity along any nontangential path where $\beta : \partial\mathbb{D} \to \mathbb{R}$ *is a function that is finite a.e. and measurable with respect to logarithmic capacity.*

Remark 5.6. Recall a subtle fact due to N. Luzin: the harmonic functions in a unit disk with continuous (even absolutely continuous!) boundary values can have conjugate harmonic functions whose boundary values are not continuous functions. Moreover, they are not even essentially bounded in a neighborhood of any point of the unit circle, see e.g. [8]. Thus, the interconnection between the boundary values of conjugate harmonic functions is a quite complicated item, see also I.E in [44].

Moreover, correspondingly to Proposition 5.1 in [69], we have the following:

Lemma 5.7. *For every function* $\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ *of the class* $\mathcal{BV}(\partial\mathbb{D})$ *there is a function* $\alpha_\lambda : \partial\mathbb{D} \to \mathbb{R}$ *of the class* $\mathcal{BV}(\partial\mathbb{D})$ *such that* $\lambda(\zeta) = \exp\{i\alpha_\lambda(\zeta)\}$*,* $\zeta \in \partial\mathbb{D}$*.*

Finally, on the basis of Theorems 5.3 and 5.5, Lemma 5.7, see also Remark 2.3, it was derived similarly to the proof of Theorem 3.1 the following significant result on the Hilbert boundary value problem, see Theorems 6.1 and 8.1 in [69].

Theorem 5.8. *Let* $\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ *be of bounded variation and* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be measurable with respect to logarithmic capacity. Then the space of all analytic functions* $f : \mathbb{D} \to \mathbb{C}$ *such that along any nontangential path*

$$\lim_{z\to\zeta} \text{Re}\,\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \qquad \textit{for a.e.} \quad \zeta \in \partial\mathbb{D} \tag{5.13}$$

with respect to logarithmic capacity has the infinite dimension.

Note that, in view of Remark 5.1, we strengthen in Theorem 5.8 in comparison with Theorem 3.1 its hypothesis as well as its conclusion. Thus, Theorem 5.8 is not a consequence of Theorem 3.1.

6. Angular limits in Hilbert problem for quasiconformal functions

Let D be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The equation of the form

$$f_{\bar{z}} = \mu(z) \cdot f_z \tag{6.1}$$

where $f_{\bar{z}} = \bar{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, f_x and f_y are partial derivatives of the function f in x and y, respectively, is said to be a **Beltrami equation**. The Beltrami equation (6.1) is said to be **nondegenerate** if $||\mu||_\infty < 1$.

Note that there were recently established a great number of new theorems on the existence and on the boundary behavior of homeomorphic solutions and, on this basis, on the Dirichlet problem for the Beltrami equations with essentially unbounded **distortion quotients** $K_\mu(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$, see e.g. papers [9–13, 33, 45–47, 65–67] and monographs [5, 32, 52] with many references therein. However, under the study of the Hilbert boundary value problem for (6.1) we restrict ourselves with the nondegenerate case because this research leads to a very delicate Lusin's problem on interconnections of the boundary data of conjugate harmonic functions, see Remark 5.6, and to the difficult problem on the distortion of boundary measures under mappings which are more general than quasiconformal mappings.

Recall that homeomorphic solutions with distributional derivatives of the nondegenerate Beltrami equations (6.1) are called **quasiconformal mappings**, see e.g. [3, 14] and [50]. The images of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ under the quasiconformal mappings $\mathbb{C}$ onto itself are called **quasidisks** and their boundaries are called **quasicircles** or **quasiconformal curves**. It is known that every smooth (or Lipschitz) Jordan curve is a quasiconformal curve and, at the same time, quasiconformal curves can be nonrectifiable as it follows from the known examples, see e.g. the point II.8.10 in [50].

Note also that a Jordan curve generally speaking has no tangents. Hence we need a replacement for the notion of a nontangential limit usually applied. In this connection, recall the Bagemihl theorem in [6], see also Theorem III.1.8 in [58], stated that, for any function $\Omega : \mathbb{D} \to \overline{\mathbb{C}}$, except at most countable set of points $\zeta \in \partial\mathbb{D}$, for all pairs of arcs γ_1 and γ_2 in $\mathbb{D}$ terminating at $\zeta \in \partial\mathbb{D}$,

$$C(\Omega, \gamma_1) \ \cap \ C(\Omega, \gamma_2) \ \neq \ \varnothing \ , \tag{6.2}$$

where $C(\Omega, \gamma)$ denotes the **cluster set of Ω at ζ along** γ, i.e.,

$$C(\Omega, \gamma) \ = \ \{w \in \overline{\mathbb{C}} \ : \ \Omega(z_n) \to w, \ z_n \to \zeta, \ z_n \in \gamma\} \ .$$

Immediately by the theorems of Riemann and Caratheodory, this result is extended to an arbitrary Jordan domain D in $\mathbb{C}$. Given a function $\Omega : D \to \overline{\mathbb{C}}$ and $\zeta \in \partial D$, denote by $P(\Omega, \zeta)$ the intersection of all cluster sets $C(\Omega, \gamma)$ for arcs γ in D terminating at ζ. Later on, we call the points of the set $P(\Omega, \zeta)$ **principal asymptotic values** of Ω at ζ. Note that if

Ω has a limit along at least one arc in D terminating at a point $\zeta \in \partial D$ with the property (6.2), then the principal asymptotic value is unique.

Recall that a mapping $f : D \to \mathbb{C}$ is called **discrete** if the pre-image $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

The **regular solution** of a Beltrami equation (6.1) is a continuous, discrete and open mapping $f : D \to \mathbb{C}$ with distributional derivatives satisfying (6.1) a.e. Note that, in the case of nondegenerate Beltrami equations (6.1), a regular solution f belongs to class $W^{1,p}_{\rm loc}$ for some $p > 2$ and, moreover, its Jacobian $J_f(z) \neq 0$ for almost all $z \in D$, and it is called a **quasiconformal function**, see e.g. Chapter VI in [50], or, in the modern manner, a **quasiregular mapping**, see [53].

Next, considering the corresponding generalization of the Hilbert boundary value problem for the Beltrami equations, we are able to formulate the following result, see Theorem 7.1 in [69].

Theorem 6.1. *Let D be a Jordan domain in $\mathbb{C}$ bounded by a quasiconformal curve, $\mu : D \to \mathbb{C}$ be a measurable (by Lebesgue) function with $||\mu||_\infty < 1$, $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$ be a function of bounded variation and let $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity.*

Then the Beltrami equation (6.1) *has a regular solution f such that in the sense of the unique principal asymptotic value*

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)} \cdot f(z)\} \ = \ \varphi(\zeta) \qquad \text{for a.e.} \quad \zeta \in \partial D \tag{6.3}$$

with respect to logarithmic capacity. If in addition ∂D is rectifiable, then the limit in (6.3) *holds a.e. with respect to the natural parameter along any nontangential path. Finally, the space of all such solutions has the infinite dimension.*

In particular, the latter conclusion of Theorem 6.1 holds in the case of smooth and, more generally, Lipschitz boundaries. We give here the complete proof of Theorem 6.1 in view its significance and to demonstrate the way of its reduction to Theorem 5.8.

Proof. Without loss of generality we may assume that $0 \in D$ and $1 \in \partial D$. Extending μ by zero everywhere outside of D, we obtain the existence of a quasiconformal mapping $\omega : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the normalization $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(\infty) = \infty$ satisfying the Beltrami equation (6.1) with the given μ, see e.g. Theorem V.B.3 in [3]. By the theorems of Riemann and Caratheodory, the Jordan domain $\omega(D)$ can be mapped by a conformal mapping g with the normalization $g(0) = 0$ and $g(1) = 1$

onto the unit disk $\mathbb{D}$. It is clear that $h := g \circ \omega$ is a quasiconformal homeomorphism with normalization $h(0) = 0$ and $h(1) = 1$ satisfying the same Beltrami equation.

By the reflection principle for quasiconformal mappings, using the conformal reflection (inversion) with respect to the unit circle in the image and quasiconformal reflection with respect to ∂D in the preimage, we can extend h to a quasiconformal mapping $H : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the normalization $H(0) = 0$, $H(1) = 1$ and $H(\infty) = \infty$, see e.g. I.8.4, II.8.2 and II.8.3 in [50]. Note that $\Lambda = \lambda \circ H^{-1}$ is a function of bounded variation, $V_\Lambda(\partial\mathbb{D}) = V_\lambda(\partial D)$.

The mappings H and H^{-1} transform sets of logarithmic capacity zero on ∂D into sets of logarithmic capacity zero on $\partial\mathbb{D}$ and vice versa because quasiconformal mappings are continuous by Hölder on ∂D and $\partial\mathbb{D}$ correspondingly, see e.g. Theorem II.4.3 in [50].

Further, the function $\Phi = \varphi \circ H^{-1}$ is measurable with respect to logarithmic capacity. Indeed, under this mapping measurable sets with respect to logarithmic capacity are transformed into measurable sets with respect to logarithmic capacity because such a set can be represented as the union of a sigma-compactum and a set of logarithmic capacity zero and compacta under continuous mappings are transformed into compacta and compacta are measurable sets with respect to logarithmic capacity.

Thus, the Hilbert boundary value problem (6.3) for the Beltrami equation (6.1) is reduced to the corresponding Hilbert problem for analytic functions F in the unit disk:

$$\lim_{z\to\zeta} \operatorname{Re} \overline{\Lambda(\zeta)} \cdot F(z) \;=\; \Phi(\zeta) \tag{6.4}$$

and by Theorem 5.8 there is an analytic function $F : \mathbb{D} \to \mathbb{C}$ for which this boundary condition holds for a.e. $\zeta \in \partial\mathbb{D}$ with respect to logarithmic capacity along any nontangential path.

So, the desired solution of the original Hilbert boundary value problem (6.3) for the Beltrami equation (6.1) exists and can be represented as $f = F \circ H$.

Finally, since the distortion of angles under the quasiconformal mapping is bounded, see e.g. [1, 2] and [71], then in the case of a rectifiable boundary of D condition (6.3) can be understood along any nontangential path a.e. with respect to the natural parameter. □

7. Angular limits in Neumann problem for $A-$harmonic functions

The Neumann problem with respect to angular limits for A-harmonic functions in the case of boundary data that are measurable with respect to logarithmic capacity was first solved in the paper [74]. The corresponding partial differential equations in the divergence form below take a significant part in many problems of mathematical physics, in particular, in anisotropic and inhomogeneous media. These equations are closely interconnected with Beltrami equations, see e.g. [5, 39] and [42].

In this connection, note that if $f = u + i \cdot v$ is a regular solution of the Beltrami equation (6.1), then the function u is a continuous generalized solution of the divergence-type equation

$$\operatorname{div} A(z) \nabla\, u = 0\ , \tag{7.1}$$

called $A-$**harmonic function**, i.e. $u \in C \cap W^{1,1}$ and

$$\int\limits_D \langle A(z)\nabla u, \nabla\varphi\rangle = 0 \qquad \forall\ \varphi \in C_0^\infty(D)\ , \tag{7.2}$$

where $A(z)$ is the matrix function:

$$A = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{7.3}$$

As we see in (7.3), the matrix $A(z)$ is symmetric and its entries $a_{ij} = a_{ij}(z)$ are dominated by the quantity

$$K_\mu(z) \ = \ \frac{1\ +\ |\mu(z)|}{1\ -\ |\mu(z)|}\ , \tag{7.4}$$

and, thus, they are bounded if the Beltrami equation (6.1) is not degenerate.

Vice verse, uniformly elliptic equations (7.1) with symmetric $A(z)$ and $\det A(z) \equiv 1$ just correspond to nondegenerate Beltrami equations (6.1) with coefficient

$$\mu \ = \ \frac{1}{\det\,(I + A)}\ (a_{22} - a_{11}\ -\ 2ia_{21})\ =\ \frac{a_{22} - a_{11}\ -\ 2ia_{21}}{1\ +\ \operatorname{Tr} A\ +\ \det A} \tag{7.5}$$

where I denotes identity 2×2 matrix, $\operatorname{Tr} A = a_{22} + a_{11}$, see e.g. theorem 16.1.6 in [5]. Following [34], call all such matrix functions $A(z)$ of the *class $\mathcal{B}$*. Recall that equation (7.1) is said to be **uniformly elliptic**, if $a_{ij} \in L^\infty$ and $\langle A(z)\eta, \eta\rangle \geq \varepsilon|\eta|^2$ for some $\varepsilon > 0$ and for all $\eta \in \mathbb{R}^2$.

First, similarly to the paper [65], see Section 4 above, it is proved theorems on Poincare and Neumann problems for harmonic functions in the unit disk $\mathbb{D}$ but in terms of logarithmic capacity on the unit circle $\partial\mathbb{D}$, see Theorems 1 and 2 in [74]:

Theorem 7.1. *Let $\nu : \partial\mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$ be a function of bounded variation, and let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there exist harmonic functions $u : \mathbb{D} \to \mathbb{R}$ such that*

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu} \ = \ \varphi(\zeta) \tag{7.6}$$

along any nontangential paths for a.e. $\zeta \in \partial\mathbb{D}$ with respect to logarithmic capacity.

Proof. Indeed, by Theorem 6.1 in [69], see Theorem 6.1 above, there exist analytic function $f : D \to \mathbb{C}$ such that

$$\lim_{z\to\zeta} \mathrm{Re}\ \overline{\nu(\zeta)\cdot f(z)} \ = \ \lim_{z\to\zeta} \mathrm{Re}\ \nu(\zeta)\cdot f(z) \ = \ \varphi(z) \tag{7.7}$$

along any nontangential paths for a.e. $\zeta \in \partial\mathbb{D}$ with respect to logarithmic capacity. Note that an indefinite integral F of f in $\mathbb{D}$ is also an analytic function and, correspondingly, the harmonic functions $u = \mathrm{Re}\, F$ and $v = \mathrm{Im}\, F$ satisfy the system of Cauchy–Riemann $v_x = -u_y$ and $v_y = u_x$. Hence

$$f \ = \ F' \ = \ F_x \ = \ u_x \ + \ i\cdot v_x \ = \ u_x \ - \ i\cdot u_y \ = \ \overline{\nabla u}$$

where $\nabla u = u_x + i\cdot u_y$ is the gradient of the function u in complex form. Thus, (7.6) follows from (7.7), i.e. u is one of the desired harmonic functions because its directional derivative

$$\frac{\partial u}{\partial \nu} \ = \ \mathrm{Re}\,\overline{\nu}\cdot\nabla u \ = \ \mathrm{Re}\,\nu\cdot\overline{\nabla u} \ = \ \langle \nu, \nabla u\rangle$$

is the scalar product of ν and the gradient ∇u interpreted as vectors in $\mathbb{R}^2$. □

Remark 7.2. We can say more in the case $\mathrm{Re}\ n\cdot\overline{\nu} > 0$ where $n = n(\zeta)$ is the unit interior normal with a tangent to $\partial\mathbb{D}$ at the point $\zeta \in \partial\mathbb{D}$. In view of (7.6), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in $\mathbb{D}$ along the straight line passing through the point ζ

and being parallel to the vector $\nu(\zeta)$ because along this line, for z and z_0 that are close enough to ζ,

$$u(z) \ = \ u(z_0) \ - \int\limits_0^1 \frac{\partial u}{\partial \nu}\,(z_0 + \tau(z-z_0))\ d\tau\ .$$

Thus, at each point with the condition (7.6), there is the directional derivative

$$\frac{\partial u}{\partial \nu}\,(\zeta) \ := \ \lim_{t\to 0} \frac{u(\zeta + t\cdot\nu) - u(\zeta)}{t} \ = \ \varphi(\zeta)\ .$$

In particular, in the case of the Neumann problem, we have by Theorem 7.1 and Remark 7.2 the following significant result.

Theorem 7.3. *For each function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ that is measurable with respect to logarithmic capacity, one can find harmonic functions $u : \mathbb{D} \to \mathbb{C}$ such that, for a.e. point $\zeta \in \partial\mathbb{D}$ with respect to logarithmic capacity, there exist:*

1) the finite radial limit

$$u(\zeta) \ := \ \lim_{r\to 1} \ u(r\zeta)$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \ := \ \lim_{t\to 0} \frac{u(\zeta + t\cdot n) - u(\zeta)}{t} \ = \ \varphi(\zeta)$$

3) the nontangential limit

$$\lim_{z\to\zeta} \frac{\partial u}{\partial n}\,(z) \ = \ \frac{\partial f}{\partial n}\,(\zeta)$$

where $n = n(\zeta)$ denotes the unit interior normal to $\partial\mathbb{D}$ at the point ζ.

Then these results are extended to smooth enough Jordan domains by reduction to the case of the unit disk, see Theorem 3 and 4 in [74].

Theorem 7.4. *Let D be an almost smooth Jordan domain in the complex plane $\mathbb{C}$, $\nu : \partial D \to \ \mathbb{C}$, $|\nu(\zeta)| \equiv 1$ be a function of bounded variation and let $\varphi : \partial D \to \mathbb{R}$ be a function that is measurable with respect to logarithmic capacity. Then there exist harmonic functions $u : D \to \mathbb{R}$ such that along any nontangential paths*

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu} \ = \ \varphi(\zeta) \tag{7.8}$$

for a.e. point $\zeta \in \partial D$ with respect to logarithmic capacity.

Recall that a Jordan domain is called **almost smooth** if its boundary is Lipschitzian and has tangent to almost all points with respect to logarithmic capacity.

Theorem 7.5. *Let D be an almost smooth Jordan domain in the complex plane $\mathbb{C}$ and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then one can find harmonic functions $u : D \to \mathbb{C}$ such that for a.e. $\zeta \in \partial D$ with respect to logarithmic capacity, there exist:*

1) the finite normal limit

$$u(\zeta) \ := \ \lim_{z\to\zeta} \ u(z)$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\cdot n) \ - \ u(\zeta)}{t} \ = \ \varphi(\zeta)$$

3) the nontangential limit

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial n}\,(z) \ = \ \frac{\partial u}{\partial n}\,(\zeta) \ .$$

And only then the author prove the corresponding results on Poincare and Neumann problems for $A-$harmonic functions, see Theorems 5 and 6 in [74]:

Theorem 7.6. *Let D be an almost smooth Jordan domain in $\mathbb{C}$, $A(z)$, $z \in D$, be a matrix function of the class $\mathcal{B}\cap C^{\alpha}$, $\alpha \in (0,1)$, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be a function of bounded variation and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there exist A-harmonic functions $u : D \to \mathbb{R}$ of the class $C^{1+\alpha}$ such that*

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial \nu}\,(z) \ = \ \varphi(\zeta) \tag{7.9}$$

along any nontangential paths for a.e. $\zeta \in \partial D$ with respect to logarithmic capacity.

Note that the last and the next theorem are based on the well–known fact that the homeomorphic solutions of the Beltrami equations with complex coefficients in the class C^{α}, $\alpha \in (0,1)$, belong to the class $C^{1+\alpha}$, see e.g. [37] and [38].

Theorem 7.7. *Let D be an almost smooth Jordan domain in $\mathbb{C}$, the interior unit normal $n = n(\zeta)$ to ∂D has bounded variation, $A(z)$, $z \in D$, be a matrix function of class $\mathcal{B} \cap C^{\alpha}$, $\alpha \in (0,1)$ and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there exist A-harmonic function $u : D \to \mathbb{R}$ of class $C^{1+\alpha}$ such that for a.e. $\zeta \in \partial D$ with respect to logarithmic capacity there exist:*

1) the finite normal limit

$$u(\zeta) \ := \ \lim_{z\to\zeta} \ u(z)$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\cdot n) \ - \ u(\zeta)}{t} \ \ = \ \varphi(\zeta)$$

3) the nontangential limit

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial n}\,(z) \ \ = \ \ \frac{\partial u}{\partial n}\,(\zeta) \ .$$

Finally, it was established by Theorem 7 in [74] that all the spaces of solutions in Theorems 7.1–7.7 have the infinite dimension.

8. Tangent limits in boundary value problems for analytic functions

In this section, we demonstrate an alternative approach making possible to obtain new results with tangent limits in the Hilbert boundary value problem, see [62]. In particular, it is shown that the spaces of the found solutions have the infinite dimension for prescribed collections of Jordan arcs terminating in almost every boundary point. Moreover, similar results are proved for the Riemann boundary value problem.

Jordan arcs $\{J_\zeta\}_{\zeta\in C}$ is of **class $\mathcal{B}$S (of Bagemihl–Seidel class)**, cf. [7], 740–741, if all J_ζ lie in a ring $\mathfrak{R}$ generated by C and a Jordan curve C_* in $\mathbb{C}$, $C_* \cap C = \emptyset$, J_ζ is joining C_* and $\zeta \in C$, every $z \in \mathfrak{R}$ belongs to a single arc J_ζ, and for a sequence of mutually disjoint Jordan curves C_n in $\mathfrak{R}$ such that $C_n \to C$ as $n \to \infty$, $J_\zeta \cap C_n$ consists of a single point for each $\zeta \in C$ and $n = 1, 2, \ldots$.

In particular, a family of Jordan arcs $\{J_\zeta\}_{\zeta\in C}$ is of class $\mathcal{B}$**S** if J_ζ is generated by an isotopy of C. For instance, every curvilinear ring $\mathfrak{R}$ one of whose boundary component is C can be mapped with a conformal mapping g onto a circular ring R and the inverse mapping $g^{-1} : R \to \mathfrak{R}$ maps radial lines in R onto suitable Jordan arcs J_ζ and centered circles in R onto Jordan curves giving the corresponding isotopy of C to other boundary component of $\mathfrak{R}$. We may also to choose in R a curve which is tangent to its boundary components and which intersects every centered circle in R only one time and to obtain the rest lines by its rotation.

Finally, if $\Omega \subset \mathbb{C}$ is an open set bounded by a finite collection of mutually disjoint Jordan curves, then we say that a family of Jordan arcs $\{J_\zeta\}_{\zeta\in\partial\Omega}$ is of class $\mathcal{B}\mathbf{S}$ if its restriction to each component of $\partial\Omega$ is so.

In these terms, it is easy to prove the following results on the Hilbert boundary value problem, see Theorem 1 and Remark 1 in [62].

Theorem 8.1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint rectifiable Jordan curves, and let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, $\varphi : \partial D \to \mathbb{R}$ and $\psi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter. Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D.*

Then there exist single-valued analytic functions $f : D \to \mathbb{C}$ such that

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \varphi(\zeta)\;, \tag{8.1}$$

$$\lim_{z\to\zeta} \mathrm{Im}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \psi(\zeta) \tag{8.2}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the natural parameter.

Remark 8.2. Thus, the space of all solutions f of the Hilbert problem (8.1) in the given sense has the infinite dimension for any prescribed φ, λ and $\{\gamma_\zeta\}_{\zeta\in D}$ because the space of all measurable functions $\psi : \partial D \to \mathbb{R}$ has the infinite dimension.

Proof. Indeed, set $\Psi(\zeta) = \varphi(\zeta) + i\psi(\zeta)$ and $\Phi(\zeta) = \lambda(\zeta)\cdot\Psi(\zeta)$ for all $\zeta \in \partial D$. Then by Theorem 2 in [7] there is a single-valued analytic function f such that

$$\lim_{z\to\zeta} f(z) \;=\; \Phi(\zeta) \tag{8.3}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the natural parameter. Then also

$$\lim_{z\to\zeta} \overline{\lambda(\zeta)}\cdot f(z) \;=\; \Psi(\zeta) \tag{8.4}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the natural parameter. □

Similar results can be formulated for arbitrary Jordan domains with respect to the harmonic measure, see Theorem 2 and Remark 2 in [62].

Theorem 8.3. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, and let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, $\varphi : \partial D \to \mathbb{R}$ and $\psi : \partial D \to \mathbb{R}$ be measurable functions with respect to the harmonic measure. Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D.*

Then there exist single-valued analytic functions $f : D \to \mathbb{C}$ such that

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \varphi(\zeta)\ , \tag{8.5}$$

$$\lim_{z\to\zeta} \mathrm{Im}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \;=\; \psi(\zeta) \tag{8.6}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the harmonic measure.

Remark 8.4. Again, the space of all solutions f of the Riemann–Hilbert problem (8.5) in the given sense has the infinite dimension for any prescribed φ, λ and $\{\gamma_\zeta\}_{\zeta\in D}$ because the space of all functions $\psi : \partial D \to \mathbb{R}$ that are measurable with respect to the harmonic measure has the infinite dimension.

Proof. Theorem 8.3 is reduced to Theorem 8.1 in the following way.

First, there is a conformal mapping ω of D onto a circular domain $\mathbb{D}_*$ whose boundary consists of a finite number of circles and points, see e.g. Theorem V.6.2 in [28]. Note that $\mathbb{D}_*$ cannot be degenerate because isolated singularities of conformal mappings are removable that is due to the well-known Weierstrass theorem, see e.g. Theorem 1.2 in [17]. Applying in the case of need the inversion with respect to a boundary circle of $\mathbb{D}_*$, we may assume that $\mathbb{D}_*$ is bounded.

Remark that ω is extended to a homeomorphism ω_* of $\overline{D}$ onto $\overline{\mathbb{D}_*}$, see e.g. point (i) of Lemma 3.1 in [64]. Set $\Lambda = \lambda\circ\Omega$, $\Phi = \varphi\circ\Omega$ and $\Psi = \psi\circ\Omega$ where $\Omega : \partial\mathbb{D}_* \to \partial D$ is the restriction of $\Omega_* := \omega_*^{-1}$ to $\partial\mathbb{D}_*$. Let us show that these functions are measurable with respect to the natural parameter on $\partial\mathbb{D}_*$.

For this goal, note first of all that the sets of the harmonic measure zero are invariant under conformal mappings between multiply connected Jordan domains because a composition of a harmonic function with a conformal mapping is again a harmonic function. Moreover, a set $E \subset \partial\mathbb{D}_*$ has the harmonic measure zero if and only if it has the length zero, say in view of the integral representation of the harmonic measure through the Green function of the domain $\mathbb{D}_*$, see e.g. Section II.4 in [57].

Hence Ω and Ω^{-1} transform measurable sets into measurable sets because every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [70], and continuous mappings transform compact sets into compact sets. Thus, the functions

λ, φ and ψ are measurable with respect to the harmonic measure on ∂D if and only if the functions Λ, Φ and Ψ are measurable with respect to the natural parameter on $\partial\mathbb{D}_*$.

Then by Theorem 8.1 there exist single-valued analytic functions $F : D \to \mathbb{C}$ such that

$$\lim_{w\to\xi} \ \mathrm{Re}\ \{\overline{\Lambda(\xi)}\cdot F(w)\} \ = \ \Phi(\xi)\ , \tag{8.7}$$

$$\lim_{w\to\xi} \ \mathrm{Im}\ \{\overline{\Lambda(\xi)}\cdot F(w)\} \ = \ \Psi(\xi) \tag{8.8}$$

along $\Gamma_\xi = \omega(\gamma_{\Omega(\xi)})$ for a.e. $\xi \in \partial\mathbb{D}_*$ with respect to the natural parameter.

Thus, by the construction the functions $f = F\circ\omega$ are the desired analytic functions $f : D \to \mathbb{C}$ satisfying the boundary conditions (8.5) and (8.6) along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the harmonic measure. $\Box$

Remark 8.5. Many investigations were devoted to the nonlinear Hilbert (Riemann–Hilbert) boundary value problems with conditions of the type

$$\Phi(\,\zeta,\, f(\zeta)\,) \ = \ 0 \qquad \forall\, \zeta\in\partial D\ , \tag{8.9}$$

see e.g. [20, 43] and [73]. It is natural also to weaken such conditions to

$$\Phi(\,\zeta,\, f(\zeta)\,) \ = \ 0 \qquad \text{for a.e.} \quad \zeta\in\partial D\ . \tag{8.10}$$

It is easy to see that the proposed approach makes possible also to reduce such problems to the algebraic and measurable solvability of the relation

$$\Phi(\,\zeta,\, v\,) \ = \ 0 \tag{8.11}$$

with respect to a complex-valued function $v(\zeta)$, cf. e.g. [30].

Through suitable modifications of Φ under the corresponding mappings of Jordan boundary curves onto the unit circle $\mathbb{S} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, we may assume that ζ belongs to $\mathbb{S}$.

The following results on the Riemann boundary value problem can be found as Theorem 3 and Lemma 1 in [62].

Theorem 8.6. *Let D be a domain in $\overline{\mathbb{C}}$ whose boundary consists of a finite number of mutually disjoint rectifiable Jordan curves, $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$ be measurable functions with respect to the natural parameter. Suppose that $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ are families of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D and $\mathbb{C}\setminus\overline{D}$, correspondingly.*

Then there exist single-valued analytic functions $f^+ : D \to \mathbb{C}$ *and* $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ *that satisfy (1.7) for a.e.* $\zeta \in \partial D$ *with respect to the natural parameter where* $f^+(\zeta)$ *and* $f^-(\zeta)$ *are limits of* $f^+(z)$ *and* $f^-(z)$ *az* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-*, correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for every couple* (A, B) *and any collections* γ_ζ^+ *and* γ_ζ^-*,* $\zeta \in \partial D$*.*

Theorem 8.6 is a special case of the following lemma on the generalized Riemann problem with shifts that can be useful for other goals, too.

Lemma 8.7. *Under the hypotheses of Theorem 8.6, let in addition* $\alpha : \partial D \to \partial D$ *be a homeomorphism keeping components of* ∂D *such that* α *and* α^{-1} *have the* $(N)-$*property of Lusin with respect to the natural parameter.*

Then there exist single-valued analytic functions $f^+ : D \to \mathbb{C}$ *and* $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ *that satisfy (1.8) for a.e.* $\zeta \in \partial D$ *with respect to the natural parameter where* $f^+(\zeta)$ *and* $f^-(\zeta)$ *are limits of* $f^+(z)$ *and* $f^-(z)$ *az* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-*, correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for every couple* (A, B) *and any collections* γ_ζ^+ *and* γ_ζ^-*,* $\zeta \in \partial D$*.*

Proof. First, let D be bounded and let $g^- : \partial D \to \mathbb{C}$ be a measurable function. Note that the function

$$g^+ \ := \ \{A \cdot g^- + B\} \circ \alpha^{-1} \tag{8.12}$$

is measurable. Indeed, $E := \{A \cdot g^- + B\}^{-1}(\Omega)$ is a measurable subset of ∂D for every open set $\Omega \subseteq \mathbb{C}$ because the function $A \cdot g^- + B$ is measurable by the hypotheses. Hence the set E is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [70]. However, continuous mappings transform compact sets into compact sets and, thus, $\alpha(E) = \alpha \circ \{A \cdot g^- + B\}^{-1}(\Omega) = (g^+)^{-1}(\Omega)$ is a measurable set, i.e. the function g^+ is really measurable.

Then by Theorem 2 in [7] there is a single-valued analytic function $f^+ : D \to \mathbb{C}$ such that

$$\lim_{z \to \xi} f^+(z) \ = \ g^+(\xi) \tag{8.13}$$

along γ_ξ^+ for a.e. $\xi \in \partial D$ with respect to the natural parameter. Note that $g^+(\alpha(\zeta))$ is determined by the given limit for a.e. $\zeta \in \partial D$ because α^{-1} also has the $(N)-$property of Lusin.

Note that $\overline{\mathbb{C}} \setminus \overline{D}$ consists of a finite number of (simply connected) Jordan domains $D_0, D_1, \ldots, D_m$ in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\infty \in D_0$. Then again by Theorem 2 in [7] there exist single-valued analytic functions $f_l^- : D_l \to \mathbb{C}$, $l = 1, \ldots, m$, such that

$$\lim_{z \to \zeta} f_l^-(z) \ = \ g_l^-(\zeta) \ , \qquad g_l^- := g^-|_{\partial D_l} \ , \tag{8.14}$$

along γ_ζ^- for a.e. $\zeta \in \partial D_l$ with respect to the natural parameter.

Now, let S be a circle that contains D and let j be the inversion of $\overline{\mathbb{C}}$ with respect to S. Set

$$D_* = j(D_0), \quad g_* = \overline{g_0 \circ j}, \quad g_0^- := g^-|_{\partial D_0}, \quad \gamma_\xi^* = j\left(\gamma_{j(\xi)}^-\right), \quad \xi \in \partial D_*.$$

Then by Theorem 2 in [7] there is a single-valued analytic function $f_* : D_* \to \mathbb{C}$ such that

$$\lim_{w \to \xi} f_*(w) \ = \ g_*(\xi) \tag{8.15}$$

along γ_ξ^* for a.e. $\xi \in \partial D_*$ with respect to the natural parameter. Note that $f_0^- := \overline{g_* \circ j}$ is a single-valued analytic function in D_0 and by construction

$$\lim_{z \to \zeta} f_0^-(z) \ = \ g_0^-(\zeta) \ , \qquad g_0^- := g^-|_{\partial D_0} \ , \tag{8.16}$$

along γ_ζ^- for a.e. $\zeta \in \partial D_0$ with respect to the natural parameter.

Thus, the functions f_l^-, $l = 0, 1, \ldots, m$, form an analytic function $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ satisfying (1.8) for a.e. $\zeta \in \partial D$ with respect to the natural parameter.

The space of all such couples (f^+, f^-) has the infinite dimension for every couple (A, B) and any collections γ_ζ^+ and γ_ζ^-, $\zeta \in \partial D$, in view of the above construction because of the space of all measurable functions $g^- : \partial D \to \mathbb{C}$ has the infinite dimension.

The case of unbounded D is reduced to the case of bounded D through the complex conjugation and the inversion of $\overline{\mathbb{C}}$ with respect to a circle S in some of the components of $\overline{\mathbb{C}} \setminus \overline{D}$ arguing as above. $\square$

Remark 8.8. Some investigations were devoted also to the nonlinear Riemann problems with boundary conditions of the form

$$\Phi(\,\zeta,\, f^+(\zeta),\, f^-(\zeta)\,) \ = \ 0 \qquad \forall\, \zeta \in \partial D \ . \tag{8.17}$$

It is natural as above to weaken such conditions to the following

$$\Phi(\,\zeta,\, f^+(\zeta),\, f^-(\zeta)\,) \ = \ 0 \qquad \text{for a.e.} \quad \zeta \in \partial D \ . \tag{8.18}$$

It is easy to see that the proposed approach makes possible also to reduce such problems to the algebraic and measurable solvability of the relations

$$\Phi(\,\zeta,\,v,\,w\,)\ =\ 0 \tag{8.19}$$

with respect to complex-valued functions $v(\zeta)$ and $w(\zeta)$, cf. e.g. [30].

Through suitable modifications of Φ under the corresponding mappings of Jordan boundary curves onto the unit circle $\mathbb{S} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, we may assume that ζ belongs to $\mathbb{S}$.

Example 8.9. For instance, correspondingly to the scheme given above, special nonlinear problems of the form

$$f^{+}(\zeta)\ =\ \varphi(\,\zeta,\,f^{-}(\zeta)\,) \qquad \text{for a.e.} \quad \zeta \in \partial\mathbb{D} \tag{8.20}$$

in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are always solved if the function $\varphi : \mathbb{S} \times \mathbb{C} \to \mathbb{C}$ satisfies the **Caratheodory conditions**: $\varphi(\zeta, w)$ is continuous in the variable $w \in \mathbb{C}$ for a.e. $\zeta \in \mathbb{S}$ and it is measurable in the variable $\zeta \in \mathbb{S}$ for all $w \in \mathbb{C}$.

Furthermore, the spaces of solutions of such problems always have the infinite dimension. Indeed, the function $\varphi(\zeta, \psi(\zeta))$ is measurable in $\zeta \in \mathbb{S}$ for every measurable function $\psi : \mathbb{S} \to \mathbb{C}$ if the function φ satisfies the Caratheodory conditions, see e.g. Section 17.1 in [48], and the space of all measurable functions $\psi : \mathbb{S} \to \mathbb{C}$ has the infinite dimension.

Problems. Finally, it is necessary to point out the open problems on solvability of Hilbert and Riemann problems along any prescribed families of arcs terminating in the boundary but not only along families of the Bagemihl–Seidel class and, more generally, along any prescribed families of paths to a.e. boundary point.

9. Tangent limits in boundary value problems for quasiconformal functions

Here we restrict ourselves by a history survey and short comments of results in [34] extending the approach from the last section to the boundary value problems for the Beltrami equations, see (6.1).

The first relevant problem is the measurement of sets on boundaries of domains. Recall that the sets of the length measure zero as well as of the harmonic measure zero are invariant under conformal mappings, however, they are not invariant under quasiconformal mappings as it follows from the famous Ahlfors-Beurling example of quasisymmetric mappings of the real axis that are not absolutely continuous, see [4]. Hence we are forced to apply here instead of them the so-called absolute harmonic measure by Nevanlinna, in other words, logarithmic capacity, see e.g. [57], or Section 5, whose zero sets are invariant under quasiconformal mappings, see comments after (5.3)–(5.4).

By the well–known Priwalow uniqueness theorem analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ coincide if they have the equal boundary values along all nontangential paths to a set E of points in $\partial\mathbb{D}$ of a positive length, see e.g. Theorem IV.2.5 in [60]. The theorem is valid also for analytic functions in Jordan domains with rectifiable boundaries, see e.g. Section IV.2.6 in [60].

However, examples of Luzin and Priwalow show that there exist nontrivial analytic functions in $\mathbb{D}$ whose radial boundary values are equal to zero on sets $E \subseteq \partial\mathbb{D}$ of a positive measure, see e.g. Section IV.5 in [60]. Simultaneously, by Theorem IV.6.2 in [60] of Luzin and Priwalow the uniqueness result is valid if E is of the second category. Theorem 1 in [7] demonstrates that the latter condition is necessary.

Theorem 1 in [7] can be formulated in the following way, see footnote 9 there. For the definition of class $\mathcal{B}\mathbf{S}$, see Section 8.

Theorem D. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves and let $\{\gamma_\zeta\}_{\zeta\in\partial D}$ be a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D.*

Suppose M is an F_σ set of first category on ∂D and $\Phi(\zeta)$ is a complex-valued function of Baire class 1 on M. Then there is a nonconstant single-valued analytic function $f : D \to \mathbb{C}$ such that, for all $\zeta \in M$, along γ_ζ

$$\lim_{z\to\zeta} f(z) \ = \ \Phi(\zeta)\ . \tag{9.1}$$

Recall Baire's terminology for categories of sets and functions. Namely, given a topological space X, a set $E \subseteq X$ is of **first category** if it can be written as a countable union of nowhere dense sets, and is of **second category** if E is not of first category. Also, given topological spaces X and X_*, $f : X \to X_*$ is said to be a **function of Baire class 1** if $f^{-1}(U)$ for every open set U in X_* is an F_σ set in X where an F_σ set is the union of a sequence of closed sets.

On the basis of Theorem D, in the case of domains D whose boundaries consist of rectifiable Jordan curves, it was formulated Theorem 2 in [7] on the existence of analytic functions $f : D \to \mathbb{C}$ such that (9.1) holds a.e. on ∂D with respect to the natural parameter for each prescribed measurable function $\Phi : \partial D \to \mathbb{C}$.

The following statement is similar to Theorem 2 in [7] but formulated in terms of logarithmic capacity instead of the natural parameter, see

Theorem 1 in [34]. It is necessary to stress that this theorem does not assume that boundary Jordan curves are rectifiable and that this fact is key.

Theorem 9.1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves and let a function $\Phi : \partial D \to \mathbb{C}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D. Then there is a nonconstant single-valued analytic function $f : D \to \mathbb{C}$ such that (9.1) *holds along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity.*

This result has made possible to solve the corresponding Dirichlet, Hilbert, Riemann, Neumann, Poincare and mixed boundary value problems for analytic functions as well as for quasiconformal mappings with an arbitrary prescribed complex dilatation μ, see Theorems 2–10 in [34]. Hence we give here only its proof.

Proof. Note first of all that $\mathcal{C} := C(\partial D) < \infty$ because ∂D is bounded and Borel, even compact, and show that there is a sigma-compact set S in ∂D of first category such that $C(S) = \mathcal{C}$. More precisely, S will be the union of a sequence of sets S_m in ∂D of the Cantor type that are nowhere dense in ∂D.

Namely, S_m is constructed in the following way. First we remove an open arc A_1 in ∂D of the logarithmic capacity $2^{-m}\mathcal{C}$ and one more open arc A_2 in $\partial D \setminus A_1$ of the logarithmic capacity $2^{-(m+1)}\mathcal{C}$ such that $\partial D \setminus (A_1 \cup A_2)$ consists of 2 segments of ∂D with the equal logarithmic capacity. Then we remove a union A_3 of 2 open arcs in each these segments of the total logarithmic capacity $2^{-(m+2)}\mathcal{C}$ such that new 4 segments in $\partial D \setminus (A_1 \cup A_2 \cup A_3)$ have the equal logarithmic capacity. Repeating by induction this construction, we obtain the compact sets $S_m = \partial D \setminus \bigcup\limits_{m=1}^{\infty} A_m$ with the logarithmic capacity $(1 - 2^{-(m-1)}) \cdot \mathcal{C} \to \mathcal{C}$ as $m \to \infty$.

Note also that the logarithmic capacity is Borel's regular measure as well as Radon's measure in the sense of points 2.2.3 and 2.2.5 in [22], correspondingly. Hence the classic Luzin theorem holds for the logarithmic capacity on $\mathbb{C}$, see e.g. Theorem 2.3.5 in [22].

By the Luzin theorem one can find a sequence of compacta K_n in S with $C(S \setminus K_n) < 2^{-n}$ such that $\Phi|_{K_n}$ is continuous for each $n = 1, 2, \ldots$, i.e., for every open set $U \subseteq \mathbb{C}$, $W_n := \Phi|_{K_n}^{-1}(U) = V_n \cap S$ for some open set V_n in $\mathbb{C}$, and W_n is sigma-compact because V_n and S are sigma-compact. Consequently, $W := \Phi|_{K}^{-1}(U) = \bigcup\limits_{n=1}^{\infty} W_n$ is also sigma-compact where

$K = \bigcup_{n=1}^{\infty} K_n$. Hence the restriction of Φ on the set K is a function of Baire's class 1. Finally, note that by the construction $C(Z) = 0$ where $Z = \partial D \setminus K = (\partial D \setminus S) \cup (S \setminus K)$ and

$$K = \bigcup_{m,\,n=1}^{\infty} (S_m \cap K_n)$$

where each set $E_{m,n} := S_m \cap K_n$, $m, n = 1, 2, \ldots$, is nowhere dense in ∂D.

Thus, the conclusion of Theorem 9.1 follows from Theorem D. □

We give, for example, only one theorem on the directional derivative problem for the Beltrami equation, see Theorem 7 in [34], because, firstly, many of the rest theorems are similar to the corresponding theorems in Section 8 and, secondly, they can be found in the given paper.

Theorem 9.2. *Let D be a Jordan domain in $\mathbb{C}$, $\mu : D \to \mathbb{C}$ be a function of the Hölder class C^α with $\alpha \in (0,1)$ and $|\mu(z)| \le k < 1$, $z \in D$, and let $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\Phi : \partial D \to \mathbb{C}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta \in \partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D. Then the Beltrami equation (6.1) *has a regular solution $f : D \to \mathbb{C}$ of the class $C^{1+\alpha}$ such that*

$$\lim_{z \to \zeta} \frac{\partial f}{\partial \nu}\,(z) \;=\; \Phi(\zeta) \tag{9.2}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity.

10. Tangent limits in boundary value problems for A-harmonic functions

Here we give short comments to results in [34] extending the approach from the last two sections to the boundary value problems for the Laplace equation and its generalizations corresponding to problems of mathematical physics in inhomogeneous and anisotropic media. For simplicity, we first give the corresponding results for harmonic functions that are simple consequences of Theorem 9.1.

Corollary 10.1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}S$ in D. Then there is a harmonic function $u : D \to \mathbb{R}$ such that

$$\lim_{z\to\zeta} u(z) \ = \ \varphi(\zeta) \tag{10.1}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity.

Corollary 10.2. *Let D be a Jordan domain in $\mathbb{C}$ and a function $\Phi : \partial D \to \mathbb{C}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D. Then there is a harmonic function $u : D \to \mathbb{R}$ such that

$$\lim_{z\to\zeta} \nabla u(z) \ = \ \Phi(\zeta) \tag{10.2}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity.

Here we use the complex writing for the gradient $\nabla u := u_x + i \cdot u_y$.

Proof. Indeed, by Theorem 9.1 there is a single-valued analytic function $f : D \to \mathbb{R}$ such that

$$\lim_{z\to\zeta} f(z) \ = \ \overline{\Phi(\zeta)} \tag{10.3}$$

along γ_ζ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity. Then any indefinite integral F of f is also a single-valued analytic function in the simply connected domain D and the harmonic functions $u = \mathrm{Re}\, F$ and $v = \mathrm{Im}\, F$ satisfy the Cauchy–Riemann system $v_x = -u_y$ and $v_y = u_x$. Hence

$$f \ = \ F' \ = \ F_x \ = \ u_x \ + \ i \cdot v_x \ = \ \ u_x \ - \ i \cdot u_y \ = \ \overline{\nabla u} \ .$$

Thus, (10.2) follows from (10.3) and, consequently, u is the desired function. □

The following statement on the directional derivative problem for $A-$harmonic functions can be derived from Theorem 9.2, see Corollary 7 in [34].

Corollary 10.3. *Let D be a Jordan domain in $\mathbb{C}$, $A(z)$, $z \in D$, be a matrix function of class $\mathcal{B} \cap C^\alpha$, $\alpha \in (0,1)$, and let $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{B}\mathbf{S}$ in D. Then there exist A-harmonic functions $u : D \to \mathbb{R}$ of the class $C^{1+\alpha}$ such that

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu}\ (z) \ = \ \varphi(\zeta) \tag{10.4}$$

along γ_ζ *for a.e.* $\zeta \in \partial D$ *with respect to the logarithmic capacity.*

Furthermore, the space of all such $A-$*harmonic functions* u *has the infinite dimension for any such prescribed* A, φ, ν *and* $\{\gamma_\zeta\}_{\zeta\in\partial D}$.

We refer the reader to the paper [34] for a great number of other consequences on boundary value problems for $A-$harmonic functions.

Finally, we recall that it was recently obtained a great number of the existence theorems for the degenerate Beltrami equations, see e.g. the papers [9–11, 15, 41, 67, 68] and the monographs [32, 40, 52]. The authors conjecture that the main part of results on boundary value problems can be extended to the corresponding degenerate equations but in terms of the so-called logarithmic measure that is a more refined measurement than logarithmic capacity, see e.g. [57]. Moreover, they should have suitable nonlinear and spatial analogues.

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