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On Hilbert boundary value problem for Beltrami equations with singularities

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Dedicated to the 80th anniversary of academician I.V. Skrypnik

Abstract. We investigate the Hilbert boundary value problem for Beltrami equations $\overline{\partial} f = \mu \partial f$ with singularities in generalized quasidisks D whose Jordan boundary ∂D consists of a countable collection of open quasiconformal arcs and, in addition, maybe a countable collection of points. Such generalized quasicircles can be nowhere even locally rectifiable but include, for instance, all piecewise smooth as well as all piecewise Lipschitz Jordan curves.

Generally speaking, generalized quasidisks do not satisfy the standard $(A)-$condition in PDE by Ladyzhenskaya–Ural'tseva, in particular, the outer cone touching condition as well as the quasihyperbolic boundary condition by Gehring–Martio that we assumed in our last paper for the uniformly elliptic Beltrami equations.

In essence, here we admit any countable collection of singularities of the Beltrami equations on the boundary and arbitrary singularities inside of the domain D of a general nature. As usual, a point in $\overline{D}$ is called a singularity of the Beltrami equation if the dilatation quotient $K_\mu := (1+|\mu|)/(1-|\mu|)$ is not essentially bounded in all its neighborhoods.

Presupposing that the coefficients of the problem are arbitrary functions of countable bounded variation and the boundary data are arbitrary measurable with respect to the logarithmic capacity, we prove the existence of regular solutions of the Hilbert boundary value problem. As a consequence, we derive the existence of nonclassical solutions of the Dirichlet, Neumann and Poincare boundary value problems for equations of mathematical physics with singularities in anisotropic and inhomogeneous media.

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1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. A **Beltrami equation** is an equation of the form

$$f_{\bar{z}} = \mu(z) f_z \tag{1.1}$$

where $f_{\bar{z}} = \bar{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, f_x and f_y are partial derivatives of the function f in x and y, respectively.

The equation (1.1) is said to be **nondegenerate** if $||\mu||_\infty < 1$, see e.g. the monographs [3, 11] and [35]. As usual, a point $z_0 \in \overline{D}$ is called a **singularity** of the Beltrami equation if the dilatation quotient

$$K_\mu(z) \ := \ \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

is not bounded in all its neighborhoods.

Recall that Hilbert [25] studied the boundary value problem formulated as follows: To find an analytic function $f(z)$ in a domain D bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z \to \zeta} \ \mathrm{Re}\, \{\overline{\lambda(\zeta)}\ f(z)\} \ = \ \varphi(\zeta) \qquad\qquad \forall\, \zeta \in C\ , \tag{1.3}$$

where both the **coefficient** λ and the **boundary date** φ of the problem are continuously differentiable with respect to the natural parameter s on C.

Moreover, it was assumed by Hilbert that $\lambda \neq 0$ everywhere on C. The latter allows us, without loss of generality, to consider that $|\lambda| \equiv 1$ on C. Note that the quantity $\mathrm{Re}\, \{\overline{\lambda}\, f\}$ in (1.3) means a projection of f into the direction λ interpreted as vectors in $\mathbb{R}^2$.

We refer the reader to a rather comprehensive treatment of the theory in the new excellent books [8, 9, 24, 56] and also recommend to make familiar with the historic surveys contained in the monographs [16, 39, 57] on the topic with an exhaustive bibliography and take a look at our recent papers [20, 23, 45].

In our last paper [22] we studied the Hilbert boundary value problem in a wider class of functions than those of analytic. Namely, instead of analytic functions we considered **quasiconformal functions** F represented as a composition of analytic functions $\mathcal{A}$ and quasiconformal mappings f, see [35], Chapter VI. Recall that a **quasiconformal mapping** is a homeomorphic solution f of a nondegenerate equation (1.1) in the class $W^{1,2}_{\rm loc}$. It is easy to see that every quasiconformal function $F = \mathcal{A} \circ f$ satisfies the same Beltrami equation as f.

Recall also that the images of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ under quasiconformal mappings of $\mathbb{C}$ onto itself are called **quasidisks** and their boundaries are called **quasicircles** or **quasiconformal curves**. Similarly, images of open intervals under quasiconformal mappings are called **open quasiconformal arcs**. We say that a Jordan curve is a **generalized quasicirle** if it consists of a countable collection of open quasiconformal arcs and, in addition, maybe a countable collection of points. The corresponding Jordan domains are called **generalized quasidisks**.

It is known that (piecewise) smooth and Lipschitz Jordan curve is a (generalized) quasiconformal curve and, at the same time, (generalized) quasiconformal curves can be locally nonrectifiable as it follows from the known examples, see e.g. the point II.8.10 in [35]. On the other hand, see Remark 3.2 in [22], quasicircles satisfied **(A)–condition** by Ladyzhenskaya–Ural'tseva, which is standard in the theory of boundary value problems for PDE, see e.g. the monograph [33]. However, generalized quasicircles, generally speaking, have not this property.

Proceeding from the above, the problem under consideration in [22] was to find quasiconformal functions satisfying both the Beltrami equation (1.1) in a Jordan domain D and the Hilbert boundary condition (1.3). There we substantially weakened the regularity conditions both on the functions λ and φ in the boundary condition (1.3) and on the boundary C of the domain D.

Namely, we dealt with the coefficients λ of **countable bounded variation** and measurable boundary data φ with respect to **logarithmic capacity**. The fundamental Becker – Pommerenke result in [7] allowed us to study the Hilbert boundary value problem in domains D with the **quasihyperbolic boundary condition** by Gehring–Martio [17]. Such domains may fail to satisfy the (A)–condition, see Remark 3.3 in [22]. Moreover, quasidisks had this property, see Remark 3.1 in [22], but generalized quasicircles, generally speaking, have not it.

Let D be a Jordan domain such that it has a tangent at a point $\zeta \in \partial D$. A path in D terminating at ζ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle in D with the vertex at ζ. The limit along all nontangential paths at ζ is called **angular** at the point. The latter notion is a standard tool for the study of the boundary behavior of analytic and harmonic functions, see e.g. [14], [30] and [42].

Further, the Hilbert boundary condition (1.3) will be understood precisely in the sense of the angular limit.

The notion of the logarithmic capacity is the important tool for our research, see e.g. [12, 40, 41], because the sets of zero logarithmic capacity are transformed under quasiconformal mappings into the sets of zero logarithmic capacity. Note that, as it follows from the classic Ahlfors–Beurling example, see [4], the sets of zero length as well as the sets of zero harmonic measure are not invariant under quasiconformal mappings. Dealing with measurable boundary date functions $\varphi(\zeta)$ with respect to the logarithmic capacity, we will use the **abbreviation q.e.** (**quasi-everywhere**) on a set $E \subset \mathbb{C}$, if a property holds for all $\zeta \in E$ except its subset of zero logarithmic capacity, see e.g. [34] for this term.

We say that a function $f : D \to \mathbb{C}$ is a **regular generalized solution of the Beltrami equation** (1.1) if f is continuous, diskrete and open, has the first generalized derivatives, satisfies (1.1) a.e. in D and, moreover, its Jacobian $J_f(z) \neq 0$ a.e. in D. We also say that f is a **regular generalized solution of the Hilbert boundary value problem** (1.3) for the Beltrami equation (1.1) if f in addition satisfies (1.3) q.e. on ∂D along nontangential paths in D.

Recall that a mapping $f : D \to \mathbb{C}$ is called **diskrete** if the pre-image $f^{-1}(z)$ consists of isolated points for every $z \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$. By the known Stoilow result, see e.g. [54], every regular generalized solution f of (1.1) has the representation $f = \mathcal{A} \circ g$ where g is a regular homeomorphic solution of (1.1) and $\mathcal{A}$ is an analytic function.

2. Logarithmic capacity and almost smooth domains

Given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_{E} \log \left| \frac{1}{z - \zeta} \right| d\nu(\zeta) \tag{2.1}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the Borel set E is the quantity

$$C(E) = e^{-V}\ , \qquad V \ = \ \inf_{\nu}\ V_{\nu}(E)\ , \qquad V_{\nu}(E) \ = \ \sup_{z}\ U^{\nu}(z)\ . \tag{2.2}$$

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [40]:

$$C(E) \;=\; \tau(E) \;:=\; \lim_{n\to\infty} V_n^{\frac{2}{n(n-1)}} \tag{2.3}$$

where V_n denotes the supremum of the product

$$V(z_1,\dots,z_n) \;=\; \prod_{k<l}^{l=1,\dots,n} |z_k - z_l| \tag{2.4}$$

taken over all collections of points $z_1,\dots,z_n$ in the set E. Following Fékete, see [15], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E.

Remark 2.1. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is continuous by Hölder and, in particular, for quasiconformal mappings on compact sets, see e.g. Theorem II.4.3 in [35].

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [12], **inner C_* and outer C^* capacities:**

$$C_*(E) \;:=\; \sup_{F\subseteq E} C(E), \qquad C^*(E) \;:=\; \inf_{E\subseteq O} C(O) \tag{2.5}$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$ and infimum is taken over all open sets $O \subset \mathbb{C}$. A set $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if $C^*(E) = C_*(E)$, and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$.

A function $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{z \in E : \varphi(z) \in O\} \tag{2.6}$$

are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [40]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [40]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [12].

Remark 2.2. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [12]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of a sigma-compactum (union of countable collection of compact sets) and a set of logarithmic capacity zero. It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. Theorem II(7.4) in [53]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [12].

Later on, we say that a Jordan curve Γ in $\mathbb{C}$ is **almost smooth** if Γ has a tangent quasi–everywhere. It is clear that this request is absolutely necessary to have a regular solution of the Hilbert boundary value problem in the sense of the angular limits in the Jordan domains. Recall that a straight line L in $\mathbb{C}$ is **tangent** to Γ at a point $z_0 \in \Gamma$ if

$$\limsup_{z \to z_0, z \in \Gamma} \frac{\operatorname{dist}(z, L)}{|z - z_0|} \; = \; 0 \; . \tag{2.7}$$

3. The Hilbert boundary value problem for analytic functions in the disk

We call $\lambda : \partial\mathbb{D} \to \mathbb{C}$ a **function of bounded variation**, write $\lambda \in \mathcal{BV}(\partial\mathbb{D})$, if

$$V_\lambda(\partial\mathbb{D}) \; : \; = \; \sup \sum_{j=1}^{j=k} |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| \; < \; \infty \tag{3.1}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial\mathbb{D}$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_\lambda(\partial\mathbb{D})$ is called the **variation of the function** λ.

Remark 3.1. It is clear by the triangle inequality that if we add new intermediate points in the collection ζ_j, $j = 1, \ldots, k$, then the sum in (3.1) does not decrease. Thus, the given supremum is attained as $\delta = \sup\limits_{j=1,\ldots k} |\zeta_{j+1} - \zeta_j| \to 0$. Note also that by the definition $V_\lambda(\partial\mathbb{D}) =$

$V_{\lambda\circ h}(\partial\mathbb{D})$, i.e., the **variation is invariant** under every homeomorphism $h : \partial\mathbb{D} \to \partial\mathbb{D}$ and, thus, the definition can be extended in a natural way to an arbitrary Jordan curve in $\mathbb{C}$.

Now, we call $\lambda : \partial\mathbb{D} \to \mathbb{C}$ a function of **countable bounded variation**, write $\lambda \in \mathcal{CBV}(\partial\mathbb{D})$, if there is a countable collection of mutually disjoint arcs γ_n of $\partial\mathbb{D}$, $n = 1, 2, \ldots$ on each of which the restriction of λ is of bounded variation V_n and the set $\partial\mathbb{D} \setminus \cup\gamma_n$ has logarithmic capacity zero. In particular, the latter holds true if $\partial\mathbb{D} \setminus \cup\gamma_n$ is countable. Choosing smaller γ_n, we may assume that $\sup\limits_n V_n < \infty$. It is clear, such functions can be singular enough, see e.g. [13]. The definition is also extended in the natural way to an arbitrary Jordan curve Γ in $\mathbb{C}$.

Recently we proved the following significant fact that is a base for further extensions, see Theorems 5.1 and 8.1 in the paper [22].

Theorem 3.1. *Let $\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ be in the class $\mathcal{CBV}(\partial\mathbb{D})$ and $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity. Then there is an analytic function $f : \mathbb{D} \to \mathbb{C}$ that has the angular limit*

$$\lim_{z\to\zeta} \mathrm{Re}[\overline{\lambda(\zeta)} f(z)] \;=\; \varphi(\zeta) \qquad\qquad q.e. \;\; on \;\partial\mathbb{D}. \tag{3.2}$$

Furthermore, the space of all such analytic functions has the infinite dimension.

4. The main lemma for the Beltrami equations

We start from the following general lemma formulated in terms of singular functional parameter whose choice, later on, makes possible to derive many effective criteria for existence of regular generalized solutions of the Hilbert boundary value problem for a wide circle of the Beltrami equations.

Given a point z_0 in $\mathbb{C}$, we apply here the more refined quantity than the dilatation quotient $K_\mu(z)$:

$$K^T_\mu(z, z_0) \;:=\; \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\,\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{4.1}$$

that is called the **tangent dilatation quotient** of the Beltrami equation (1.1) with respect to z_0, see, e.g., [50], cf. the corresponding terms and notations in [5,19,36] and [43]. The given term was first introduced in [50] and its geometric sense was described in [47], see also [37], Section 11.3. Note that

$$K_{\mu}^{-1}(z) \leqslant K_{\mu}^{T}(z,z_0) \leqslant K_{\mu}(z) \qquad \forall\, z\in D \quad \forall\, z_0\in\mathbb{C} \tag{4.2}$$

and the given estimates are precise. The quantity (4.1) takes into account not only the modulus of the complex coefficient μ but also its argument.

Lemma 4.1. *Let D be a generalized quasidisk, ∂D have a tangent q.e., $\lambda:\partial D\to\mathbb{C}$, $|\lambda(\zeta)|\equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi:\partial D\to\mathbb{R}$ be measurable with respect to the logarithmic capacity. Suppose that $\mu:D\to\mathbb{C}$, $|\mu(z)|<1$ a.e., $K_{\mu}\in L^1(D)$ and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu}^{T}(z,z_0)\cdot\psi_{z_0,\varepsilon}^2(|z-z_0|)\; dm(z) \;=\; o\left(I_{z_0}^2(\varepsilon)\right) \qquad \forall\, z_0\in\overline{D} \tag{4.3}$$

as $\varepsilon\to 0$ where $0<\varepsilon_0<\sup\limits_{z\in D}|z-z_0|$ and $\psi_{z_0,\varepsilon}(t):(0,\infty)\to[0,\infty]$, $\varepsilon\in(0,\varepsilon_0)$, is a two-parametric family of (Lebesgue) measurable functions such that

$$0<I_{z_0}(\varepsilon) \;:=\; \int\limits_{\varepsilon}^{\varepsilon_0}\psi_{z_0,\varepsilon}(t)\; dt \;<\; \infty \qquad \forall\,\varepsilon\in(0,\varepsilon_0)\;. \tag{4.4}$$

If Beltrami equation (1.1) *has not more than countable singularities on ∂D, then its Hilbert boundary value problem* (1.3) *has a space of regular generalized solutions of the infinite dimension.*

Here and later on, we set $K_{\mu}(z,z_0)=1$ for all z outside of the domain D.

Remark 4.1. In particular, in view of the right inequality in (4.2), the conclusion of Lemma 4.1 holds if $K_{\mu}(z,z_0)$ is replaced by $K_{\mu}(z)$ in (4.3).

Note also that, if $\psi_{z_0,\varepsilon}\equiv\psi_{z_0}$, then the conditions (4.3) and (4.4) imply that $I_{z_0}(\varepsilon)\to\infty$ as $\varepsilon\to 0$. In other words, for the regular solvability of the Hilbert boundary value problem it suffices the controlled divergence of the singular integral from the left in (4.3) in the sense of the principal value by Cauchy.

Finally, note that (4.3) holds, in particular, if

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu}(z,z_0)\cdot\psi_{z_0}^2(|z-z_0|)\; dm(z) \;<\; \infty \qquad \forall\, z_0\in\overline{D} \tag{4.5}$$

where $\psi_{z_0}(t) : (0,\infty) \to [0,\infty]$ is a locally integrable function with $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the solvability of the Hilbert boundary value problem it suffices for the singular integrals in (4.5) to be convergent for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable on $(0,\infty)$ but that has a non-integrable singularity at zero.

Proof. Let us extend the complex coefficient μ of the Beltrami equation by zero outside of the domain D. Then $K_\mu(z,z_0) \equiv 1$ outside of D.

By Lemma 3 in [49] there is a regular homeomorphic solution $f_\mu : \mathbb{C} \to \mathbb{C}$ of the extended Beltrami equation. Let g be a conformal mapping of the domain $D_* := f_\mu(D)$ onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see e.g. Theorem II.2.1 in [18]. Then the mapping $G := g \circ f_\mu|_D : D \to \mathbb{D}$ is a regular homeomorphic solution of the same Beltrami equation in the domain D.

Now, by the Caratheodory theorem, see e.g. Theorem II.3.4 in [18], g is extended to a homeomorphism $\tilde{g}$ of $\overline{D_*}$ onto $\overline{\mathbb{D}}$ and $\tilde{g}(\partial D_*) = \partial\mathbb{D}$. Thus, the mapping G is extended to a homeomorphism $\tilde{G}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$ and $\tilde{G}(\partial D) = \partial\mathbb{D}$. Furthermore, G is extended to a quasiconformal mapping in a neighborhood of each open quasiconformal arc of the generalized quasicircle ∂D, see e.g. Theorem II.8.2 in [35].

Next, both functions $G_* := \tilde{G}|_{\partial D}$ and G_*^{-1} are Hölder continuous on the corresponding arcs, see [10], Theorem 3.5, and also [35], Theorem II.4.3. Hence $\Lambda := \lambda \circ G_*^{-1} \in \mathcal{CBV}(\partial\mathbb{D})$ and $\Phi := \varphi \circ G_*^{-1}$ is measurable with respect to logarithmic capacity by Remarks 2.1. Moreover, by Theorem 3.1 there exist analytic functions $\mathcal{A} : \mathbb{D} \to \mathbb{C}$ that have the angular limit

$$\lim_{\omega \to \eta} \operatorname{Re}\, \{\overline{\Lambda(\eta)}\, \mathcal{A}(\omega)\} \;=\; \Phi(\eta) \qquad \text{q.e. on } \partial\mathbb{D}\,, \tag{4.6}$$

furthermore, the space of all such analytic functions has the infinite dimension.

Let us consider the function $f := \mathcal{A} \circ G$. Since $f_z = \mathcal{A}' \circ G(z)\, G_z$ and $f_{\bar{z}} = \mathcal{A}' \circ G(z)\, G_{\bar{z}}$ a.e. in D, we see that f satisfies the equation (1.1). On the other hand, the mapping f is continuous, open and discrete, and therefore f is a regular solution of (1.1). It remains to show that f satisfies also the boundary condition (1.3) q.e.

Indeed, it is known that the distortion of angles under a quasiconformal mapping is bounded, see e.g. [1, 2] and [55]. Hence the mapping $\tilde{G}$ and its inverse transform nontangential paths into nontangential paths and G_* and G_*^{-1} keep sets of logarithmic capacity zero along the arcs mentioned above. Consequently, $G : D \to \mathbb{D}$ and $G^{-1} : \mathbb{D} \to D$ also transform nontangential paths into nontangential paths q.e. on ∂D and $\partial\mathbb{D}$, respectively. Thus, (4.6) implies the existence of the angular limit (1.3) q.e. on ∂D. $\square$

Remark 4.2. As it follows from the given proof, the regular generalized solutions f of the Hilbert boundary value problem for the Beltrami equation in Lemma 4.1 has the representation $f = \mathcal{A} \circ g \circ f_\mu|_D$. Here $f_\mu : \mathbb{C} \to \mathbb{C}$ is a regular homeomorphic solution of the Beltrami equation with μ extended by zero outside of D, g is a conformal mapping of the domain $D_* := f_\mu(D)$ onto $\mathbb{D}$ and $\mathcal{A}$ are analytic solutions of the Hilbert boundary value problem with the coefficient $\Lambda = \lambda \circ G_*^{-1}$ and the boundary data $\Phi = \varphi \circ G_*^{-1}$ where $G_* : \partial D \to \partial\mathbb{D}$ is the boundary correspondence under the homeomorphism $G := g \circ f_\mu$.

5. On BMO and FMO functions

Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B|\, dm(z) < \infty\,, \tag{5.1}$$

where the supremum is taken over all discs B in D, $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$ and

$$u_B = \frac{1}{|B|} \int\limits_B u(z)\, dm(z)\,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [29] and soon became an important concept in harmonic analysis, partial differential equations and related areas; see, e.g., [24] and [44].

Following [26], we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \;\; ⨍_{B(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)|\, dm(z) < \infty\,, \tag{5.2}$$

where

$$\widetilde{\varphi}_\varepsilon(z_0) = ⨍_{B(z_0,\varepsilon)} \varphi(z)\, dm(z) \tag{5.3}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Note that the condition (5.2) includes the assumption that φ is integrable in some neighborhood of the point z_0. The following statement is obvious by the triangle inequality.

Proposition 5.1. *If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \;\; -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty\,, \tag{5.4}$$

then φ is of finite mean oscillation at z_0.

In particular choosing here $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we obtain the following.

Corollary 5.1. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \;\; -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} |\varphi(z)| \, dm(z) < \infty\,, \tag{5.5}$$

then φ has finite mean oscillation at z_0.

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon\to 0} \;\; -\!\!\!\!\!\!\int_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, dm(z) = 0\,. \tag{5.6}$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus we have by Proposition 5.1 the following corollary.

Corollary 5.2. *Every locally integrable function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at almost every point in D.*

The following lemma plays a key role in many investigations, see e.g. Lemma 5.3 in the monograph [21]. Here we use the standard notation of the ring:

$$A(z_0, \varepsilon, \varepsilon_0) \;:=\; \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}\,. \tag{5.7}$$

Lemma 5.1. *Let $\varphi : D \to \mathbb{R}$ be a nonnegative function with finite mean oscillation at $z_0 \in D$. Then*

$$\int\limits_{A(z_0,\varepsilon,\varepsilon_0)} \frac{\varphi(z)\, dm(z)}{\left(|z - z_0| \log \frac{1}{|z-z_0|}\right)^2} \;=\; O\left(\log\log \frac{1}{\varepsilon}\right)\,. \tag{5.8}$$

Versions of this lemma were first established for BMO functions in [52], Lemma 3.2, and then for FMO functions in [26], Corollary 2.3.

Remark 5.1. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ, see, e.g., [44], p. 5, and hence also to FMO. However, $\widetilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (5.5) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [29] or [44]. However, FMO is not a subclass of L^p_{loc} for any $p > 1$ but only of L^1_{loc}, see the corresponding example in the monograph [21]. Thus, the class FMO is much more wide than $\mathrm{BMO}_{\mathrm{loc}}$.

6. Criteria of regular solvability for Beltrami equations

In this section we demonstrate that Lemma 4.1 leads to a series of nice integral criteria of the type of Lavrentiev-Lehto-Zorich, Orlicz, Calderon–Zygmund, John-Nierenberg and many others for solvability of the Hilbert boundary value problems to the Beltrami equations with singularities.

For instance, choosing in Lemma 4.1 $\psi(t) = 1/\left(t \log\left(1/t\right)\right)$, see also Lemma 5.1, we obtain the following result.

Theorem 6.1. *Let D be a generalized quasidisk, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose $\mu : D \to \mathbb{C}$, $|\mu(z)| < 1$ a.e., $K_\mu \in L^1(D)$ and $K^T_\mu(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in a neighborhood U_{z_0} of every point $z_0 \in \overline{D}$ with a function $Q_{z_0} : U_{z_0} \to [0, \infty]$ in the class $\mathrm{FMO}(z_0)$.

If Beltrami equation (1.1) has not more than countable singularities on ∂D, then its Hilbert boundary value problem (1.3) has a space of regular generalized solutions of the infinite dimension.

By Proposition 5.1 and Corollaries 5.1 and 5.2 we have the next consequences of Theorem 6.1.

Corollary 6.1. *In particular, the conclusion of Theorem 6.1 holds if every point $z_0 \in \overline{D}$ is the Lebesgue point of a function $Q_{z_0} : \mathbb{C} \to [0, \infty]$ which is integrable in a neighborhood U_{z_0} of z_0 and $K^T_\mu(z, z_0) \leqslant Q_{z_0}(z)$ a.e. in U_{z_0}.*

Corollary 6.2. *In particular, the conclusion of Theorem 6.1 holds if*

$$\overline{\lim_{\varepsilon \to 0}} \;\; \fint_{B(z_0, \varepsilon)} K^T_\mu(z, z_0)\, dm(z) < \infty \qquad \forall\ z_0 \in \overline{D}\,. \tag{6.1}$$

Similarly, choosing in Lemma 4.1 the function $\psi(t) = 1/t$, we come to the following statement.

Theorem 6.2. *Let D be a generalized quasidisk, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\mu : D \to \mathbb{C}$, $|\mu(z)| < 1$ *a.e.*, $K_\mu \in L^1(D)$ *and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall\ z_0 \in \overline{D} \qquad (6.2)$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \delta(z_0) \in (0, d(z_0))$ *where* $d(z_0) = \sup_{z\in D}|z - z_0|$.

If Beltrami equation (1.1) *has not more than countable singularities on* ∂D, *then its Hilbert boundary value problem* (1.3) *has a space of regular generalized solutions of the infinite dimension.*

Next, choosing in Lemma 4.1 $\psi(t) = 1/\left(t\, k_{z_0}(t)\right)$ where $k_{z_0}(t)$ is the average of $K^T_\mu(z,z_0)$ over the circle $|z - z_0| = t$, we obtain the following conclusion.

Theorem 6.3. *Let* D *be a generalized quasidisk,* ∂D *have a tangent q.e.,* $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, *be in* $\mathcal{CBV}(\partial D)$ *and* $\varphi : \partial D \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity.*

Suppose that $\mu : D \to \mathbb{C}$, $|\mu(z)| < 1$ *a.e.*, $K_\mu \in L^1(D)$ *and*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{||K^T_\mu||_1(z_0,r)} = \infty \qquad \forall\ z_0 \in \overline{D} \qquad (6.3)$$

for some $\delta(z_0) \in (0, d(z_0))$ *where* $d(z_0) = \sup\limits_{z\in D}|z - z_0|$ *and*

$$||K^T_\mu||_1(z_0,r) \ := \int\limits_{|z-z_0|=r)} K^T_\mu(z,z_0)\,|dz|\,. \qquad (6.4)$$

If Beltrami equation (1.1) *has not more than countable singularities on* ∂D, *then its Hilbert boundary value problem* (1.3) *has a space of regular generalized solutions of the infinite dimension.*

Corollary 6.3. *In particular, the conclusion of Theorem 6.3 holds if*

$$k_{z_0}(\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad as\ \varepsilon \to 0 \qquad \forall\ z_0 \in \overline{D} \qquad (6.5)$$

where $k_{z_0}(\varepsilon)$ *is the average of the function* $K^T_\mu(z,z_0)$ *over the circle* $|z - z_0| = \varepsilon$.

Remark 6.1. For instance, the conclusion of Corollary 6.3 holds if

$$K^T_\mu(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in \overline{D}\,. \qquad (6.6)$$

Finally, combining Theorems 6.3 above and Theorem 3.2 in [48], see also Theorem 2.4 in [21], we obtain the following.

Theorem 6.4. *Let D be a generalized quasidisk, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\mu : D \to \mathbb{C}$, $|\mu(z)| < 1$ a.e., and

$$\int\limits_D \Phi\left(K_\mu(z)\right)\ dm(z)\ <\ \infty \tag{6.7}$$

for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ such that

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \qquad \text{for some } \delta > \Phi(0)\ . \tag{6.8}$$

If Beltrami equation (1.1) *has not more than countable singularities on ∂D, then its Hilbert boundary value problem* (1.3) *has a space of regular generalized solutions of the infinite dimension.*

Remark 6.2. By the Stoilow theorem on factorization, see e.g. [54], any regular generalized solution f of the Hilbert boundary value problem (1.3) for the Beltrami equation (1.1) can be represented in the form of the composition $f = \mathcal{A} \circ F$ where $\mathcal{A}$ is an analytic function and F is a regular homeomorphic solution of (1.1). Thus, by Theorem 5.1 in [48], the condition (6.8) is not only sufficient but also necessary for the existence of regular generalized solutions of the Hilbert boundary value problems (1.3) to any Beltrami equation (1.1) with the integral restriction (6.7).

Note also that by Theorem 2.1 in [49] the condition (6.8) is equivalent to the following condition:

$$\int\limits_{\Delta}^{\infty} \log \Phi(t)\, \frac{dt}{t^2} = \infty \qquad \text{for some } \Delta > 0\ . \tag{6.9}$$

Corollary 6.4. *In particular, the conclusion of Theorem 6.4 holds if*

$$\int\limits_D e^{\alpha K_\mu(z)}\ dm(z)\ <\ \infty \qquad \text{for some } \alpha > 0\ . \tag{6.10}$$

7. On Dirichlet, Neumann and Poincare problems

We reduce the given boundary value problems to suitable Hilbert boundary value problems studied above and start with the Laplace equation. For instance, choosing $\mu \equiv 0$ and $\lambda \equiv 1$ in Theorem 6.3, we immediately obtain the following consequence on solutions of the Dirichlet boundary value problem.

Corollary 7.1. *Let D be a generalized quasidisk and ∂D have a tangent q.e. Suppose $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then there exists a harmonic function $u : D \to \mathbb{C}$ that has the angular limit*

$$\lim_{z\to\zeta} u(z) \ = \ \varphi(\zeta) \qquad\qquad q.e.\ on\ \partial D. \tag{7.1}$$

The space of such harmonic functions has the infinite dimension.

We proceed to the study of nonclassical solutions of the Neumann boundary value problem. For this goal, we will study the more general **problem on directional derivatives** that is due to Poincare.

First of all, let us recall the classical setting of the problem on directional derivatives for the Laplace equation in the unit disk $\mathbb{D}$: To find a twice continuously differentiable function $u : \mathbb{D} \to \mathbb{R}$ that admits a continuous extension to the boundary $\partial\mathbb{D}$ together with its first partial derivatives, satisfies the Laplace equation

$$\Delta u \ := \ \frac{\partial^2 u}{\partial x^2} \ + \ \frac{\partial^2 u}{\partial y^2} \ = \ 0 \qquad \forall\ z \in \mathbb{D} \tag{7.2}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu} \ = \ \varphi(\zeta) \qquad \forall\ \zeta \in \partial\mathbb{D}. \tag{7.3}$$

Here $\varphi : \partial\mathbb{D} \to \mathbb{R}$ stands for a prescribed continuous function and $\frac{\partial u}{\partial \nu}$ denotes the derivative of u at the point ζ in the direction $\nu = \nu(\zeta)$, $|\nu(\zeta)| = 1$, i.e.,

$$\frac{\partial u}{\partial \nu} \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\,\nu) - u(\zeta)}{t} \ . \tag{7.4}$$

The Neumann boundary value problem for the Laplace equation is a special case of the above problem with the following boundary condition

$$\frac{\partial u}{\partial n} \ = \ \varphi(\zeta) \qquad \forall\ \zeta \in \partial\mathbb{D}\ , \tag{7.5}$$

where n denotes the unit interior normal to $\partial\mathbb{D}$ at the point ζ.

It is well known, that the Neumann problem, in general, has no classical solution. The necessary condition for the solvability is that the integral of the function φ over $\partial\mathbb{D}$ is equal zero, see e.g. [38]. Recently, it was established the existence of nonclassical solutions of the Neumann problem for the Laplace equation in rectifiable Jordan domains for arbitrary measurable data with respect to the natural parameter, see [46]. Then the results have been extended to linear divergence equations in Lipschitz domains with arbitrary measurable data with respect to the logarithmic capacity in [58] and then to domains with the quasihyperbolic boundary condition. By the results of the last section, we obtain, in particular, the following simple consequences.

Theorem 7.1. *Let D be a generalized quasidisk and ∂D have a tangent q.e. Suppose that $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, is in the class $\mathcal{CBV}$ and $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then there exist harmonic functions $u : D \to \mathbb{R}$ that have the angular limits*

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu} = \varphi(\zeta) \qquad \text{q.e. on } \partial D. \tag{7.6}$$

Furthermore, the space of such harmonic functions has the infinite dimension.

Proof. For instance, by Theorem 6.3 there exist the space of analytic functions $f : D \to \mathbb{C}$ of the infinite dimension that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\,[\nu(\zeta)\, f(z)] \ = \ \varphi(\zeta) \qquad \text{q.e. on } \partial D\ . \tag{7.7}$$

Note that an indefinite integral F of f in D is also an analytic function and, correspondingly, the harmonic functions $u = \mathrm{Re}\, F$ and $v = \mathrm{Im}\, F$ satisfy the Cauchy-Riemann system $v_x = -u_y$ and $v_y = u_x$. Hence

$$f \ = \ F' \ = \ F_x \ = \ u_x \ + \ i\, v_x \ = \ u_x \ - \ i\, u_y \ = \ \overline{\nabla u}$$

where $\nabla u = u_x + i\, u_y$ is the gradient of the function u in the complex form. Thus, (7.6) follows from (7.7), i.e. u is the desired harmonic function, because its directional derivative

$$\frac{\partial u}{\partial \nu} \ = \ \mathrm{Re}\, \overline{\nu}\, \nabla u \ = \ \mathrm{Re}\, \nu\, \overline{\nabla u} \ = \ \langle \nu, \nabla u \rangle$$

is the scalar product of ν and the gradient ∇u. □

Remark 7.1. We are able to say more in the case $\mathrm{Re}[n\overline{\nu}] > 0$ where $n = n(\zeta)$ is the unit interior normal at the point $\zeta \in \partial D$. In view of (7.6), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $u(\zeta)$ of $u(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector $\nu(\zeta)$. Indeed, along this line, for z and z_0 that are close enough to ζ,

$$u(z) \;=\; u(z_0) \;-\; \int\limits_0^1 \frac{\partial u}{\partial \nu}\,(z_0 + \tau(z - z_0))\;d\tau\;.$$

Thus, at each point with the condition (7.6), there is the directional derivative

$$\frac{\partial u}{\partial \nu}\,(\zeta) \;:=\; \lim_{t\to 0}\;\frac{u(\zeta + t\,\nu) - u(\zeta)}{t} \;=\; \varphi(\zeta)\;.$$

In particular, $\mathrm{Re}[n\overline{\nu}] = 1$ in the case of the Neumann problem and, thus, we arrive, by Theorem 7.1 and Remark 7.1, at the following result.

Corollary 7.2. *Let D be a generalized quasidisk and the unit interior normal $n(\zeta)$ to the boundary ∂D be in the class $\mathcal{CBV}$. Suppose that $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then one can find harmonic functions $u : D \to \mathbb{C}$ such that q.e. on ∂D there exist:*

1) the finite limit along the normal $n(\zeta)$

$$u(\zeta) \;:=\; \lim_{z\to\zeta}\; u(z)\;,$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \;:=\; \lim_{t\to 0}\;\frac{u(\zeta + t\,n) - u(\zeta)}{t} \;=\; \varphi(\zeta)\;,$$

3) the angular limit

$$\lim_{z\to\zeta}\;\frac{\partial u}{\partial n}\,(z) \;=\; \frac{\partial u}{\partial n}\,(\zeta)\;.$$

Furthermore, the space of such harmonic functions has the infinite dimension.

Now, recall that, see e.g. Theorem 16.1.6 in [6], if $f = u + i\,v$ is a regular generalized solution of the Beltrami equation (1.1) in a domain

D in $\mathbb{C}$, then the function u is a continuous generalized solution of the divergence type equation

$$\operatorname{div} A(z)\nabla u = 0 \ , \tag{7.8}$$

called **A-harmonic function**, see [24], i.e., $u \in C \cap W^{1,1}_{\rm loc}(D)$ and

$$\int\limits_D \langle A(z)\nabla u, \nabla\varphi\rangle = 0 \qquad \forall\, \varphi \in C_0^\infty(D) \ ,$$

where $A(z)$ is the matrix function:

$$A = \left(\begin{array}{cc} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} \\ \frac{-2{\rm Im}\,\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{array} \right) . \tag{7.9}$$

As we see, the matrix function $A(z)$ in (7.9) is symmetric and its entries $a_{ij} = a_{ij}(z)$ are dominated by the quantity

$$K_\mu(z) \ = \ \frac{1 \ + \ |\mu(z)|}{1 \ - \ |\mu(z)|} \ ,$$

and, thus, they are bounded if Beltrami's equation (1.1) is not degenerate.

Vice verse, uniformly elliptic equations (7.8) with symmetric $A(z)$ and $\det A(z) \equiv 1$ just correspond to nondegenerate Beltrami equations (1.1) with the coefficient

$$\mu \ = \ \frac{1}{\det\,(I+A)}\ (a_{22} - a_{11} \ - \ 2ia_{21}) \ = \ \frac{a_{22} - a_{11} \ - \ 2ia_{21}}{1 \ + \ {\rm Tr}\, A \ + \ \det A} \ . \tag{7.10}$$

Recall that the equation (7.8) is said to be **uniformly elliptic**, if $a_{ij} \in L^\infty$ and $\langle A(z)\eta, \eta\rangle \geq \varepsilon|\eta|^2$ for some $\varepsilon > 0$ and for all $\eta \in \mathbb{R}^2$.

Given a domain D in $\mathbb{C}$, denote by $M^{2\times 2}(D)$ the class of all 2×2 symmetric matrix function $A(z) = \{a_{jk}(z)\}$ with measurable real-valued entries and $\det A(z) = 1$, satisfying the **ellipticity condition** (positive definiteness)

$$\langle\, A(z)\,\xi,\, \xi\,\rangle \ > \ 0 \qquad \text{a.e. in } D \quad \forall\, \xi \in \mathbb{R}^2 \setminus \{0\} \ . \tag{7.11}$$

We call a point $z_0 \in \overline{D}$ **singularity of the equation** (7.8) or $A \in M^{2\times 2}(D)$ if (7.8) is not uniformly elliptic in all neighborhoods of z_0.

Given $M^{2\times 2}(D)$, we say that a continuous function $u : D \to \mathbb{R}$ is a **regular (A)–harmonic function** if u has the first generalized derivatives by Sobolev, satisfies (7.8) a.e. in D and can be represented as a composition of a harmonic function h and a regular homeomorphic solution of the Beltrami equation (1.1) with the coefficient μ given by (7.10).

Corollary 7.3. *Let D be a generalized quasidisk and ∂D have a tangent q.e. and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $A \in M^{2\times 2}(D)$ has not more than countable singularities on ∂D and the corresponding μ in (7.10) satisfies one of the integral conditions of theorems from Section 6.

Then there exists a space of regular $A-$harmonic functions $u : D \to \mathbb{R}$ of the infinite dimension satisfying the Dirichlet boundary condition (7.1).

Theorem 7.2. *Let D be a generalized quasidisk and ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be in the class $\mathcal{CBV}$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $A \in M^{2\times 2}(D)$ is in the class C^{α}, $\alpha \in (0,1)$ about ∂D and the corresponding μ in (7.10) satisfies one of the integral conditions of theorems from Section 6.

Then there exists a space of regular $A-$harmonic functions $u : D \to \mathbb{R}$ of the infinite dimension that have the angular limits

$$\lim_{z\to\zeta} \frac{\partial u}{\partial \nu}(z) \ = \ \varphi(\zeta) \quad q.e. \ on\ \partial D \ . \tag{7.12}$$

Proof. By the definition, desired regular (A)–harmonic functions u are real parts of regular solutions f of the Beltrami equation (1.1) with μ given by the formula (7.10) which belongs to the class C^{α}, $\alpha \in (0,1)$ in an open neighborhood U of ∂D inside of D. With no loss of generality, we may assume that $k = \max |\mu(z)| < 1$ in U. By Lemma 1 in [23] μ is extended to a Hölder continuous function $\mu_* : \mathbb{C} \setminus (D \setminus U) \to \mathbb{C}$ of the class C^{α}. Then, for every $k_* \in (k,1)$, there is an open neighborhood U_* of ∂D in $\mathbb{C} \setminus (D \setminus U)$ where $|\mu_*(z)| \le k_*$. Let $D_* = D \cup U^*$ where U^* is a connected component of U_* containing ∂D. Note that by the construction D_* is a domain in $\mathbb{C}$.

By the existence theorems in [49], there is a regular homeomorphic solution $h : D_* \to \mathbb{C}$ of the Beltrami equation (1.1) with the complex coefficient $\mu^* := \mu_*|_{D_*}$ in D_*. Note that the mapping h has the Hölder continuous first partial derivatives in U^* with the same order of the Hölder continuity as μ, see e.g. [27] and also [28]. Moreover, its Jacobian

$$J_h(z) \neq 0 \qquad\qquad \forall\ z \in U^* \ , \tag{7.13}$$

see e.g. Theorem V.7.1 in [35]. Thus, the directional derivative

$$h_{\omega}(z) = \frac{\partial h}{\partial \omega}(z) \ := \ \lim_{t\to 0} \ \frac{h(z \ + \ t\,\omega) \ - \ h(z)}{t} \ \neq \ 0 \quad \ \forall\ z \in U^*\ \forall\ \omega \in \partial\mathbb{D}$$

and it is continuous by the collection of the variables $\omega \in \partial\mathbb{D}$ and $z \in U^*$. Thus, the functions

$$\nu_*(\zeta) \ := \ \frac{|h_{\nu(\zeta)}(\zeta)|}{h_{\nu(\zeta)}(\zeta)} \quad \textit{and} \quad \varphi_*(\zeta) \ := \ \frac{\varphi(\zeta)}{|h_{\nu(\zeta)}(\zeta)|}$$

are measurable with respect to the logarithmic capacity, see e.g. convergence arguments in [31], Section 17.1.

The restriction of h onto the domain U^* is a quasiconformal mapping. Hence by Remark 2.1 the mappings h and h^{-1} transform sets of logarithmic capacity zero on ∂D into sets of logarithmic capacity zero on ∂D^*, where $D^* := h(D)$, and vice versa.

Further, the functions $\mathcal{N} := \nu_* \circ h^{-1}|_{\partial D^*}$ and $\Phi := (\varphi_*/h_\nu) \circ h^{-1}|_{\partial D^*}$ are measurable with respect to the logarithmic capacity. Indeed, a measurable set with respect to the logarithmic capacity is transformed under the mappings h and h^{-1} into measurable sets with respect to the logarithmic capacity. Really, such a set can be represented as the union of a sigma-compactum and a set of logarithmic capacity zero. On the other hand, the compacta are transformed under continuous mappings into compacta and the compacta are measurable with respect to the logarithmic capacity.

Recall that the distortion of angles under quasiconformal mappings h and h^{-1} is bounded, see e.g. [1], [2] and [55]. Thus, nontangential paths to ∂D are transformed into nontangential paths to ∂D^* for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity and inversely.

By Theorem 7.1, there is a space of harmonic functions $U : D^* \to \mathbb{R}$ of the infinite dimension that have the angular limits

$$\lim_{w \to \xi} \frac{\partial U}{\partial \mathcal{N}}(w) \ = \ \Phi(\xi) \qquad \text{q.e. on } \partial D^* \ . \tag{7.14}$$

Moreover, one can find a harmonic function V in the simply connected domain D^* such that $F = U + iV$ is an analytic function and, thus, $u := \mathrm{Re}\, f = U \circ h$, where $f := F \circ h$, is a desired regular A-harmonic function in Theorem 7.2 because f is a regular generalized solution of the corresponding Beltrami equation (1.1) and also

$$u_\nu = \langle\ \nabla U \circ h\ ,\ h_\nu\ \rangle = \langle\ \nu_*\, \nabla U \circ h\ ,\ \nu_*\ h_\nu\ \rangle =$$

$$= \langle\ \frac{\partial U}{\partial \mathcal{N}}\ \circ h\ ,\ \nu_*\ h_\nu\ \rangle = \frac{\partial U}{\partial \mathcal{N}}\ \circ h \ \ \mathrm{Re}\,(\nu_* h_\nu).$$

$\Box$

The following statement concerning to the Neumann problem for A-harmonic functions is a partial case of Theorem 7.2.

Corollary 7.4. *Let D be a generalized quasidisk and ∂D have a tangent q.e., the interior unit normal $n = n(\zeta)$ to ∂D be in the class $\mathcal{CBV}$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $A \in M^{2\times 2}(D)$ is in the class C^{α}, $\alpha \in (0,1)$ about ∂D and the corresponding μ in (7.10) satisfies one of the integral conditions of theorems from Section 6.

Then there exists a space of regular $A-$harmonic functions $u : D \to \mathbb{R}$ of the infinite dimension such that q.e. on ∂D there exist:

1) the finite limit along the normal $n(\zeta)$

$$u(\zeta) \ := \ \lim_{z\to\zeta} \ u(z)$$

2) the normal derivative

$$\frac{\partial u}{\partial n}\,(\zeta) \ := \ \lim_{t\to 0} \ \frac{u(\zeta + t\,n) \ - \ u(\zeta)}{t} \ = \ \varphi(\zeta)$$

3) the angular limit

$$\lim_{z\to\zeta} \ \frac{\partial u}{\partial n}\,(z) \ = \ \frac{\partial u}{\partial n}\,(\zeta) \ .$$

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