Український математичний вісник
Том 18 (**2021**), № 1, 12 – 36
https://doi.org/10.37069/1810-3200-2021-18-1-2

Logarithmic potential and generalized analytic functions

V. Gutlyanskiĭ, O. Nesmelova,
V. Ryazanov, A. Yefimushkin

Abstract. The study of the Dirichlet problem in the unit disk $\mathbb{D}$ with arbitrary measurable data for harmonic functions is due to the famous dissertation of Luzin [31]. Later on, the known monograph of Vekua [48] has been devoted to boundary value problems (only with Hölder continuous data) for the generalized analytic functions, i.e., continuous complex valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form $\partial_{\bar{z}} h + ah + b\overline{h} = c$, where it was assumed that the complex valued functions a, b and c belong to the class L^p with some $p > 2$ in smooth enough domains D in $\mathbb{C}$.

The present paper is a natural continuation of our previous articles on the Riemann, Hilbert, Dirichlet, Poincare and, in particular, Neumann boundary value problems for quasiconformal, analytic, harmonic and the so-called $A-$harmonic functions with boundary data that are measurable with respect to logarithmic capacity. Here we extend the corresponding results to the generalized analytic functions $h : D \to \mathbb{C}$ with the sources $g : \partial_{\bar{z}} h = g \in L^p$, $p > 2$, and to generalized harmonic functions U with sources $G : \triangle U = G \in L^p$, $p > 2$.

This paper contains various theorems on the existence of nonclassical solutions of the Riemann and Hilbert boundary value problems with arbitrary measurable (with respect to logarithmic capacity) data for generalized analytic functions with sources. Our approach is based on the geometric (theoretic-functional) interpretation of boundary values in comparison with the classical operator approach in PDE. On this basis, it is established the corresponding existence theorems for the Poincare problem on directional derivatives and, in particular, for the Neumann problem to the Poisson equations $\triangle U = G$ with arbitrary boundary data that are measurable with respect to logarithmic capacity. These results can be also applied to semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

2010 MSC. Primary 30C65, 31A05, 31A20, 31A25, 31B25, 35J61; Secondary 30E25, 31C05, 34M50, 35F45, 35Q15.

Received 30.12.2020

Key words and phrases. Logarithmic potential, generalized analytic and harmonic functions; Poisson equations; Riemann, Hilbert, Dirichlet, Neumann and Poincare boundary value problems.

1. Introduction

The well–known monograph of Vekua [48] has been devoted to the theory of the **generalized analytic functions**, i.e., continuous complex valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form

$$\partial_{\bar{z}} h \;+\; ah \;+\; b\overline{h} \;=\; c\;, \qquad \partial_{\bar{z}} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x} \;+\; i\cdot\frac{\partial}{\partial y}\right)\;, \tag{1.1}$$

where it was assumed that the complex valued functions a, b and c belong to the class L^p with some $p > 2$ in the corresponding domains $D \subseteq \mathbb{C}$.

The present paper is a natural continuation of the articles [9, 17–22, 40–44, 49] and [50] devoted to the Riemann, Hilbert, Dirichlet, Poincare and, in particular, Neumann boundary value problems for quasiconformal, analytic, harmonic and the so–called $A-$harmonic functions with arbitrary boundary data that are measurable with respect to logarithmic capacity, see relevant definitions with history notes and necessary comments on the previous results below. Here we extend the corresponding results to **generalized** analytic and harmonic functions.

The first part of the paper is devoted to the proof of existence of nonclassical solutions of Riemann, Hilbert and Dirichlet boundary-value problems with arbitrary measurable boundary data with respect to logarithmic capacity for the equations

$$\partial_{\bar{z}} h(z) \;=\; g(z) \tag{1.2}$$

with the real valued functions g in the class L^p, $p > 2$. We will call continuous solutions h of the equations (1.2) with the generalized first partial derivatives by Sobolev **generalized analytic functions with sources g**.

The second part of the paper contains the proof of existence of nonclassical solutions to the Poincare problem on the directional derivatives and, in particular, to the Neumann problem with arbitrary measurable boundary data with respect to logarithmic capacity for the Poisson equations

$$\triangle U(z) \;=\; G(z) \tag{1.3}$$

with real valued functions G of a class $L^p(D)$, $p > 2$, in the corresponding domains $D \subset \mathbb{C}$. For short, we will call continuous solutions to (1.3) of the class $W^{2,p}_{\rm loc}(D)$ **generalized harmonic functions with the sources G**. Note that by the Sobolev embedding theorem, see Theorem I.10.2 in [46], such functions belong to the class C^1.

The research of boundary value problems with arbitrary measurable data is due to the famous dissertation of Luzin, see its original text [31], and its reprint [32] with comments of his pupils Bari and Men'shov. Namely, he has established that, for each measurable a.e. finite $2\pi-$periodic function $\varphi(\vartheta):\mathbb{R}\to\mathbb{R}$, there is a harmonic function U in the unit disk $\mathbb{D}$ such that $U(z)\to\varphi(\vartheta)$ for a.e. ϑ as $z\to\zeta:=e^{i\vartheta}$ along all nontangential paths to $\partial\mathbb{D}$. The latter was based on his other deep result on the antiderivatives stated that, for any measurable function $\psi:[0,1]\to\mathbb{R}$, there is a continuous function $\Psi:[0,1]\to\mathbb{R}$ with $\Psi'=\psi$ a.e., see e.g. his papers [30] and [33].

Later on, the Luzin theorem on harmonic functions was strengthened in the paper [40], Corollary 5.1, see also [42], by the statement that, for each (Lebesgue) measurable function $\varphi:\partial\mathbb{D}\to\mathbb{R}$, the space of all harmonic functions $u:\mathbb{D}\to\mathbb{R}$ with the angular limits $\varphi(\zeta)$ for a.e. $\zeta\in\partial\mathbb{D}$ has the infinite dimension. Recall, it is well–known the uniqueness theorem to the Dirichlet problem in terms of the angular limits e.g. for bounded harmonic functions u, see Corollary IX.1.1 and Theorem IX.2.3 in [38]. However, in general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under a.e. zero boundary data, see e.g. Theorem 2.1 in [42].

The Luzin theorem was key to establish the corresponding result on the Hilbert boundary value problem in [40], Theorems 2.1 and 5.2: for arbitrary measurable functions $\lambda:\partial\mathbb{D}\to\mathbb{C}$, $|\lambda(\zeta)|\equiv 1$, and $\varphi:\partial\mathbb{D}\to\mathbb{R}$, the space of all analytic functions $f:\mathbb{D}\to\mathbb{C}$ with the angular limits

$$\lim_{z\to\zeta}\ {\rm Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\}\ =\ \varphi(\zeta)\qquad\text{for a.e.}\quad \zeta\in\partial\mathbb{D}\tag{1.4}$$

has the infinite dimension. Then this theorem was extended to arbitrary Jordan domains with rectifiable boundaries in terms of the natural parameter, see Theorem 3.1 in [40].

In turn, these results have been applied in the paper [43] to the study of the Poincare problem on directional derivatives and, in particular, of the Neumann problem for harmonic functions with arbitrary boundary data that are measurable with respect to natural parameter in arbitrary Jordan domains with rectifiable boundaries. Similarly, the results on the Hilbert and Riemann problems for analytic functions along the so–called **Bagemihl–Seidel systems** of Jordan arcs terminating at the boundary in [41] have been applied to the Poincare and Neumann problems for harmonic functions.

Moreover, a series of the corresponding results have been formulated and proved in terms of logarithmic capacity, see its definition and properties in the next section. The base is the following analog of the Luzin theorem in [9], see also [50], where the abbreviation **q.e. means quasi–everywhere** with respect to logarithmic capacity.

Theorem A. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e.*

Furthermore, the function Φ can be chosen in such a way that $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \le \varepsilon$ for any prescribed $\varepsilon > 0$ and all $x \in [a, b]$.

On the basis of Theorem A, it was proved there the following analog of the second Luzin theorem:

Theorem B. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, measurable with respect to logarithmic capacity and finite q.e. Then the space of all harmonic functions u in $\mathbb{D}$ with the angular limits $u(z) \to \varphi(\vartheta)$ as $z \to e^{i\vartheta}$ q.e. on $\mathbb{R}$ has the infinite dimension.*

In turn, on the basis of Theorem B, it was obtain the result on the Hilbert boundary value problem:

Theorem C. *Let $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be of bounded variation and $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there is a space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \ \mathrm{Re}\,\{\overline{\lambda(\zeta)}\cdot f(z)\} \ = \ \varphi(\zeta) \qquad q.e.\ on\ \partial\mathbb{D}\,. \tag{1.5}$$

Then this result was extended to domains with the so–called quasiconformal boundaries and, in particular, to arbitrary smooth (C^1) domains, see [9] and [50], that was applied to the Poincare and Neumann problems for harmonic and $A-$harmonic functions, see [49]. Moreover, it was proved in [21] the next result:

Theorem D. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be of countable bounded variation and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity. Then there is a space of analytic functions $f : D \to \mathbb{C}$ of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \ \mathrm{Re}\{\overline{\lambda(\zeta)}f(z)\} \ = \ \varphi(\zeta) \qquad q.e.\ on\ \partial D. \tag{1.6}$$

See the further sections for definitions. This theorem on the Hilbert boundary value problem for analytic functions implied the corresponding theorems on the Poincare and Neumann problems for harmonic functions in [21]. Finally, notice a wide circle of the corresponding results on boundary value problems in terms of the Bagemihl–Seidel systems in [22].

2. On the logarithmic potential and capacity

Given a bounded Borel set E in the plane $\mathbb{C}$, a **mass distribution** on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_{E} \log\left|\frac{1}{z-\zeta}\right| \, d\nu(\zeta) \tag{2.1}$$

is called a **logarithmic potential** of the mass distribution ν at a point $z \in \mathbb{C}$. A **logarithmic capacity** $C(E)$ of the bounded Borel set E is the quantity

$$C(E) = e^{-V} \ , \qquad V \ = \ \inf_{\nu} \ V_{\nu}(E) \ , \qquad V_{\nu}(E) \ = \ \sup_{z} \ U^{\nu}(z) \ . \tag{2.2}$$

It is also well-known the following geometric characterization of the logarithmic capacity, see e.g. the point 110 in [35]:

$$C(E) \ = \ \tau(E) \ := \ \lim_{n\to\infty} \ V_n^{\frac{2}{n(n-1)}} \tag{2.3}$$

where V_n denotes the supremum of the product

$$V(z_1, \ldots, z_n) \ = \ \prod_{k<l}^{l=1,\ldots,n} \ |z_k - z_l| \tag{2.4}$$

taken over all collections of points $z_1, \ldots, z_n$ in the set E. Following Fékete, see [11], the quantity $\tau(E)$ is called the **transfinite diameter** of the set E.

Remark 1. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is Hölder continuous.

In order to introduce sets that are measurable with respect to logarithmic capacity, we define, following [6], **inner C_* and outer C^* capacities:**

$$C_*(E) \ : \ = \ \sup_{F\subseteq E} \ C(E), \qquad C^*(E) \ : \ = \ \inf_{E\subseteq O} \ C(O) \tag{2.5}$$

where supremum is taken over all compact sets $F \subset \mathbb{C}$ and infimum is taken over all open sets $O \subset \mathbb{C}$. A set $E \subset \mathbb{C}$ is called **measurable with respect to the logarithmic capacity** if $C^*(E) = C_*(E)$, and the common value of $C_*(E)$ and $C^*(E)$ is still denoted by $C(E)$.

A function $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called **measurable with respect to logarithmic capacity** if, for all open sets $O \subseteq \mathbb{C}$, the sets

$$\Omega = \{z \in E : \varphi(z) \in O\} \tag{2.6}$$

are measurable with respect to logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to logarithmic capacity.

Note also that sets of logarithmic capacity zero coincide with sets of the so-called **absolute harmonic measure** zero introduced by Nevanlinna, see Chapter V in [35]. Hence a set E is of (Hausdorff) length zero if $C(E) = 0$, see Theorem V.6.2 in [35]. However, there exist sets of length zero having a positive logarithmic capacity, see e.g. Theorem IV.5 in [6].

Remark 2. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to logarithmic capacity, see e.g. Lemma I.1 and Theorem III.7 in [6]. Moreover, as it follows from the definition, any set $E \subset \mathbb{C}$ of finite logarithmic capacity can be represented as a union of a sigma-compactum (union of countable collection of compact sets) and a set of logarithmic capacity zero. It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to measure of length, see e.g. Theorem II(7.4) in [45]. Consequently, any set $E \subset \mathbb{C}$ of finite logarithmic capacity is measurable with respect to measure of length. Thus, on such a set any function $\varphi : E \to \mathbb{C}$ being measurable with respect to logarithmic capacity is also measurable with respect to measure of length on E. However, there exist functions that are measurable with respect to measure of length but not measurable with respect to logarithmic capacity, see e.g. Theorem IV.5 in [6].

Later on, we use the **abbreviation q.e.** (**quasi-everywhere**) on a set $E \subset \mathbb{C}$ if the corresponding property holds only for all $\zeta \in E$ except its subset of zero logarithmic capacity, see e.g. [29] for this term.

3. Hilbert problem and angular limits

In this section, we prove the existence of nonclassical solutions of the Hilbert boundary value problem for generalized analytic functions with arbitrary boundary data that are measurable with respect to logarithmic capacity. The result is formulated in terms of the angular limit that is a traditional tool of the geometric function theory, see e.g. monographs [8, 26, 32, 37] and [38].

Recall that the classic boundary value **problem of Hilbert**, see [24], was formulated as follows: To find an analytic function $f(z)$ in a domain D bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\,\{\overline{\lambda(\zeta)}\ f(z)\} \ = \ \varphi(\zeta) \qquad\qquad \forall\ \zeta\in C\ , \tag{3.1}$$

where the **coefficient** λ and the **boundary date** φ of the problem are continuously differentiable with respect to the natural parameter s and $\lambda \neq 0$ everywhere on C. The latter allows to consider that $|\lambda| \equiv 1$ on C. Note that the quantity $\mathrm{Re}\,\{\overline{\lambda}\, f\}$ in (3.1) means a projection of f into the direction λ interpreted as vectors in $\mathbb{R}^2$.

The reader can find a rather comprehensive treatment of the theory in the new excellent books [3, 4, 23, 47]. We also recommend to make familiar with the historic surveys contained in the monographs [12, 34, 48] on the topic with an exhaustive bibliography and take a look at our recent papers, see Introduction.

Next, recall that a straight line L is **tangent** to a curve Γ in $\mathbb{C}$ at a point $z_0 \in \Gamma$ if

$$\limsup_{z\to z_0, z\in\Gamma} \frac{\mathrm{dist}\,(z,L)}{|z-z_0|} \ = \ 0\ . \tag{3.2}$$

Let D be a Jordan domain in $\mathbb{C}$ with a tangent at a point $\zeta \in \partial D$. A path in D terminating at ζ is called **nontangential** if its part in a neighborhood of ζ lies inside of an angle with the vertex at ζ. The limit along all nontangential paths at ζ is called **angular** at the point.

Following [21], we say that a Jordan curve Γ in $\mathbb{C}$ is **almost smooth** if Γ has a tangent q.e. In particular, Γ is almost smooth if Γ has a tangent at all its points except a countable collection. The nature of such a Jordan curve Γ can be complicated enough because this countable collection can be everywhere dense in Γ, see e.g. [7].

Recall that the **quasihyperbolic distance** between points z and z_0 in a domain $D \subset \mathbb{C}$ is the quantity

$$k_D(z,z_0)\ :=\ \inf_{\gamma} \int\limits_{\gamma} ds/d(\zeta,\partial D)\ ,$$

where $d(\zeta, \partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D, see [14].

Further, it is said that a domain D satisfies the **quasihyperbolic boundary condition** if there exist constants a and b and a point $z_0 \in D$ such that

$$k_D(z,z_0) \ \leq \ a \ + \ b \ \ln \frac{d(z_0,\partial D)}{d(z,\partial D)} \qquad \forall \ z \in D \ . \tag{3.3}$$

The latter notion was introduced in [13] but, before it, was first implicitly applied in [5]. By the discussion in [21], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition but such boundaries can be nowhere locally rectifiable.

Note that it is well–known the so–called $(A)-$condition by Ladyzhenskaya–Ural'tseva, which is standard in the theory of boundary value problems for PDE, see e.g. [28]. Recall that a domain D in $\mathbb{R}^n$, $n \geq 2$, is called satisfying **(A)-condition** if

$$\mathrm{mes}\ D \cap B(\zeta,\rho) \ \leq \ \Theta_0 \,\mathrm{mes}\ B(\zeta,\rho) \qquad \forall \ \zeta \in \partial D \ , \ \rho \leq \rho_0 \tag{3.4}$$

for some Θ_0 and $\rho_0 \in (0,1)$, where $B(\zeta,\rho)$ denotes the ball with the center $\zeta \in \mathbb{R}^n$ and the radius ρ, see 1.1.3 in [28].

Recall also that a domain D in $\mathbb{R}^n$, $n \geq 2$, is said to be satisfying the **outer cone condition** if there is a cone that makes possible to be touched by its top to every boundary point of D from the completion of D after its suitable rotations and shifts. It is clear that the outer cone condition implies (A)–condition.

Probably one of the simplest examples of an almost smooth domain D with the quasihyperbolic boundary condition and without (A)–condition is the union of 3 open disks with the radius 1 centered at the points 0 and $1 \pm i$. It is clear that this domain has zero interior angle at its boundary point 1.

Given a Jordan domain D in $\mathbb{C}$, we call $\lambda : \partial D \to \mathbb{C}$ a **function of bounded variation**, write $\lambda \in \mathcal{BV}(\partial D)$, if

$$V_\lambda(\partial D) \ : \ = \ \sup \sum_{j=1}^{k} \ |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| \ < \ \infty \tag{3.5}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial D$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_\lambda(\partial D)$ is called the **variation of the function** λ.

Now, we call $\lambda : \partial D \to \mathbb{C}$ a function of **countable bounded variation**, write $\lambda \in \mathcal{CBV}(\partial D)$, if there is a countable collection of mutually disjoint arcs γ_n of ∂D, $n = 1, 2, \ldots$ on each of which the restriction of

λ is of bounded variation and the set $\partial D \setminus \cup \gamma_n$ has logarithmic capacity zero. In particular, the latter holds true if the set $\partial D \setminus \cup \gamma_n$ is countable. It is clear that such functions can be singular enough.

Theorem 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and let $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $g : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\, \left\{ \overline{\lambda(\zeta)} \cdot h(z) \right\} = \varphi(\zeta) \qquad q.e.\ on\ \partial D. \tag{3.6}$$

Furthermore, the space of such functions h has the infinite dimension.

Later on, we often apply the **logarithmic (Newtonian) potential** $\mathcal{N}_G$ **of sources** $G \in L^p(\mathbb{C})$, $p > 2$, with compact supports given by the formula:

$$\mathcal{N}_G(z) := \frac{1}{2\pi} \int\limits_{\mathbb{C}} \ln|z-w|\, G(w)\, dm(w)\ . \tag{3.7}$$

By Lemma 3 in [19], $\mathcal{N}_G \in W^{2,p}_{\mathrm{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$, $\alpha := (p-2)/p$, $\triangle \mathcal{N}_G = G$ a.e.

Proof. Extending the function g by zero outside of D and setting $P = \mathcal{N}_G$ with $G = 2g$, $U = P_x$ and $V = -P_y$, we have that $U_x - V_y = G$ and $U_y + V_x = 0$. Thus, elementary calculations show that $H := U + iV$ is just a generalized analytic function with the source g. Moreover, the function

$$\varphi_*(\zeta) := \lim_{z\to\zeta} \mathrm{Re}\, \left\{ \overline{\lambda(\zeta)} \cdot H(z) \right\} = \mathrm{Re}\, \left\{ \overline{\lambda(\zeta)} \cdot H(\zeta) \right\}\ , \quad \forall\, \zeta \in \partial D\ , \tag{3.8}$$

is measurable with respect to logarithmic capacity because the function H is continuous in the whole plane $\mathbb{C}$.

By Theorems 5.1 and 6.1 in [21], see Theorem D in Introduction, there exist analytic functions $\mathcal{A}$ in D with the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\, \{\overline{\lambda(\zeta)} \cdot \mathcal{A}(z)\} = \Phi(\zeta) \qquad \text{q.e. on } \partial D \tag{3.9}$$

for the function $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$, $\zeta \in \partial D$. The space of such analytic functions $\mathcal{A}$ has the infinite dimension, see e.g. Corollary 8.1 in [21].

Finally, it is clear that the functions $h := \mathcal{A}+H$ are desired generalized analytic functions with the source g satisfying the Hilbert condition (3.6). Thus, the space of such functions h has really the infinite dimension. $\Box$

Remark 3. As it follows from the proof of Theorems 1, the generalized analytic functions h with a source $g \in L^p$, $p > 2$, satisfying the Hilbert boundary condition (3.6) q.e. in the sense of the angular limits can be represented in the form of the sums $\mathcal{A}+H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (3.9) and a generalized analytic function $H = U+iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential $\mathcal{N}_G$ with $G = 2g$ in the class $W^{2,p}_{\rm loc}(\mathbb{C}) \cap C^{1,\alpha}_{\rm loc}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\triangle P = G$.

In particular, for the case $\lambda \equiv 1$, we obtain the following consequence of Theorem 1 on the Dirichlet problem for the generalized analytic functions.

Corollary 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity and let $g : D \to \mathbb{R}$ be in $L^p(D)$ for some $p > 2$.*

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta} \mathrm{Re}\; h(z) \;=\; \varphi(\zeta) \qquad \textit{q.e. on } \partial D\,. \tag{3.10}$$

Furthermore, the space of all such functions h has the infinite dimension.

4. Hilbert problem and Bagemihl–Seidel systems

Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite collection of mutually disjoint Jordan curves. A family of mutually disjoint Jordan arcs $J_\zeta : [0,1] \to \overline{D}$, $\zeta \in \partial D$, with $J_\zeta([0,1)) \subset D$ and $J_\zeta(1) = \zeta$ that is continuous in the parameter ζ is called a **Bagemihl–Seidel system** or, in short, of **class** $\mathcal{BS}$.

Theorem 2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, and let functions $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, $\varphi : \partial D \to \mathbb{R}$ and $\psi : \partial D \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$ for some $p > 2$.

Then there is a generalized analytic function $f : D \to \mathbb{C}$ with the source g such that

$$\lim_{z\to\zeta} \ \mathrm{Re}\ \{\overline{\lambda(\zeta)} \cdot h(z)\} \ = \ \varphi(\zeta) \ , \tag{4.1}$$

$$\lim_{z\to\zeta} \ \mathrm{Im}\ \{\overline{\lambda(\zeta)} \cdot h(z)\} \ = \ \psi(\zeta) \tag{4.2}$$

along γ_ζ q.e. on ∂D.

Proof. As in the proof of Theorem 1, the function $H = U + iV$ with $U = P_x$ and $V = -P_y$, where $P = \mathcal{N}_G$ with $G = 2g$ is a generalized analytic function with the source g. Moreover, the functions

$$\varphi_*(\zeta) \ := \ \lim_{z\to\zeta} \ \mathrm{Re}\ \left\{\overline{\lambda(\zeta)} \cdot H(z)\right\} \ = \ \mathrm{Re}\ \left\{\overline{\lambda(\zeta)} \cdot H(\zeta)\right\} \ , \quad \forall\, \zeta \in \partial D \ , \tag{4.3}$$

$$\psi_*(\zeta) \ := \ \lim_{z\to\zeta} \ \mathrm{Im}\ \left\{\overline{\lambda(\zeta)} \cdot H(z)\right\} \ = \ \mathrm{Im}\ \left\{\overline{\lambda(\zeta)} \cdot H(\zeta)\right\} \ , \quad \forall\, \zeta \in \partial D \ , \tag{4.4}$$

are measurable with respect to logarithmic capacity because the function H is continuous in the whole plane $\mathbb{C}$.

Next, by Theorem 3 in [22] there is an analytic function $\mathcal{A}$ in D that has along γ_ζ q.e. on ∂D the limits

$$\lim_{z\to\zeta} \ \mathrm{Re}\ \{\overline{\lambda(\zeta)} \cdot \mathcal{A}(z)\} \ = \ \Phi(\zeta) \ , \tag{4.5}$$

$$\lim_{z\to\zeta} \ \mathrm{Im}\ \{\overline{\lambda(\zeta)} \cdot \mathcal{A}(z)\} \ = \ \Psi(\zeta) \tag{4.6}$$

for the functions $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$ and $\Psi(\zeta) := \psi(\zeta) - \psi_*(\zeta)$, $\zeta \in \partial D$. Thus, the function $h := \mathcal{A} + H$ is a desired generalized analytic function with the source g. $\Box$

Remark 4. As it follows from the proof of Theorems 2, the generalized analytic functions h with a source $g \in L^p$, $p > 2$, satisfying the Hilbert boundary condition (4.1) q.e. in the sense of the limits along γ_ζ can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (4.5) and a generalized analytic function $H = U + iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential $\mathcal{N}_G$ with $G = 2g$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\triangle P = G$.

The space of all solutions h of the Hilbert problem (4.1) in the given sense has the infinite dimension for any such prescribed φ, λ and $\{\gamma_\zeta\}_{\zeta\in D}$ because the space of all functions $\psi : \partial D \to \mathbb{R}$ which are measurable with respect to the logarithmic capacity has the infinite dimension.

The latter is valid even for its subspace of continuous functions ψ : $\partial D \to \mathbb{R}$. Indeed, by the Riemann theorem every Jordan domain Ω can be mapped with a conformal mapping c onto the unit disk $\mathbb{D}$ and by the Caratheodory theorem c can be extended to a homeomorphism of $\overline{G}$ onto $\overline{\mathbb{D}}$. By the Fourier theory, the space of all continuous functions $\tilde{\psi}$: $\partial\mathbb{D} \to \mathbb{R}$, equivalently, the space of all continuous 2π-periodic functions $\psi_* : \mathbb{R} \to \mathbb{R}$, has the infinite dimension.

Corollary 2. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, and λ : $\partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the logarithmic capacity.*

Suppose also that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$, $p > 2$.

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g that have the limits (4.1) *along γ_ζ q.e. on ∂D. Furthermore, the space of such functions h has the infinite dimension.*

In particular, for the case $\lambda \equiv 1$, we obtain the corresponding consequence on the Dirichlet problem for the generalized analytic functions with the source g along any prescribed Bagemihl–Seidel system:

Corollary 3. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves and $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity.*

Suppose also that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$, $p > 2$.

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source g such that

$$\lim_{z\to\zeta} \operatorname{Re} h(z) \ = \ \varphi(\zeta) \qquad \textit{along } \gamma_\zeta \textit{ q.e. on } \ \partial D \ . \tag{4.7}$$

Furthermore, the space of such functions h has the infinite dimension.

5. Riemann problem and Bagemihl–Seidel systems

Recall that the classical setting of the **Riemann problem** in a smooth Jordan domain D of the complex plane $\mathbb{C}$ is to find analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ that admit continuous extensions to ∂D and satisfy the boundary condition

$$f^+(\zeta) \ = \ A(\zeta)\cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\ \zeta \in \partial D \tag{5.1}$$

with prescribed Hölder continuous functions $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$.

Recall also that the **Riemann problem with shift** in D is to find analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ satisfying the condition

$$f^+(\alpha(\zeta)) \ = \ A(\zeta) \cdot f^-(\zeta) \ + \ B(\zeta) \qquad \forall\, \zeta \in \partial D \tag{5.2}$$

where $\alpha : \partial D \to \partial D$ was a one-to-one sense preserving correspondence having the non-vanishing Hölder continuous derivative with respect to the natural parameter on ∂D. The function α is called a **shift function**. The special case $A \equiv 1$ gives the so–called **jump problem** and $B \equiv 0$ gives the **problem on gluing** of analytic functions.

Arguing similarly to the proof of Theorem 1, we obtain by Theorem 8 in [22] on the Riemann problem for analytic functions the following statement.

Theorem 3. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $A : \partial D \to \mathbb{C}$ and $B : \partial D \to \mathbb{C}$ be functions that are measurable with respect to the logarithmic capacity and let $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ be families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C} \setminus \overline{D}$, correspondingly.*

Suppose that $g : \mathbb{C} \to \mathbb{R}$ is a function with compact support in the class $L^p(\mathbb{C})$ with some $p > 2$. Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ with the source g that satisfy (5.1) q.e. on $\zeta \in \partial D$, where $f^+(\zeta)$ and $f^-(\zeta)$ are limits of $f^+(z)$ and $f^-(z)$ az $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.

Furthermore, the space of all such couples (f^+, f^-) has the infinite dimension for every couple (A, B) and any collections γ^+_ζ and γ^-_ζ, $\zeta \in \partial D$.

Theorem 3 is a special case of the following lemma based on Lemma 3 in [22] on the Riemann problem with shift that may have of independent interest.

Lemma 1. *Under the hypotheses of Theorem 3, let in addition $\alpha : \partial D \to \partial D$ be a homeomorphism keeping components of ∂D such that α and α^{-1} have the $(N)-$property of Luzin with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \overline{\mathbb{C}} \setminus \overline{D} \to \mathbb{C}$ with the source g that satisfy (5.2) *for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity, where $f^+(\zeta)$ and $f^-(\zeta)$ are limits of $f^+(z)$ and $f^-(z)$ az $z \to \zeta$ along γ^+_ζ and γ^-_ζ, correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) has the infinite dimension for every couple (A, B) and any collections γ^+_ζ and γ^-_ζ, $\zeta \in \partial D$.

Remark 5. Some investigations were devoted also to the nonlinear Riemann problems with boundary conditions of the form

$$\Phi(\,\zeta,\, f^{+}(\zeta),\, f^{-}(\zeta)\,) \;=\; 0 \qquad \forall\, \zeta\in\partial D\,. \tag{5.3}$$

It is natural as above to weaken such conditions to the following

$$\Phi(\,\zeta,\, f^{+}(\zeta),\, f^{-}(\zeta)\,) \;=\; 0 \qquad \text{q.e. on } \zeta\in\partial D\,. \tag{5.4}$$

It is easy to see that the proposed approach makes possible also to reduce such problems to the algebraic measurable solvability of the relations

$$\Phi(\,\zeta,\, v,\, w\,) \;=\; 0 \tag{5.5}$$

with respect to complex-valued functions $v(\zeta)$ and $w(\zeta)$, cf. e.g. [16].

Later on, we sometimes say in short "$C-$measurable" instead of the expression "measurable with respect to the logarithmic capacity".

Example 1. For instance, correspondingly to the scheme given above, special nonlinear problems of the form

$$f^{+}(\zeta) \;=\; \varphi(\,\zeta,\, f^{-}(\zeta)\,) \qquad \text{q.e. on} \quad \zeta\in\partial D \tag{5.6}$$

are always solved if the function $\varphi : \partial D\times\mathbb{C}\to\mathbb{C}$ satisfies the **Caratheodory conditions** with respect to the logarithmic capacity, that is if $\varphi(\zeta, w)$ is continuous in the variable $w\in\mathbb{C}$ for a.e. $\zeta\in\partial D$ with respect to the logarithmic capacity and it is $C-$measurable in the variable $\zeta\in\partial D$ for all $w\in\mathbb{C}$.

The spaces of solutions of such problems always have the infinite dimension. Indeed, by the Egorov theorem, see e.g. Theorem 2.3.7 in [10], see also Section 17.1 in [27], the function $\varphi(\zeta,\psi(\zeta))$ is $C-$measurable in $\zeta\in\partial D$ for every $C-$measurable function $\psi : \partial D\to\mathbb{C}$ if the function φ satisfies the Caratheodory conditions, and the space of all $C-$measurable functions $\psi : \partial D\to\mathbb{C}$ has the infinite dimension, see e.g. arguments in Remark 4 above.

Furthermore, applying Lemma 1 with $A\equiv 0$ in (5.2), we able to resolve nonlinear boundary-value problems with shifts of the type (even with arbitrary measurable $f^{-}(\zeta)$ with respect to the logarithmic capacity)

$$f^{+}(\alpha(\zeta)) \;=\; \varphi(\,\zeta,\, f^{-}(\zeta)\,) \qquad \text{q.e. on} \quad \zeta\in\partial D\,. \tag{5.7}$$

6. On mixed boundary-value problems

Remark 5 makes possible to formulate a series of nonlinear boundary-value problems in terms of Bagemihl–Seidel systems for generalized analytic functions including mixed boundary value problems. In order to demonstrate the potentiality of our approach, we give here a couple of results.

Namely, arguing similarly to the proof of Theorem 1, see also Theorem 1.10 in [48], we obtain for instance by Theorem 10 and Lemma 5 in [22] the following statement on mixed boundary value problems.

Theorem 4. *Let D be a domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $\varphi : \partial D \times \mathbb{C} \to \mathbb{C}$ satisfy the Caratheodory conditions and $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be measurable with respect to the logarithmic capacity. Suppose also that $g : \mathbb{C} \to \mathbb{R}$ is in $C^{\alpha}(\mathbb{C})$, $\alpha \in (0,1)$, with compact support, $\{\gamma_\zeta^+\}_{\zeta\in\partial D}$ and $\{\gamma_\zeta^-\}_{\zeta\in\partial D}$ are families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C}\setminus\overline{D}$, correspondingly.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \mathbb{C}\setminus\overline{D} \to \mathbb{C}$ with the source g such that

$$f^+(\zeta) \;=\; \varphi\left(\zeta,\, \left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)\right) \qquad q.e. \text{ on } \partial D\,, \tag{6.1}$$

where $f^+(\zeta)$ and $\left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)$ are limits of the functions $f^+(z)$ and $\frac{\partial f^-}{\partial \nu}(z)$ as $z \to \zeta$ along γ_ζ^+ and γ_ζ^-, correspondingly.

Furthermore, the space of all such couples (f^+, f^-) has the infinite dimension for any such prescribed functions g, φ, ν and collections γ_ζ^+ and γ_ζ^-, $\zeta \in \partial D$.

Theorem 4 is a special case of the following lemma on the mixed problem with shift.

Lemma 2. *Under the hypotheses of Theorem 4, let in addition $\beta : \partial D \to \partial D$ be a homeomorphism keeping components of ∂D such that β and β^{-1} have the $(N)-$property of Luzin with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^+ : D \to \mathbb{C}$ and $f^- : \overline{\mathbb{C}}\setminus\overline{D} \to \mathbb{C}$ with the source g such that

$$f^+(\beta(\zeta)) \;=\; \varphi\left(\zeta,\, \left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)\right) \qquad q.e. \text{ on } \partial D\,, \tag{6.2}$$

where $f^+(\zeta)$ and $\left[\frac{\partial f}{\partial \nu}\right]^-(\zeta)$ are limits of the functions $f^+(z)$ and $\frac{\partial f^-}{\partial \nu}(z)$ as $z \to \zeta$ along γ_ζ^+ and γ_ζ^-, correspondingly.

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for any such prescribed* g, φ, ν, β *and collections* $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ *and* $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$.

Proof. Indeed, by relations (2.21) in [18] and Theorem 1.10 in [48] the logarithmic (Newtonian) potential $\mathcal{N}_G$ with the source $G = 2g$, see (3.7), is in $C^{2,\alpha}(\mathbb{C})$. Setting $P := \mathcal{N}_G$, we conclude by elementary calculations that the function

$$H(z) := \overline{\nabla P(z)}\ ,\ z \in \mathbb{C}\ , \qquad \nabla P := P_x + iP_y\ ,\ z = x + iy\ , \tag{6.3}$$

is a generalized analytic function in the class $C^{1,\alpha}(\mathbb{C})$ with the source g. Hence the function

$$h(\zeta) \ :=\ \frac{\partial H}{\partial \nu}\,(\zeta)\ , \qquad \zeta \in \partial D\ , \tag{6.4}$$

belongs to the class $C^{\alpha}(\partial D)$ is correctly definite and measurable with respect to the logarithmic capacity.

Now, let $a : \partial D \to \mathbb{C}$ be an arbitrary function that is measurable with respect to the logarithmic capacity. Then by Theorem 6 in [22] there exist analytic functions $\mathcal{A}^- : \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ such that

$$\lim_{z\to\zeta} \frac{\partial \mathcal{A}^-}{\partial \nu}\,(z) \ =\ a(\zeta) \qquad \text{q.e. on } \partial D\ . \tag{6.5}$$

Setting $f^- = H + \mathcal{A}^-$ on $\mathbb{C} \setminus \overline{D}$ and $\psi = h + a$ on ∂D, we see that the function $\psi : \partial D \to \mathbb{C}$ can be arbitrary measurable with respect to the logarithmic capacity, f^- is a generalized analytic function with the source g in $\mathbb{C} \setminus \overline{D}$ and

$$\lim_{z\to\zeta} \frac{\partial f^-}{\partial \nu}\,(z) \ =\ \psi(\zeta) \qquad \text{q.e. on } \partial D\ . \tag{6.6}$$

Next, the function $\Psi(\zeta) \ :=\ \varphi(\zeta, \psi(\zeta))$ is measurable with respect to the logarithmic capacity on ∂D, see Example 1 to Remark 4. Then the function $\Phi = \Psi \circ \beta^{-1}$ is also measurable with respect to the logarithmic capacity because the homeomorphism β has the $(N)-$property by Luzin with respect to the logarithmic capacity.

Consequently, by Theorem 1 in [22] there exist analytic functions $\mathcal{A}^+ : D \to \mathbb{C}$ such that $\mathcal{A}^+(z) \to \Phi(\zeta) - H(\zeta)$ as $z \to \zeta$ along γ_ζ q.e. on ∂D. Setting $f^+ = H + \mathcal{A}^+$ on D, we see that f^+ is a generalized analytic function with the source g in D such that $f^+(z) \to \Phi(\zeta)$ as $z \to \zeta$ along γ_ζ q.e. on ∂D.

Thus, f^+ and f^- are the desired functions because β^{-1} also has the $(N)-$property. It remains to note that the space of all such couples (f^+, f^-) has the infinite dimension because the space of all functions $\psi : \partial D \to \mathbb{C}$ which are measurable with respect to the logarithmic capacity has the infinite dimension, see arguments in Remark 4. $\Box$

Remark 6. In the case of Jordan domains D, following the same scheme, namely, applying once more Theorem 6 in [22] instead of Theorem 1 in [22] in the final stage of the above proof, the similar statement can be derived for the boundary gluing conditions of the form

$$\left[\frac{\partial f^+}{\partial \nu_*}\right](\beta(\zeta)) \ = \ \varphi\left(\zeta, \left[\frac{\partial f^-}{\partial \nu}\right](\zeta)\right) \qquad \text{q.e. on } \partial D\ . \tag{6.7}$$

7. Poincare and Neumann problems in terms of angular limits

In this section, we consider the Poincare boundary value problem on the directional derivatives and, in particular, the Neumann problem for the Poisson equations

$$\triangle U(z) \ = \ G(z) \tag{7.1}$$

with real valued functions G of classes $L^p(D)$ with $p > 2$ in the corresponding domains $D \subset \mathbb{C}$. Recall that a continuous solution U of (7.1) in the class $W^{2,p}_{\rm loc}$ is called a **generalized harmonic function with the source G** and that by the Sobolev embedding theorem such a solution belongs to the class C^1.

Theorem 5. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, be in $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then there exist generalized harmonic functions $U : D \to \mathbb{R}$ with the source G that have the angular limits

$$\lim_{z\to\zeta} \frac{\partial U}{\partial \nu}\,(z) \ = \ \varphi(\zeta) \qquad \textit{q.e. on } \partial D\ . \tag{7.2}$$

Furthermore, the space of such functions U has the infinite dimension.

Proof. Indeed, let us extend the function G by zero outside of D and let P be the logarithmic potential $\mathcal{N}_G$ with the source G, see (3.7). Then by Lemma 3 in [19] $P \in W^{2,p}_{\rm loc}(\mathbb{C}) \cap C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$ and $\triangle P = G$ a.e. in $\mathbb{C}$. Set

$$\varphi_*(\zeta) \ = \ {\rm Re}\,\nu(\zeta)H(\zeta)\ , \qquad \zeta\in\partial D\ , \tag{7.3}$$

where

$$H(z) := \overline{\nabla P(z)}\ ,\ z\in\mathbb{C}\ , \qquad \nabla P := P_x + iP_y\ ,\ z = x+iy\ . \tag{7.4}$$

Then by Theorem 1 with $g = G/2$ in D and $\lambda = \overline{\nu}$ on ∂D, there exist generalized analytic functions h with the source g that have the angular limits

$$\lim_{z\to\zeta}\ {\rm Re}\,\nu(\zeta)h(z) \ = \ \varphi(\zeta) \qquad\quad \text{q.e. on} \ \ \partial D \tag{7.5}$$

and, moreover, by the proof of Theorem 4 the given functions h can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ in D that have the angular limits

$$\lim_{z\to\zeta}\ {\rm Re}\ \nu(\zeta)\mathcal{A}(z) \ = \ \Phi(\zeta) \qquad\quad \text{q.e. on}\ \partial D \tag{7.6}$$

with $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta)$, $\zeta\in\partial D$, and the space of such analytic functions $\mathcal{A}$ has the infinite dimension.

Note that any indefinite integral $\mathcal{F}$ of such $\mathcal{A}$ in the simply connected domain D is also a single-valued analytic function and the harmonic functions $u := {\rm Re}\ \mathcal{F}$ and $v := {\rm Im}\ \mathcal{F}$ satisfy the Cauchy-Riemann system $u_x = v_y$ and $u_y = -v_x$. Hence

$$\mathcal{A} \ = \ \mathcal{F}' \ = \ \mathcal{F}_x \ = \ u_x \ + \ i\cdot v_x \ = \ \ u_x \ - \ i\cdot u_y \ = \ \overline{\nabla u}\ . \tag{7.7}$$

Consequently, setting $U_* = u+P$, we see that U_* is a generalized harmonic function with the source G and, moreover, by the construction $h = \overline{\nabla U_*}$.

Note also that the directional derivative of U_* along the unit vector ν is the projection of its gradient ∇U_* into ν, i.e., the scalar product of ν and ∇U_* interpreted as vectors in $\mathbb{R}^2$ and, consequently,

$$\frac{\partial U_*}{\partial\nu} \ = \ (\nu,\nabla U_*) \ = \ {\rm Re}\ \nu\cdot\overline{\nabla U_*} \ = \ {\rm Re}\ \nu\cdot h\ . \tag{7.8}$$

Thus, (7.5) implies (7.2) and the proof is complete. $\Box$

Remark 7. We are able to say more in the case of Re $n(\zeta)\overline{\nu(\zeta)} > 0$, where $n(\zeta)$ is the inner normal to ∂D at the point ζ. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector ν into n. In view of (7.2), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in D along the straight line passing through the point ζ and being parallel to the vector ν because along this line

$$U(z) \;=\; U(z_0) \;-\; \int\limits_0^1 \frac{\partial U}{\partial \nu}\,(z_0+\tau(z-z_0))\;d\tau\;. \tag{7.9}$$

Thus, at each point with condition (7.2), there is the directional derivative

$$\frac{\partial U}{\partial \nu}\,(\zeta) \;:=\; \lim_{t\to 0}\,\frac{U(\zeta+t\cdot\nu)-U(\zeta)}{t} \;=\; \varphi(\zeta)\;. \tag{7.10}$$

In particular, in the case of the Neumann problem, Re $n(\zeta)\overline{\nu(\zeta)} \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to ∂D at the point ζ, and we have by Theorem 5 the following significant result.

Corollary 4. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, the unit inner normal $n(\zeta)$, $\zeta \in \partial D$, belong to the class $\mathcal{CBV}(\partial D)$ and $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to logarithmic capacity.*

Suppose that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then one can find generalized harmonic functions $U : D \to \mathbb{R}$ with the source G such that q.e. on ∂D there exist:

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) \;:=\; \lim_{z\to\zeta}\, U(z)\;,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}\,(\zeta) \;:=\; \lim_{t\to 0}\,\frac{U(\zeta+t\cdot n(\zeta))-U(\zeta)}{t} \;=\; \varphi(\zeta)\;,$$

3) the angular limit

$$\lim_{z\to\zeta}\,\frac{\partial U}{\partial n}\,(z) \;=\; \frac{\partial U}{\partial n}\,(\zeta)\;.$$

Furthermore, the space of such functions U has the infinite dimension.

8. Poincare and Neumann problems and Bagemihl–Seidel systems

Arguing similarly to the last section, we obtain by Theorem 6 in [22], as well as Theorem 2 and Remark 4 above, the following statement.

Theorem 6. *Let D be a Jordan domain in $\mathbb{C}$, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{C}$ be measurable functions with respect to the logarithmic capacity and let $\{\gamma_\zeta\}_{\zeta\in\partial D}$ be a family of Jordan arcs of class $\mathcal{BS}$ in D.*

Suppose also that $G : D \to \mathbb{R}$ is in $L^p(D)$, $p > 2$. Then there exist generalized harmonic functions $U : D \to \mathbb{C}$ with the source G that have the limits along γ_ζ

$$\lim_{z\to\zeta} \frac{\partial U}{\partial \nu}\,(z) \ = \ \varphi(\zeta) \qquad \text{q.e. on } \ \partial D\ . \tag{8.1}$$

Furthermore, the space of such functions U has the infinite dimension.

Remark 8. As it follows from the proofs of Theorems 5 and 6, the generalized harmonic functions U with a source $G \in L^p$, $p > 2$, satisfying the Poincare boundary conditions can be represented in the form of the sums $\mathcal{N}_G + U_*$ of the logarithmic (Newtonian) potential $\mathcal{N}_G$ that is a generalized harmonic function with the source G and harmonic functions U_* satisfying the corresponding Poincare boundary conditions.

9. On the Riemann–Poincare type problems for the Poisson equations

Finally, arguing similarly to the proof of Lemma 2 in a much more simple manner, we obtain by Theorem 6, see also Remark 5 and Example 1, for example, the following consequence.

Corollary 5. *Let D be a Jordan domain in $\mathbb{C}$, $\varphi : \partial D \times \mathbb{R} \to \mathbb{R}$ satisfy the Caratheodory conditions, ν and $\nu_* : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, $|\nu_*(\zeta)| \equiv 1$, are measurable with respect to the logarithmic capacity, and let $\{\gamma^+_\zeta\}_{\zeta\in\partial D}$ and $\{\gamma^-_\zeta\}_{\zeta\in\partial D}$ be families of Jordan arcs of class $\mathcal{BS}$ in D and $\mathbb{C} \setminus \overline{D}$, correspondingly.*

Suppose also that $G : \mathbb{C} \to \mathbb{R}$ is in the class $L^p(D)$, $p > 2$, with compact support. Then there exist generalized harmonic functions $U^+ : D \to \mathbb{R}$ and $U^- : \mathbb{C} \setminus \overline{D} \to \mathbb{R}$ with the source G such that

$$\left[\frac{\partial U}{\partial \nu_*}\right]^+ (\zeta) \ = \ \varphi\left(\zeta,\, \left[\frac{\partial U}{\partial \nu}\right]^- (\zeta)\right) \qquad \text{q.e. on } \ \partial D\ , \tag{9.1}$$

where $\left[\frac{\partial U}{\partial \nu_*}\right]^+(\zeta)$ *and* $\left[\frac{\partial U}{\partial \nu}\right]^-(\zeta)$ *are limits of the directional derivatives* $\frac{\partial U^+}{\partial \nu_*}(z)$ *and* $\frac{\partial U^-}{\partial \nu}(z)$ *as* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-*, correspondingly.*

Furthermore, the function $\left[\frac{\partial U}{\partial \nu}\right]^-(\zeta)$ *can be arbitrary measurable with respect to the logarithmic capacity and, correspondingly, the space of all such couples* (U^+, U^-) *has the infinite dimension for any such prescribed functions* φ*,* ν*,* ν_* *and collections* γ_ζ^+ *and* γ_ζ^-*,* $\zeta \in \partial D$*.*

10. Conclusions

In this connection, it is necessary to note that all the above results are valid in terms of the length measure in Jordan domains with rectifiable boundaries, see [44]. However, by the well–known Ahlfors-Beurling example, see [1], the sets of length zero as well as of harmonic measure zero are not invariant with respect to quasiconformal changes of variables. The latter circumstance does not make it is possible to apply the results of the paper [44] in the future for the extension of the statements to generalizations in anisotropic and inhomogeneous media. Hence we have preferred here to use logarithmic capacity.

In comparison with the paper [21], here we have considered the boundary value problems of Riemann, Hilbert, Dirichlet, Poincare and Neumann with arbitrary measurable boundary data with respect to logarithmic capacity for the simplest equations with sources describing the case of the isotropic homogeneous media. The corresponding results on the boundary value problems for semi–linear equations of mathematical physics in anisotropic and inhomogeneous media with arbitrary measurable boundary data with respect to logarithmic capacity can be obtained similarly to the Dirichlet problem in [18] and [20] on the basis of the results of the present paper and the factorization theorem in the paper [17]. Thus, the present paper creates the basis and opens a whole series of articles on the corresponding results for nonlinear equations (with nonlinear sources) that will be published elsewhere.

Acknowledgements. This work was partially supported by grants of Ministry of Education and Science of Ukraine, project number is 0119U100421.

References

[1] Ahlfors, L., Beurling, A. (1956). The boundary correspondence under quasiconformal mappings. *Acta Math., 96*, 125–142.

[2] Bagemihl, F., Seidel, W. (1955). Regular functions with prescribed measurable boundary values almost everywhere. *Proc. Nat. Acad. Sci. U.S.A., 41*(10), 740–743.

[3] Begehr, H. (1994). *Complex analytic methods for partial differential equations. An introductory text.* River Edge, NJ: World Scientific Publishing Co., Inc.

[4] Begehr, H., Wen, G.Ch. (1996). *Nonlinear elliptic boundary value problems and their applications.* Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 80. Harlow: Longman.

[5] Becker, J., Pommerenke, Ch. (1982). Hölder continuity of conformal mappings and nonquasiconformal Jordan curves. *Comment. Math. Helv., 57*(2), 221–225.

[6] Carleson, L. (1971). *Selected Problems on Exceptional Sets.* Princeton etc.: Van Nostrand Co., Inc.

[7] Dovgoshey, O., Martio, O., Ryazanov, V., Vuorinen, M. (2006). The Cantor function. *Expo. Math., 24*(1), 1–37.

[8] Duren, P.L. (1970). *Theory of Hp spaces.* Pure and Applied Mathematics, Vol. 38. New York-London: Academic Press.

[9] Efimushkin, A.S., Ryazanov, V.I. (2015). On the Riemann–Hilbert problem for the Beltrami equations in quasidisks. *Ukr. Mat. Bull., 12*(2), 190–209; transl. in (2015). *J. Math. Sci., 211*(5), 646–659.

[10] Federer, H. (1969). *Geometric Measure Theory.* Springer-Verlag. Berlin.

[11] Fékete, M. (1923). Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z., 17*, 228–249.

[12] Gakhov, F.D. (1990). *Boundary value problems.* Dover Publications. Inc. New York.

[13] Gehring, F.W., Martio, O. (1985). Lipschitz classes and quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math. 10*, 203–219.

[14] Gehring, F.W., Palka, B.P. (1976). Quasiconformally homogeneous domains. *J. Analyse Math., 30*, 172–199.

[15] Goluzin, G.M. (1969). *Geometric theory of functions of a complex variable.* Transl. of Math. Monographs. 26. American Mathematical Society, Providence, R.I.

[16] Gromov, M. (1986). *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Springer-Verlag. Berlin.

[17] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2017). On quasiconformal maps and semi-linear equations in the plane. *Ukr. Mat. Bull., 14*(2), 161–191; transl. in (2018). *J. Math. Sci., 229*(1), 7–29.

[18] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2019). To the theory of semi-linear equations in the plane. *Ukr. Mat. Bull., 16*(1), 105–140; transl. in (2019). *J. Math. Sci., 242*(6), 833–859.

[19] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2020). On a quasilinear Poisson equation in the plane. *Anal. Math. Phys., 10*(1), Paper No. 6, 1–14.

[20] Gutlyanskii, V., Nesmelova, O., Ryazanov, V. (2020). Semi-linear equations and quasiconformal mappings. *Complex Var. Elliptic Equ., 65*(5), 823–843.

[21] Gutlyanskii, V.Ya., Ryazanov, V.I., Yakubov, E., Yefimushkin, A.S. (2020). On Hilbert boundary value problem for Beltrami equation. *Ann. Acad. Sci. Fenn. Math., 45*(2), 957–973.

[22] Gutlyanskii, V., Ryazanov, V., Yefimushkin, A. (2015). On the boundary value problems for quasiconformal functions in the plane. *Ukr. Mat. Bull., 12*(3), 363–389; transl. in (2016). *J. Math. Sci., 214*(2), 200–219.

[23] Heinonen, J., Kilpeläinen, T., Martio, O. (1993). *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York.

[24] Hilbert, D. (1904). *Über eine Anwendung der Integralgleichungen auf eine Problem der Funktionentheorie.* Verhandl. des III Int. Math. Kongr., Heidelberg.

[25] Hörmander, L. (1983). *The analysis of linear partial differential operators. I.* Distribution theory and Fourier analysis. Grundlehren der Mathematischen Wissenschaften. 256. Springer-Verlag. Berlin.

[26] Koosis, P. (1998). *Introduction to H^p spaces.* Cambridge Tracts in Mathematics. 115. Cambridge Univ. Press, Cambridge.

[27] Krasnosel'skii, M.A., Zabreiko, P.P., Pustyl'nik, E.I., Sobolevskii, P.E. (1976). *Integral operators in spaces of summable functions.* Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. Noordhoff International Publishing. Leiden.

[28] Ladyzhenskaya, O.A., Ural'tseva, N.N. (1968). *Linear and quasilinear elliptic equations.* Academic Press, New York-London; transl. from (1964) Lineinye i kvazilineinye uravneniya ellipticheskogo tipa. Nauka, Moscow.

[29] Landkof, N.S. (1972). *Foundations of modern potential theory.* Die Grundlehren der mathematischen Wissenschaften. 180. Springer-Verlag: New York–Heidelberg.

[30] Luzin, N.N. (1912). On the main theorem of integral calculus. *Mat. Sb., 28*, 266–294 (in Russian).

[31] Luzin, N.N. (1915). *Integral and trigonometric series*, Dissertation. Moskwa (in Russian).

[32] Luzin, N.N. (1951). *Integral and trigonometric series.* Editing and commentary by N.K. Bari and D.E. Men'shov. Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow–Leningrad (in Russian).

[33] Luzin, N. (1917). Sur la notion de l'integrale. *Annali Mat. Pura e Appl., 26*(3), 77–129.

[34] Muskhelishvili, N.I. (1992) *Singular integral equations. Boundary problems of function theory and their application to mathematical physics.* Dover Publications Inc. New York.

[35] Nevanlinna, R. (1944). *Eindeutige analytische Funktionen.* Ann Arbor. Michigan.

[36] Noshiro, K. (1960). *Cluster sets.* Springer-Verlag. Berlin etc.

[37] Pommerenke, Ch. (1992). *Boundary behaviour of conformal maps. Grundlehren der Mathematischen Wissenschaften.* Fundamental Principles of Mathematical Sciences, 299. Springer-Verlag. Berlin.

[38] Priwalow, I.I. (1956). *Randeigenschaften analytischer Funktionen.* Hochschulbücher für Mathematik. 25. Deutscher Verlag der Wissenschaften. Berlin.

[39] Ransford, T. (1995). *Potential theory in the complex plane.* London Mathematical Society Student Texts. 28. Cambridge University Press. Cambridge.

[40] Ryazanov, V. (2014). On the Riemann–Hilbert problem without index. *Ann. Univ. Buchar. Math. Ser. 5(LXIII)*(1), 169–178.

[41] Ryazanov, V. (2015). On Hilbert and Riemann problems. An alternative approach. *Ann. Univ. Buchar. Math. Ser. 6(LXIV)*(2), 237–244.

[42] Ryazanov, V. (2015). Infinite dimension of solutions of the Dirichlet problem. *Open Math. (the former Central European J. Math.), 13*(1), 348–350.

[43] Ryazanov, V. (2017). On Neumann and Poincare problems for Laplace equation. *Anal. Math. Phys., 7*(3), 285–289.

[44] Ryazanov, V. (2021). On Hilbert and Riemann problems for generalized analytic functions and applications. *Anal. Math. Phys., 11*(5). Published online: 22 November 2020.

[45] Saks, S. (1964). *Theory of the integral.* Warsaw; Dover Publications Inc., New York.

[46] Sobolev, S.L. (1963). *Applications of functional analysis in mathematical physics.* Transl. of Math. Mon. 7. AMS, Providence, R.I.

[47] Trogdon, Th., Olver, Sh. (2016). *Riemann–Hilbert problems, their numerical solution, and the computation of nonlinear special functions.* Society for Industrial and Applied Mathematics (SIAM). Philadelphia.

[48] Vekua, I.N. (1962). *Generalized analytic functions.* Pergamon Press. London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.

[49] Yefimushkin, A. (2016). On Neumann and Poincare problems in A–harmonic analysis. *Advances in Analysis, 1*(2), 114–120.

[50] Yefimushkin, A., Ryazanov, V. (2016). On the Riemann-Hilbert problem for the Beltrami equations. *Complex analysis and dynamical systems VI. Part 2*, 299–316. Contemp. Math. 667. Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI.

Contact information

Vladimir Gutlyanskiĭ
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: vgutlyanskii@gmail.com

Olga Nesmelova
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: star-o@ukr.net

Vladimir Ryazanov
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine,
Bogdan Khmelnytsky National
University of Cherkasy,
Cherkasy, Ukraine
E-Mail: Ryazanov@nas.gov.ua,
vl.ryazanov1@gmail.com

Artyem Yefimushkin
Institute of Applied Mathematics
and Mechanics of the NAS of Ukraine,
Slavyansk, Ukraine
E-Mail: a.yefimushkin@gmail.com