UDC 519.718

B.Ye. Rytsar

New minimization method of logical functions in polynomial set-theoretical format. 1. Generalized rules of conjuncterms simplification

Рассмотрены обобщенные правила упрощения конъюнктермов в полиномиальном теоретико-множественном формате, основанные на предложенных теоремах для разных начальных условий преобразования пары конъюнктермов, хеммингово расстояние между которыми может быть произвольным. Упомянутые правила могут быть полезными для минимизации в полиномиальном теоретико-множественном формате произвольных логических функций от *n* переменных. Преимущества предложенных правил проиллюстрированы примерами.

A generalized simplify rules of conjuncterms in polynomial set-theoretical format is considered. These rules are based on the proposed theorems for different initial transform condition of pair conjuncterms where hamming distance between them can be arbitrary. These rules may be useful to minimize in polynomial set-theoretical format of arbitrary logic functions of n variables. Advantages of the proposed rules are illustrated by the examples.

Розглянуто узагальнені правила спрощення кон'юнктермів у поліноміальному теоретико-множинному форматі, які ґрунтуються на запропонованих теоремах для різних початкових умов перетворення пари кон'юнктермів, геммінгова відстань між якими може бути довільна. Зазначені правила можуть бути корисні для мінімізації у поліноміальному теоретико-множинному форматі довільних логічних функцій від *n* змінних. Переваги запропонованих правил проілюстровано прикладами.

Introduction. The problem of minimization of logical functions in polynomial format caused the practical interest because of many advantages of realization of digital devices and systems in comparison with realization in disjunctive format. The investigations [1–8] have shown that it is economically profitable to build such digital devices as arithmetic, communication, coding error detectors as well as devices of programmed logic and others on logical elements *AND-EXOR* which realize polynomial basis $\{\&, \oplus, \mathbf{1}\}$, that is *AND*, *EXCLUSIVE OR* (*EXOR*) logical operations and constant **1**. It is easier to test and diagnose [9–11] digital devices on *AND-EXOR* if compared to the devices built on *AND-OR*. However, inspite of the mentioned advantages it is more difficult to minimize a function in polynomial format than in disjunctive format.

The precise solutions of a minimization problem in polynomial format generally is based on analytical transformations of positive *p*Davio-expansion and/or negative *n*Davio-expansion [2] or on visual *K*-map method [1–3]. Correspondingly, such methods are suitable only for functions from not great number of variables [5, 7, 10–13] and only for special classes of functions with up to 10 variables [14]. Heuristic methods have comparatively wider practical application [1, 8, 16–23]. Among them two should be singled out. One of them, a minimization method based on a coefficient of generalized canonical Reed–Muller forms which makes use of matrix transformations. The second method being more efficient involves iterative carring out of operations with conjuncterms of different ranks and polynomial forms of the given function. To the last belongs the algorithm [17], which after transformation of the given function in Zhegalkin polynomial (that is Positive Polarity Reed–Muller expression) minimizes it on the basis of three operations with conjuncterms. Better results have been shown by algorithm based on the procedure of so-called linked product terms [18, 19]. Later, on the basis of this procedure, the algorithms have been worked out which were completed with more perfect operations (that is *primary xlink*, *secondary xlink*, *unlink*, *exorlink*), which can be used for minimization of a system of complete and incomplete functions [20, 21].

However, the mentioned above algorithms have one drawback in common. They involve the procedure of linking in pairs only conjuncterms of the same rank $r \in \{1,2,...,n\}$, which differ by binary positions. Correspondingly, this limits the use of such algorithms to functions given in *SOP* (Sum-Of-Product) or

ESOP (EXOR Sum-Of-Product), which can have triple values in the different part. In these cases to conjuncterms that differ in ranks certain procedures of transformation (for example into canonical form) are applied which lead to an increase of steps and time of procedures. Besides the above mentioned operations of conjuncterms linking and other rules of simplification [24–26] do not have generalized character as to Hamming distance between any two conjuncterms of different ranks of a given function that does not guarantee final minimized result. Because of this, as one can judge from the literature, the search is still under way of such operations and procedures with conjuncterms which would guarantee precise or at least close to precise result of solving the problem of minimization of arbitrarily given function in polynomial format.

In this paper we consider a new method of minimization of logical functions with n variables in polynomial set-theoretical format, in the basis of which there is a procedure of splitting of conjuncterms of complete and incomplete function and also their system [27–29], and generalized set-theoretical rules of conjuncterms simplification. The suggested rules contrary to the known [21–25], guarantee generally better (as to costs of realization and number of procedure steps) results of minimization of different forms of given functions what is proved by the given in the paper numerous examples that are borrowed from publications by other authors for comparison. The suggested method of minimization on the basis of conjuncterms splitting in polynomial set-theoretical format has been described in three papers by the author: the first one is focuses on generalized set-theoretical rules of conjuncterms simplification (this paper), the second one – on minimization of complete and incomplete functions and the third one – on minimization of functions system.

1.1. Formulation of problem

The complexity of minimization problem of logical functions in the polynomial format consists of the fact that contrary to minimization in disjunctive format based on an operation of adjacency of neighbouring conjuncterms, here, except for the last one, other operations can be applied which can also simplify conjuncterms of a given function [1, 2].

The simplification of conjuncterms of the rank $r \in \{1,2,...,n\}$ of any logical function $f(x_1, x_2,..., x_n)$ in polynomial set-theoretical form (PSTF) is based on analytical axioms of a logical operation of sum for mod 2:

$$0 \oplus a = a,\ 0 \oplus \bar{a} = \bar{a},\ 1 \oplus a = \bar{a},\ 1 \oplus \bar{a} = a,\ a \oplus a = 0,\ a \oplus \bar{a} = 1,\ a \in \{0,1\}.$$

Such sets of PSTF $Y^{\oplus}$ [30] correspond to the below given expressions:

$$\{\varnothing,\sigma\}^{\oplus} = \sigma,\ \{\varnothing,\bar{\sigma}\}^{\oplus} = \bar{\sigma},\ \{(-),\sigma\}^{\oplus} = \bar{\sigma},\ \{(-),\bar{\sigma}\}^{\oplus} = \sigma,\ \{\sigma,\sigma\}^{\oplus} = \varnothing,\ \{\sigma,\bar{\sigma}\}^{\oplus} = (-),$$

where $\sigma \in \{0,1\}$, $\varnothing$ – an empty set that reflects the function-constant $f(x) = \mathbf{0}$, and symbol dash $(-)$ reflects absorbed variable x, that is complete set $\mathbf{E}_2^1 = \{0,1\}$, that reflects the function-constant $f(x) = \mathbf{1}$.

On this ground, for example, the function $f(x_1, x_2) = x_2 \oplus x_1 x_2 = (1 \oplus x_1) x_2 = \bar{x}_1 x_2$ will have PSTF $Y^{\oplus} = \{(-1),(11)\}^{\oplus} = (01)$, and the function $f = \bar{x}_1 x_2 \oplus \bar{x}_1 x_2 = 0$ will have PSTF $Y^{\oplus} = \{(01),(01)\}^{\oplus} = \varnothing$.

The suggested generalized set-theoretical rules of simplification of a conjuncterm set of any logical function $f(x_1, x_2,..., x_n)$, given in PSTF $Y^{\oplus}$, are based on iterative process of simplification of two conjuncterms $\theta_1^{r_1} = (\sigma_1 \sigma_2 \cdots \sigma_n)$ and $\theta_2^{r_2} = (\sigma_1 \sigma_2 \cdots \sigma_n)$, $\sigma_i \in \{0,1,-\}$, $r_1, r_2 \in \{1,2,...,n\}$, which differ in (Hamming) difference $d = 1,2,...,n$ – number of different in value $\alpha, \beta, \gamma, \delta,... \in \{0,1,-\}$ of onename positions. Here the different part $\alpha, \beta, \gamma, \delta,...$ of these conjuncterms can have a different total number of literals k_l. For example, two pairs of conjuncterms $\begin{pmatrix} 11-01 \\ 01-10 \end{pmatrix}$ and $\begin{pmatrix} 11--1 \\ 01-10 \end{pmatrix}$ have $d = 3$, but the first has $k_l = 6$, and the second – $k_l = 5$. In connection with this different initial conditions of transformation are possible. We will consider the following conditions:

• when $k_l = 2d$, here two conjuncterms are of the same rank θ_1^r and θ_2^r but differ in *d* onename binary positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1\}$;

• when $k_l = 2d - 1$, here one conjuncterm of $(r-1)$-rank θ_1^{r-1}, and the second of *r*-rank θ_2^r, differ in *d* onename positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1,-\}$, where dash (–) belongs to θ_1^{r-1};

• when $k_l = 2(d-1)$, here two conjuncterms are of the same *r*-rank θ_1^r and θ_2^r differ in *d* onename positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1,-\}$ each of them has one dash (–).

As a result of transformation of two conjuncterms in polynomial set-theoretical format some set of conjuncterms of certain ranks will appear that is *transformed PSTF* $Y^{\oplus}$. Depending on distance *d*, as it is further shown, different number of transformed PSTF $Y^{\oplus}$, will appear which will be designated as k_Y. The very fact of creation of different transformed PSTF $Y^{\oplus}$ will have decisive meaning for minimization in polynomial format of a given function *f*.

Efficiency of simplification of two different conjuncterms for the mentioned above initial conditions will be estimated on the basis of comparison of interrelation k_θ^* / k_l^*, obtained on the ground of data of transformed PSTF $Y^{\oplus}$, with initial interrelation k_θ / k_l, where $k_\theta = 2$.

1.2. Theoretical part

If $d = 0$, then conjuncterms θ_1^r and θ_2^r are the same, that is $\theta_1^r = \theta_2^r = \theta^r$. Then according to the expression $\theta^r \oplus \theta^r = 0$, transformed PSTF $Y^{\oplus} = \{\theta^r, \theta^r\}^{\oplus} = \varnothing$, that corresponds to removal of θ_1^r and θ_2^r from the given function *f*. In this case $k_Y = 1$ and $k_\theta^* / k_l^* = 0/0$.

In the case $d \geq 1$ conjuncterms are different $\theta_1^r \neq \theta_2^r$. Correspondingly every transformed PSTF $Y^{\oplus}$ will have different interrelation k_θ^* / k_l^* and their set will have different power k_Y.

Let us consider the first initial condition (see p. 1.1) when two conjuncterms of *r*-rank θ_1^r and θ_2^r differ in values of *d* onename binary position $\alpha, \beta, \gamma, \delta, ... \in \{0,1\}$.

Theorem 1. Two conjuncterms of *r*-rank θ_1^r and θ_2^r, $r \in \{1,2,...,n\}$, of the function $f(x_1, x_2, ..., x_n)$, that differ in values *d* of onename binary positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1\}$, in polynomial set-theoretical format form a set of different PSTF $Y^{\oplus}$ of power $k_Y = d!$, each of them consists of $k_\theta^* = d$ conjuncterms of $(r-1)$-rank and has in different part the total number of literals $k_l^* = d(d-1)$.

Proof. To determine k_θ^* / k_l^* and k_Y let us consider transformation of conjuncterms θ_1^r and θ_2^r for $d = 1, 2, 3, 4$. Here it should be mentioned that initial interrelation $k_\theta / k_l = 2/2d$.

• Let $d = 1$. Then $\theta_1^r = (\sigma_1 \cdots \overline{\alpha}_i \cdots \sigma_n)$, $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \sigma_n)$, $\alpha_i \in \{0,1\}$, respectively for analytical expression $\overline{a} \oplus a = 1$ we can write:

$$Y^{\oplus} = \{(\sigma_1 \cdots \overline{\alpha}_i \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots \sigma_n)\}^{\oplus} = (\sigma_1 \cdots -_i \cdots \sigma_n), \quad (1)$$

where the transformed PSTF $Y^{\oplus} = \{(\sigma_1 \cdots -_i \cdots \sigma_n)\}^{\oplus} = \theta^{r-1}$ that is a triple conjuncterm of $(r-1)$-rank.

For (1) interrelation $k_\theta^* / k_l^* = 1/0$, and as initial interrelation $k_\theta / k_l = 2/2$, then it is indicative of the fact that as a result of transformation (1) simplification took place due to replacement of two conjuncterms of *r*-rank by one conjuncterm of $(r-1)$-rank; $k_Y = 1$.

With the aim of simpler writing the conjuncterms of the given and transformed PSTF $Y^{\oplus}$ will be considered only for their different positions which will be written down in a column. In (1) such position is α_i, so, simplified writing down (1) with taking into account $\alpha_i \equiv \alpha \in \{0,1\}$ will look like:

$$\begin{pmatrix}\overline{\alpha}\\ \alpha\end{pmatrix}\overset{\oplus}{\Rightarrow}(-),\qquad (2)$$

where $\overset{\oplus}{\Rightarrow}$ – operator of the given conjuncterms θ_1^r and θ_2^r in polynomial format of the function *f*. In particular examples of transformation the same in meaning onename positions of conjuncterms will be rewritten without any change. For example, $\begin{pmatrix}1-01\\ 1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}(1--1)$, that in decimal format corresponds to $\begin{pmatrix}9,13\\ 11,15\end{pmatrix}\overset{\oplus}{\Rightarrow}(9,11,13,15)$ and in analytical form is $x_1\bar{x}_3x_4 \oplus x_1x_3x_4 = x_1x_4$.

• Let $d=2$. Then $\theta_1^r=(\sigma_1\cdots\overline{\alpha}_i\cdots\overline{\beta}_j\cdots\sigma_n)$, $\theta_2^r=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\sigma_n)$, $\alpha_i,\beta_j\in\{0,1\}$, and respectively to analytical expressions $\bar{a}\bar{b}\oplus ab=\begin{cases}\bar{a}\oplus b\\ a\oplus\bar{b}\end{cases}$ (if $\alpha_i=\beta_i$) and $\bar{a}b\oplus a\bar{b}=\begin{cases}\bar{a}\oplus\bar{b}\\ a\oplus b\end{cases}$ (if $\alpha_i\neq\beta_i$), in simplified way ($\alpha_i\equiv\alpha$, $\beta_i\equiv\beta$, $\alpha,\beta\in\{0,1\}$) we will get

$$\begin{pmatrix}\overline{\alpha}\overline{\beta}\\ \alpha\beta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\overline{\alpha}\,-\\ -\beta\end{pmatrix},\begin{pmatrix}\alpha\,-\\ -\overline{\beta}\end{pmatrix}\right\}.\qquad (3)$$

For (3) we have $k_\theta^*/k_l^*=2/2$, that are indicative of simplification of the given conjuncterms due to reduction of their rank from *r* до $(r-1)$, as initial interrelation $k_\theta/k_l=2/4;\ k_Y=2$.

The set of conjuncterms PSTF $Y^\oplus$ for $d>2$ can be easily formed visually with the help of a pattern of a given function *f*, vertices of which are interpreted by minterms and edges – conjuncterms of $(n-1)$-rank [31]. As the degree of each vertex of the pattern is equal to *n*, then possible ways of passing from initial vertex to final one will have points of branching the number of which will be directly proportional to distance *d*. Here, a change of an edge in the point of branching will determine a change of way of passing and respectively a position of dash (–) in a formed conjuncterm of $(n-1)$-rank. So, placement of dash (–) in certain position of a conjuncterm of $(n-1)$-rank has combinative character and it is a permutation without any repetition. It will be that what determine number of k_Y of transformed PSTF $Y^\oplus$. For $d=2$ one can be sure about this.

• Let $d=3$. Then $\theta_1^r=(\sigma_1\cdots\overline{\alpha}_i\cdots\overline{\beta}_j\cdots\overline{\gamma}_k\cdots\sigma_n)$ and $\theta_2^r=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)$, $\alpha_i,\beta_j,\gamma_k\in\{0,1\}$.

In the Fig. 1 six patterns of the function $f(\alpha,\beta,\gamma)$, $\alpha,\beta,\gamma\in\{0,1\}$, are shown where marked edges illustrate all possible ways of passing from initial vertex 0 (it is minterm (000)) to final vertex 7 (it is minterm (111)). As every such way consists of three edges, every transformed PSTF $Y^\oplus$ will have three conjuncterms of 2-rank.

From Fig. 1 we get the set $\begin{pmatrix}0\\7\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\underset{a}{\begin{pmatrix}0,1\\1,3\\3,7\end{pmatrix}},\underset{b}{\begin{pmatrix}0,1\\1,5\\5,7\end{pmatrix}},\underset{c}{\begin{pmatrix}0,2\\2,6\\6,7\end{pmatrix}},\underset{d}{\begin{pmatrix}0,2\\2,3\\3,7\end{pmatrix}},\underset{e}{\begin{pmatrix}0,4\\4,5\\5,7\end{pmatrix}},\underset{f}{\begin{pmatrix}0,4\\4,6\\6,7\end{pmatrix}}\right\}$, to which corresponds the set

of the transformed PSTF $Y^\oplus$: $\begin{pmatrix}000\\111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}00-\\0-1\\-11\end{pmatrix},\begin{pmatrix}00-\\-01\\1-1\end{pmatrix},\begin{pmatrix}0-0\\-10\\11-\end{pmatrix},\begin{pmatrix}0-0\\01-\\-11\end{pmatrix},\begin{pmatrix}-00\\10-\\1-1\end{pmatrix},\begin{pmatrix}-00\\1-0\\11-\end{pmatrix}\right\}$.

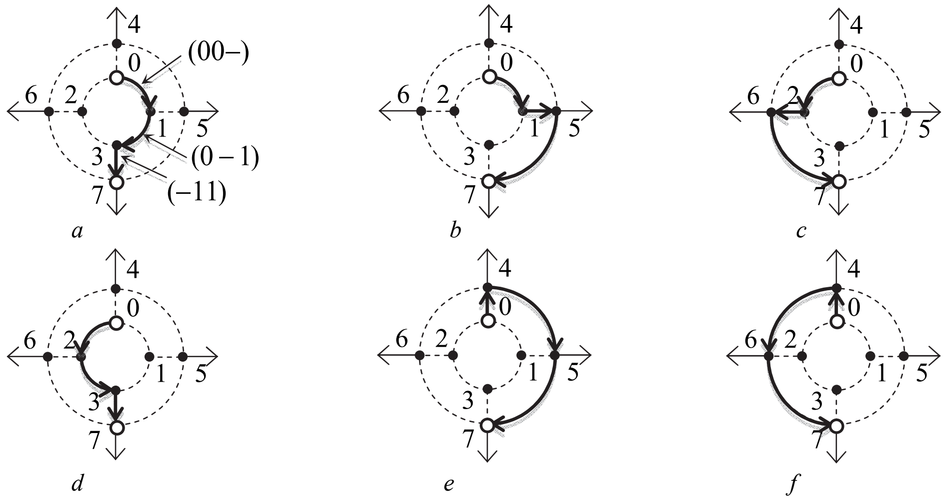

Fig. 1

To form the conjuncterms of $(n-1)$-rank of the set of transformed PSTF $Y^{\oplus}$ for any pair of minterms of the function *f*, that have $d=3$, is possible with the help of *matrix of disposition of dashes* (–) of the scale $3\times 6:\begin{bmatrix}0&0&1&1&2&2\\1&2&2&0&0&1\\2&1&0&2&1&0\end{bmatrix}$, where in every column the numbers 0, 1, 2 – values of degrees of scales of binary positions $\langle 2^2,2^1,2^0\rangle$ in a cortege of conjuncterms of the transformed PSTF $Y^{\oplus}$, where a dash (–) must be. For example, let the generating element of the pair $\begin{pmatrix}\overline{\alpha}\overline{\beta}\overline{\gamma}\\ \alpha\beta\gamma\end{pmatrix}$ be the minterm $(\overline{\alpha}\overline{\beta}\overline{\gamma})$. Then, if the first element of the matrix is $(\overline{\alpha}\overline{\beta}-)$ and this number 0 (for dash in 2^0), we will have two variants of disposition of dashes: the first one will be represented by the column $\begin{pmatrix}0\\1\\2\end{pmatrix}$, and so PSTF $Y^{\oplus}=\begin{pmatrix}\overline{\alpha}\overline{\beta}-\\ \overline{\alpha}-\gamma\\ -\beta\gamma\end{pmatrix}$, the second – $\begin{pmatrix}0\\2\\1\end{pmatrix}$, and so PSTF $Y^{\oplus}=\begin{pmatrix}\overline{\alpha}\overline{\beta}-\\ -\overline{\beta}\gamma\\ \alpha-\gamma\end{pmatrix}$. We will show creation of a set of transformed PSTF $Y^{\oplus}$ on the examples of two conjuncterms that have $d=3$ and common part in position 2^1 and 2^4:

$$\begin{pmatrix}101-0\\110-1\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}101--\\10--1\\1-0-1\end{pmatrix},\begin{pmatrix}101--\\1-1-1\\11--1\end{pmatrix},\begin{pmatrix}10--0\\1-0-0\\110--\end{pmatrix},\begin{pmatrix}10--0\\100--\\1-0-1\end{pmatrix},\begin{pmatrix}1-1-0\\111--\\11--1\end{pmatrix},\begin{pmatrix}1-1-0\\11--0\\110--\end{pmatrix}\right\},\text{ that is}$$

$$\begin{pmatrix}20,22\\25,27\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}20,21,22,23\\17,19,21,23\\17,19,25,27\end{pmatrix},\begin{pmatrix}20,21,22,23\\21,23,29,31\\25,27,29,31\end{pmatrix},\begin{pmatrix}16,18,20,22\\16,18,24,26\\24,25,26,27\end{pmatrix},\begin{pmatrix}16,18,20,22\\16,17,18,19\\17,19,25,27\end{pmatrix},\begin{pmatrix}20,22,28,30\\28,29,30,31\\25,27,29,31\end{pmatrix},\begin{pmatrix}20,22,28,30\\24,26,28,30\\24,25,26,27\end{pmatrix}\right\}.$$

So, for $d=3$ in general case one can write:

$$\begin{pmatrix}\overline{\alpha}\overline{\beta}\overline{\gamma}\\ \alpha\beta\gamma\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\overline{\alpha}\overline{\beta}-\\ \overline{\alpha}-\gamma\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}\overline{\alpha}\overline{\beta}-\\ -\overline{\beta}\gamma\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}\overline{\alpha}-\overline{\gamma}\\ -\beta\overline{\gamma}\\ \alpha\beta-\end{pmatrix},\begin{pmatrix}\overline{\alpha}-\overline{\gamma}\\ \overline{\alpha}\beta-\\ -\beta\gamma\end{pmatrix},\begin{pmatrix}-\overline{\beta}\overline{\gamma}\\ \alpha\overline{\beta}-\\ \alpha-\gamma\end{pmatrix},\begin{pmatrix}-\overline{\beta}\overline{\gamma}\\ \alpha-\overline{\gamma}\\ \alpha\beta-\end{pmatrix}\right\}. \quad (4)$$

For (4) $k_\theta^* / k_l^* = 3/6$ is indicative of an increase of power of each transformed PSTF $Y^\oplus$ and unchangeability of number of their literals as initial interrelation $k_\theta / k_l = 2/6$; $k_Y = 6$.

The following analytical expressions correspond to the transformed PSTF $Y^\oplus$ (4):

$$\bar{a}\bar{b}\bar{c} \oplus abc = \begin{cases} \bar{a}\bar{b} \oplus \bar{a}c \oplus bc \\ \bar{a}\bar{b} \oplus \bar{b}c \oplus ac \\ \bar{a}\bar{c} \oplus b\bar{c} \oplus ab \\ \bar{a}\bar{c} \oplus \bar{a}b \oplus bc \\ \bar{b}\bar{c} \oplus a\bar{b} \oplus ac \\ \bar{b}\bar{c} \oplus a\bar{c} \oplus ab \end{cases}$$

, where the first equation, for example, can be got this way:

$$\bar{a}\bar{b}\bar{c} \oplus abc = \bar{a}\bar{b}(1 \oplus c) \oplus abc = \bar{a}\bar{b} \oplus \bar{a}\bar{b}c \oplus abc =$$

$$= \bar{a}\bar{b} \oplus \bar{a}(1 \oplus b)c \oplus abc = \bar{a}\bar{b} \oplus \bar{a}c \oplus \bar{a}bc \oplus abc = \bar{a}\bar{b} \oplus \bar{a}c \oplus (1 \oplus a)bc \oplus abc = \bar{a}\bar{b} \oplus \bar{a}c \oplus bc.$$

• Let $d = 4$. Then for $\theta_1^r = (\sigma_1 \cdots \bar{\alpha}_i \cdots \bar{\beta}_j \cdots \bar{\gamma}_k \cdots \bar{\delta}_l \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots \beta_j \cdots \gamma_k \cdots \delta_l \cdots \sigma_n)$, where $\alpha_i, \beta_j, \gamma_k, \delta_l \in \{0,1\}$, matrix of disposition of dashes will have the scale 4×24, namely:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 1 & 1 & 2 & 3 & 0 & 0 & 2 & 2 & 3 & 0 & 1 & 1 & 3 & 3 & 0 & 1 & 2 & 2 & 0 & 0 & 1 \\ 3 & 2 & 1 & 3 & 2 & 1 & 0 & 3 & 2 & 0 & 3 & 2 & 1 & 0 & 3 & 1 & 0 & 3 & 2 & 1 & 0 & 2 & 1 & 0 \end{bmatrix},$$

where 0, 1, 2, 3 – values of degrees of scales of binary positions $\langle 2^3, 2^2, 2^1, 2^0 \rangle$ in the cortege of conjuncterms of each transformed PSTF $Y^\oplus$, what is reflected by columns of matrix.

On the ground of the matrix of disposition of dashes for the case $d = 4$ we will get a set from 24 transformed PSTF $Y^\oplus$, each of them consists of 4 conjuncterms of 3-rank:

$$\begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha\beta\gamma\delta \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ \bar{\alpha}\bar{\beta}-\delta \\ \bar{\alpha}-\gamma\delta \\ -\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ \bar{\alpha}\bar{\beta}-\delta \\ -\bar{\beta}\gamma\delta \\ \alpha-\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ \bar{\alpha}-\bar{\gamma}\delta \\ -\beta\bar{\gamma}\delta \\ \alpha\beta-\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ \bar{\alpha}-\bar{\gamma}\delta \\ \bar{\alpha}\beta-\delta \\ -\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ -\bar{\beta}\bar{\gamma}\delta \\ \alpha\bar{\beta}-\delta \\ \alpha-\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}\bar{\gamma}- \\ -\bar{\beta}\bar{\gamma}\delta \\ \alpha-\bar{\gamma}\delta \\ \alpha\beta-\delta \end{pmatrix}, \right.$$

$$\begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ \bar{\alpha}-\gamma\bar{\delta} \\ -\beta\gamma\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ \bar{\alpha}-\gamma\bar{\delta} \\ \bar{\alpha}\beta\gamma- \\ -\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ -\bar{\beta}\gamma\bar{\delta} \\ \alpha\bar{\beta}\gamma- \\ \alpha-\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ -\bar{\beta}\gamma\bar{\delta} \\ \alpha-\gamma\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ \bar{\alpha}\bar{\beta}\gamma- \\ \bar{\alpha}-\gamma\delta \\ -\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}\bar{\beta}-\bar{\delta} \\ \bar{\alpha}\bar{\beta}\gamma- \\ -\bar{\beta}\gamma\delta \\ \alpha-\gamma\delta \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ -\beta\bar{\gamma}\bar{\delta} \\ \alpha\beta\bar{\gamma}- \\ \alpha\beta-\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ -\beta\bar{\gamma}\bar{\delta} \\ \alpha\beta-\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ \bar{\alpha}\beta\bar{\gamma}- \\ \bar{\alpha}\beta-\delta \\ -\beta\gamma\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ \bar{\alpha}\beta\bar{\gamma}- \\ -\beta\bar{\gamma}\delta \\ \alpha\beta-\delta \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ \bar{\alpha}\beta-\bar{\delta} \\ -\beta\gamma\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix}, \begin{pmatrix} \bar{\alpha}-\bar{\gamma}\bar{\delta} \\ \bar{\alpha}\beta-\bar{\delta} \\ \bar{\alpha}\beta\gamma- \\ -\beta\gamma\delta \end{pmatrix},$$

$$\left. \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha\bar{\beta}\bar{\gamma}- \\ \alpha\bar{\beta}-\delta \\ \alpha-\gamma\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha\bar{\beta}\bar{\gamma}- \\ \alpha-\bar{\gamma}\delta \\ \alpha\beta-\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha\bar{\beta}-\bar{\delta} \\ \alpha-\gamma\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha\bar{\beta}-\bar{\delta} \\ \alpha\bar{\beta}\gamma- \\ \alpha-\gamma\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha-\bar{\gamma}\bar{\delta} \\ \alpha\beta\bar{\gamma}- \\ \alpha\beta-\delta \end{pmatrix}, \begin{pmatrix} -\bar{\beta}\bar{\gamma}\bar{\delta} \\ \alpha-\bar{\gamma}\bar{\delta} \\ \alpha\beta-\bar{\delta} \\ \alpha\beta\gamma- \end{pmatrix} \right\}.$$

For example, let $\alpha=\beta=\gamma=\delta=1$. Then for function $f(\alpha,\beta,\gamma,\delta)$ the set (5) will look like:

$$\begin{pmatrix}0000\\1111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}000-\\00-1\\0-11\\-111\end{pmatrix},\begin{pmatrix}000-\\00-1\\-011\\1-11\end{pmatrix},\begin{pmatrix}000-\\0-01\\-101\\11-1\end{pmatrix},\begin{pmatrix}000-\\0-01\\01-1\\-111\end{pmatrix},\begin{pmatrix}000-\\-001\\10-1\\1-11\end{pmatrix},\begin{pmatrix}000-\\-001\\1-01\\11-1\end{pmatrix},\right. \qquad (5')$$

$$\begin{pmatrix}00-0\\0-10\\-110\\111-\end{pmatrix},\begin{pmatrix}00-0\\0-10\\011-\\-111\end{pmatrix},\begin{pmatrix}00-0\\-010\\101-\\1-11\end{pmatrix},\begin{pmatrix}00-0\\-010\\1-10\\111-\end{pmatrix},\begin{pmatrix}00-0\\001-\\0-11\\-111\end{pmatrix},\begin{pmatrix}00-0\\001-\\-011\\1-11\end{pmatrix}, \qquad (5'')$$

$$\begin{pmatrix}0-00\\-100\\110-\\11-1\end{pmatrix},\begin{pmatrix}0-00\\-100\\11-0\\111-\end{pmatrix},\begin{pmatrix}0-00\\010-\\01-1\\-111\end{pmatrix},\begin{pmatrix}0-00\\010-\\-101\\11-1\end{pmatrix},\begin{pmatrix}0-00\\01-0\\-110\\111-\end{pmatrix},\begin{pmatrix}0-00\\01-0\\011-\\-111\end{pmatrix}, \qquad (5''')$$

$$\left.\begin{pmatrix}-000\\100-\\10-1\\1-11\end{pmatrix},\begin{pmatrix}-000\\100-\\1-01\\11-1\end{pmatrix},\begin{pmatrix}-000\\10-0\\1-10\\111-\end{pmatrix},\begin{pmatrix}-000\\10-0\\101-\\1-11\end{pmatrix},\begin{pmatrix}-000\\1-00\\110-\\11-1\end{pmatrix},\begin{pmatrix}-000\\1-00\\11-0\\111-\end{pmatrix}\right\}. \qquad (5'''')$$

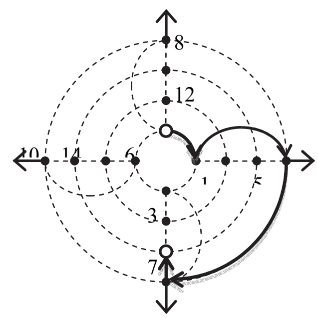

Fig. 2

In Fig. 2 we see a pattern of the function $f(\alpha,\beta,\gamma,\delta)$, on which we have one of examples (5′) of formation of a set of edges starting with vertex 0 up to vertex 15, that corresponds to formation of a set of conjuncterms of 3-rank PSTF $Y^{\oplus}$ in case of transformation of a pair of minterms (0000) and (1111), distance between them $d=4$, i.e. $\begin{pmatrix}0\\15\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}0,1\\1,9\\9,11\\11,15\end{pmatrix}$, that is $\begin{pmatrix}0000\\1111\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}000-\\-001\\10-1\\1-11\end{pmatrix}$.

So, for (5) $k_{\theta}^{*}/k_{l}^{*}=4/12$ is indicative of an increase of power of transformed PSTF $Y^{\oplus}$ and the number of literals, as the initial interrelation $k_{\theta}/k_{l}=2/8;\ k_{Y}=24$.

In the case of necessity for any pair of conjuncterms of *r*-rank of a function *f* , that have distance $d>4$, one can in analogical way form a set of *d*! transformed PSTF $Y^{\oplus}$; $d=1,2,...,n$.

So, on the ground of the considered above one can state that two conjuncterms of *r*-rank θ_1^r and θ_2^r function *f*, that differ $d=1,2,...,n$ in different by values onename binary positions $\alpha,\beta,\gamma,\delta,...\in\{0,1\}$, form in polynomial format a set with 3 $k_Y = d!$ of transformed PSTF $Y^{\oplus}$, each of them consists of different conjuncterms of $(r-1)$-rank with interrelation $k_\theta^* / k_l^* = d / d(d-1)$, that proves truth of theorem 1. □

Efficiency of application of theorem 1 for simplification of a set of conjuncterms and obtaining minimal PSTF $Y^{\oplus}$ of a given function *f* is illustrated by examples given further. Here the cost of realization of minimized function *f* will be estimated by comparison of numeric interrelation k_θ^* / k_l^* and k_θ / k_l.

Example 1. To apply theorem 1 to the function $f(x_1, x_2, x_3, x_4)$, given by perfect STF $Y^1 = \{0,6,14,15\}^1$, which in polynomial format has been minimized by *xlinking* method [21, p. 28], as the result of what we have got solution PSTF $Y^{\oplus} = \{(-11-),(0111),(0000)\}^{\oplus}$, where $k_\theta / k_l = 3/10$.

Solution . For minterms (0111),(0000) we will apply theorem 1 for $d = 3$ rule (4):

$$\begin{pmatrix}0000\\0111\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix}000-\\00-1\\0-11\end{pmatrix}, \begin{pmatrix}000-\\0-01\\01-1\end{pmatrix}, \underline{\begin{pmatrix}00-0\\0-10\\011-\end{pmatrix}}, \begin{pmatrix}00-0\\001-\\0-11\end{pmatrix}, \begin{pmatrix}0-00\\010-\\01-1\end{pmatrix}, \underline{\begin{pmatrix}0-00\\01-0\\011-\end{pmatrix}} \right\}.$$

Having changed (0111) and (0000) by marked (underlined) PSTF $Y^{\oplus}$, we will get two solutions:

$$Y^{\oplus} = \{(-11-),(0111),(0000)\}^{\oplus} \Rightarrow \left\{ (\underline{-11-}), \begin{Bmatrix} 1.\ (00-0),(0-10),(\underline{011-}) \\ 2.\ (0-00),(01-0),(\underline{011-}) \end{Bmatrix} \right\}^{\oplus}.$$

Underlined conjuncterms are simplified by merging operation [30]: $\{(-11-),(011-)\}^{\oplus} = (111-)$, that corresponds to the expression $x_2 x_3 \oplus \bar{x}_1 x_2 x_3 = x_1 x_2 x_3$. So the given function *f* has two solutions of minimization that reflect minimal PSTF:

1. $Y^{\oplus} = \{(111-),(00-0),(0-10)\}^{\oplus}$;
2. $Y^{\oplus} = \{(111-),(01-0),(0-00)\}^{\oplus}$.

Answer. The cost of realization of minimized function *f* is equal to $k_\theta^* / k_l^* = 3/9$ and is better than in [21].

Let us consider the case when one conjuncterm of $(r-1)$-rank θ_1^{r-1} differs from another conjuncterm of *r*-rank θ_2^r in one dash (–), and other onename positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1\}$.

Theorem 2. Two conjuncterms of the function $f(x_1, x_2, ..., x_n)$, one of which of $(r-1)$-rank θ_1^{r-1} differs from another *r*-rank θ_2^r in the number of *d* different in values onename positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1,-\}$, among which the dash (–) belongs to the conjuncterm θ_1^{r-1}, $r \in \{1,2,...,n\}$, in polynomial set-theoretical format create $k_Y = (d-1)!$ of sets PSTF $Y^{\oplus}$, each of them has power $k_\theta^* = d$ and the total number of literals in different part $k_l^* = d(d-1) - (d-2)$, here:

- if $d = 1$, then $$\begin{pmatrix}-\\\tilde{\alpha}\end{pmatrix} \overset{\oplus}{\Rightarrow} (\bar{\tilde{\alpha}});\qquad (6)$$
- if $d = 2$, then $$\begin{pmatrix}\bar{\alpha}-\\\alpha\tilde{\beta}\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix}--\\\alpha\bar{\tilde{\beta}}\end{pmatrix},\ \begin{pmatrix}-\bar{\beta}\\\tilde{\alpha}\beta\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix}--\\\bar{\tilde{\alpha}}\beta\end{pmatrix};\qquad (7)$$
- if $d = 3$, then

$$\begin{pmatrix}\bar{\alpha}\bar{\beta}-\\\alpha\beta\tilde{\gamma}\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix}\bar{\alpha}--\\-\beta-\\\alpha\beta\bar{\tilde{\gamma}}\end{pmatrix}, \begin{pmatrix}-\bar{\beta}-\\\alpha--\\\alpha\beta\bar{\tilde{\gamma}}\end{pmatrix} \right\},\ \begin{pmatrix}\bar{\alpha}-\bar{\gamma}\\\alpha\tilde{\beta}\gamma\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix}\bar{\alpha}--\\--\gamma\\\alpha\bar{\tilde{\beta}}\gamma\end{pmatrix}, \begin{pmatrix}--\bar{\gamma}\\\alpha--\\\alpha\bar{\tilde{\beta}}\gamma\end{pmatrix} \right\},\ \begin{pmatrix}-\bar{\beta}\bar{\gamma}\\\tilde{\alpha}\beta\gamma\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix}-\bar{\beta}-\\--\gamma\\\bar{\tilde{\alpha}}\beta\gamma\end{pmatrix}, \begin{pmatrix}--\bar{\gamma}\\-\beta-\\\bar{\tilde{\alpha}}\beta\gamma\end{pmatrix} \right\},\ (8), (9), (10)$$

• if $d=4$, then

$$\begin{pmatrix}\bar\alpha\bar\beta\bar\gamma-\\ \alpha\beta\gamma\tilde\delta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta--\\ \bar\alpha-\gamma-\\ -\beta\gamma-\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta--\\ -\bar\beta\gamma-\\ \alpha-\gamma-\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma-\\ \bar\alpha\beta--\\ -\beta\gamma-\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma-\\ -\beta\bar\gamma-\\ \alpha\beta--\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma-\\ \alpha\bar\beta--\\ \alpha-\gamma-\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma-\\ \alpha-\bar\gamma-\\ \alpha\beta--\\ \alpha\beta\gamma\bar{\tilde\delta}\end{pmatrix}\right\}, \tag{11}$$

$$\begin{pmatrix}\bar\alpha\bar\beta-\bar\delta\\ \alpha\beta\tilde\gamma\delta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha\bar\beta--\\ \bar\alpha--\delta\\ -\beta-\delta\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha\bar\beta--\\ -\bar\beta-\delta\\ \alpha--\delta\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ \bar\alpha\beta--\\ -\beta-\delta\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ -\beta-\bar\delta\\ \alpha\beta--\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ \alpha\bar\beta--\\ \alpha--\delta\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ \alpha--\bar\delta\\ \alpha\beta--\\ \alpha\beta\bar{\tilde\gamma}\delta\end{pmatrix}\right\}, \tag{12}$$

$$\begin{pmatrix}\bar\alpha-\bar\gamma\bar\delta\\ \alpha\tilde\beta\gamma\delta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\bar\alpha-\bar\gamma-\\ \bar\alpha--\delta\\ --\gamma\delta\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha-\bar\gamma-\\ --\bar\gamma\delta\\ \alpha--\delta\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ --\gamma\bar\delta\\ \alpha-\gamma-\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}\bar\alpha--\bar\delta\\ \bar\alpha-\gamma-\\ --\gamma\delta\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ \alpha-\bar\gamma-\\ \alpha--\delta\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ \alpha--\bar\delta\\ \alpha-\gamma-\\ \alpha\bar{\tilde\beta}\gamma\delta\end{pmatrix}\right\}, \tag{13}$$

$$\begin{pmatrix}-\bar\beta\bar\gamma\bar\delta\\ \tilde\alpha\beta\gamma\delta\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-\bar\beta\bar\gamma-\\ -\bar\beta-\bar\delta\\ --\bar\gamma\bar\delta\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta\bar\gamma-\\ --\bar\gamma\delta\\ -\beta-\delta\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ -\bar\beta\gamma-\\ --\gamma\delta\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}-\bar\beta-\bar\delta\\ --\gamma\bar\delta\\ -\beta\gamma-\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ -\beta\bar\gamma-\\ -\beta-\delta\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix},\begin{pmatrix}--\bar\gamma\bar\delta\\ -\beta-\bar\delta\\ -\beta\gamma-\\ \bar{\tilde\alpha}\beta\gamma\delta\end{pmatrix}\right\}, \tag{14}$$

where $\tilde\alpha,\tilde\beta,\tilde\gamma,\tilde\delta$ – binary positions of any value 0 or 1.

Proof. In this case the given PSTF $Y^{\oplus}$ has interrelation $k_\theta / k_l = 2/(2d-1)$.

• Let $d=1$. Then $\theta_1^{r-1}=(\sigma_1\cdots -_i \cdots\sigma_n)$, $\theta_2^{r}=(\sigma_1\cdots\tilde\alpha_i\cdots\sigma_n)$, $\tilde\alpha_i\in\{0,1\}$, and respectively for the expression $1\oplus\tilde a=\bar{\tilde a}$, $\tilde a\in\{a,\bar a\}$, we can write down such PSTF $Y^{\oplus}$:

$Y^{\oplus}=\{(\sigma_1\cdots -_i\cdots\sigma_n),(\sigma_1\cdots\tilde\alpha_i\cdots\sigma_n)\}^{\oplus}=\{(\sigma_1\cdots\bar{\tilde\alpha}_i\cdots\sigma_n)\}^{\oplus}$.

As interrelation $k_\theta^* / k_l^* = 1/1$, then compared with $k_\theta / k_l = 2/1$ we have simplification of the given PSTF $Y^{\oplus}$ due to removal of one conjuncterm; $k_Y = 1$.

For example, $\begin{pmatrix}1--1\\ 1-01\end{pmatrix}\overset{\oplus}{\Rightarrow}(1-11)$ corresponds to $\begin{pmatrix}9,11,13,15\\ 9,13\end{pmatrix}\overset{\oplus}{\Rightarrow}(11,15)$ and $x_1x_4\oplus x_1\bar x_3x_4 = x_1x_3x_4$.

• Let $d=2$. Then for $\theta_1^{r-1}=(\sigma_1\cdots\bar\alpha_i\cdots -_j\cdots\sigma_n)$ and $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\tilde\beta_j\cdots\sigma_n)$ and $\theta_1^{r-1}=(\sigma_1\cdots -_i\cdots\bar\beta_j\cdots\sigma_n)$ and $\theta_2^{r}=(\sigma_1\cdots\tilde\alpha_i\cdots\beta_j\cdots\sigma_n)$, $\alpha_i,\beta_j\in\{0,1\}$, and respectively to the expressions $\bar a\oplus a\tilde b=\bar a\oplus a(1\oplus\bar{\tilde b})=1\oplus a\bar{\tilde b}$ and $\bar b\oplus\tilde a b=\bar b\oplus(\bar{\tilde a}\oplus 1)b=1\oplus\bar{\tilde a}b$, $\tilde a\in\{a,\bar a\}$, $\tilde b\in\{b,\bar b\}$, we will get

$Y^{\oplus}=\{(\sigma_1\cdots\bar\alpha_i\cdots -_j\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\tilde\beta_j\cdots\sigma_n)\}^{\oplus}=\{(\sigma_1\cdots -_i\cdots -_j\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\bar{\tilde\beta}_j\cdots\sigma_n)\}^{\oplus}$ and $Y^{\oplus}=\{(\sigma_1\cdots -_i\cdots\bar\beta_j\cdots\sigma_n),(\sigma_1\cdots\tilde\alpha_i\cdots\beta_j\cdots\sigma_n)\}^{\oplus}=\{(\sigma_1\cdots -_i\cdots -_j\cdots\sigma_n),(\sigma_1\cdots\bar{\tilde\alpha}_i\cdots\beta_j\cdots\sigma_n)\}^{\oplus}$.

Comparing the obtained interrelation $k_\theta^* / k_l^* = 2/2$ with $k_\theta / k_l = 2/3$, we see that formed PSTF $Y^{\oplus}$ is simpler than the given PSTF $Y^{\oplus}$ for one literal.

For example, having applied the rule (7) of theorem 2 to $f(\alpha,\beta)$, $\alpha,\beta\in\{0,1\}$, we will get:

$$\left.\begin{matrix}\begin{pmatrix}0-\\10\end{pmatrix}\\ \begin{pmatrix}-0\\01\end{pmatrix}\end{matrix}\right\}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\11\end{pmatrix},\quad \left.\begin{matrix}\begin{pmatrix}0-\\11\end{pmatrix}\\ \begin{pmatrix}-1\\00\end{pmatrix}\end{matrix}\right\}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\10\end{pmatrix},\quad \left.\begin{matrix}\begin{pmatrix}1-\\00\end{pmatrix}\\ \begin{pmatrix}-0\\11\end{pmatrix}\end{matrix}\right\}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\01\end{pmatrix},\quad \left.\begin{matrix}\begin{pmatrix}1-\\01\end{pmatrix}\\ \begin{pmatrix}-1\\10\end{pmatrix}\end{matrix}\right\}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--\\00\end{pmatrix}.$$

It should be mentioned that a number of transformed PSTF $Y^{\oplus}$ is determined by a number of binary positions in different part θ_1^{r-1} and θ_2^{r}, which conforms to theorem 1. Therefore for $d=2$ we have $k_Y=1$.

• Let $d=3$. Then for $\theta_1^{r-1}=(\sigma_1\cdots\bar{\alpha}_i\cdots\bar{\beta}_j\cdots -_k\cdots\sigma_n)$ and $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\tilde{\gamma}_k\cdots\sigma_n)$, $\theta_1^{r-1}=(\sigma_1\cdots\bar{\alpha}_i\cdots -_j\cdots\bar{\gamma}_k\cdots\sigma_n)$ and $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\tilde{\beta}_j\cdots\gamma_k\cdots\sigma_n)$ and $\theta_1^{r-1}=(\sigma_1\cdots -_i\cdots\bar{\beta}_j\cdots\bar{\gamma}_k\cdots\sigma_n)$ and $\theta_2^{r}=(\sigma_1\cdots\tilde{\alpha}_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)$, and respectively to the expressions $\bar{a}\bar{b}\oplus ab\tilde{c}=\bar{a}\bar{b}\oplus ab(1\oplus\bar{\tilde{c}})=ab\overline{\tilde{c}}\oplus\begin{cases}\bar{a}\oplus b\\ a\oplus\bar{b}\end{cases}$, $\bar{a}\bar{c}\oplus a\tilde{b}c=a\overline{\tilde{b}}c\oplus\begin{cases}\bar{a}\oplus c\\ a\oplus\bar{c}\end{cases}$ and $\bar{b}\bar{c}\oplus\tilde{a}bc=\overline{\tilde{a}}bc\oplus\begin{cases}\bar{b}\oplus c\\ b\oplus\bar{c}\end{cases}$, $\tilde{a}\in\{a,\bar{a}\}$, $\tilde{b}\in\{b,\bar{b}\}$, $\tilde{c}\in\{c,\bar{c}\}$, we have:

$$Y^{\oplus}=\{(\sigma_1\cdots\bar{\alpha}_i\cdots\bar{\beta}_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\tilde{\gamma}_k\cdots\sigma_n)\}^{\oplus}=$$
$$=\begin{Bmatrix}(\sigma_1\cdots\bar{\alpha}_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots\beta_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\overline{\tilde{\gamma}}_k\cdots\sigma_n)\\ (\sigma_1\cdots -_i\cdots\bar{\beta}_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\overline{\tilde{\gamma}}_k\cdots\sigma_n)\end{Bmatrix}^{\oplus},$$
$$Y^{\oplus}=\{(\sigma_1\cdots\bar{\alpha}_i\cdots -_j\cdots\bar{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\tilde{\beta}_j\cdots\gamma_k\cdots\sigma_n)\}^{\oplus}=$$
$$=\begin{Bmatrix}(\sigma_1\cdots\bar{\alpha}_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots -_j\cdots\gamma_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\overline{\tilde{\beta}}_j\cdots\gamma_k\cdots\sigma_n)\\ (\sigma_1\cdots -_i\cdots -_j\cdots\bar{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots\overline{\tilde{\beta}}_j\cdots\gamma_k\cdots\sigma_n)\end{Bmatrix}^{\oplus},$$
$$Y^{\oplus}=\{(\sigma_1\cdots -_i\cdots\bar{\beta}_j\cdots\bar{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots\tilde{\alpha}_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)\}^{\oplus}=$$
$$=\begin{Bmatrix}(\sigma_1\cdots -_i\cdots\bar{\beta}_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots -_j\cdots\gamma_k\cdots\sigma_n),(\sigma_1\cdots\overline{\tilde{\alpha}}_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)\\ (\sigma_1\cdots -_i\cdots -_j\cdots\bar{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots\beta_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\overline{\tilde{\alpha}}_i\cdots\beta_j\cdots\gamma_k\cdots\sigma_n)\end{Bmatrix}^{\oplus}.$$

The obtained interrelation $k_\theta^*/k_l^*=3/5$ is indicative of an increase in one conjuncterm for $k_\theta/k_l=2/5$. Whereas $k_Y=2$ and as a result of the fact that the given conjuncterms (8) have two binary positions in common which are transformed for $d=2$ according to the rule (3) of theorem 1. This is illustrated (look at dotted lines) $\begin{pmatrix}00-\\110\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0--\\-1-\\111\end{pmatrix},\begin{pmatrix}1--\\-0-\\111\end{pmatrix}\right\}$.

• Let $d=4$. Then on the ground of the considered above, taking into account the rule (4) of theorem 1 for $d=3$ (three binary positions are common), it is not hard to state that for

$\theta_1^{r-1}=(\sigma_1\cdots\bar{\alpha}_i\cdots\bar{\beta}_j\cdots\bar{\gamma}_k\cdots -_l\cdots\sigma_n)$ i $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\gamma_k\cdots\tilde{\delta}_l\cdots\sigma_n)$,

$\theta_1^{r-1}=(\sigma_1\cdots\bar{\alpha}_i\cdots\bar{\beta}_j\cdots -_k\cdots\bar{\delta}_l\cdots\sigma_n)$ i $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots\tilde{\gamma}_k\cdots\delta_l\cdots\sigma_n)$,

$\theta_1^{r-1}=(\sigma_1\cdots\bar{\alpha}_i\cdots -_j\cdots\bar{\gamma}_k\cdots\bar{\delta}_l\cdots\sigma_n)$ i $\theta_2^{r}=(\sigma_1\cdots\alpha_i\cdots\tilde{\beta}_j\cdots\gamma_k\cdots\delta_l\cdots\sigma_n)$,

$\theta_1^{r-1}=(\sigma_1\cdots -_i\cdots\bar{\beta}_j\cdots\bar{\gamma}_k\cdots\bar{\delta}\cdots\sigma_n)$ i $\theta_2^{r}=(\sigma_1\cdots\tilde{\alpha}_i\cdots\beta_j\cdots\gamma_k\cdots\delta_k\cdots\sigma_n)$, the interrelation $k_\theta^* / k_l^* = 4/10$, which compared with $k_\theta / k_l = 2/7$ is indicative of an increase of number of conjuncterms as well as their literals; $k_Y = 6$.

For example, $\begin{pmatrix} 000- \\ 1111 \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} 00-- \\ 0-1- \\ -11- \\ 1110 \end{pmatrix}, \begin{pmatrix} 00-- \\ -01- \\ 1-1- \\ 1110 \end{pmatrix}, \begin{pmatrix} 0-0- \\ 01-- \\ -11- \\ 1110 \end{pmatrix}, \begin{pmatrix} 0-0- \\ -10- \\ 11-- \\ 1110 \end{pmatrix}, \begin{pmatrix} -00- \\ 10-- \\ 1-1- \\ 1110 \end{pmatrix}, \begin{pmatrix} -00- \\ 1-0- \\ 11-- \\ 1110 \end{pmatrix} \right\}$, that corresponds to

$$\begin{pmatrix} 0,1 \\ 15 \end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} 0,1,2,3 \\ 2,3,6,7 \\ 6,7,14,15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0,1,2,3 \\ 2,3,10,11 \\ 10,11,14,15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0,1,4,5 \\ 4,5,6,7 \\ 6,7,14,15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0,1,4,5 \\ 4,5,12,13 \\ 12,13,14,15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0,1,8,9 \\ 8,9,10,11 \\ 10,11,14,15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0,1,8,9 \\ 8,9,12,13 \\ 12,13,14,15 \\ 14 \end{pmatrix} \right\}.$$

So, if a conjuncterm of $(r-1)$-rank θ_1^{r-1} differs from a conjuncterm of r-rank θ_2^r in the number d of different by value onename positions $\alpha, \beta, \gamma, \delta, ... \in \{0,1,-\}$, here dash (–) belongs to θ_1^{r-1}, $r \in \{1,2,...,n\}$, then $(d-1)!$ of sets PSTF $Y^\oplus$ will be formed, each of which has the interrelation $k_\theta^* / k_l^* = d/(d(d-1)-(d-2))$, that proves the truth of theorem 2. □

For example, let the function $f(x_1, x_2, x_3, x_4)$ has PSTF $Y^\oplus = \{(1-1-), (10-0), (\underline{000-}), (\underline{0110})\}^\oplus$, where distance between any pair of conjuncterms $d = 3$, a $k_\theta / k_l = 4/12$. If the function that has canonical STF $Y^1 = \{0,1,6,8,11,14,15\}^1$, is minimized in disjunctive format, then we get STF $Y^1 = \{(000-), (-000), (-110), (1-11)\}^1$, which also has $k_\theta / k_l = 4/12$. But such interrelation can be improved if theorem 2 is applied to the underlined pair PSTF $Y^\oplus$ namely rule (8), that is $\begin{pmatrix} 000- \\ 0110 \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ 0-1- \\ 0111 \end{pmatrix}$, and having got $Y^\oplus = \left\{ (\underline{1-1-}), (\underline{\underline{10-0}}), \begin{pmatrix} \underline{\underline{00--}} \\ \underline{0-1-} \\ 0111 \end{pmatrix} \right\}^\oplus$, one should apply to the underlined pairs of this set the rule (2) $\begin{pmatrix} 1-1- \\ 0-1- \end{pmatrix} \overset{\oplus}{\Rightarrow} (--1-)$ and the rule (7) $\begin{pmatrix} 10-0 \\ 00-- \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} -0-- \\ 10-1 \end{pmatrix}$. Then we get minimal PSTF $Y^\oplus = \{(--1-), (-0--), (10-1), (0111)\}^\oplus$, which if compared with the previous result has better interrelation $k_\theta / k_l = 4/9$.

Application of rules of theorems 1 and 2 is further illustrated by the examples.

Example 2. To apply the rules of theorems 1 and 2 to the function given by $f(x_1, x_2, x_3, x_4, x_5)$, PSTF $Y^\oplus$ (distances d between all pairs of conjuncterms are shown on the right of $Y^\oplus$)

$$Y^\oplus = \begin{Bmatrix} 0--00 \\ 0-0-0 \\ 1--11 \\ --110 \\ 00000 \\ 11110 \end{Bmatrix}^\oplus \begin{matrix} 2 & & & & \\ 3 & 4 & & & \\ 3 & 3 & 3 & & \\ 3 & 2 & 5 & 4 & \\ 4 & 4 & 3 & 5 & 4 \end{matrix}, \text{ where } k_\theta / k_l = 6/22.$$

Solution. We apply theorem 1 for $d=4$ to the minterms (00000) and (11110) having chosen from (5‴) the six transformed PSTF $Y^{\oplus}$: $\begin{pmatrix}00000\\11110\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}0-000\\01-00\\011-0\\-1110\end{pmatrix}$. Then after corresponding change we apply the rule (6) of theorem 2 to the pairs of elements of the formed set that have $d=1$, namely: $\begin{pmatrix}0--00\\01-00\end{pmatrix}\overset{\oplus}{\Rightarrow}(00-00)$, $\begin{pmatrix}0-0-0\\0-000\end{pmatrix}\overset{\oplus}{\Rightarrow}(0-010)$, $\begin{pmatrix}--110\\-1110\end{pmatrix}\overset{\oplus}{\Rightarrow}(-0110)$. As a result of this we get PSTF $Y^{\oplus}=\begin{Bmatrix}00-00\\0-010\\1--11\\-0110\\011-0\end{Bmatrix}^{\oplus}$, where $k_{\theta}^{*}/k_{l}^{*}=5/19$ is better than the initial $k_{\theta}/k_{l}=6/22$.

Example 3. To apply the rules of theorems 1 and 2 to the function $f(x_1,x_2,x_3,x_4)$, given by PSTF $Y^{\oplus}=\{(0-00),(1011),(--11),(-00-),(011-)\}^{\oplus}$, that has $k_{\theta}/k_{l}=5/14$.

Solution. All pairs of conjuncterms in given PSTF $Y^{\oplus}$ have $d\geq 3$, except $(1011),(--11)$, for $d=2$, transformation of which does not simplify the given set. Let us apply, for example, to the pair $(0-00),(1011)$, that has $d=4$, theorem 2, having chosen from the rule (13) the fifth set:

$\begin{pmatrix}0-00\\1011\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--00\\1-0-\\1--1\\1111\end{pmatrix}$. Now $Y^{\oplus}=\left\{\begin{pmatrix}\underline{--00}\\1-0-\\1--1\\1111\end{pmatrix},(\underline{--11}),(-00-),(011-)\right\}^{\oplus}$ and after the rule (3) $\begin{pmatrix}--00\\--11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}--0-\\---1\end{pmatrix}$ we have $Y^{\oplus}=\left\{\begin{pmatrix}\underline{--0-}\\\underline{1-0-}\\\underline{1--1}\\1111\end{pmatrix},(\underline{---1}),(-00-),(011-)\right\}^{\oplus}\Rightarrow\{(0-0-),(0--1),(1111),(\underline{-00-}),(\underline{011-})\}^{\oplus}$. Further, according to the rules (10) $\begin{pmatrix}-00-\\011-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-1--\\--0-\\111-\end{pmatrix}$ and (7) $\begin{pmatrix}0-0-\\--0-\end{pmatrix}\overset{\oplus}{\Rightarrow}(1-0-)$ and $\begin{pmatrix}1111\\111-\end{pmatrix}\overset{\oplus}{\Rightarrow}(1110)$ we will get a minimal PSTF $Y^{\oplus}=\{(1-0-),(0--1),(-1--),(1110)\}^{\oplus}$.

Answer. Cost of realization of the minimized function $k_{\theta}^{*}/k_{l}^{*}=4/9$ is better than $k_{\theta}/k_{l}=5/14$.

Let us consider now the situation when two conjuncterms of *r*-rank θ_1^r and θ_2^r differ in *d* onename binary positions $\alpha,\beta,\gamma,\delta,...\in\{0,1,-\}$, here each of them has one dash (–).

Theorem 3. Two conjuncterms of *r*-rank θ_1^r and θ_2^r, $r\in\{1,2,...,n\}$, of the function $f(x_1,x_2,...,x_n)$ differ in *d* onename binary positions $\alpha,\beta,\gamma,\delta,...\in\{0,1,-\}$, where each conjuncterm has one (–), in polynomial set-theoretical format starting with $d=2$, create $k_Y=(d-2)!$ of sets PSTF $Y^{\oplus}$, each of which has power $k_{\theta}^{*}=d$ and the total number of literals in the different part $k_{l}^{*}=d(d-1)-2(d-2)$, here:

• if $d=2$, than $\begin{pmatrix}\tilde\alpha & - \\ - & \tilde\beta\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix}\overline{\tilde\alpha} & - \\ - & \overline{\tilde\beta}\end{pmatrix}$, (15)

• where $d=3$, than $\begin{pmatrix}\tilde\alpha & \bar\beta & - \\ - & \beta & \tilde\gamma\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} - & - & - \\ \overline{\tilde\alpha} & \bar\beta & - \\ - & \beta & \overline{\tilde\gamma}\end{pmatrix}$, $\begin{pmatrix}\bar\alpha & \tilde\beta & - \\ \alpha & - & \tilde\gamma\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} - & - & - \\ \bar\alpha & \overline{\tilde\beta} & - \\ \alpha & - & \overline{\tilde\gamma}\end{pmatrix}$, $\begin{pmatrix}\tilde\alpha & - & \bar\gamma \\ - & \tilde\beta & \gamma\end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} - & - & - \\ \overline{\tilde\alpha} & - & \bar\gamma \\ - & \overline{\tilde\beta} & \gamma\end{pmatrix}$, (16), (17), (18)

• if $d=4$, than $\begin{pmatrix}\tilde\alpha & \bar\beta & \bar\gamma & - \\ - & \beta & \gamma & \tilde\delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} - & \bar\beta & - & - \\ - & - & \gamma & - \\ \overline{\tilde\alpha} & \bar\beta & \bar\gamma & - \\ - & \beta & \gamma & \overline{\tilde\delta}\end{pmatrix}, \begin{pmatrix} - & \beta & - & - \\ - & - & \bar\gamma & - \\ \overline{\tilde\alpha} & \bar\beta & \bar\gamma & - \\ - & \beta & \gamma & \overline{\tilde\delta}\end{pmatrix} \right\}$, $\begin{pmatrix}\bar\alpha & \tilde\beta & \bar\gamma & - \\ \alpha & - & \gamma & \tilde\delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar\alpha & - & - & - \\ - & - & \gamma & - \\ \bar\alpha & \overline{\tilde\beta} & \bar\gamma & - \\ \alpha & - & \gamma & \overline{\tilde\delta}\end{pmatrix}, \begin{pmatrix} \alpha & - & - & - \\ - & - & \bar\gamma & - \\ \bar\alpha & \overline{\tilde\beta} & \bar\gamma & - \\ \alpha & - & \gamma & \overline{\tilde\delta}\end{pmatrix} \right\}$, (19), (20)

$\begin{pmatrix}\bar\alpha & \bar\beta & \tilde\gamma & - \\ \alpha & \beta & - & \tilde\delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar\alpha & - & - & - \\ - & \beta & - & - \\ \bar\alpha & \bar\beta & \overline{\tilde\gamma} & - \\ \alpha & \beta & - & \overline{\tilde\delta}\end{pmatrix}, \begin{pmatrix} \alpha & - & - & - \\ - & \bar\beta & - & - \\ \bar\alpha & \bar\beta & \overline{\tilde\gamma} & - \\ \alpha & \beta & - & \overline{\tilde\delta}\end{pmatrix} \right\}$, $\begin{pmatrix}\tilde\alpha & \bar\beta & - & \bar\delta \\ - & \beta & \tilde\gamma & \delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} - & \bar\beta & - & - \\ - & - & - & \delta \\ \overline{\tilde\alpha} & \bar\beta & - & \bar\delta \\ - & \beta & \overline{\tilde\gamma} & \delta\end{pmatrix}, \begin{pmatrix} - & \beta & - & - \\ - & - & - & \bar\delta \\ \overline{\tilde\alpha} & \bar\beta & - & \bar\delta \\ - & \beta & \overline{\tilde\gamma} & \delta\end{pmatrix} \right\}$, (21), (22)

$\begin{pmatrix}\bar\alpha & \tilde\beta & - & \bar\delta \\ \alpha & - & \tilde\gamma & \delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} \bar\alpha & - & - & - \\ - & - & - & \delta \\ \bar\alpha & \overline{\tilde\beta} & - & \bar\delta \\ \alpha & - & \overline{\tilde\gamma} & \delta\end{pmatrix}, \begin{pmatrix} \alpha & - & - & - \\ - & - & - & \bar\delta \\ \bar\alpha & \overline{\tilde\beta} & - & \bar\delta \\ \alpha & - & \overline{\tilde\gamma} & \delta\end{pmatrix} \right\}$, $\begin{pmatrix}\tilde\alpha & - & \bar\gamma & \bar\delta \\ - & \tilde\beta & \gamma & \delta\end{pmatrix} \overset{\oplus}{\Rightarrow} \left\{ \begin{pmatrix} - & - & \bar\gamma & - \\ - & - & - & \delta \\ \overline{\tilde\alpha} & - & \bar\gamma & \bar\delta \\ - & \overline{\tilde\beta} & \gamma & \delta\end{pmatrix}, \begin{pmatrix} - & - & \gamma & - \\ - & - & - & \bar\delta \\ \overline{\tilde\alpha} & - & \bar\gamma & \bar\delta \\ - & \overline{\tilde\beta} & \gamma & \delta\end{pmatrix} \right\}$, (23),(24)

where $\tilde\alpha, \tilde\beta, \tilde\gamma, \tilde\delta$ – binary positions of any value 0 or 1.

Proof. Given PSTF $Y^{\oplus}$ has the initial interrelation $k_\theta / k_l = 2/2(d-1)$.

• Let $d=2$. Then $\theta_1^r = (\sigma_1 \cdots \tilde\alpha_i \cdots -_j \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots -_i \cdots \tilde\beta_j \cdots \sigma_n)$, $\tilde\alpha_i, \tilde\beta_j \in \{0,1\}$. For $f(a,b)$ respectively to (15) we have $\tilde a \oplus \tilde b = (\overline{\tilde a} \oplus 1) \oplus (\overline{\tilde b} \oplus 1) = \overline{\tilde a} \oplus \overline{\tilde b}$, that corresponds to PSTF

$$Y^{\oplus} = \{(\sigma_1 \cdots \tilde\alpha_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \tilde\beta_j \cdots \sigma_n)\}^{\oplus} = \{(\sigma_1 \cdots \overline{\tilde\alpha}_i \cdots -_j \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \overline{\tilde\beta}_j \cdots \sigma_n)\}^{\oplus}.$$

Here the interrelation $k_\theta / k_l = k_\theta^* / k_l^* = 2/2$ that is indicative of unchangeability of parameters of the transformed PSTF $Y^{\oplus}$, in which only inversion of different positions took place; $k_Y = 1$.

• Let $d=3$. Then $\theta_1^r = (\sigma_1 \cdots \tilde\alpha_i \cdots \bar\beta_j \cdots -_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots -_i \cdots \beta_j \cdots \tilde\gamma_k \cdots \sigma_n)$, $\theta_1^r = (\sigma_1 \cdots \bar\alpha_i \cdots \tilde\beta_j \cdots -_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \tilde\gamma_k \cdots \sigma_n)$ and $\theta_1^r = (\sigma_1 \cdots \tilde\alpha_i \cdots -_j \cdots \bar\gamma_k \cdots \sigma_n)$ and $\theta_2^r = (\sigma_1 \cdots -_i \cdots \tilde\beta_j \cdots \gamma_k \cdots \sigma_n)$, $\tilde\alpha_i, \tilde\beta_j, \tilde\lambda_k \in \{0,1\}$. For $f(a,b,c)$ respectively to (16–18) we have: $\tilde a \bar b \oplus b \tilde c = (\overline{\tilde a} \oplus 1)\bar b \oplus b(\overline{\tilde c} \oplus 1) = \overline{\tilde a}\bar b \oplus b\overline{\tilde c} \oplus 1$, $\bar a \tilde b \oplus a \tilde c = \bar a \overline{\tilde b} \oplus a\overline{\tilde c} \oplus 1$ and $\tilde a \bar c \oplus \tilde b c = \overline{\tilde a}\bar c \oplus \overline{\tilde b} c \oplus 1$, $\tilde a \in \{a, \bar a\}$, $\tilde b \in \{b, \bar b\}$, $\tilde c \in \{c, \bar c\}$. So, corresponding PSTF $Y^{\oplus}$ will look like:

$$Y^{\oplus} = \{(\sigma_1 \cdots \tilde\alpha_i \cdots \bar\beta_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots \tilde\gamma_k \cdots \sigma_n)\}^{\oplus} =$$

$$= \{(\sigma_1 \cdots -_i \cdots -_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \overline{\tilde\alpha}_i \cdots \bar\beta_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots -_i \cdots \beta_j \cdots \overline{\tilde\gamma}_k \cdots \sigma_n)\}^{\oplus},$$

$$Y^{\oplus} = \{(\sigma_1 \cdots \bar\alpha_i \cdots \tilde\beta_j \cdots -_k \cdots \sigma_n), (\sigma_1 \cdots \alpha_i \cdots -_j \cdots \tilde\gamma_k \cdots \sigma_n)\}^{\oplus} =$$

$$=\{(\sigma_1\cdots -_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\overline{\alpha}_i\cdots\overline{\widetilde{\beta}}_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\alpha_i\cdots -_j\cdots\overline{\widetilde{\gamma}}_k\cdots\sigma_n)\}^{\oplus},$$

$$Y^{\oplus}=\{(\sigma_1\cdots\widetilde{\alpha}_i\cdots -_j\cdots\overline{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots\widetilde{\beta}_j\cdots\gamma_k\cdots\sigma_n)\}^{\oplus}=$$

$$=\{(\sigma_1\cdots -_i\cdots -_j\cdots -_k\cdots\sigma_n),(\sigma_1\cdots\overline{\widetilde{\alpha}}_i\cdots -_j\cdots\overline{\gamma}_k\cdots\sigma_n),(\sigma_1\cdots -_i\cdots\overline{\widetilde{\beta}}_j\cdots\gamma_k\cdots\sigma_n)\}^{\oplus}.$$

Compared with $k_\theta / k_l = 2/4$ here the interrelation $k^*_\theta / k^*_l = 3/4$ means that the transformed PSTF $Y^{\oplus}$ has one more conjuncterm, here its rank is $(r-3)$; $k_Y = 1$. For example, for some values of variables $\alpha,\beta,\gamma\in\{0,1\}$ of the function $f(\alpha,\beta,\gamma)$ we will get such PSTF $Y^{\oplus}$:

$$(16)\ \begin{pmatrix}00-\\-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\10-\\-10\end{pmatrix},\ \begin{pmatrix}-00\\11-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\-01\\01-\end{pmatrix};\qquad (17)\ \begin{pmatrix}00-\\1-1\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\01-\\1-0\end{pmatrix},\ \begin{pmatrix}0-0\\11-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\0-1\\10-\end{pmatrix};$$

$$(18)\ \begin{pmatrix}0-0\\-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\1-0\\-01\end{pmatrix},\ \begin{pmatrix}-00\\1-1\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}---\\-10\\0-1\end{pmatrix}.$$

- Let $d=4$. Then $\theta^r_1=(\sigma_1\cdots\widetilde{\alpha}_i\cdots\overline{\beta}_j\cdots\overline{\gamma}_k\cdots -_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots -_i\cdots\beta_j\cdots\gamma_k\cdots\widetilde{\delta}_l\cdots\sigma_n)$,

$\theta^r_1=(\sigma_1\cdots\overline{\alpha}_i\cdots\widetilde{\beta}_j\cdots\overline{\gamma}_k\cdots -_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots\alpha_i\cdots -_j\cdots\gamma_k\cdots\widetilde{\delta}_l\cdots\sigma_n)$,

$\theta^r_1=(\sigma_1\cdots\overline{\alpha}_i\cdots\overline{\beta}_j\cdots\widetilde{\gamma}_k\cdots -_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots\alpha_i\cdots\beta_j\cdots -_k\cdots\widetilde{\delta}_l\cdots\sigma_n)$,

$\theta^r_1=(\sigma_1\cdots\widetilde{\alpha}_i\cdots\overline{\beta}_j\cdots -_k\cdots\overline{\delta}_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots -_i\cdots\beta_j\cdots\widetilde{\gamma}_k\cdots\delta_l\cdots\sigma_n)$,

$\theta^r_1=(\sigma_1\cdots\overline{\alpha}_i\cdots\widetilde{\beta}_j\cdots -_k\cdots\overline{\delta}_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots\alpha_i\cdots -_j\cdots\widetilde{\gamma}_k\cdots\delta_l\cdots\sigma_n)$,

$\theta^r_1=(\sigma_1\cdots\widetilde{\alpha}_i\cdots -_j\cdots\overline{\gamma}_k\cdots\overline{\delta}_l\cdots\sigma_n)$ and $\theta^r_2=(\sigma_1\cdots -_i\cdots\widetilde{\beta}_j\cdots\gamma_k\cdots\delta_l\cdots\sigma_n)$.

The transformations (19–24) are formed on the ground of respective expressions for $f(a,b,c,d)$:

$$\widetilde{a}\overline{b}\overline{c}\oplus bc\widetilde{d}=(\overline{\widetilde{a}}\oplus 1)\overline{b}\overline{c}\oplus bc(\overline{\widetilde{d}}\oplus 1)=\overline{\widetilde{a}}\overline{b}\overline{c}\oplus bc\overline{\widetilde{d}}\oplus\overline{b}\overline{c}\oplus bc=\overline{\widetilde{a}}\overline{b}\overline{c}\oplus bc\overline{\widetilde{d}}\oplus\begin{cases}\overline{b}\oplus c\\ b\oplus\overline{c}\end{cases},$$

$$\overline{a}\widetilde{b}\overline{c}\oplus ac\widetilde{d}=\overline{a}\overline{\widetilde{b}}\overline{c}\oplus ac\overline{\widetilde{d}}\oplus\begin{cases}\overline{a}\oplus c\\ a\oplus\overline{c}\end{cases},\ \overline{a}\overline{b}\widetilde{c}\oplus ab\widetilde{d}=\overline{a}\overline{b}\overline{\widetilde{c}}\oplus ab\overline{\widetilde{d}}\oplus\begin{cases}\overline{a}\oplus b\\ a\oplus\overline{b}\end{cases},$$

$$\widetilde{a}\overline{b}\overline{d}\oplus b\widetilde{c}d=\overline{\widetilde{a}}\overline{b}\overline{d}\oplus b\overline{\widetilde{c}}d\oplus\begin{cases}\overline{b}\oplus d\\ b\oplus\overline{d}\end{cases},\ \overline{a}\widetilde{b}\overline{d}\oplus a\widetilde{c}d=\overline{a}\overline{\widetilde{b}}\overline{d}\oplus a\overline{\widetilde{c}}d\oplus\begin{cases}\overline{a}\oplus d\\ a\oplus\overline{d}\end{cases},$$

$$\widetilde{a}\overline{c}\overline{d}\oplus\widetilde{b}cd=\overline{\widetilde{a}}\overline{c}\overline{d}\oplus\overline{\widetilde{b}}cd\oplus\begin{cases}\overline{c}\oplus d\\ c\oplus\overline{d}\end{cases}\ \text{and,}\ \widetilde{a}\in\{a,\overline{a}\},\ \widetilde{b}\in\{b,\overline{b}\},\ \widetilde{c}\in\{c,\overline{c}\},\ \widetilde{d}\in\{d,\overline{d}\}.$$

Not giving general expressions of the formed PSTF $Y^{\oplus}$, that are evident from the considered above, we will illustrate the transformations (19–24) for some values $\alpha,\beta,\gamma,\delta\in\{0,1\}$ of the function $f(\alpha,\beta,\gamma,\delta)$:

$$(19\text{–}21)\ \begin{pmatrix}000-\\-111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-0--\\--1-\\100-\\-110\end{pmatrix},\begin{pmatrix}-1--\\--0-\\100-\\-110\end{pmatrix}\right\};\ \begin{pmatrix}000-\\1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0---\\--1-\\010-\\1-10\end{pmatrix},\begin{pmatrix}1---\\--0-\\010-\\1-10\end{pmatrix}\right\};\ \begin{pmatrix}000-\\11-1\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0---\\-1--\\001-\\11-0\end{pmatrix},\begin{pmatrix}1---\\-0--\\001-\\11-0\end{pmatrix}\right\};$$

$$(22\text{–}24)\ \begin{pmatrix}00-0\\-111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}-0--\\---1\\10-0\\-101\end{pmatrix},\begin{pmatrix}-1--\\---0\\10-0\\-101\end{pmatrix}\right\};\ \begin{pmatrix}00-0\\1-11\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}0---\\---1\\01-0\\1-01\end{pmatrix},\begin{pmatrix}1---\\---0\\01-0\\1-01\end{pmatrix}\right\};\ \begin{pmatrix}0-00\\-111\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}--0-\\---1\\1-00\\-011\end{pmatrix},\begin{pmatrix}--1-\\---0\\1-00\\-011\end{pmatrix}\right\}.$$

Dotted lines in (19) show general tendency of the formed PSTF $Y^{\oplus}$, which consists in formation of two subsets: one respectively to theorem 1 (for $d=4$ according to the rule (3)), and another one that repeats given conjuncterms with certain inverse positions. This is illustrated by the example of formation of PSTF $Y^{\oplus}$ for $d=5$:

$$\begin{pmatrix}0000-\\-1111\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-00--\\-0-1-\\--11-\\1000-\\-1110\end{pmatrix},\ \text{that corresponds to}\ \begin{pmatrix}0,1\\15,31\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}0,1,2,3,16,17,18,19\\2,3,6,7,18,19,22,23\\6,7,14,15,22,23,30,31\\16,17\\14,30\end{pmatrix}.$$

Here, for $d=4$ the interrelation $k_{\theta}^{*}/k_{l}^{*}=4/8$ is greater than the initial one $k_{\theta}/k_{l}=2/6$; $k_{Y}=2$.

So, if two conjuncterms of r-rank θ_1^r i θ_2^r, $r\in\{1,2,...,n\}$, of the function $f(x_1,x_2,...,x_n)$ differ in d different by values onename positions $\alpha,\beta,\gamma,\delta,...\in\{0,1,-\}$, among which each of these conjuncterms has one dash (–), then in polynomial set-theoretical format, starting with $d=2$, they form $k_Y=(d-2)!$ of the sets PSTF $Y^{\oplus}$, each of them has $k_{\theta}^{*}/k_{l}^{*}=d/(d(d-1)-2(d-2))$, and this is proved by theorem 3. □

Application of the rules of theorems 1, 2 and 3 will be illustrated by the examples.

Let PSTF of the function $f(x_1,x_2,x_3,x_4)$ $Y^{\oplus}=\{(111-),(0-00),(01--),(--1-)\}^{\oplus}$, the cost of realization of which $k_{\theta}/k_{l}=4/9$. If the given PSTF $Y^{\oplus}$ is simplified only according to the rules for $d\le 3$, for example, to the pair (111–) i (01––), that has $d=2$, the rule (7) of theorem 2, is applied, namely $\begin{pmatrix}111-\\01--\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-1--\\110-\end{pmatrix}$, we will get PSTF $Y^{\oplus}=\left\{(0-00),(--1-),\begin{pmatrix}-1--\\110-\end{pmatrix}\right\}^{\oplus}$ with k_{θ}^{*}/k_{l}^{*} = 4/8. However, if to the pair (111–) and $(0-00)$, that has $d=4$, the second set of the rule (20) of theorem 3 is applied $\begin{pmatrix}111-\\0-00\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}0---\\--1-\\101-\\0-01\end{pmatrix}$, then after respective transformation we will get

$$Y^{\oplus}=\{(\underline{111-}),(\underline{0-00}),(01--),(--1-)\}^{\oplus}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}\underline{0---}\\\underline{\underline{--1-}}\\101-\\0-01\end{pmatrix},(\underline{01--}),(\underline{\underline{--1-}})\right\}\overset{\oplus}{\Rightarrow}\{(\underline{00--}),(\underline{101-}),(0-01)\}^{\oplus}.$$

Here, having applied to the pair $(00--)$ and (101–), that has $d=3$, the rule (7) of theorem 2, namely $\begin{pmatrix}00--\\101-\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}-0--\\100-\end{pmatrix}$, we will get the final minimal PSTF of the given function f

$Y^{\oplus}=\{(-0--),(100-),(0-01)\}^{\oplus}$, The cost of realization of which $k_\theta^* / k_l^* = 3/7$ is better than the previous one.

Example 4. To apply the theorems 1, 2 and 3 to the function $f(x_1, x_2, x_3, x_4)$, that is given by perfect STF $Y^1 = \{0,3,5,6,7,8,9,10,12,15\}^1$, which is minimized in polynomial format by *K*-maps method to the expression $f = x_1 \oplus x_2x_3 \oplus x_2x_4 \oplus x_3x_4 \oplus x_1x_2x_3x_4 \oplus \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4$ [32, c. 97].

Solution. This function has PSTF $Y^\oplus = \{(1---),(-11-),(-1-1),(--11),(1111),(0000)\}^\oplus$. To the pair (1111) i (0000), that has $d = 4$, we will apply, for example the fourth PSTF from the rule (5′):

$$Y^\oplus = \left\{ (1---),(-11-),(\underline{-1-1}),(\underline{\underline{--11}}),\begin{pmatrix} 000- \\ 0-01 \\ \underline{01-1} \\ \underline{\underline{-111}} \end{pmatrix} \right\}^\oplus .$$

Applying to the underlined pairs that have $d = 1$, the rule (6) of theorem 2, namely $\begin{pmatrix} -1-1 \\ 01-1 \end{pmatrix}^\oplus \Rightarrow (11-1)$, $\begin{pmatrix} --11 \\ -111 \end{pmatrix}^\oplus \Rightarrow (-011)$, and the rule (7), in the formed set namely $\begin{pmatrix} -011 \\ -11- \end{pmatrix}^\oplus \Rightarrow \begin{pmatrix} --1- \\ -010 \end{pmatrix}$, we will get PSTF $Y^\oplus = \{(1---),(--1-),(\underline{-010}),(\underline{000-}),(11-1),(\underline{\underline{0-01}})\}^\oplus$. Doing further transformations according to the rules (16) and (17) of theorems 3, namely $\begin{pmatrix} -010 \\ 000- \end{pmatrix}^\oplus \Rightarrow \begin{pmatrix} -0-- \\ \underline{\underline{100-}} \\ -011 \end{pmatrix}$ and $\begin{pmatrix} 100- \\ 0-01 \end{pmatrix}^\oplus \Rightarrow \begin{pmatrix} --0- \\ 0-00 \\ 110- \end{pmatrix}$, we will get the final minimal PSTF $Y^\oplus = \{(1---),(--1-),(-0--),(--0-),(-011),(0-00),(110-),(11-1)\}^\oplus \Rightarrow$

$\Rightarrow (1---),(-1--),(-011),(0-00),(110-),(11-1)\}^\oplus$.

Now the cost of realization of the minimized function $f = x_1 \oplus x_2 \oplus \bar{x}_2x_3x_4 \oplus \bar{x}_1\bar{x}_3\bar{x}_4 \oplus x_1x_2\bar{x}_3 \oplus x_1x_2x_4$ is equal to $k_\theta^* / k_l^* = 6/14$ that is a better result if compared to [32], where $k_\theta / k_l = 6/15$.

1.3. Estimate of efficiency of application of the suggested rules

Let us estimate efficiency of application of theorems 1, 2 i 3 for simplification of the sets of conjuncterms of the function *f* in polynomial set-theoretical format. On the ground of the considered above one can draw such a conclusion:

- the number of the conjuncterms composing the transformed k_θ^*, PSTF $Y^\oplus$ of the function *f*, is directly proportional to distance *d* between the pair of the given conjuncterms which in the different part have the total number of the literals $k_l = \{2d, (2d-1), 2(d-1)\}$, that is $k_\theta^* = d$;
- the power k_Y of the set of the transformed PSTF $Y^\oplus$ in the combinative way depends on distance *d*, here if $k_l = 2d$, then $k_Y = d!$; if $k_l = 2d-1$, then $k_Y = (d-1)!$; starting with $d = 2$, if $k_l = 2(d-1)$, then $k_Y = (d-2)!$;
- the quantitative estimation of efficiency of application of theorem 1 (*T*1), theorem 2 (*T*2) and theorem 3 (T3) is shown in the table of dependence of the interrelation of $k_l / k_l^* / k_Y$ on distance *d*:

d	*T*1	*T*2	*T*3
1	2/0/1	1/1/1	–
2	4/2/2	3/2/1	2/2/1
3	6/6/6	5/5/2	4/4/1
4	8/12/24	7/10/6	6/8/2
5	10/20/120	9/17/24	8/14/6
6	12/30/720	11/26/120	10/22/24
...	...	...	...

From the data of the table it is seen that starting with $d = 3$, simplification of two conjuncterms of the function *f* does not take place. This is eventually stated by the authors [7,14,15,21,24], who on this ground draw a conclusion that it is not expedient to do further minimization in polynomial format of the function *f*, if between any pair of its conjuncterms distance $d \geq 3$. However, it is evident from the considered above that this statement is false. The given examples in this and next papers on this theme are indicative of the fact that application of the suggested theorems to the function *f*, which is not given by two but greater number of conjuncterms of different ranks between any pairs of which distance *d* is different it is quite possible to do further simplification. The explanation is set that the set of the transformed PSTF $Y^{\oplus}$, by which the chosen pair with distance $d \geq 3$ is replaced, can have elements which together with other elements of the given function *f* will form pairs with distance $d < 3$. Here, in spite of an increase of power of the newformed set, one can get a minimal PSTF $Y^{\oplus}$ after application of respective rules of the suggested theorems to the pairs with little *d*. Besides with an increase of *d* the probability of simplifications is greater as it is seen from the table here the power of k_Y formed PSTF $Y^{\oplus}$ increases, so there is a better choice of useful for simplification elements. The procedure of search of such elements has a combinative character – after each replacement of a chosen pair of conjuncterms of the given PSTF $Y^{\oplus}$ for certain set of the transformed PSTF $Y^{\oplus}$ we get a new set in which it is necessary to determine distance *d* between new pairs and having chosen from them the elements with minimal *d*, to apply the rules of respective theorem and build again a new set and so on and so forth (*algorithm description of suggested minimization method in the next paper 2*).

It should also be noted that application of the suggested rules of simplification of a set of conjuncterms of the given PSTF $Y^{\oplus}$ to the pairs with distance $d \geq 3$ makes it possible to get the searched result with fewer number of steps of simplification procedure. Let us illustrate this by the examples.

Example 5. To apply the suggested theorems to the function $f(x_1, x_2, x_3, x_4)$, given by PSTF $Y^{\oplus} = \{(000-), (0-11), (-11-), (1010)\}^{\oplus}$, on the example of which [20*, example 8*, p. 388] the authors show efficiency of *exorlink method.*

Solution. Let the difference $d = 3$ between any pair of conjuncterms of the given PSTF $Y^{\oplus}$. The minimal PSTF $Y^{\oplus} = \{(00--), (--10), (1111)\}^{\oplus}$ is got within five procedure steps with the help of *exorlink* method. We will show that the same result can be got within 4 steps

$$Y^{\oplus} = \begin{pmatrix} \mathbf{000-} \\ 0-11 \\ -11- \\ \mathbf{1010} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ \mathbf{-01-} \\ 1011 \\ 0-11 \\ \mathbf{-11-} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ --1- \\ \mathbf{1011} \\ \mathbf{0-11} \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ \mathbf{--1-} \\ \mathbf{--11} \\ 1111 \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ --10 \\ 1111 \end{pmatrix},$$

where in bold font the elements are highlited which are formed according to certain rule of one from the suggested theorems. For example, for the first step the chosen pair $\begin{pmatrix} 000- \\ 1010 \end{pmatrix}$ is transformed according to the rule (8) of theorem 2 so: $\begin{pmatrix} 000- \\ 1010 \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} 00-- \\ -01- \\ 1011 \end{pmatrix}$. After respective change in the second step we apply the rule (2) of theorem 1 to the pair $\begin{pmatrix} -01- \\ -11- \end{pmatrix} \overset{\oplus}{\Rightarrow} (--1-)$, in the third step the rule (7) of theorem 2 to the pair $\begin{pmatrix} 1011 \\ 0-11 \end{pmatrix} \overset{\oplus}{\Rightarrow} \begin{pmatrix} --11 \\ 1111 \end{pmatrix}$ and in the fourth step the rule (2) of theorem 1 to $\begin{pmatrix} --1- \\ --11 \end{pmatrix} \overset{\oplus}{\Rightarrow} (--10)$.

Example 6. To apply the suggested theorems to the function $f(a,b,c,d)=\overline{a}\overline{c}\oplus\overline{a}b\overline{c}\overline{d}\oplus ab\oplus a\overline{c}d$, on the example of which the authors [15, p. 6] show on the *K*-maps efficiency of their suggested procedure (*look-ahead strategies*of the improved *exorlink method*.

Solution. The given function *f* has PSTF $Y^{\oplus}=\{(0-0-),(0100),(11--),(1-01)\}^{\oplus}$. If we apply in the first step the rule (9) of theorem 2 to the elements $\begin{pmatrix}0100\\1-01\end{pmatrix}$, that have $d=3$, namely $\begin{pmatrix}0100\\1-01\end{pmatrix}\overset{\oplus}{\Rightarrow}\left\{\begin{pmatrix}1-0-\\--00\\0000\end{pmatrix},\begin{pmatrix}0-0-\\--01\\0000\end{pmatrix}\right\}$, then we get two variants:

$$1)\ Y^{\oplus}=\begin{pmatrix}0-0-\\\mathbf{0100}\\11--\\\mathbf{1-01}\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}\mathbf{0-0-}\\11--\\\mathbf{1-0-}\\--00\\0000\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}11--\\--\mathbf{0}-\\--\mathbf{00}\\0000\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}11--\\--01\\0000\end{pmatrix}\ \text{and}\ 2)\ Y^{\oplus}=\begin{pmatrix}0-0-\\\mathbf{0100}\\11--\\\mathbf{1-01}\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}\mathbf{0-0-}\\11--\\\mathbf{0-0-}\\--01\\0000\end{pmatrix}\overset{\oplus}{\Rightarrow}\begin{pmatrix}11--\\--01\\0000\end{pmatrix},$$

out of which the second solution, as we see it, is simpler and shorter. So, the given function has the minimal PSTF $Y^{\oplus}=\{(11--),(--01),(0000)\}^{\oplus}$, to which corresponds $f(a,b,c,d)=\overline{a}\overline{b}\overline{c}\overline{d}\oplus ab\oplus\overline{c}d$. In [15] the same result is got in a more complicated way and within a greater number of steps.

Conclusions

The generalized set-theoretical rules of simplification in the polynomial format of a set of conjuncterms of different ranks of a logic function with *n* variables have been suggested. These rules are based on three theorems for certain initial conditions of transformation of a pair of conjuncterms with any Hamming distance between them. Efficiency of the suggested rules is proved by given in the paper examples which have been borrowed from the papers by well-known authors with the aim of comparison and which give the ground to confirm their expediency in application for minimization of any logic function from *n* variables in polynomial format.

1. *Besslish P.W*. Efficient computer method for EXOR logic design // IEEE Proc. Pt. E. – 1983. – **130**. – P. 203–206.
2. *Sasao T.* Switching Theory for Logic Synthesis. – Kluwer Acad. Publ. – 1999. – 361 p.
3. *Papakonstantinou G*. A Parallel algorithm for minimizing ESOP expressions // J. Circuits Syst. Comp. – 2014. – **23**, issue 01. – 17 p.
4. *Saul J.* Logic synthesis for arithmetic circuits using the Reed–Muller representation / Proc. of Europ. Conf. on Design Automation // IEEE Comp. Society Press, March 1992.
5. *Perkowski M.*, *Chrzanowska–Jeske M.* An exact algorithm to minimize mixed–radix exclusive sums of products for incompletely specified boolean functions // Proc. Int. Symp. Circuits Syst. – New Orleans, LA, May 1990. – P. 1652–1655.
6. *Tsai C.*, *Marek-Sadowska M.* Multilevel Logic Synthesis for Arithmetic Functions // Proc. DAC'96. – June 1996. – P. 242–247.
7. *Takashi Hirayama*, *Yasuaki Nishitani*. Exact minimization of AND-EXOR expressions of practical benchmark functions // J. of Circuits, Systems and Computers. – 2009. – **18**, N 3. – P. 465–486.
8. *Debnath D.*, *Sasao T.* Output phase optimization for AND-OR-EXOR PLAs with decoders and its application to design of adders // IEICE Trans. Inf. & Syst. – July 2005. – **E88-D**, N 7. – P. 1492–1500.
9. *Fujiwara H.* Logic testing and design for testability // Comp. Syst. Series. – Cambridge, MA: Mass. Inst. Tech., 1986.
10. *Sasao T.* Easily testable realizations for generalized Reed–Muller expressions // IEEE Trans. On Computers. – 1997. – **46**, N 6. – P. 709–716.
11. *Faraj Khalid*. Design Error Detection and Correction System based on Reed–Muller Matrix for Memory Protection // J. of Comp. Appl. (0975-8887). – Nov. 2011. – **34**, N 8. – P. 42–55.
12. *Exact* ESOP expressions for incompletely specified functions / M. Sampson, M. Kalathas, D. Voudouris et al. // VLSI J. – 2012. – **45**, issue 2. – P. 197–204.

13. *Stergiou S.*, *Papakonstantinou G.* Exact minimization of ESOP expressions with less than eight product terms // J. of Circuits, Systems and Comp. – 2004. – **13**, N 1. – P. 1–15.
14. *Debnath D.*, *Sasao T.* A New Relation of Logic Functions and Its Application in the Desing of AND-OR-EXOR Networks // IEICE Trans. Fundamentals. – May 2007. – **E90-A**, N 5. – P. 932 – 939.
15. *Mishchenko A.*, *Perkowski M*. Fast Heuristic Minimization of Exclusive-Sums-of-Products // Proc. Reed–Muller Inter. Workshop'01. – 2001. – P. 242–250.
16. *Wu X.*, *Chen X.*, *Hurst S. L*. Mapping of Reed–Muller coefficients and the minimization of exclusive-OR switching function // IEEE Proc. Pt. E. – Jan. 1982. – **129**. – P. 5–20.
17. *Fleisher H.*, *Tavel M.*, *Yager J.* A computer algorithm for minimizing Reed–Muller cannonical forms // IEEE Trans. Comput. – Feb. 1987. – **C-36**. – P. 247–250.
18. *Even S.*, *Kohavi I.*, *Paz A*. On minimal modulo-2-sum of products for switching functions // Ibid. – Oct. 1967. – **EC-16**. – P. 671–674.
19. *Helliwell M.*, *Perkowski M.* A fast algorithm to minimize mixed polarity generalized Reed–Muller forms. In Proceedings of the 25th ACM/IEEE Design Automation Conference // IEEE Comp. Society Press. – 1988. – P. 427–432.
20. *Song N.*, *Perkowski M*. Minimization of Exclusive Sum-of-Products Expressions for Multiple-Valued Input, Incompletely Specified Functions // IEEE Trans. Comput.-Aided Design of Integrated Circuits and Systems. – April, 1996. – **15**, N 4. – P. 385–395.
21. *Saul J.* Logic synthesis based on the Reed–Muller representation, 1991 – http://citeseer.uark.edu:8080/citeerx/viewdoc/summary;jsessionid=A765F8A29F4FB9D2143A1DB7CDC91593?doi=10.1.1.45.8570
22. *Zakrenskij A*. Minimum Polynomial Implementation of Systems of Incompletely Specified Functions // Proc. of IFIP WG 10.5 Workshop on Applications of Reed–Muller Expansion in Circuits Design 1995, Japan. – P. 250–256.
23. *Brand D*., *Sasao T*. Minimization of AND-EXOR Using Rewrite Rules // IEEE Trans. on Comp. – May 1993. – **42**, N 5.
24. *Knysh D.*, *Dubrova E*. Rule-Based Optimization of AND-XOR Expressions // Facta Universitatis (Nis), Ser.: Elec. Energ. – Dec. 2011. – **24**, N 3. – P. 437–449.
25. *Wang L*. Automated Synthesis and Optimization of Multilevel Logic Circuits, 2000 – http://researchrepository.napier.ac.uk/4342/1/Wang.pdf
26. *Stergiou S.*, *Daskalakis K.*, *Papakonstantinou G.* A Fast and Efficient Heuristic ESOP Minimization Algorithm // GLSVLSI'04. – Boston, Mass., USA, April 26–28 2004.
27. *Рицар Б.Є*. Мінімізація бульових функцій методом розчеплення кон'юнктермів // УСИМ. – 1998. – № 5. – С. 14–22.
28. *Рицар Б.Є*. Теоретико-множинні оптимізаційні методи логікового синтезу комбінаційних мереж: Дис. д. т.н. – Львів, 2004. – 348 с.
29. *Рицар Б.Є.* Мінімізація системи логікових функцій методом паралельного розчеплення кон'юнктермів // Вісн. НУ ЛП «Радіоелектроніка та телекомунікації». – 2013. – № 766. – С. 18–27.
30. *Рицар Б.Є.* Числова теоретико-множинна інтерпретація полінома Жеґалкіна // УСИМ. – 2013. – № 1. – С. 11–26.
31. *Рицар Б.Є.* Візерунки бульових функцій: метод мінімізації // УСИМ. – 2007. – № 3. – С. 34–51.
32. *Tran A.* Graphical method for the conversion of minterms to Reed–Muller coefficients and the minimization of exclusive-OR switching functions // IEEE Proc. – March 1987. – **134**, Pt.E, N 2. – P. 93–99.
33. *Tinder R.F*. Engineering digital design. – Academic Press, 2000. – 884 p.

Поступила 30.12.2014
E-mail: *bohdanrytsar@gmail.com*

●