UDC 517.9

A. Chichurin, Full Doctor, Prof.
KUL, Lublin, Poland, Lesya Ukrainka Eastern European National University, Lutsk, Ukraine,
G. Stepaniuk, researcher
Lesya Ukrainka Eastern European National University, Lutsk, Ukraine

GENERAL SOLUTION TO THE FOURTH ORDER LINEAR DIFFERENTIAL EQUATION WITH COEFFICIENTS SATISFYING THE SYSTEM OF THREE THE FIRST ORDER DIFFERENTIAL EQUATIONS

In this paper we obtain the general solution of the fourth order linear differential equation with coefficients satisfying the system of three the first order differential equations.

1. Introduction

The analytical method of the integration of the fourth order linear differential equation, which is considered in this paper, was proposed by N. A. Lukashevich [7] while studying the linear differential equation of the third order. The gist of this method is as follows: the general solution of the linear equation is searched in the form of product

$$y = \xi(x)\cdot y_1(x), \tag{1}$$

where $y_1(x)$ is arbitrary partial solution of the linear equation, $\xi(x)$ is sufficiently smooth a function. Then, using a special procedure [2,7,8], the linear differential equation of the third order is reduced to the second order nonlinear differential equation of the form

$$(12i + b(x))i'' = 15i'^2 - h_0(x)i' - 8i^3 - h_1(x)i^2 - h_2(x)i - h_3(x) \tag{2}$$

with respect to the new unknown function $i(x)$, which is connected with the function $\xi(x)$ by means of Schwarzian derivative using the relation

$$\frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = i(x)\,. \tag{3}$$

Remark 1. The using Schwarzian derivative is an effective means for solving many mathematical problems. Among them let's mention a classical problem of conformal mapping of the polygons, sides of which are the arcs of circles, the theory of univalent functions, the theory of quadratic differentials, problems of the nonlinear dynamics, etc. In the papers [5, 9] the applications of Schwarzian derivative to the studying of the dynamics of mappings of the segments are given. Application examples of the generalized Schwarzian derivative in solving the problems about softness or stiffness of the Andronov-Hopf bifurcations are given in the paper [13]. Some other applications of Schwarzian derivative are described in the papers [1, 10].

Remark 2. Under certain coefficient ratios the equation (2) can be reduced to the XXV Painlevé equation in the Ince classification [6].

Detailed study of the homogeneous linear differential equation of the third order by the instrumentality of the relations (1), (3) and the equation (2) is given in the paper [2]. There's also given the generalization of studying method on the fourth order linear equations of the form

$$y^{(IV)} + p(x)y''' + q(x)y'' + r(x)y' + s(x)y = 0\,. \tag{4}$$

In this case the appropriate nonlinear differential equation with respect to the function i(x) is the fourth order equation and is quite cumbersome (denote it (A)). The explicit form of the equation (A) is given in the paper [2].

It should be noted, that if the functions $\xi(x)$ and $i(x)$ are known, then partial solution of the equation (4) can be found as the general solution of the first order linear differential equation of the form [3]

$$\begin{aligned}
&y_1'(6ip^4 - 3rp^3 + 3q'p^3 + 36i^2p^2 + q^2p^2 - 32iqp^2 - 6sp^2 - 3r'p^2 + 9i''p^2 + 3q''p^2 + 20irp + 14qrp + 20iq'p -\\
&-10qq'p - 30ip''p - 3qp''p + 80i^3 - 4q^3 + 36iq^2 - 18r^2 - 150i'^2 + 6(6i-q)p'^2 - 96i^2q - 80is + 16qs +\\
&+12rq' - 40ir' + 8qr' + 120ii'' - 24qi'' + 15i'(p^3 - 4qp + 3p'p + 8r - 4q' - 2p'') + 18rp'' - 12q'p'' + 40iq'' -\\
&-8qq'' + p'(144i^2 + 18p^2i - 68qi + 10q^2 - 3p^2q - 3pr - 24s + 6pq' - 12r' + 36i'' + 12q''))2\xi' +\\
&+y_1((6ip^4 - 3rp^3 + 3q'p^3 + 36i^2p^2 + q^2p^2 - 32iqp^2 - 6sp^2 - 3r'p^2 + 9i''p^2 + 3q''p^2 + 20irp + 14qrp + 20iq'p -\\
&-10qq'p - 30ip''p - 3qp''p + 80i^3 - 4q^3 + 36iq^2 - 18r^2 - 150i'^2 + 6(6i-q)p'^2 - 96i^2q - 80is + 16qs +\\
&+12rq' - 40ir' + 8qr' + 120ii'' - 24qi'' + 15i'(p^3 - 4qp + 3p'p + 8r - 4q' - 2p'') + 18rp'' - 12q'p'' +\\
&+40iq'' - 8qq'' + p'(144i^2 + 18p^2i - 68qi + 10q^2 - 3p^2q - 3pr - 24s + 6pq' - 12r' + 36i'' + 12q''))\xi'' +\\
&+\xi'(40pi^3 + 4p^3i^2 + 8pqi^2 - 112ri^2 + 80q'i^2 - 8p''i^2 - 14pq^2i + 4p^3qi - 14p^2ri + 64qri - 40psi + 6p^2q'i -\\
&-16qq'i + 20pr'i - 160s'i + 60pi''i - 20qp''i + 40r''i + 40i'''i + 3pr^2 - 75pi'^2 - 6rp'^2 - 4q^2r + p^2qr -\\
&-9p^3s + 32pqs - 48rs + 3p^3r' - 10pqr' + 12rr' - 12p^2s' + 32qs' + 6p^3i'' - 18pqi'' + 12ri'' - 3prp'' +\\
&+48sp'' - 12r'p'' - 12i''p'' + i'(3p^4 + 24ip^2 - 6qp^2 - 15p''p + 80i^2 - 12q^2 + 18p'^2 - 64iq + 240s + (9p^2 +\\
&+96i + 6q)p' - 60r' - 60i'') + 3p^2r'' - 8qr'' + 3p^2i''' - 8qi''' + p'(12pi^2 + 6pqi - 56ri + 24q'i - 3p^2r +\\
&+10qr - 12ps + 6pr' - 48s' + 18pi'' + 12r'' + 12i'''))) = 0.
\end{aligned} \tag{5}$$

2. Problem statement and preliminaries

In this paper we'll solve the problem of integration of the fourth order linear differential equation (4), coefficients of which satisfy the conditions

$$p' + \frac{1}{4}p^2 - \frac{2}{3}q = 0,\ \ q' + \frac{1}{4}pq - \frac{3}{2}r = 0,\ \ r' + \frac{1}{4}pr - 4s = 0. \tag{6}$$

Let use the result of the paper [2] according to which the equation (A) for the equations (4), (6) has the form

$$\begin{gathered} 20 \cdot \left(8\, i^3 - 15\, i'^2 + 12\, i \cdot i''\right) \cdot i^{(IV)} - 280 i \cdot i'''^2 + 20 \cdot \left(42\, i' \cdot i'' - 56\, i^2 i'\right) \cdot i''' + 504\, i''^3 + \\ +192\, i^2 \cdot i''^2 + \left(448\, i^4 + 2040\, i \cdot i'^2\right) \cdot i'' - 1275 i'^4 - 560 i^3 \cdot i'^2 + 64 i^6 = 0. \end{gathered} \tag{7}$$

The equation (7) was studied in detail in [2, 4,11,12]. In particular, there was proved, that equation (7) has one- and two-parametric families of solutions, which are contained among the functions

$$i(x) = \frac{P_2(x)}{P_4(x)} \text{ or } i(x) = \frac{P_4(x)}{P_8(x)},$$

where $P_2(x), P_4(x), P_8(x)$ are polynomials of the second, fourth and eighth degrees with certain coefficients. Respectively for example, such family of solutions is the function

$$\begin{gathered} i = (6(-24x(6C_1^2 + 6C_2^2 - 3C_1(2C_2 + C_3) - 2C_2C_4 + C_3C_4) - 4x^3(18C_1(2C_2 - C_3) + \\ +9C_3^2 - 6C_3C_4 + 4C_4(-3C_2 + C_4)) - 12x^2(18C_1^2 + C_2(9C_3 - 6C_4) + 2C_4^2 - 3C_1(3C_3 + 2C_4)) - \\ -3x^4(12C_2^2 + 3C_3^2 - 2C_2(3C_3 + 2C_4) + C_1(-6C_3 + 4C_4)) + 3(-12C_2^2 - 3C_3^2 + C_1(6C_3 - 4C_4) + \\ +C_2(6C_3 + 4C_4)))) \times (6(1 + 2x + 3x^2)C_1 + 6(-1 + x^2 + 2x^3)C_2 + x(3(-2 - x + x^3)C_3 - 2x(3 + 2x + x^2)C_4))^{-2}. \end{gathered}$$

Note also, that the equation (7), using substitutions

$$h(i) = u(t) \cdot \exp(\frac{3}{2}t),\ \ i = \exp(\frac{3}{2}t),\ \ u'(t) = g(u),$$

can be reduced to the differential equation of the second order of the form [2]

$$\begin{gathered} (1+u)^2(8+3u^2)^2 + (56u \cdot (8+3u^2) + 36u^3(8+3u^2))g + (1072u^2 + 422u^4)g^2 + \\ (160u + 596u^3)g^3 - 40u^2g^4 + (10u^3(8+3u^2)g + 80(8u^2+u^4)g^2 + 400u^3g^3)g' + \\ (20u^3(8+3u^2)g - 40u^4g^2)g'^2 + 20((8u^3+3u^5)g^2 + 12u^4g^3)g'' = 0. \end{gathered}$$

Let consider the function ξ in the form of the polynomial of the third degree

$$\xi(x) = C_1x^3 + C_2x^2 + C_3x + C_4\,, \tag{8}$$

where $C_i\ (i = \overline{1,4})$ are arbitrary constants. Such choice of the functions $\xi(x)$ allows us to find the family of solutions of the equation (7). Really, substituting the equality (8) into the equation (3), we find the function

$$i(x) = -\frac{6(6C_1^2x^2 + C_2^2 + C_1(4C_2x - C_3))}{(3C_1x^2 + 2C_2x + C_3)^2}. \tag{9}$$

Theorem 1. Function (9) defines the two-parametric family of solutions to the equation (7).

Proof. Fact, that function (9) is the solution is verified by substituting it into the equation (7). Fact, that it's the two-parametric family is proved by the fact, that the function (9) defines the general solution of the second order nonlinear differential equation in the form

$$2048i^6 + 5000i^3i'^2 + 375i'^4 - 1024i^4i'' + 1500ii'^2i'' - 96i^2i''^2 + 72i''^3 = 0\,. \tag{10}$$

3. Solution of the problem

Let find the partial solution $y_1(x)$ for the equations (4), (6). To do this we use the formula (5). Substituting (6), (8) and (9) into the equality (5) we obtain the differential equation of the form

$$p(x)y_1(x) + 4y_1'(x) = 0\,. \tag{11}$$

Integrating the equation (11) and substituting found function $y_1(x)$ into the relation (1) we obtain

$$y = (C_1x^3 + C_2x^2 + C_3x + C_4)\exp(-\frac{1}{4}\int_1^x p(\tau)d\tau)\,. \tag{12}$$

Theorem 2. The general solution of the equation (4), (6) has the form (12).

Proof. Functions $x^i \exp(-\frac{1}{4}\int_1^x p(\tau)d\tau),\ i = 0,1,2,3$ are the partial solutions of the equation (4), (6), which is easily verified by direct calculations. These functions form the fundamental system of solutions. Therefore, their linear combination of the form (12) gives the general solution of the equation (4), (6).

Example. Let the first coefficient of the equation (4), (6) has the form

$$p(x) = \mathrm{sn}(x/m)\,, \tag{13}$$

where $\mathrm{sn}(x/m)$ is the Jacobi elliptic function with the parameter m. Solving (6), we find the other three coefficients of the equation, namely

$$q(x) = \frac{3}{8}(4\,\mathrm{cn}(x/m)\,\mathrm{dn}(x/m) + \mathrm{sn}(x/m)^2)\,, \qquad (14)$$

$$r(x) = \frac{1}{16}\mathrm{sn}(x/m)\,(12\,\mathrm{cn}(x/m)\,\mathrm{dn}(x/m) - 16\,(2\,\mathrm{dn}(x/m)^2 + m - 1) + \mathrm{sn}(x/m)^2)\,, \qquad (15)$$

$$s(x) = \frac{1}{256}(8\,\mathrm{cn}(x/m)\,\mathrm{dn}(x/m)\,(6\,\mathrm{dn}(x/m)\,(\mathrm{cn}(x/m) - 8\,\mathrm{dn}(x/m)) + 3\,\mathrm{sn}(x/m)^2 - 8m + 40) - -64(2\,\mathrm{dn}(x/m)^2 + m - 1)\,\mathrm{sn}(x/m)^2 + \mathrm{sn}(x/m)^4)\,, \qquad (16)$$

where $\mathrm{cn}(x/m)$, $\mathrm{dn}(x/m)$ are the Jacobi elliptic functions.

Substituting (13) into the general solution (12), we obtain

$$y(x) = (C_1x^3 + C_2x^2 + C_3x + C_4)\,(\mathrm{dn}(x/m) - \sqrt{m}\,\mathrm{cn}(x/m))^{-\frac{1}{4\sqrt{m}}}\,. \qquad (17)$$

Expression (17) defines the general solution of the equation (4) with the coefficients (13) – (16).

Remark 3. As noted above, the choice of the function $\xi(x)$ in the form of (8) is not random. It is connected to the fact, that obtained from the equation (3) function $i(x)$ is the solution of equation (7). Following functions $\xi(x)$ of the form (18) – (20) also satisfy the similar property

$$\xi(x) = \frac{C_1x^3 + C_2x^2 + C_3x + C_4}{x}, \qquad (18)$$

$$\xi(x) = \frac{C_1x^3 + C_2x^2 + C_3x + C_4}{x^2}, \qquad (19)$$

$$\xi(x) = \frac{C_1x^3 + C_2x^2 + C_3x + C_4}{x^3}. \qquad (20)$$

These functions correspond to the functions $i(x)$ of the form

$$i(x) = -\frac{6(C_1{}^2x^4 + 4C_1C_4x + C_2C_4)}{(C_4 - C_2x^2 - 2C_1x^3)^2}, \qquad (21)$$

$$i(x) = -\frac{6(C_1C_3x^4 + 4C_1C_4x^3 + C_4{}^2))}{x^2(2C_4 + C_3x - C_1x^3)^2}, \qquad (22)$$

$$i(x) = -\frac{6((C_3{}^2 - C_2C_4)x^2 + 4C_3C_4x + 6C_4{}^2)}{x^2(C_2x^2 + 2C_3x + 3C_4)^2}, \qquad (23)$$

which are also the solutions of the equation (7).

Let prove, that the choice of the function $\xi(x)$ in the form of the one of the functions (18) – (20) doesn't change the structure of the general solution (12). Really, let choose, for example, the function $\xi(x)$ in the form (18). Then corresponding function $i(x)$ has form (21). Substituting relations (6), (18) and (21) into (5), we obtain the differential equation

$$y_1{}'(x) + \frac{(xp(x) - 4)y_1(x)}{4x} = 0\,. \qquad (24)$$

Integrating the equation (24) and substituting found function $y_1(x)$ into the relation (1), where the function $\xi(x)$ has the form (18), we obtain equality (12), which was to be proved above. In the other two cases (19) and (20), carrying out analogous reasoning, we get the general solution of the equation (4), (6) again in the form (12).

4. Conclusions

1) Relation (9) from the theorem 1 defines two-parametric family of the solutions of the equation (7).
2) Relation (9) from the theorem 1 also defines the general solution of the differential equation (10).
3) Theorem 2 indicates the general solution of the equation (4), (6).
4) Found functions (21) – (23) are also two-parametric families of the solutions of the equation (7).
5) Considered method of finding the solutions can be applied not only to the equation (7), which is connected to the linear equation (4), (6). It can be used for investigating the subclasses of the fourth order nonlinear equation of the form (A) (given in [2, p.70-71]), which is connected to the linear equation (4).

References

1. Bihn Zhou, Chuan-Jie Zhu. An Application of the Schwarzian Derivative // arXiv:hep-th/9907193v1 27 Jul 1999.
2. Chichurin A.V. The Chazy equation and the linear equations of the Fuchs' class. Russian University of Peoples' Friendship Publishing House, Moscow – 2003. – 163 p.
3. Chichurin A., Lukashevich N. To the theory of the fourth order linear differential equations // Proc. of the Intern. Conf. "Computer Algebra Systems in Teaching and Research" (31.01-3.02.2007, Siedlce, Poland), Siedlce, Wyd. AP, 2007. – P. 47-51.
4. Chichurin A.V. About one non-linear equation of the IV-th order with constant coefficients // Vestik of Brest University. – 2000. – № 4. – P. 33-38.
5. Guckenheimer J., Holmes F. Nonlinear oscillations, dynamical systems and bifurcations of vector fields. Moscow, Izhevsk, IKI, 2002.
6. E.L. Ince. Ordinary differential equations, Dover Publications, New York. – 1956. – 558 p.
7. Lukashevich N.A. About linear equations of the third order // Differential equations – 1999. – Vol.35, № 10. – P. 1366-1371.
8. Lukashevich N.A., Chichurin A.V. Differential Equations of the first order. Belarusian State University Publishing House, Minsk. – 1999. – 210 p.
9. Sataev E.A. Schwartzian derivative for multidimensional maps and flows // Mat. Sb. 1999. Vol. 190. № 1. P. 139-160.

10. Seong-a Kim, Toshiyuki Sugava. Invariant Schwarzian derivatives of higher order // arXiv:0911.2663v1 [math.CV] 13 Nov 2009.
11. Stepanyuk G.P., Chichurin A.V. About one condition of the integrability of the linear differential equation of the fourth order // Vesnik of Brest University. Series 4: Physics. Mathematics. – 2011. – № 2. – P. 99-103.
12. Stepanyuk G.P., Chichurin A.V. About the solutions of the nonlinear differential equation of the fourth order, associated to the linear equations of the fourth order by the instrumentality of Schwarzian derivative // Collection of articles of the Intern. Conf. "Fundamental and applied problems of solid mechanics, mathematical modeling and information technologies", 12-15.08.2013), Cheboksary. Part 2, Mathematical Modeling and Information Technologies. P. 60-66.
13. Yakushkin N.A. Generalized Schwarzian derivative and its applications // Proceedings of ISA RAS, The dynamics of inhomogeneous systems, 2008, issue 12, P.139-158.

Надійшла до редколегії 16.12.13

О.Чичурін, д-р фіз.-мат. наук, доц., проф.,
КУЛ, Люблін, Польща, СНУ імені Лесі Українки, Луцьк, Україна
Г. Степанюк, ст. лаб.
СНУ імені Лесі Українки, Луцьк, Україна

ПРО ЗАГАЛЬНИЙ РОЗВ'ЯЗОК ЛІНІЙНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ЧЕТВЕРТОГО ПОРЯДКУ, КОЕФІЦІЄНТИ ЯКОГО ЗАДОВОЛЬНЯЮТЬ СИСТЕМУ ТРЬОХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ

У роботі знайдено загальний розв'язок лінійного диференціального рівняння четвертого порядку, коефіцієнти якого задовольняють систему трьох диференціальних рівнянь першого порядку.

А.Чичурін, д-р физ.-мат. наук, доц., проф.
КУЛ, Люблин, Польша, ВНУ имени Леси Українки, Луцк, Украина
Г. Степанюк, ст. лаб.
ВНУ имени Леси Українки, Луцк, Украина

ОБ ОБЩЕМ РЕШЕНИИ ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА, КОЭФФИЦИЕНТЫ КОТОРОГО УДОВЛЕТВОРЯЮТ СИСТЕМЕ ТРЕХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА

В работе найдено общее решение линейного дифференциального уравнения четвертого порядка, коэффициенты которого удовлетворяют системе трех дифференциальных уравнений первого порядка.

УДК 517.98

М. Городній, д-р фіз.-мат. наук,
КНУ імені Тараса Шевченка, Київ
А. Сиротенко, асист.,
НТУУ "КПІ", Київ

ІНТЕГРОВНІ ЗІ СТЕПЕНЕМ p РОЗВ'ЯЗКИ РІЗНИЦЕВОГО РІВНЯННЯ З НЕПЕРЕРВНИМ АРГУМЕНТОМ

Отримано необхідні і достатні умови, при виконанні яких різницеве рівняння з неперервним аргументом має єдиний інтегровний зі степенем p (обмежений) розв'язок для спеціального класу "вхідних" функцій.

ВСТУП. Нехай B – комплексний банахів простір з нормою $\|\cdot\|$ і нульовим елементом $\overline{0}$, A – лінійний неперервний оператор, що діє із B в B. Покладемо при $p \in [1, \infty)$

$$l_p(B) := \left\{ \overline{x} = \{x_n : n \in \mathbb{Z}\} \subset B \,\middle|\, |\overline{x}|_p := \left(\sum_{k \in \mathbb{Z}} \|x_k\|^p \right)^{\frac{1}{p}} < \infty \right\} \text{ і } l_\infty(B) := \left\{ \overline{x} = \{x_n : n \in \mathbb{Z}\} \subset B \,\middle|\, |\overline{x}|_\infty := \sup_{n \in \mathbb{Z}} \|x_n\| < \infty \right\}.$$

Зафіксуємо $p \in [1, \infty]$. Відомо [1,2,4], що різницеве рівняння

$$x_{n+1} = Ax_n + y_n, \ n \in \mathbb{Z}, \tag{1}$$

має для довільного $\overline{y} = \{y_n : n \in \mathbb{Z}\} \in l_p(B)$ єдиний розв'язок $\overline{x} = \{x_n : n \in \mathbb{Z}\}$ у просторі $l_p(B)$ тоді і лише тоді, коли для спектра $\sigma(A)$ оператора A виконується умова

$$\sigma(A) \cap \{z \in C \,\|z| = 1\} = \varnothing. \tag{2}$$

У випадку, коли умова (2) не виконується, у [6] отримано результат про існування та властивості розв'язків різницевого рівняння (1). Сформулюємо цей результат, оскільки він використовується в подальшому.

Нехай V_d – набір усіх таких елементів $y \in B$, що різницеве рівняння (1) має при $y_0 = y, \ y_n = \overline{0}, \ n \neq 0$, єдиний розв'язок в просторі $l_p(B)$. Цей розв'язок у подальшому позначатимемо $\overline{x}_y$.

Теорема 1 [6]. Нехай множина V_d містить хоча б один ненульовий елемент та виконуються наступні умови:

1) якщо $\{y; y_m : m \geq 1\} \subset V_d$ і $\|y_m - y\| \to 0$ при $m \to \infty$, то для розв'язків $\overline{x}_y$, $\overline{x}_{y_m}$ рівняння (1), що відповідають y та y_m, виконується $\left|\overline{x}_y - \overline{x}_{y_m}\right|_p \to 0$ при $m \to \infty$.

2) для довільної послідовності $\{y_n : n \in \mathbb{Z}\} \subset V_d$, що належить $l_p(B)$, рівняння (1) має єдиний розв'язок в $l_p(B)$.