O. Limarchenko, Dr Hab., Prof.
O. Nefedov, PhD,
O. Sirenko, Applicant
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

VERIFICATION OF THE CONTROL ALGORITHM FOR RESERVOIRS WITH LIQUID BASED ON THE COMPENSATION OF THE FORCE RESPONSE IN DIFFERENT RANGES OF MANIFESTATION OF NONLINEARITIES

Problem about motion of a reservoir with liquid with a free surface is considered based on the compensation of a force response of the liquid on reservoir walls. Such an approach is selected since usual methods of control of mechanical system motion are mostly intended for linear systems of relatively small dimension. However, models of dynamics of the combined motion of reservoirs with liquid are described with relatively high-dimensional nonlinear systems ordinary differential equations. For obtaining the mathematical model of combined motion of a reservoir with liquid with a free surface we use the Hamilton–Ostrogradskiy variational principle, for which it is possible to determine analytically all internal forces of interaction of system component parts. Namely using this algorithm, we determine the main vector of forces of the liquid pressure on reservoir walls (force response of liquid). The algorithm of the motion control of the reservoir with liquid is based on the inclusion of the compensation of the liquid force response to controlling actions, this reduces the motion of the system reservoir–liquid, where the effect of forces from oscillating liquid on the reservoir motion is eliminated. This algorithm was tested for problems of impulse and vibration disturbance of the translational motion of the system in the horizontal plain. We consider the disturbance of the system motion by a force rectangular impulse applied to the reservoir wall, the duration of the impulse is lesser than a quarter of the period of a liquid free oscillations according to the first normal mode. Amplitudes of the impulse were selected with the purpose of analysis of the behavior of the controlled system in different ranges of manifestation of nonlinearities. We state the problem to verify the accuracy of this algorithm for three ranges of manifestation of nonlinear properties in the system, namely, for the linear range (amplitudes of waves on a free surface h do not exceed 0,1 of the radius of a free surface (ξ <0,1R); for the weakly nonlinear range (ξ <0,2R) and for the strongly nonlinear range with maximum amplitudes of waves about ξ =0,32R. Numerical modeling enables the determination of errors of developed algorithm, which does not exceed 0,5 %, although they insignificantly increase with the increase of amplitudes of oscillations on a free surface of liquid. At the same time perturbations on a free surface of liquid for the controlled motion are always greater than for the uncontrolled motion.
Keywords: reservoir with liquid, combined motion, control, force response of liquid, compensation.

УДК 517.9
DOI https://doi.org/10.17721/1684-1565.2020.01-41.13.56-59

K. Elgondiyev, PhD., Associate Prof.,
S. Matmuratova, Student
Karakalpak State University, Nukus city, Respublic of Karakalpakstan, Uzbekistan,
E-mail: elgondiev.61@gmail.com
V. Borodin, PhD, Associate Prof.,
E-mail: viktorborodinmm@gmail.com; borodin@univ.kiev.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine,
L. Vovk, PhD, Associate Prof.
E-mail: ludmylavera@gmail.com
Kyiv National University of Culture and Arts, Kyiv, Ukraine

TOTAL ENERGY OF HARMONIC OSCILLATOR WITH IMPULSE ACTION

The problem of finding the total energy of a harmonic oscillator with pulsed action at fixed moments of time is considered. Both for the case of the homogeneous equation of harmonic oscillations and for the case of the equation of harmonic oscillations in the presence of external perturbation, formulas for the total energy of the oscillatory system are obtained. The case of periodic impulse effects is analyzed. The conditions under which in this oscillatory system there are periodic modes are specified. It is shown that under the fulfillment of these conditions on the values of impulse action and external perturbation, the total energy of the vibrational system is also a periodic function of the time variable.
Keywords: harmonic oscillator, systems with pulsed action, total energy, external perturbation, periodic oscillations.

1. Introduction. Harmonic vibrations are of great importance in the study of many different problems of physics, since the motion of any system in the vicinity of the minimum of its potential energy can be described using the equation of harmonic vibrations

$$\ddot{x} + \omega^2 x = 0 . \qquad (1)$$

Harmonic oscillations are one of their simplest types. Well-known examples of harmonic oscillations are small vibrations of a mathematical pendulum, oscillations in molecules, small oscillations of a finite string (with appropriate initial conditions), oscillations in various electrical engineering, for example, in RLC circuits, and other systems. If we take into account that an arbitrary periodic function under rather general conditions can be represented in the form of its Fourier series, it can also be noted that almost any periodic motion is a superposition of harmonic oscillations.

Mathematically, harmonic oscillations can be described using the formula [7, 8] of the following form $x(t) = x_0 \cos\omega(t-t_0) + \frac{\dot{x}_0}{\omega}\sin\omega(t-t_0)$, where x_0 , $\dot{x}_0$ are initial state of the system (1), t_0 is an initial moment of time. In the case of a material particle of unit mass, its total energy is represented by the formula

$$E = E(t) = \frac{1}{2}\left[\omega^2 x^2(t) + \dot{x}^2(t)\right] \qquad (2)$$

and is a constant, i.e., $E = E(t_0) = \frac{1}{2}\left(\omega^2 x_0^2 + \dot{x}_0^2\right)$.

In the presence of an external disturbance $f(t)$, i.e. when the motion of a material particle is described by a differential equation of the form

$$\ddot{x} + \omega^2 x = f(t), \tag{3}$$

the energy introduced into a given oscillatory system by an external disturbance can be represented [3] as $\mathrm{E}(t) = \frac{1}{2}\left|\int_{-\infty}^{+\infty} f(t)e^{-iwt}dt\right|^2$, where it is assumed that at the initial moment of time the system is in the state $x_0 = \dot{x}_0 = 0$.

On the other hand, many oscillatory systems, phenomena and processes are impulsive in nature, when certain characteristics of the system change over a very short period of time, that is, instantly [5, 6, 11]. Differential equations with impulse action are most often used as mathematical models of such systems [4]. There are systems with fixed moments of impulse action, when the moments of time of impulse action are determined a priori, and systems with non-fixed moments of impulse action, when moments of impulse action are determined from some (additional) conditions.

It is known [4–8, 10] that the qualitative behavior of solutions of differential equations with impulsive action has, generally speaking, a rather non-trivial character and is nonlinear. While studying such systems, qualitative methods of the theory of differential equations are often applied, based on the using expressions for the total energy of the system.

In this article, using the example of a harmonic oscillator, we study the problem of finding the total energy of an oscillatory system with an impulse effect. The case of impulse action at fixed times is considered.

2. Oscillator with impulse action. Let us firstly consider the case when the external force in (3) is of an impulsive nature and its action occurs in a very short period of time. We assume that the duration of the action of the force is much less than $\frac{1}{\omega}$. Then the energy introduced by an external force into this system can be calculated using formula $\left|\int_{-\infty}^{+\infty} f(t)dt\right|^2$, where it is taken into account that in this case $e^{-iwt} \approx 1$. These arguments are used further in the study of a more general case.

Let $t = t_0$ be initial moment of time and $x_0, \dot{x}_0$ be initial state of the system. The total energy of the oscillatory system for values $t \in [t_0, \infty)$ in the case when the external force $f(t)$, $t \in [t_0, \infty)$, has an impulsive character and acts according to the law

$$F(t) = \begin{cases} f_1(t), & if \quad t \in [t_1 - \delta,\ t_1], \\ 0, & if \quad t \notin [t_1 - \delta,\ t_1], \end{cases} \tag{4}$$

can be calculated by formula (2), where one should put $x(t) = x_0 \cos \omega(t - t_0) + \frac{\dot{x}_0}{\omega} \sin \omega(t - t_0) + \frac{1}{\omega}\int_{t_0}^{t} F(\tau) \sin \omega\,(t - \tau)\, d\tau.$

Assuming the value $\delta > 0$ to be sufficiently small and taking into account formula (4), for $t > t_1$ we find

$$E(t) = \frac{1}{2}\left[\left(\dot{x}_0 + \int_{t_1-\delta}^{t_1} f_1(\tau) \cos \omega\,(t-\tau)\, d\tau\right)^2 + \omega^2\left(x_0 + \frac{1}{\omega}\int_{t_1-\delta}^{t_1} f_1(\tau) \sin\omega\,(t-\tau) d\tau\right)^2\right].$$

From this it follows that with an external disturbance of a pulse type, the total energy of a harmonic oscillator is expressed by a formula similar to (2) but with slightly different (corrected) initial data. In other words, we can assume that during the action of external forces of an impulsive nature, some correction of the initial data occurs. This conclusion also follows from the formula for the difference in the total energy of a harmonic oscillator after the end and before the beginning of the impulse force, namely:

$$E(t_1) - E(t_0) = 2\sin\frac{\omega(t_1 - t_0)}{2}\int_{t_1-\delta}^{t_1} f(\tau)\left[\omega x_0 \cos\frac{\omega(t_1 + t_0 - 2\tau)}{2} - \dot{x}_0 \sin\frac{\omega(t_1 + t_0 - 2\tau)}{2}\right] d\tau.$$

While studying the influence of impulsive forces on oscillatory systems the assumption of an instantaneous change in the momentum of the system at the moment of impulse action $t = \tau_1$ is also often used. In similar cases, in addition to considering differential equation (1) at $t \ne \tau_1$, an additional condition of the form $\Delta\dot{x}\big|_{t=\tau_1} = \dot{x}(\tau_1 + 0) - \dot{x}(\tau_1 - 0) = I_1$ is used, which is called the impulse action condition [4]. For the general case, for example, in the presence of an infinite number of moments of impulse action $t_0 < \tau_1 < \tau_2 < ....,$ where $\tau_k \to +\infty$, the problem is formulated as follows: it is necessary to study the dynamics of the system, the behavior of which at $t \ne \tau_k$ is described by the differential equation (1), and at $t = \tau_k$ it is regulated by the conditions of impulse action [4, 7, 8]

$$\Delta\dot{x}\big|_{t=\tau_k} = \dot{x}(\tau_k + 0) - \dot{x}(\tau_k - 0) = I_k, \tag{5}$$

where $I_k,\ k \in N,$ are some given values.

Equation (1) with conditions (5) is called the differential equation of a harmonic oscillator with impulse action. Oscillations in such a system occur according to the law [7, 10]

$$x(t) = x_0 \cos \omega(t - t_0) + \frac{\dot{x}_0}{\omega}\sin \omega(t - t_0) + \frac{1}{\omega}\sum_{t_0 < \tau_k \le t} I_p \sin\omega(t - \tau_k). \tag{6}$$

This formula gives an explicit solution $x(t)$ to problem (1), (5) for an arbitrary (fixed) moment of time t and takes into account the effect of forces of an impulsive nature of the form (5).

Taking into account (6), the total energy of system (1), (5) can be written using the following formula:

$$E(t)=\frac{1}{2}\left[\dot{x}^2(t)-\omega^2x^2(t)\right]=E(t_0)=\frac{1}{2}\left[\left(\dot{x}_0+\sum_{t_0<\tau_k\le t}I_k\cos\omega(t_0-\tau_k)\right)^2+\omega^2\left(x_0-\frac{1}{\omega}\sum_{t_0<\tau_k\le t}I_k\sin\omega(t_0-\tau_k)\right)^2\right].$$

Putting $x_0=\dot{x}_0=0$ (to simplify the corresponding expressions) one can analyze the effect of one impulse action on the change in the total energy of the system. We find:

$$E(\tau_1)-E(t_0)=\frac{1}{2}I_1^2,\quad E(\tau_2)-E(\tau_1)=\frac{1}{2}I_2^2+I_1\cdot I_2\cos\omega(\tau_2-\tau_1),\ \dots\ ,\ E(\tau_k)-\mathrm{E}(\tau_{k-1})=\frac{1}{2}I_k^2+I_k\cdot\sum_{p=1}^{k-1}I_p\cos\omega(\tau_k-\tau_p).$$

The question naturally arises: is the equality $E(\tau_k)-E(\tau_{k-1})=\frac{1}{2}I_k^2$ possible ? Obviously, a necessary condition for this equality to hold is a condition of the form $\omega\cdot(\tau_k-\tau_p)=\frac{\pi}{2}+r_{kp}\cdot\pi,$ where $r_{kp},\ p=\overline{1,k-1}$, are some integers. However, it is easy to show that the last condition cannot be satisfied.

If, in addition to impulse forces of the form (5), an external force $f(t)$ of impulsive nature of the form (4) acts on the harmonic oscillator (1), then for $t>t_1$ the total energy of system (3), (5) is calculated by the formula

$$E(t)=\frac{1}{2}\left[\omega^2x^2(t)+\dot{x}^2(t)\right]=\frac{1}{2}\left[\omega^2\left(x_0\cos\omega(t-t_0)+\frac{\dot{x}_0}{\omega}\sin\omega(t-t_0)+\frac{1}{\omega}\sum_{t_0<\tau_k\le t}I_k\sin\omega(t-\tau_k)+\frac{1}{\omega}\int_{t_0}^{t}f_1(\tau)\sin\omega(t-\tau)\,d\tau\right)^2+\right.$$

$$\left.+\left(-x_0\omega\sin\omega(t-t_0)+\dot{x}_0\cos\omega(t-t_0)+\sum_{t_0<\tau_k\le t}I_k\cos\omega(t-\tau_k)+\int_{t_0}^{t}f_1(\tau)\cos\omega(t-\tau)\,d\tau\right)^2\right]=E(t_0)=$$

$$=\frac{1}{2}\left[\left(\omega x_0+\sum_{t_0<\tau_k\le t}I_k\sin\omega(t_0-\tau_k)+\int_{t_0}^{t}f_1(\tau)\sin\omega(t_0-\tau)d\tau\right)^2+\left(\dot{x}_0+\sum_{t_0<\tau_k\le t}I_k\cos\omega(t_0-\tau_k)+\int_{t_0}^{t}f_1(\tau)\cos\omega(t_0-\tau)d\tau\right)^2\right].$$

For various practical problems, the case of periodic oscillations is often important [7, 10, 11]. A number of conditions are required for the existence of periodic modes in system (3). Let system (1), (5) satisfy the following conditions [9, 12, 13]:

a) $\omega T=2\pi q_0$ for some natural number q_0;

b) $I_{n+k}=I_k,\ t_{n+k}=t_k+T$ for some natural number $n\in N$ and all $k\in N$ (periodicity conditions);

c) $\sum_{k=1}^{n}I_k\cos\omega(t_0-t_k)=0,\quad \sum_{k=1}^{n}I_k\sin\omega(t_0-t_k)=0$.

Then problem (1), (5) has a two-parameter family of periodic solutions, and the initial values of the problem can be considered as parameters. Note that otherwise, when condition a) is not satisfied, problem (1), (5) will have a quasiperiodic solution [7]. Under conditions a), b), c), the total energy of system (1), (5) is determined by the formula

$$E(t)=\frac{1}{2}\left[\dot{x}^2(t)+\omega^2x^2(t)\right]=\frac{1}{2}\left[\left(\dot{x}_0+\sum_{t_0<\tau_k\le t}I_k\cos\omega(t_0-\tau_k)\right)^2+\omega^2\left(x_0-\frac{1}{\omega}\sum_{t_0<\tau_k\le t}I_k\sin\omega(\tau_k-t_0)\right)^2\right].$$

Similarly, for harmonic oscillator (3) with impulsive action (5) under assumptions a), b) and conditions

d) $f(t+T)=f(t)$ for all $t\in[t_0,\infty)$;

e) $\sum_{k=1}^{n}I_k\cos\omega(t_0-t_k)+\int_{t_0}^{t_0+T}f(\tau)\cos\omega(t_0-\tau)\,d\tau=0,\qquad \sum_{k=1}^{n}I_k\sin\omega(t_0-t_k)+\int_{t_0}^{t_0+T}f(\tau)\sin\omega(t_0-\tau)\,d\tau=0$

the total energy of system (3), (5) is determined by the formula

$$E(t)=\frac{1}{2}\left[\omega^2x^2(t)+\dot{x}^2(t)\right]=\frac{1}{2}\left[\omega^2\left(x_0\cos\omega(t-t_0)+\frac{\dot{x}_0}{\omega}\sin\omega(t-t_0)+\frac{1}{\omega}\sum_{t_0<\tau_k\le t}I_k\sin\omega(t-\tau_k)+\frac{1}{\omega}\int_{t_0}^{t}f(\tau)\sin\omega(t-\tau)\,d\tau\right)^2+\right.$$

$$\left.+\left(-x_0\omega\sin\omega(t-t_0)+\dot{x}_0\cos\omega(t-t_0)+\sum_{t_0<\tau_k\le t}I_k\cos\omega(t-\tau_k)+\int_{t_0}^{t}f(\tau)\cos\omega(t-\tau)\,d\tau\right)^2\right].$$

It is easy to show that under conditions a), b), c) in the case of system (1), (5) or conditions a), b) d), e) in the case of system (3), (5), the total energy of the corresponding oscillatory system is a periodic function of the time variable, i.e., $E(t+T)=E(t)$ for all $t\in[t_0,\infty)$.

3. Conclusions. In this article, expressions are obtained for the total energy of a harmonic oscillator with an impulse action at fixed times. Both the case of the homogeneous equation of harmonic oscillations and the case of the equation of harmonic oscillations in the presence of an external disturbance, including an impulse one, are considered. The case of periodic impulse influences is analyzed. Conditions are given under which periodic modes are present in the considered oscillatory system. It is shown that under the conditions of periodicity on the values of the impulse action and external disturbance, the total energy of the oscillatory system is also a periodic function of the time variable. The results obtained can be used in the study of problems of mathematical physics [1, 2, 14].

References:

1. Елгондиев К. К., Пильтяй М. М., Хомченко Л. В. Распространение тепла в однородном стержне с импульсным воздействием // Краевые задачи для дифференциальных уравнений. – 2003. – Вып. 10. – С. 59–65.
2. Елгондиев К. К., Пильтяй М. М., Хомченко Л. В. Распространение тепла в прямоугольной пластинке с импульсным воздействием // Вест. Киев. нац. ун-та имени Тараса Шевченко. Сер. : физико-математические науки, 2003. – Вып. 1. – С. 69–72.
3. Ландау Л. Д., Лифшиц Е. М. Теоретическая физика. Т. 1. Механика. 4-е изд. – М. : Наука, 1988. – 216 с.
4. Самойленко А. М., Перестюк Н. А. Дифференциальные уравнения с импульсным воздействием. – Киев : Вища шк., 1987. – 287 с.
5. Самойленко В. Г., Бородін В. А. Вплив імпульсної дії на асимптотичні властивості розв'язків диференціальних рівнянь з імпульсною дією // Вісник Київського національного університету імені Тараса Шевченка. Математика. Механіка, 2004. – Вип. 11-12. – С. 30–35.
6. Самойленко В. Г., Бородін В. А. Асимптотичні властивості розв'язків диференціального рівняння n-го порядку з імпульсною дією // Математичні методи та фізико-механічні поля, 2004. – Т. 47, № 4. – С. 76–81.
7. Самойленко В. Г., Елгондиев К. К. Исследование линейных дифференциальных уравнений с импульсным воздействием в R^2. – Препринт. № 89.59. – Киев : Ин-т математики АН УССР. – 32 с.
8. Самойленко В. Г., Елгондиев К. К. О решениях дифференциального уравнения, описывающего движение осциллятора под воздействием мгновенных сил // Асимптотические методы в уравнениях математической физики. – Киев: Институт математики АН УССР, 1989. – С.145–148.
9. Самойленко В. Г., Елгондиев К. К. Периодические и почти-периодические решения линейных дифференциальных уравнений с импульсным воздействием // Математична фізика та нелінійна механіка. – 1991. – Вып. 15. – С. 13–19.
10. Самойленко В. Г., Елгондиев К. К. Качественное поведение решений дифференциальных уравнений с импульсным воздействием // Дифференц. уравнения, 1999. – Т. 35, № 6. – С. 852.
11. Samoilenko V. H., Borodin V. A. Asymptotic properties of solutions of a differential equation of order n with almost constant coefficients and an impulsive effect // Nonlinear Oscillations, 2005. – Vol. 8, No. 4. – P. 444–455.
12. Samoilenko V. G., Elgondiyev K. K. On periodic solutions of linear differential equations with pulsed influence // Ukr. Math. Jour., 1997. – Vol. 49, No. 1. – P. 156–164.
13. Samoilenko V. G., Elgondiyev K. K. On existence of periodical solutions for differential equations with impulse effects // Facta Universitatis. Ser.: Mechanics, Autonomic Control and Robotics, 1998. – Vol. 2, № 8. – P. 635–640.
14. Samoilenko V. H., Khomchenko L. V. A von Neumann boundary-value problem for a singularly perturbed heat equation with an impulsive effect // Nonlinear Oscillations, 2005. – Vol. 8, No. 1. – P. 89–122.

Надійшла до редколегії 20.09.20

К. Єлгондієв, канд. фіз.-мат. наук, доц.,
С. Матмуратова, студ.
Каракалпакський державний університет ім. Бердаха, Республіка Каракалпакія, Узбекістан,
В. Бородін, канд. фіз.-мат. наук, доц.
Київський національний університет імені Тараса Шевченка, Київ, Україна,
Л. Вовк, канд. фіз.-мат. наук, доц.
Київський національний університет культури і мистецтв, Київ, Україна

ПОВНА ЕНЕРГІЯ ГАРМОНІЧНОГО ОСЦИЛЯТОРА З ІМПУЛЬСНОЮ ДІЄЮ

Розглянуто задачу про знаходження повної енергії гармонічного осцилятора з імпульсною дією у фіксовані моменти часу. Як для випадку однорідного рівняння гармонійних коливань, так і для випадку рівняння гармонічних коливань за наявності зовнішнього збурення отримано формули для повної енергії коливальної системи. Проаналізовано випадок періодичних імпульсних впливів. Вказано умови, при виконанні яких у цій коливальній системі наявні періодичні режими. Показано, що при виконанні цих умов на величини імпульсного впливу і зовнішнього збурення, повна енергія коливальної системи також є періодичною функцією часової змінної.

Ключові слова: гармонічний осцилятор, системи з імпульсною дією, повна енергія, зовнішнє збурення, періодичні коливання.