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Про одне узагальнення слабкого принципу Кондорсе

Для задачі колективного вибору в постановці К.Ерроу запропоновано узагальнення слабкого принципу Кондорсе. Вивчено питання обґрунтованості за цим принципом деяких відомих правил колективного вибору.

Ключові слова: прийняття рішень, принцип Кондорсе, правила колективного вибору.

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About one generalization of weak Condorcet's principle

One generalization of weak Condorcet's principle was proposed for the problem of collective choice in the formulation of Arrow. The question of consistency of some well-known rules of collective choice was explored according to this principle.

Key Words decision making, Condorcet's principle, the rules of group choice.

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Introduction. One of the topical issues in the field of collective decision making is a problem of optimality principle definition – the principle of alternatives comparing, in the result of which a decision in favor of one or another alternative is made. On the set of the two alternatives the only possible and correct principle of optimality is a choice according to relative majority rule [1]. However, it is known [2] that on the set of three or more alternatives arises a problem of constructing the rule of collective choice, which would be adequate continuation of collective choice according to majority principle of pairs of alternatives.

To date the principle proposed in the late XVIII century by French philosopher and mathematician Condorcet one way or another is taken into consideration in all rational models of collective decision making [1-2]. The rules that satisfy this principle are called reasonably (wealthy) Condorcet. The above said makes actual the generalization problem of the concept of Condorcet reasonability.

In this paper we propose one generalization of the weak version of Condorcet's principle.

Statement of the problem and the basic concepts. Let a finite set of alternatives be $A=\{a_1,...,a_{n_A}\}$ (n_A – number of alternatives), and also a set (profile) $\Pi=\{P_1,...,P_{n_E}\}$ of estimations of alternatives set A, where P_l is strict linear order on a set A, given by individual l of collective (group) and which corresponds to their individual preferences on a set of alternatives A ($l \in N_E=\{1,...,n_E\}$, n_E – quantity of individuals). The problem of collective (group, resulting, etc.) order definition which in the "best" possible way displays the preferences on set A of group of individuals in general is set.

The only possible and correct solution on the set of the two alternatives is a collective choice according to the relative majority rule: the best alternative is that one which was given a strong preference for at least half of individuals. On the set of three or more alternatives arises a problem of constructing of collective choice rule, which would be adequate continuation of voting according to majority principle for a couple of alternatives.

Definition 1. Let's denote for any profile of individual $c_{ij}=Card(\{l \in N_E:(a_i,a_j)\in P_l\})$, $\forall\, i,j \in N_A=\{1,...,n_A\}$. Value c_{ij} – is the number of individuals which gave strong prefernence to alternative a_i over alternative a_j in corresponding profile.

Definition 2. For any profile of individual preferences value $m_{ij}=c_{ij}-c_{ji}$ is called majority margin of alternatives a_i over aternative a_j, $\forall\, i,j \in N_A$, and matrix $M=(m_{ij})_{i,j=1,...,n_A}$ is called matrix of majority margin.

In the futher if you need to point out correspondence of mentioned notations to certain profile $\Pi \in \mathrm{P}$ (P – set of all possible profiles), then we will write down $c_{ij}(\Pi)$, $m_{ij}(\Pi)$ and $M(\Pi)$.

Definition 3. Relation of simpe majority (majority relation), which was generated by profile $\Pi \in \mathrm{P}$, is called connected binary relation $R_{M(\Pi)}$ on a set A, which is defined as following

$$(a_i, a_j) \in R_{M(\Pi)} \Leftrightarrow m_{ij}(\Pi) \geq 0, \quad \forall\, a_i, a_j \in A.$$

Definition 4. The tournament is called a pair (A,T), where T is assymetric and connected binary relation on a set of alternatives A.

Definition 5. Let q – is non-negive real number. q-weighted tournament is called a pair (A,G), where G – is matrix, $G = \left(g(a_i, a_j)\right)_{i,j=1,\ldots,n_A}$, is such, that $g(a_i,a_j) + g(a_j,a_i) = q$, $\forall\, a_i, a_j \in A$ and $g(a_i,a_j) \neq g(a_j,a_i)$, for different arbitraries $i,j \in N_A$. From definition 5 it follows that $g(a_i,a_i) = q/2$, $\forall\, a_i \in A$.

Every q-weighted tournament (A,G) generates tournament (A,T_G) in the following way: $(a_i,a_j) \in T_G \Leftrightarrow m(a_i,a_j) > m(a_j,a_i), \ \forall\, a_i, a_j \in A.$

Every q-weighted tournament (A,G) generates 0-weighted tournament (A,G_0), elements of matrix G_0 of which are determined in the following way: $g_0(a_i,a_j) = g(a_i,a_j) - g(a_j,a_i) = 2g(a_i,a_j) - q$, $\forall\, a_i, a_j \in A.$

It is easy to see if the number of individuals n_E is odd, then arbitrary profile of individual preferece $\Pi \in \mathrm{P}$ generates n_E-weighted tournament (A,G), where $g(a_i,a_j) = c_{ij}$, $\forall\ i \neq j$. In general case when a number of individuals is even such statement is not always true. In case when profile $\Pi \in \mathrm{P}$ generates n_E-weighted tournament (A,G), then it generates 0-weighted tournament (A,G_0), where $G_0 = M(\Pi)$ and $T_G = R_M$.Taking into consideration all said above we choose on a P subset $\mathrm{T} \subset \mathrm{P}$ of profiles which generate 0-weighted tournament.

Smith's principle [2]. *A set of profiles* T *is considered. Let there exists some non-empty set* W_{Smith} *which is such that:*

1. $W_{Smith} \subset A$;
2. $\forall\, a \in W_{Smith}, \forall\, b \in A \setminus W_{Smith} \Rightarrow (a,b) \in R_M$.

The winner shall be chosen from the set W_{Smith}.

Analogue of Smith's principle. A set of profiles $\mathrm{P} \setminus \mathrm{T}$ *is considered. Let there exists non-empty set* W_C *which is such, that:*

1. $W_C \subset A$;

2. $\forall\, a \in W_C \Rightarrow \neg\exists\, b \in A \setminus W_C : (b,a) \in R_M^s$, R_M^s – *is strict component of majority relation;*

3. $\neg\exists\, b \in A \setminus W_C : W_C \cup \{b\}$ *satisfies 1 and 2.*

The winner shall be chosen from the set W_C.

It is easy to see that for each profile $\Pi \in \mathrm{P} \setminus \mathrm{T}$, set W_C contains all weak winners according to Condorcet (alternative is a weak winner according to Condorcet, if there is no other alternative, which would win from it according to the rule of relative majority) for this profile.

Kemeny rule. Let $R_1, R_2 \in \Omega$ (Ω –set of all connected, asymmetric, transitive binary relations) are two arbitrary strict linear orders. Let's define the distance between theese two relations as the distance beween the sets:

$$\delta(R_1,R_2) = \frac{Card\left((R_1 \setminus R_2) \cup (R_2 \setminus R_1)\right)}{2}.$$

Binary relation R_{Kemeny} is called collective order of Kemeny (Kemeny's median) if and only if $R_{Kemeny} = arg \min\limits_{R \in \Omega} \sum\limits_{l=1}^{n_E} \delta(R, P_l)$.

Slater's rule is to find collective orders, which are closest to the corresponding relation of simple majority. Let R_M be relation of simple majority generated by some profile. Binary relation R_{Slater} is called Slater's collective order if and only if

$$R_{Slater} = arg \min_{R \in \Omega} \delta(R, R_M).$$

Prudent order. Suppose $\lambda \in \{-n_E, -n_E + 2, \ldots, n_E - 2, n_E\}$ and let's define relation $R_{>\lambda}$: $(a_i,a_j) \in R_{>\lambda} \Leftrightarrow m_{ij} > \lambda,$ $\forall\, i,j \in N_A, i \neq j.$ Prudent oreder is called strict linear order R_{PO}, which completes acyclic relation $R_{>\beta}$, that is $R_{>\beta} \subseteq R_{PO}$,

$\beta = min\{\lambda \in \{-n_E, -n_E + 2, \ldots, n_E - 2, n_E\}:$
$R_{>\lambda}$ – acyclic$\}$.

Theorem 1. *Kemeny's rule satisfies an analogue of Smith's principle. Slater's rule and prudent order satisfy the Smith's principle and its analogue.*

Proof. Truth of statement concerning the rules of Kemeny and Slater follows from their equivalence to linear ordering problem of alternatives [4], and as a result from decomposition procedures for a linear ordering of alternatives. [5]

Let's choose arbitrary alternatives $a_i \in W_C$, $a_j \in A \setminus W_C$. Let $m_{ij} > \beta$. Then for every prudent order is performed $(a_i, a_j) \in R_{>\beta}$, and from asymmetry of acyclic relation it follows that $(a_j, a_i) \notin R_{>\beta}$. If $m_{ij} \le \beta$, then from $(a_i, a_j) \in R_M$ and $m_{ij} = -m_{ji}$ it follows $m_{ji} \le 0 \le m_{ij} \le \beta$, this is because $(a_i, a_j) \notin R_{>\beta}$ and $(a_j, a_i) \notin R_{>\beta}$. In this case a strict partial order $R_{>\beta}$ can be always completed to a strict linear order $R_{>}$, so that $(a_i, a_j) \in R_{>}$, which makes it impossible to choose any alternative from the set $A \setminus W_C$. Similarly we can show also another part of the theorem.

Egalitarian Simpson's rule. To win according to this rule it is necessary that alternative does not collect against it a large majority according to this rule. Simpson's score of alternative $a_i \in A$ is called value $S(a_i) = \min\limits_{z \in N_i^-} m_{iz}$.

Simpson's collective order is called ordening $R_{Simpson}$, which is defined as follows:

$$(a_i, a_j) \in R_{Simpson} \Leftrightarrow S(a_i) \ge S(a_j), \ \forall a_i, a_j \in A.$$

Utilitarian Tideman's rule. As alternative to the rule (1) it is possible to define the rule which is based on an utilitarian criterion. Utilitarian score of alternative $a_i \in A$ is called value $U(a_i) = \sum\limits_{z \in N_i^-} m_{iz}$.

Utilitarian collective order is called ordening R_{UC}, which is defined as follows:

$$(a_i, a_j) \in R_{UC} \Leftrightarrow U(a_i) \ge U(a_j), \quad \forall a_i, a_j \in A.$$

Copland's rule. In order to defeat Copland's rule it is necessary to win on the basis of simple majority from the greatest number of other alternatives. Copland's score of alternative $a_i \in A$ is called value

$$C(a_i) = 2Card\big(\{z \in N_A \setminus \{i\} : w_{iz} > 0\}\big) + \\ + Card\big(\{z \in N_A \setminus \{i\} : w_{iz} = 0\}\big).$$

Copland's collective rule is called ordening $R_{Copeland}$, which is defined as follows:

$$(a_i, a_j) \in R_{Copeland} \Leftrightarrow C(a_i) \ge C(a_j), \ \forall a_i, a_j \in A.$$

Borda's rule puts in order alternatives according to the sum of the ranks of alternatives in a profile of individual preferences. We use equivalent method of points calculation which is based on the majority margin. Borda's score of alternative $a_i \in A$ is called value $B(a_i) = \sum\limits_{z=1}^{n_A} w_{iz}$. Borda's collective order is called ordening R_{Borda}, which is defined as follows:

$$(a_i, a_j) \in R_{Borda} \Leftrightarrow B(a_i) \ge B(a_j), \ \forall a_i, a_j \in A.$$

Theorem 2. *If a strong winner according to Condorcet is absent the Smith's principle is violated at the determination of collective order according to the rules of Simpson and Tiedemann, if* $n_A \ge 4$*; to Borda's rule if* $n_A \ge 5$*. On the set of 4 alternatives if a strong winner according to Condorcet is absent Borda's rule satisfies Smith's principle.*

Proof. Let matrix M of majority margin:

$$|m_{ij}| > x \ge 1, 1 \le i < j \le n_A - 1, m_{in_A} = x, \ 1 \le i \le n_A - 1.$$

It is easy to see for a profile which generates matrix M, an inclusion $W_{Smith} \subseteq A \setminus \{a_{n_A}\}$ is performed. We also have the following Simpson's score: $S(a_{n_A}) = -x$, and for alternative from the set $A \setminus \{a_{n_A}\}$, due to the fact that there is no strong winner according to Condorcet $S(a_i) < -x$, $1 \le i \le n_A - 1$ is performed.

Let

$$m_{12} = u, \ m_{13} = -v, \ m_{23} = g, \ m_{14} = x, \ m_{24} = y, \ m_{34} = z,$$
$$u > 0, \ v > 0, \ g > 0, \ x > 0, \ y > 0, \ z > 0,$$
$$u > x + y + z, \ v > x + y + z, \ g > x + y + z,$$
$$m_{1j} > 0, \ m_{2j} > 0, \ m_{3j} > 0, \ 4 < j \le n_A,$$
$$m_{ij} > 0, \ 4 \le i < j \le n_A.$$

For a profile which defines the above showed matrix of majority margin we have $W_{Smith} = \{a_1, a_2, a_3\}$ and the following utilitarian estimations are true: $UC(a_1) = -v$, $UC(a_2) = -u$, $UC(a_3) = -g$, $UC(a_4) = -x - y - z$, which taking into consideration introducted restrictions makes it impossible to choose any alternative from the set W_{Smith} according to utilitarian criterion.

Let's define the matrix of majority margin according to the following rule:

$$m_{12} = m_{23} = u, \ m_{13} = -u, \ m_{45} = v, \ v > 5u, \ u > 0,$$
$$m_{14} = m_{15} = m_{24} = m_{25} = m_{34} = m_{35} = u,$$
$$m_{1j} = m_{2j} = m_{3j} = m_{4j} = x, \ x > 0, \ 5 < j \le n_A,$$
$$m_{ij} > 0, \ 5 < i < j \le n_A.$$

Let's consider the following profile which generates such matrix. We have $W_{Smith} = \{a_1, a_2, a_3\}$ and the following Borda's scores are true:

$$B(a_1) = B(a_2) = B(a_3) = 2u + (n_A - 5)x,$$
$$B(a_4) = -3u + v + (n_A - 5)x.$$

Then taking into consideration the restrictions on the elements value of majority margin matrix the following scores are performed

$B(a_4)-B(a_i)=-5u+v>0$, $i=1,2,3$, which makes impossible the choice of any alternative of the set W_{Smith} according to Borda's rule.

Matrix of majority margin M of arbitrary profile of individual preferences on the set of four alternatives, for which $Card(W_{Smith})=3$, with some renumbering of alternatives satisfies the following restrictions:

$$w_{12}=u,\ w_{13}=-v,\ w_{23}=g,$$
$$u>0,\ v>0,\ g>0,\ w_{i4}>0,\ i=1,2,3.$$

For such matrix we have $B(a_4)<0$, $B(a_1)=u-v$, $B(a_2)=-u+g$, $B(a_3)=v-g$. If suppose that $B(a_i)<0$, $\forall\, i\in\{1,2,3\}$, then from the corresponding Borda's scores we get contradiction in the form of inequality $u<v<g<u$. So is found $i\in\{1,2,3\}$, for which $B(a_i)\ge 0$, which makes impossible the choice of alternative a_4 according to Borda's rule. Theorem is proved.

Top cycle. Let (A,T) – is tournament. Let us denote by T^* transitive closure T. Top cycle of tournament (A,T) is called non-empty subset of top elements for relation T^*. Let us denote $TC(T)=\{a\in A:(a,b)\in T^*,\forall\, b\in A\setminus\{a\}\}$.

Uncovered set. Let (A,T) – is tournament. Let us define for relation T the relation of covering

$$R_{covering}:\begin{array}{l}(b,a)\in R_{covering}\Leftrightarrow a\ne b,(b,a)\in T\ \wedge\\ \forall\, c\in A:(a,c)\in T\Rightarrow (b,c)\in T.\end{array}$$

Uncovered set $US(T)$ of tournament (A,T) is called non-empty subset of top elements of covering relation $R_{covering}$:

$$US(T)=\{a\in A:\neg\exists\, b\in A:(b,a)\in R_{covering}\}.$$

Minimal covering set. Covering set of tournament (A,T) is called subset $B\subset A$, which satisfies the following condition:

$$US(A,T\mid B)=B\ u\ \forall\, b\notin B: b\notin US(A,T\mid B\cup\{b\}),$$

where $T\mid B$ are those and only those pairs T, the elements of which belong to B. Minimal covering set $MC(T)$ of tournament (A,T) is called a single covered set of this tournament, which does not contain any other covering set.

Bipartisan set. Tournament game, which is generated by tournament (A,T) is called symmetric (A,A,u_T) of two persons with zero-sum with gain

$$u_T(a,b)=\begin{cases}1, & if\ (a,b)\in T;\\ -1, & if\ (b,a)\in T;\\ 0, & if\ a=b.\end{cases}$$

Set $BS(T)$ of tournament (A,T) is called a single Nash equilibrium in mixed strategies of game (A,A,u_T), which is generated by this tournament.

The approach of [3-4] consists in the use of rules of choice, which would provide the choice from the sets $TC(T_M)$, $US(T_M)$, $MC(T_M)$, $BS(T_M)$ respectively, where T_M is tournament, generated by majority relation R_M.

Let us denote by W_{Kemeny}, $W_{Copeland}$, $W_{Simpson}$, W_{UC}, W_{Borda} a set of minimal elements of respective collective orders obtained according to existing rules. It is known [2], that for every tournament (A,T) we have the following:

- $BP(T)\subseteq MC(T)\subseteq US(T)\subseteq TC(T)$;
- if $W_{Smith}\ne\varnothing$, then $TC(T)\subseteq W_{Smith}$;
- $W_{Copeland}\subseteq US(T)$;
- $W_{Kemeny}\subseteq TC(T)$.

From theorem 2 we get.

Conclusion 1. If $n_A\ge 4$, then there exists a profile $\Pi\in\mathrm{T}$, for which does not exist a strong winner according to Condorcet and

$$TC(R_{M(\Pi)})\cap W_{Simpson}(\Pi)=US(R_{M(\Pi)})\cap W_{Simpson}(\Pi)=\varnothing,$$
$$MC(R_{M(\Pi)})\cap W_{Simpson}(\Pi)=BS(R_{M(\Pi)})\cap W_{Simpson}(\Pi)=\varnothing.$$

Conclusion 2. If $n_A\ge 4$, then there exists profile $\Pi\in\mathrm{T}$, for which does not exist a strong winner according to Condorcet and

$$TC(R_{M(\Pi)})\cap W_{UC}(\Pi)=US(R_{M(\Pi)})\cap W_{UC}(\Pi)=\varnothing,$$
$$MC(R_{M(\Pi)})\cap W_{UC}(\Pi)=BS(R_{M(\Pi)})\cap W_{UC}(\Pi)=\varnothing.$$

Conclusion 3. If $n_A\ge 5$, then there exists a profile $\Pi\in\mathrm{T}$, fir which does not exist a strong winner according to Condorcet and

$$TC(R_{M(\Pi)})\cap W_{Borda}(\Pi)=US(R_{M(\Pi)})\cap W_{Borda}(\Pi)=\varnothing,$$
$$MC(R_{M(\Pi)})\cap W_{Borda}(\Pi)=BS(R_{M(\Pi)})\cap W_{Borda}(\Pi)=\varnothing.$$

Theorem 3. *If the weak winners according to Condorcet are absent the analogue of Smith's principle is violated at the determination of collective order according to Copland's rule, if* $n_A\ge 8$ *; according to Simpson's and Tideman's rules if* $n_A\ge 4$ *; according to Borda's rule if* $n_A\ge 6$.

Proof. Let us define the following matrix of majority margin:

$$m_{12}>0,\ m_{23}>0,\ m_{34}>0,\ m_{13}<0,\ m_{14}=0,\ m_{24}=0,$$
$$m_{ii+1}<0,\ i=4,5,\ldots,n_A-1,$$
$$m_{ij}\geq 0,\ 4\leq i\leq n_A-2,\ i+2\leq j\leq n_A,$$
$$m_{ij}=0,\ 1\leq i\leq 3,\ 5\leq j\leq n_A.$$

For the corresponding profile which generates the considered matrix of majority margin is valid: $W_C=\{a_1,a_2,a_3\}$, $C(a_3)=2\cdot 2+n_A-4=n_A$, $C(a_5)=3+2(n_A-5)=2n_A-7$. Hence taking into consideration the restrictions on the number of alternatives we obtain score $C(a_5)-C(a_3)=2n_A-7-n_A=n_A-7>0$, which makes it impossible to choose any alternative from W_C according to Copland's rule.

Let the following matrix of majority margin:

$$|m_{ij}|>x\geq 0,\ 1\leq i<j\leq n_A-1,\ x>m_{in_A}\geq 0,\ 1\leq i\leq n_A-1.$$

For a profile which corresponds to this matrix the following inclusion is performed $W_C\subseteq A\setminus\{a_{n_A}\}$ and, taking into consideration weak winners according to Condorcet the following inequalities are performed $S(a_i)<-x<S(a_{n_A})$, $1\leq i\leq n_A-1$, which makes it impossible to choose any alternative from set W_C according to Simpson's rule.

For the next part of the statement we define the following matrix M:

$$m_{12}=u,\ m_{13}=-v,\ m_{23}=g,$$
$$m_{14}=x,\ m_{24}=y,\ m_{34}=0,\ m_{45}=-z,$$
$$u>0,\ v>0,\ g>0,\ x>0,\ y>0,\ z>0,$$
$$u>x+y+z,\ v>x+y+z,\ g>x+y+z,$$
$$m_{1j}>0,\ m_{2j}>0,\ m_{3j}>0,\ 4<j\leq n_A,$$
$$m_{ii+1}<0,\ i=5,\ldots,n_A-1,$$
$$m_{ij}>0,\ i=4,\ldots,n_A-2,\ j=i+2,\ldots,n_A.$$

Then by analogy with the corresponding case of the previous theorem and taking into consideration the introduced restrictions, we obtain inability to choose any of the alternatives of a set W_C according to utilitarian criterion.

For part of the statement, which refers to the negative result concerning the Borda's rule we consider the profile, which leads to the following matrix of majority margin:

$$m_{12}=m_{23}=u>0,\ m_{13}=-u,\ m_{in_A}=x>0,\ i=1,2,3,4$$
$$m_{ij}=0,\ i=1,2,3,3<j<n_A,\ m_{4j}=0,\ 5<j<n_A-1,$$
$$m_{ii+1}=-y,\ y>5x,\ 3<i\leq n_A-1,$$
$$m_{ij}=0,\ 3<i\leq n_A-2,\ i+2\leq j\leq n_A.$$

For such profile $W_C=\{a_1,a_2,a_3\}$ we have: $B(a_1)=B(a_2)=B(a_3)=x$, $B(a_{n_A})=-4x+y$. Then taking into consideration the restrictions on the value of elements of majority margin matrix such scores are true $B(a_{n_A})-B(a_i)=-4x+y-x>0$, $i=1,2,3$, which makes it impossible to choose any alternative from a set W_C according to Bordas' rule.

Conclusions. A generalization for one principle of rational collective choice is suggested for a problem of collective choice in the classical formulation in this paper. If there is a weak winner according to Condercet for some profile of individual preferences then the proposed principle coincides with Condorcet's principle, and in the absence thereof − serve as its reasonable substitution (continuation). Study of consistency according to this principle of some well-known rules of collective choice proves once again the complexity and paradox of the theory of collective decision making.

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