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An approach for solving of knapsack problem with fuzzy priorities

In this paper knapsack problem with fuzzy priorities of objects to be placed in a knapsack was discussed and solved. The values are give as the fuzzy numbers with linear membership function. The method of solution based on the use of the dual optimization problem was request. The real example of fuzzy knapsack problem solving was given.

Key words: knapsack problem, fuzzy number, dual linear programming problem, method of solving of linear programming problem with fuzzy constraints

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Об одном подходе к решению задачи про рюкзак с нечеткими приоритетами

В статье рассмотрена и решена задача про рюкзак с нечетко заданными приоритетами объектов, размещаемых в рюкзаке. Величины задаются в виде нечетких чисел с линейной функцией принадлежности. Предложен метод решения на основе использования двойственной задачи оптимизации, приведен пример решения реальной задачи.

Ключевые слова: задача про рюкзак, нечеткое число, двойственная задача линейного программирования, метод решения ЗЛП с нечеткими ограничениями

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1. INTRODUCTION

The knapsack problem, also known as the capital budgeting or cargo loading problem, is a famous integer programming formulation [1]. The knapsack context refers to a hiker selecting the most valuable items to carry, subject to a weight or capacity limit. Partial items are not allowed, thus choices are depicted by zero-one variables. The capital budgeting context involves selection of the most valuable investments from a set of available, but indivisible, investments subject to limited capital availability. The cargo loading context involves maximization of cargo value subject to hold capacity and indivisibility restrictions.

A traditional knapsack problem is attempts to maximize utility while adhering to a major weight or cost constraint. In more mathematical terms, there are *n* items, with each item *j* having a cost (weight), and a value,. The goal is to maximize the sum of the values while staying within the cost constraint. There are many variations on knapsack problems. Binary constraints may be placed on each item prohibiting dividing up items. Also, multiple other constraints can be added beyond one major cost constraint. The knapsack problem has seen a number of variations over the years. In its basic form, the problem is to maximize the value of items placed in a knapsack without going over a weight limit. This is the 0/1 Knapsack Problem, defined more formally as [2]

$$\sum_{j=1}^{n} v_i x_i \rightarrow Max \qquad (1)$$

subject to

$$\sum_{j=1}^{n} w_j x_j \le W , \; x_j \in \{0,1\} , \; j = \overline{1,n}, \qquad (2)$$

where v_j is the value of item's priority, w_j is item's weight and W is the maximum weight allowed in the knapsack, $j = \overline{1,n}$.

The knapsack problem is a simple example of a type of integer programming problem which frequently in the field of mathematics known as operations research, and among intensively studies knapsack problem hard combinatorial optimization problem, the application of these problems span a wide canvas from industrial applications and financial management to electronic commerce and personal health care. The common flavor in most of these problems is resource allocation. The allocation of specific amount of a single resource among competitive alternatives is often modeled as knapsack problem or its variants with, in addition, the unknown or fuzzy coefficients of objective function. Usually we may to know only some approximation of these coefficients. In this case the values of coefficients may be considered as the fuzzy numbers, for example in triangle form. We obtain the knapsack problem with fuzzy coefficients $\widetilde{v_j}$, $j=\overline{1,n}$, of objective function as next:

$$\sum_{j=1}^{n} \widetilde{v_j}\, x_j \ \to Max \qquad (3)$$

subject to

$$\sum_{j=1}^{n} w_j^i x_j \ \le W^i, \quad i=\overline{1,m}, \qquad (4)$$

$$x_j \in \{0,1\},\ j=\overline{1,n},$$

where $\tilde{v}_j$, $j=\overline{1,n}$, - the fuzzy numbers which describe the unknown priority values of items j, $j=\overline{1,n}$. The item's weights and the maximum weight of the knapsack are define by vectors $w_j = (w_j^1,\dots,w_j^m)^T$ and $W=(W^1,\dots,W^m)^T$ respectively, $j=\overline{1,n}$.

The task above is linear optimizing problem with fuzzy objective function and integer variables. The method of linear programming problem solving with fuzzy right-hand side (resources) was considered in [3]. We apply this method to our task (3), (4).

2. GENERAL INFORMATION

The General fuzzy single-objective Linear Programming Problem when constrain goals are triangular fuzzy number (TFN) may be written as follows:

$$\sum_{j=1}^{n} c_j x_j \ \to Max \qquad (5)$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \ge \widetilde{b_i}, \qquad 1\le i \le m_1,$$
$$\sum_{j=1}^{n} a_{ij}x_j \le \widetilde{b_i}, \quad m_1+1\le i \le m_2,$$
$$\sum_{j=1}^{n} a_{ij}x_j = \widetilde{b_i}, \quad m_2+1\le i \le m,$$
$$x_j \ge 0,\ 1\le j\le n.$$

The problem (5) becomes with the extreme tolerance as :

$$\sum_{j=1}^{n} c_j x_j \ \to Max \qquad (6)$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^0, \qquad 1\le i \le m_1,$$
$$\sum_{j=1}^{n} a_{ij}x_j \le b_i + b_i^0, \quad m_1+1\le i \le m_2,$$
$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^l, \quad m_2+1\le i \le m_3,$$
$$\sum_{j=1}^{n} a_{ij}x_j \le b_i - b_i^r, \quad m_3+1\le i \le m,$$
$$x_j \ge 0,\ 1\le j\le n.$$

Fuzzy values $\widetilde{b_i}$, $i=\overline{1,m}$, are considering as [4]

· Left Triangular Fuzzy Number (LTFN): $\widetilde{b_i}=(b_i - b_i^0, b_i, b_i)$ or $\widetilde{b_i}= b_i - b_i^0 + \lambda b_i^0$, $1\le i \le m_1$, with tolerance $b_i^0(< b_i)$ for $\sum_{j=1}^{n} a_{ij}x_j \ge \widetilde{b_i}$, $1\le i\le m_1$;

· Right Triangular Fuzzy Number (RTFN): $\widetilde{b_i}=(b_i, b_i, b_i + b_i^0)$ or $\widetilde{b_i}= b_i + b_i^0 - \lambda b_i^0$, $m_1+1\le i \le m_2$, with tolerance $b_i^0(>0)$ for $\sum_{j=1}^{n} a_{ij}x_j \le \widetilde{b_i}$, $m_1+1\le i\le m_2$;

· Triangular Fuzzy Number (TFN): $\widetilde{b_i}=(b_i - b_i^l, b_i, b_i + b_i^r)$ or $\widetilde{b_i}=b_i - b_i^l + \lambda b_i^l$, $\widetilde{b_i}=b_i + b_i^r - \lambda b_i^r$, $m_2+1\le i \le m$, with tolerances $b_i^l(< b_i)$, $b_i^r(>0)$ for $\sum a_{ij}x_j = \widetilde{b_i}$, $m_2+1\le i \le m$.

Let L and U be the lower and upper bound for objective function:

$$L=\sum_{j=1}^{n} c_j x_j^{*1},\ U=\sum_{j=1}^{n} c_j x_j^{*0},$$

where x_j^{*0}, x_j^{*1}, $j=\overline{1,n}$ - optimal solutions of task optimization (5) with Triangular Fuzzy values $\widetilde{b_i}$, $1\le i \le m$ for $\lambda=0$ and $\lambda=1$ respectively.

When the aspiration levels have been obtained, we form a fuzzy model which is as follows:

find x_j, $1\le j \le n$, so as to satisfy

$$Z \ge \tilde{s}, \qquad (7)$$
$$\tilde{s}=(L,U,U),\ \tilde{s}=L+\lambda(U-L),$$

or

$$Z \le \tilde{s}, \qquad (8)$$
$$\tilde{s}=(L,L,U),\ \tilde{s}=U-\lambda(U-L),$$

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^0 + \lambda b_i^0, \quad 1\le i\le m_1,$$
$$\sum_{j=1}^{n} a_{ij}x_j \le b_i + b_i^0 - \lambda b_i^0, \quad m_1+1\le i\le m_2,$$
$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^l + \lambda b_i^l,$$
$$\sum_{j=1}^{n} a_{ij}x_j \le b_i + b_i^r - \lambda b_i^r, \quad m_2+1\le i\le m,$$
$$x_{ij} \ge 0,\ 1\le j\le n.$$

The membership functions for fuzzy constraints (7), (8) are defined as:

$$\mu_G(x)=\begin{cases} 0, & \text{if } \sum_{j=1}^{n} c_j x_j < L,\\ \dfrac{\sum_{j=1}^{n} c_j x_j - L}{U-L}, & \text{if } L\le \sum_{j=1}^{n} c_j x_j < U,\\ 1, & \text{if } \sum_{j=1}^{n} c_j x_j \ge U,\end{cases}$$

or

$$\mu_G(x)=\begin{cases} 1, & \text{if } \sum_{j=1}^{n} c_j x_j < L,\\ \dfrac{U-\sum_{j=1}^{n} c_j x_j}{U-L}, & \text{if } L\le \sum_{j=1}^{n} c_j x_j < U,\\ 0, & \text{if } \sum_{j=1}^{n} c_j x_j \ge U,\end{cases}$$

for the i-th constrain $\tilde{F}_i$, $1 \le i \le m_1$,

$$\mu_{\tilde{F}_i}(x) = \begin{cases} 0, & \text{if } \sum_{j=1}^{n} a_{ij}x_j < b_i - b_i^0, \\ \dfrac{\sum_{j=1}^{n} a_{ij}x_j - b_i + b_i^0}{b_i^0}, & \text{if } b_i - b_i^0 \le \sum_{j=1}^{n} a_{ij}x_j < b_i, \\ 1, & \text{if } \sum_{j=1}^{n} a_{ij}x_j \ge b_i; \end{cases}$$

for the i-th constrain $\tilde{F}_i$, $m_1 + 1 \le i \le m_2$,

$$\mu_{\tilde{F}_i}(x) = \begin{cases} 1, & \text{if } \sum_{j=1}^{n} a_{ij}x_j < b_i, \\ \dfrac{b_i + b_i^0 - \sum_{j=1}^{n} a_{ij}x_j}{b_i^0}, & \text{if } b_i \le \sum_{j=1}^{n} a_{ij}x_j < b_i + b_i^0, \\ 0, & \text{if } \sum_{j=1}^{n} a_{ij}x_j \ge b_i + b_i^0; \end{cases}$$

for the i-th constrain $\tilde{F}_i$, $m_2 + 1 \le i \le m$,

$$\mu_{\tilde{F}_i}(x) = \begin{cases} 0, & \text{if } \sum_{j=1}^{n} a_{ij}x_j < b_i - b_i^l, \\ \dfrac{\sum_{j=1}^{n} a_{ij}x_j - b_i + b_i^l}{b_i^l}, & \text{if } b_i - b_i^l \le \sum_{j=1}^{n} a_{ij}x_j < b_i, \\ \dfrac{b_i + b_i^r - \sum_{j=1}^{n} a_{ij}x_j}{b_i^r}, & \text{if } b_i \le \sum_{j=1}^{n} a_{ij}x_j < b_i + b_i^r, \\ 0, & \text{if } \sum_{j=1}^{n} a_{ij}x_j \ge b_i + b_i^r. \end{cases}$$

Using the *max-min* operator (as [4]) crisp linear programming problems for (6) is formulated as follows:

$$\lambda \to Max \tag{9}$$

subject to (for example)

$$\sum_{j=1}^{n} c_j x_j + \lambda(U - L) \le U,$$

$$\begin{aligned} &\sum_{j=1}^{n} c_j x_j - \lambda b_i^0 \ge b_i - b_i^0, \ 1 \le i \le m_1, \\ &\sum_{j=1}^{n} c_j x_j + \lambda b_i^0 \le b_i + b_i^0, \ m_1 + 1 \le i \le m_2 \\ &\sum_{j=1}^{n} c_j x_j - \lambda b_i^l \ge b_i - b_i^l, \\ &\sum_{j=1}^{n} c_j x_j + \lambda b_i^r \le b_i + b_i^r, \ m_2 + 1 \le i \le m, \\ &0 \le \lambda \le 1, \ x_j \ge 0, \ 1 \le j \le n. \end{aligned} \tag{10}$$

Suppose that the tolerance of constraints on resources are independent and can be formalized in the form of inequalities with different parameters $\lambda_i \in [0,1]$, $1 \le i \le m$,

$$\begin{aligned} &\sum_{j=1}^{n} c_j x_j + \lambda(U - L) \le U, \\ &\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^0 + \lambda_i b_i^0, \ \text{for } 1 \le i \le m_1, \\ &\sum_{j=1}^{n} a_{ij}x_j \le b_i + b_i^0 - \lambda_i b_i^0 \ \text{for } m_1 + 1 \le i \le m_2, \\ &\sum_{j=1}^{n} a_{ij}x_j \ge b_i - b_i^l + \lambda_i b_i^l, \\ &\sum_{j=1}^{n} a_{ij}x_j \le b_i + b_i^r - \lambda_i b_i^r, \ \text{for } m_2 + 1 \le i \le m, \\ &0 \le \lambda_i \le 1, \ 1 \le i \le m, \ x_j \ge 0, \ 1 \le j \le n. \end{aligned} \tag{11}$$

In this case we must add m inequalities

$$\lambda \le \lambda_i, \ 1 \le i \le m. \tag{12}$$

Finally the optimization problem with Triangular Fuzzy Numbers and different λ_i (see [5]) may be written in form (9), (11), (12).

3. DUAL APPROACH

By using dual linear programming problem [6] for initial task in form (3),(4) we rewrite a new optimization model with fuzzy right-hand side

$$\sum_{i=1}^{m} W^i y_i \to Min \tag{13}$$

subject to

$$\sum_{i=1}^{m} w_j^i y_i = \tilde{v}_j, \ j = \overline{1,n}, \tag{14}$$

$$y_i \ge 0, \ i = \overline{1,m},$$

where $\tilde{v}_j$, $j = \overline{1,n}$, $w_j = (w_j^1, \dots, w_j^m)^T$, $j = \overline{1,n}$, $W = (W^1, \dots, W^m)^T$ are the values which were define in initial task (3),(4).

For this model we apply method described in part 2 and find the optimal solution.

4. THE RESULTS

Let we consider the fuzzy knapsack problem in the following form [6]:

$$f(x) = \widetilde{170}\, x_1 + \widetilde{1300} x_2 \to Max \tag{15}$$

subject to

$$\begin{aligned} &5.04\, x_1 + 96\, x_2 \le 248.4 \\ &24.48\, x_1 + 350\, x_2 \le 753.1 \\ &2.16\, x_1 + 16\, x_2 \le 48.1 \\ &150\, x_1 + 944\, x_2 \le 2896.9, x_j \in \{0,1\}, j = \overline{1,2}. \end{aligned} \tag{16}$$

By using dual we obtain the following model

$$f(y) \to Min, \tag{17}$$

$$f(y) = 248.4 y_1 + 753.1 y_2 + 48.1 y_3 + 2896.9 y_4,$$

subject to

$5.04\,y_1 + 24.48\,y_2 + 2.16\,y_3 + 150\,y_4 = \widetilde{170}$,
$96\,y_1 + 350\,y_2 + 16\,y_3 + 944\,y_4 = \widetilde{1300}$, (18)
$y_j \geq 0$, $j = \overline{1,4}$,

where fuzzy values $\widetilde{170}, \widetilde{1300}$ are the triangle fuzzy numbers which define as $\widetilde{170} = 170 \pm 10\lambda_1$ with $b_1^l = b_1^r = 10$, $\widetilde{1300} = 1300 \pm 100\lambda_2$ with $b_2^l = b_2^r = 100$, $0 \leq \lambda_i \leq 1, i = \overline{1,2}$.

The lower and upper bounds for objective function are L=3366.24 and U=3858.56. According to (11) the model become:

$$\lambda \rightarrow Max \quad (19)$$

subject to

$248.4y_1 + 753.1y_2 + 48.4y_3 + 2896.9y_4 - (3858.56 - 3366.24)\lambda \geq 3366.24$,
$5.04\,y_1 + 24.48\,y_2 + 2.16\,y_3 + 150\,y_4 \leq 180 - 10\,\lambda_1$,
$5.04\,y_1 + 24.48\,y_2 + 2.16\,y_3 + 150\,y_4 \geq 160 + 10\,\lambda_1$, (20)
$96\,y_1 + 350\,y_2 + 16\,y_3 + 944\,y_4 \leq 1400 - 100\,\lambda_2$,
$96\,y_1 + 350\,y_2 + 16\,y_3 + 944\,y_4 \geq 1200 + 100\,\lambda_2$,
$\lambda_i \geq \lambda$, $0 \leq \lambda_i \leq 1$, $i = \overline{1,2}$, $y_j \geq 0$, $j = \overline{1,4}$.

The optimal solution of this task is $y_1 = 0, y_2 = 1.112, y_3 = 0, y_4 = 0.929, \lambda = \lambda_1 = \lambda_2 = 0.667$ and $f(y) = 3530.4$.

Check all constraints (20) for this solution. We obtained five corresponding unequalities

$3694{,}617 \geq 3366.24$,
$166{,}67 \leq 173.33$,
$166{,}67 \geq 166.67$,
$1266{,}7 \leq 1333.3$,
$1266{,}7 \geq 1266.7$,

which are correct. By dual theorem $f(y) > f(x)$ and value 3530.4 is the maximal level of objective function (15).

The task (17), (18) with crisp constraints

$5.04\,y_1 + 24.48\,y_2 + 2.16\,y_3 + 150\,y_4 \geq 170$,
$96\,y_1 + 350\,y_2 + 16\,y_3 + 944\,y_4 \geq 1300$,

was solved in [7] using the unfuzzy method. The optimal solution was $y_1 = 1$, $y_2 = 0.958$, $y_3 = 0$, $y_4 = 0.999$ and the objective function value $f(y) = 3864.98$. This result is comparing with solution which was obtained by approach based on traditional fuzzy optimization model (9),(10) with one parameter λ. In this case optimal solution was $y_1 = 2.902$, $y_2 = 0$, $y_3 = 0$, $y_4 = 1.128$ and value of objective function was $f(y) = 3988.25$. As can be seen from the solution leeway has been provided with respect to all constraints and at additional cost of 3.2 percent.

Using our method we obtained the upper bound for optimal value of knapsack objective function with fuzzy priorities by using dual approach and modified the solving scheme for fuzzy linear optimization problem which can give better solution than the traditional fuzzy solution.

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