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Т. П. Гой, к. ф.-м. наук,¹
Р. А. Заторський, д. ф.-м. наук²

Про одну неелементарну функцію типу інтеграла Доусона

Досліджена нова неелементарна функція типу інтеграла Доусона, побудована у вигляді степеневого ряду при допомозі зростаючих факторіальних степенів. Встановлений її зв'язок з функцією помилок (функцією ймовірностей). Показано, що нова функція є розв'язком рівняння Ріккати.

Ключові слова: зростаючий факторіальний степінь, інтеграл Доусона, функція помилок, рівняння Ріккати.

Прикарпатський національний університет імені Василя Стефаника, 76018, Івано-Франківськ, вул. Шевченка, 57.

E-mail: ¹tarasgoy@yahoo.com ²romazz@rambler.ru
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Introduction

Duality of rising and falling factorial powers is a common feature in the combinatorial analysis. In other words, if a problem leads to some combinatorial identity constructed with the help of falling factorial powers, then often there is a dual combinatorial problem, which leads to a dual combinatorial identity involving rising factorial powers. One can find some examples of these dual combinatorial identities in [1], [2].

The classic exponential e^x is given by the corresponding power series with factorials, which can be written as the falling factorial power $n^{\underline{x}}$. Replacing the falling factorial powers by the corresponding rising factorial powers $n^{\overline{x}}$, we get the function $\text{Exp}(x)$ [3].

Now if in the Dawson's integral [4]

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

we replace the exponentials by $\text{Exp}(x)$, then we get a new nonelementary function

$$D(x) = (\text{Exp}(x^2))^{-1} \int_0^x \text{Exp}(t^2) dt,$$

the basic properties of which are to be studied in this article.

T. P. Goy, Ph. D.,
R. A. Zatorsky, D. Sc.

On a nonelementary function of the Dawson's integral type

A new nonelementary function of the Dawson's integral type is studied. It is constructed as a power series with the help of rising factorial powers. Its connection with the error function (probability function) is determined. It is proved that the new function is a solution of the Riccati equation.

Key Words: rising factorial power, Dawson's integral, error function, Riccati equation.

Precarpathian Vasyl Stefanyk National University, 76018, Ivano-Frankivsk, 57 Shevchenko Str.

The Dawson's integral and its generalization are widely applied in astrophysics, spectroscopy, theory of electric oscillation, processes of heat conduction, viscosity mechanics, finance, applied mathematics [5]–[12].

1 Preliminaries and Notations

Definition 1. [13] For an arbitrary $x \in \mathbb{R}$ and $m \in \mathbb{N}$, the factorial power m with the step of $k \in \mathbb{R}$ is the expression

$$x^{m\{k\}} = x(x+k)(x+2k) \cdot \dots \cdot (x+(m-1)k).$$

By definition $x^{0\{k\}} := 1$.

If $k = 0$, then we have a simple power, i.e. $x^{m\{0\}} = x^m$.

Most often, there are rising factorial powers with the step of 1 and falling factorial powers with the step of -1 , which we will denote respectively by $x^{\overline{m}} = x^{m\{1\}} = x(x+1) \cdot \dots \cdot (x+m-1)$, $x^{\underline{m}} = x^{m\{-1\}} = x(x-1) \cdot \dots \cdot (x-m+1)$.

Let us assume $x^{\overline{0}} = x^{\underline{0}} = 1$. It is obvious that $1^{\overline{m}} = m^{\underline{m}} = m!$.

In analogy to the known power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which can be treated as the series constructed with the help of falling factorial powers ($n! = n^{\underline{n}}$). The "dual" function $\text{Exp}(x)$, constructed with the help of rising factorial powers, it is studied in [3].

Definition 2. By $\text{Exp}(x)$ we will denote the function defined with the help of the power series

$$\begin{aligned} \text{Exp}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n^{\overline{n}}} = \\ &= 1 + \frac{x}{1} + \frac{x^2}{2 \cdot 3} + \dots + \frac{x^n}{n \cdot (n+1) \cdot \dots \cdot (2n-1)} + \dots \end{aligned}$$

It is obvious that

$$\text{Exp}(x) = 1 + \sum_{n=1}^{\infty} \frac{(n-1)!}{(2n-1)!} x^n \quad (1)$$

and the series in (1) converges on the real axis.

The graph of the function $y = \text{Exp}(x)$ is shown in Figure 1.

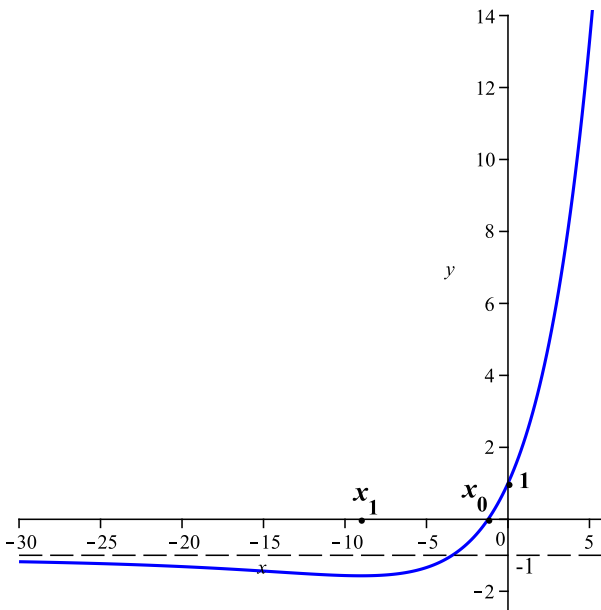


Fig. 1. The graph of the function $y = \text{Exp}(x)$

The only zero of the function $\text{Exp}(x)$ is

$$x_0 \approx -1, 22041009,$$

and at $x_1 \approx -9, 02371883$ it reaches its minimum.

In [3] it is proved that

$$\begin{aligned} \text{Exp}(x) &= 1 + \sum_{n=0}^{\infty} \frac{x^n}{4^n n!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n+1) 4^n n!} = \\ &= 1 + \sqrt{\pi x} e^{\frac{x}{4}} \Phi\left(\frac{\sqrt{x}}{2}\right), \quad (2) \end{aligned}$$

where

$$\Phi(x) = \text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the probability function (error function). [4]

2 On operations with formal power series

Denote by $\left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leq r \leq s \leq j}$ the paraderminant of the triangular matrix of order j , assuming that $\left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leq r \leq s \leq 0} = 1$. [2]

Let us formulate two theorems from [2], [14] to be used hereafter.

Theorem 1. Let $A(z)$, $B(z)$, $C(z)$ be the notations of the following formal power series

$$\sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} b_n z^n, \quad \sum_{n=0}^{\infty} c_n z^n,$$

respective, where $a_0 = b_0 = c_0 = 1$ and $C(z) = \frac{A(z)}{B(z)}$. Then for all $n = 1, 2, \dots$

$$c_n = \sum_{j=0}^{n-1} (-1)^j (a_{n-j} - b_{n-j}) \cdot \left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leq r \leq s \leq j}.$$

Theorem 2. If $C(z) = \frac{1}{A(z)}$, then

$$c_n = (-1)^n \left\langle \begin{array}{cccc} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \dots & \dots & \ddots & \\ \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \dots & a_1 \end{array} \right\rangle_n,$$

for all $n = 1, 2, \dots$

3 Function of the Dawson's integral type

Replacing in the Dawson's integral

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

the exponential with the function $\text{Exp}(x)$, we obtain a nonelementary function, which we will denote by $D(x)$, i.e. let

$$D(x) := (\text{Exp}(x^2))^{-1} \int_0^x \text{Exp}(t^2) dt. \quad (3)$$

It is easy to verify that

$$\int_0^x \text{Exp}(t^2) dt = -x + 2\sqrt{\pi} e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right).$$

Thus from (2), (3) we obtain the formula

$$D(x) = \frac{2\sqrt{\pi} e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right) - x}{1 + \sqrt{\pi} x e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right)}.$$

The graphs of the function $y = D(x)$ (solid graph) and the Dawson's integral $y = F(x)$ (dotted graph) are shown in Figure 2.

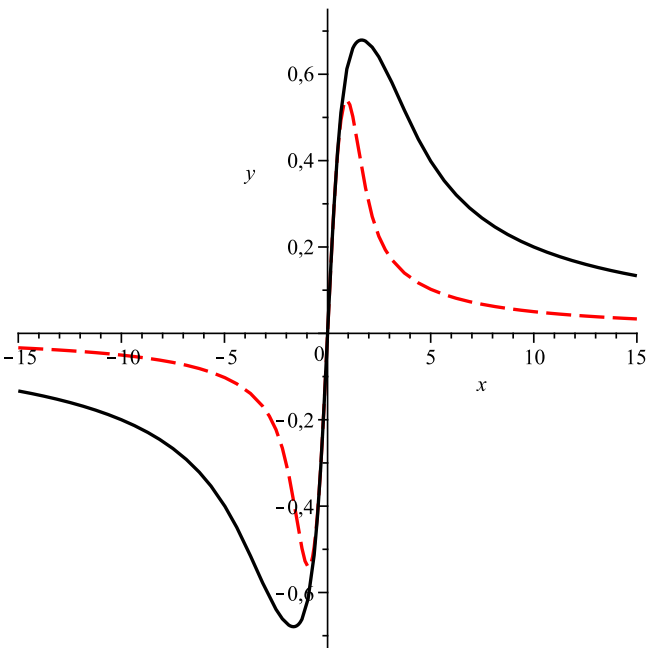


Fig. 2. The graphs of the functions $y = D(x)$ and $y = F(x)$

The function $D(x)$ can also be represented as the quotient of two power series

$$\begin{aligned} &\int_0^x \text{Exp}(t^2) dt = \\ &= x + \sum_{j=1}^{\infty} \frac{(j-1)!}{(2j-1)!(2j+1)} x^{2j+1} \end{aligned}$$

and

$$\text{Exp}(x^2) = 1 + \sum_{j=1}^{\infty} \frac{(j-1)!}{(2j-1)!} x^{2j}.$$

To prove this we apply Theorems 1 and 2. If we denote

$$A(x) = 1 + x + \sum_{j=1}^{\infty} \frac{(j-1)!}{(2j-1)!(2j+1)} x^{2j+1},$$

$$B(x) = 1 + \sum_{j=1}^{\infty} \frac{(j-1)!}{(2j-1)!} x^{2j},$$

then $D(x) = R(x) - S(x)$, where

$$R(x) = \frac{A(x)}{B(x)}, \quad S(x) = \frac{1}{B(x)}.$$

We denote the coefficients of the above series by small Latin letters.

Using Theorems 1 and 2, it is easy to verify that $s_{2k-1} = 0$ and

$$r_{2k} = s_{2k} = (-1)^k B_k,$$

where

$$B_k = \left\langle \begin{array}{cccccccc} b_2 & & & & & & & \\ \frac{b_4}{b_2} & & & & & & & \\ & b_2 & & & & & & \\ \frac{b_6}{b_4} & & \frac{b_4}{b_2} & & b_2 & & & \\ & \vdots & & & \dots & & \dots & \\ & & & & & \dots & \dots & \\ \frac{b_{2k-2}}{b_{2k-4}} & & \frac{b_{2k-4}}{b_{2k-6}} & & \frac{b_{2k-6}}{b_{2k-8}} & & \dots & b_2 \\ & \frac{b_{2k}}{b_{2k-2}} & \frac{b_{2k-2}}{b_{2k-4}} & & \frac{b_{2k-4}}{b_{2k-6}} & & \dots & \frac{b_4}{b_2} & b_2 \end{array} \right\rangle_k,$$

for all $k = 1, 2, \dots$

Thus $d_{2k} = 0$ and for all $k = 1, 2, \dots$

$$d_{2k-1} = r_{2k-1} =$$

$$= a_{2k-1}r_0 + a_{2k-3}r_2 + \dots + a_1r_{2k-2}.$$

Notice, that the parapermanent B_k can be calculated with the help of the linear recurrent equation

$$\begin{aligned} B_0 &= 1, \quad B_k = b_2 B_{k-1} - b_4 B_{k-2} + \dots + \\ &+ (-1)^{k-2} b_{2k-2} B_1 + (-1)^{k-1} b_{2k} B_0. \end{aligned}$$

Hence, performing necessary calculations, the function $D(x)$ is represented as the power series

$$\begin{aligned} D(x) &= x - \frac{2}{3} x^3 + \frac{8}{15} x^5 - \frac{55}{126} x^7 + \frac{338}{945} x^9 - \\ &- \frac{121861}{415800} x^{11} + \frac{486781}{2027025} x^{13} - \dots \quad (4) \end{aligned}$$

Let us note that the coefficients of the power series (4) can also be found using the recurrence formulas

$$d_1 = 1, \quad d_{2n-1} = \frac{(n-2)!}{(2n-3)!(2n-1)} -$$

$$- \sum_{n=2}^{\infty} \sum_{j=1}^n \frac{(n-1-j)!}{(2n-2j-1)!} d_{2j-1}, \quad n = 2, 3, \dots$$

4 Differential equation of the function $D(x)$

The Dawson's integral is a solution of the linear nonhomogeneous equation $y' + 2xy = 1$ [4]. Let us prove that the function $D(x)$ is a solution of the Riccati equation.

Theorem 3. *The function $y = D(x)$ is a solution of the Cauchy problem*

$$\left. \begin{aligned} y' &= \frac{x^2-2}{2(x^2+2)} y^2 - \frac{x(x^2+6)}{2(x^2+2)} y + 1, \\ y(0) &= 0. \end{aligned} \right\} \quad (5)$$

Proof. It follows from the formula (3) that the integral curve $y = D(x)$ passes through the origin.

Let us prove that the function $y = D(x)$ is a solution of the Riccati equation (5). Indeed, since

$$\begin{aligned} y' &= \frac{1}{2} \left(1 + \sqrt{\pi} x e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right) \right)^{-2} \times \\ &\times \left(2\sqrt{\pi} x e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right) + e^{\frac{x^2}{4}} x^3 \Phi\left(\frac{x}{2}\right) + 2x^2 - \right. \\ &\left. - 4\pi e^{\frac{x^2}{2}} \Phi^2\left(\frac{x}{2}\right) + 2 \right), \end{aligned}$$

excluding from this formula and from (3) the expression $\sqrt{\pi} e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right)$, we obtain the relation

$$y' = \frac{x^2 - 2}{2(x^2 + 2)} y^2 - \frac{x(x^2 + 6)}{2(x^2 + 2)} y + 1.$$

The Theorem 3 is proved.

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