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Про узагальнений частковий вінцевий добуток напівгруп

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On a generalized partial wreath product of semigroups

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У статті запропоноване узагальнення часткового вінцевого добутку. Доведено, що узагальнений частковий вінцевий добуток напівгруп є напівгрупою. Показано, що узагальнений частковий вінцевий добуток інверсних напівгруп є інверсною напівгрупою. Знайдено вигляд ідемпотентів узагальненого часткового вінцевого добутку напівгруп.

Ключові слова: напівгрупа, частковий вінцевий добуток, узагальнений частковий вінцевий добуток, ідемпотент.

The construction of wreath product is well–known and widely used both for groups and semigroups. It gives a possibility to get new group (semigroups) from old. The semigroup of partial automorphisms of a rooted tree of a special kind is isomorphic to partial wreath product of inverse symmetric semigroups. However, sometimes it is necessary to consider a construction similar to wreath product, but which allows to get wreath product of a semigroup (group) with a set of semigroups (groups). It is shown In the present paper a generalization of partial wreath product of semigroups is introduced. It is proved that the generalized partial wreath product of semigroups is a semigroup. Besides, it is shown that the generalized wreath product of inverse semigroups is an inverse semigroup. It is also shown that under certain conditions the generalized wreath product of semigroup is isomorphic to the direct product of wreath products of semigroups. The description of idempotents of a generalized partial wreath product is provided.

Key Words: semigroup, partial wreath product, generalized partial wreath product, idempotent.
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1 Introduction

The wreath product construction is a useful tool to obtain new groups or semigroups from old. However, it is interesting object of study. Depending on type of semigroups, slightly different modification of wreath product are considered. The most comprehensive survey is contained in [3]. In certain cases it is convenient to have a possibility construct a wreath product of a group (semigroup) with a set of groups (semigroups), for example, when describing closed subsemigroups of a semigroup of partial automorphisms of a rooted tree [1]. The generalized wreath product introduced in this paper provides such a possibility. We consider a generalized wreath product of semigroups and study its basic properties.

2 Partial wreath product of semigroups

We follow notations from [3].

Let P be a semigroup and (H, M) be a semigroup of partial transformations of a finite set M. Consider the set S^{PX} of partial functions from M to P:

$$S^{PX} = \{f : A \to P \mid \mathrm{dom}(f) = A, A \subseteq M\}.$$

If $f, g \in S^{PX}$, then product fg is defined in a following way

$$\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g),$$
$$(fg)(x) = f(x)g(x) \text{ for each } x \in \mathrm{dom}(fg).$$

If $h \in H, f \in S^{PX}$, then f^h is defined as follows:

$$\mathrm{dom}(f^h) = \{x \in \mathrm{dom}(h) : x^h \in \mathrm{dom}(f)\} = (\mathrm{ran}(h) \cap \mathrm{dom}(f))h^{-1}, (f^h)(x) = f(x^h).$$

Definition 2.1. Partial wreath product of a semigroup P with a semigroup of partial transformations (H, M) of a finite set M is a set

$$\{(f, h) \in S^{PX} \times (H, M) \mid \mathrm{dom}(f) = \mathrm{dom}(h)\}$$

with multiplication defined by

$$(f, a)(g, b) = (fg^a, ab).$$

Partial wreath product of semigroups P and (H, M) is denoted by $P \wr_p H$.

Proposition 1. [2]
1. Partial wreath product of semigroup $P \wr_p H$ is a semigroup.
2. If semigroups P and (H, M) are inverse then the semigroup $P \wr_p H$ is inverse. The inverse element to (f, h) is the element $(f, h)^{-1} = ((f^{h^{-1}})^{-1}, h^{-1})$.

3 Generalized partial wreath product of semigroups

Let H be a semigroup which acts (possibly, partially) on a set M, $\{P_x\}_{x \in M}$ be any set of semigroups satisfying condition:

if $y \in x^H$ (that is, $y = x^h$ for certain $h \in H$), then $P_y \subset P_x$.

Definition 3.1. Generalized partial wreath product of a group (semigroup) (H, M) with a set of semigroups $\{P_x\}_{x \in M}$ is a set

$$GWr(H, M, \{P_x\}) = \\ = \{(f, h) \mid h \in H,\, f\colon M \to \bigcup_x P_x, \\ f(x) \in P_x, \mathrm{dom}(f) = \mathrm{dom}(h)\}$$

with multiplication defined by

$$(f_1, h_1)(f_2, h_2) = (f_1 f_2^{h_1}, h_1 h_2), \qquad (1)$$

where $(f_1, h_1), (f_2, h_2) \in GWr(H, M, \{P_x\})$.

We define the product of functions f_1 and f_2 in the same way as for usual wreath product:

$$(f_1 f_2)(x) = f_1(x) f_2(x),$$

and we also define

$$f^h(x) = f(x^h).$$

Theorem 3.1. *The generalized partial wreath product $GWr(H, M, \{P_x\})$ of semigroup (H, M) with the set of semigroups $\{P_x\}_{x \in M}$ forms a semigroup under multiplication defined by* (1).

Proof. We show that the set $GWr(H, M, \{P_x\})$ is closed under multiplication.

Let $(f_1, h_1), (f_2, h_2) \in GWr(H, M, \{P_x\})$. Note that

$$\mathrm{dom}(f_1) = \mathrm{dom}(h_1)$$

and

$$\mathrm{dom}(f_2) = \mathrm{dom}(h_2).$$

Then

$$(f_1, h_1)(f_2, h_2) = (f_1 f_2^{h_1}, h_1 h_2).$$

Clearly, $h_1 h_2 \in H$.

Since $P_{x^{h_1}} \subset P_x$, then $f_2(x^{h_1}) \in P_x$ and

$$f_1 f_2^{h_1}(x) = f_1(x) f_2(x^{h_1}) \in P_x.$$

And we need to check that equality

$$\mathrm{dom}(f_1 f_2^{h_1}) = \mathrm{dom}(h_1 h_2)$$

holds. We have

$$\mathrm{dom}(h_1 h_2) = (\mathrm{ran}(h_1) \cap \mathrm{dom}(h_2))h_1^{-1},$$

$$\mathrm{dom}(f_2^{h_1}) = (\mathrm{ran}(h_1) \cap \mathrm{dom}(f_2))h_1^{-1}.$$

Since

$$\mathrm{dom}(f_1 f_2^{h_1}) = \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2^{h_1})$$

and

$$\mathrm{dom}(f_1) = \mathrm{dom}(h_1), \quad \mathrm{dom}(f_2) = \mathrm{dom}(h_2),$$

then

$$\mathrm{dom}(f_1 f_2^{h_1}) = \\ = \mathrm{dom}(h_1) \cap (\mathrm{ran}(h_1) \cap \mathrm{dom}(f_2))h_1^{-1} = \\ = (\mathrm{ran}(h_1) \cap \mathrm{dom}(h_2))h_1^{-1} = \mathrm{dom}(f_2^{h_1}).$$

To complete the proof, we note that every set Ah_1^{-1} is contained in $\mathrm{dom}(h_1)$.

Thus, $(f_1, h_1)(f_2, h_2) \in GWr(H, M, \{P_x\})$, and generalize wreath product of semigroups is a semigroup.

□

Consider now the semigroup $GWr(H, M, \{P_x\})$. On the set M we introduce a relation $\sim$:

$$x \sim y \overset{def}{\Leftrightarrow} x \in y^H,$$

that is, there exists $h \in H$ such that $x = y^h$.

Lemma 1. *If H is an inverse group, then relation $\sim$ is the equivalence relation on the set M.*

Proof. Clearly, $x \in x^H$.

Since H is an inverse semigroup, then $x = y^h$ implies $y = x^{h^{-1}}$. Thus, $x \in y^H$ implies $y \in x^H$.

Let $x \sim y$ and $y \sim z$. Then there exists $h_1, h_2 \in H$ such that $x = y^{h_1}$, $y = z^{h_2}$. It follows $x = y^{h_1} = z^{h_2 h_1}$, and $x \sim z$. □

Lemma 2. *For equivalence relation $\sim$ the following holds: if $x \sim y$, then $P_x = P_y$.*

Proof. It follows from the definition of the generalized wreath product that $x \in y^H$ implies $P_x \subset P_y$. Since $\sim$ is symmetric, then $P_y \subset P_x$. □

Theorem 3.2. *If semigroup H and all the semigroups of the set $\{P_x\}_{x \in M}$ are inverse, then semigroup $GWr(H, M, \{P_x\})$ is inverse.*

Proof. Let H and $\{P_x\}_{x \in M}$ be inverse semigroups. If $(f, h) \in GWr(H, M, \{P_x\})$, then $h^{-1} \in H$ and $(f^{h^{-1}})^{-1} \in P_x$.

Thus, $((f^{h^{-1}})^{-1}, h^{-1}) \in GWr(H, M, \{P_x\})$.

The element $((f^{h^{-1}})^{-1}, h^{-1})$ is inverse for (f, h). Indeed,

$$\begin{aligned}(f,h)(f,h)^{-1}(f,h) &= (f,h)((f^{h^{-1}})^{-1}, h^{-1}) = \\ &= (f,h) = (f((f^{h^{-1}})^{-1})^h, hh^{-1})(f,h) = \\ &= (f((f^{h^{-1}})^{-1})^h f^{hh^{-1}}, hh^{-1}h) = (f,h).\end{aligned}$$

Similarly,

$$(f,h)^{-1}(f,h)(f,h)^{-1} = ((f^{h^{-1}})^{-1}, h^{-1}).$$

As H and $\{P_x\}_{x \in M}$ are inverse, the uniqueness of the element $(f,h)^{-1}$ follows from the uniqueness of the elements h^{-1} and $(f^{h^{-1}})^{-1}$. □

Since $\sim$ is equivalence relation, the set M is partitioned into disjoint union of equivalence clases:

$$M = M_1 \cup M_2 \cup \ldots \cup M_k.$$

Theorem 3.3. *If the semigroup H is inverse, then generalized partial wreath product is isomorphic to the direct product of partial wreath products:*

$$GWr(H, M, \{P_{x_i}\}_{i=1}^k) \simeq \prod_{i=1}^k (P_{x_i} \wr_p (H, M_i))$$

where $x_i \in M_i$, $i = 1, 2, \ldots, k$.

Proof. Let $(f, h) \in GWr(H, M, \{P_{x_i}\}_{i=1}^k)$. Consider the map

$$\varphi : GWr(H, M, \{P_{x_i}\}_{i=1}^k) \to \prod_{i=1}^k (P_{x_i} \wr_p (H, M_i)),$$

which acts as

$$\begin{aligned}&(f,h) \mapsto \\ &\mapsto \left((f|_{M_1}, h|_{M_1}), (f|_{M_2}, h|_{M_2}), \ldots, (f|_{M_k}, h|_{M_k}) \right).\end{aligned}$$

First, we check that for all $i = 1, 2, \ldots, k$ $(f|_{M_i}, h|_{M_i})$ is in $P_{x_i} \wr_p (H, M_i)$.

Let $x \in M_i$ for some $i = 1, \ldots, k$. Since M_i is an equivalence class, then for every h from H we have $x^{h|_{M_i}} \in M_i$. Therefore, $h|_{M_i} \in (H, M_i)$.

From the definition of a generalized wreath product it follows that $f(x) \in P_{x_i}$ for any $x \in M_i$. Denote $\mathrm{dom}(f_i) = B_i$. Clearly, $B_i \subseteq M_i$. From above we have

$$f_i \colon B_i \to P_{x_i}$$

is a function from $B_i \subset M_i$ to P_{x_i} as required. Therefore, $(f|_{M_i}, h|_{M_i}) \in P_{x_i} \wr_p (H, M_i)$

Show that the map given above is a homomorphism. On one hand, we have

$$\begin{aligned}&(fg^h, hb) \mapsto \\ &\mapsto \left(((fg)^h|_{M_1}, hb|_{M_1}), \ldots, ((fg)^h|_{M_k}, hb|_{M_k}) \right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\left((f|_{M_1}, h|_{M_1}), \ldots, (f|_{M_k}, h|_{M_k}) \right) \times \\ &\times \left((g|_{M_1}, b|_{M_1}), \ldots, (g|_{M_k}, b|_{M_k}) \right) = \\ &= \Big(((fg^{h|_{M_1}})|_{M_1}, hb|_{M_1}), \ldots, \\ &\qquad ((fg^{h|_{M_k}})|_{M_k}, hb|_{M_k}) \Big) = \\ &= \left(((fg)^h|_{M_1}, hb|_{M_1}), \ldots, ((fg)^h|_{M_k}, hb|_{M_k}) \right).\end{aligned}$$

It is left to prove that this map is bijective. The injectivity follows from the definition of equivalence classes and from uniqueness of inverse element. Clearly, it is also surjective. So, given map is indeed an isomorphism. □

4 Idempotents in generalized partial wreath product of semigroups

Theorem 4.1. *Element $(f, h) \in GWr(H, M, \{P_x\})$ is idempotent if and only if $h \in H$ is idempotent and f satisfies the condition: $f(x)f(x^h) = f(x)$ for each $x \in M$.*

Proof. Ann element $(f, h) \in GWr(H, M, \{P_x\})$ is an idempotent if and only if $(f, h)(f, h) = (f, h)$. According to multiplication rule in the semigroup $GWr(H, M, \{P_x\})$ it means $(f, h)(f, h) = (ff^h, hh) = (f, h)$. Therefore, h is idempotent in H and $f(x)f(x^h) = f(x)$. □

Theorem 4.2. *Let $(f, h) \in GWr(H, M, \{P_x\})$ be idempotent, then $f(y) \in P_x$ is an idempotent, where $y = x^h$ for some $x \in M$.*

Proof. Let $h \in H$ be idempotent and $x, y \in M$ such that $x^h = y$. Then $y = x^h = x^{h^2} = (x^h)^h = y^h$ and $f(y) = f(y^h) = f^h(y)$. Therefore from the previous theorem we have $f(x)f^h(x) = f(x)$ and

$$f(y)f^h(y) = f(y)f(y) = f(y),$$

so $f(y) \in P_x$ is an idempotent. □

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