УДК 519.21

М.П. Моклячук, *д.ф.-м.н., професор*
В.І. Остапенко, *аспірант*

Мінімаксна екстраполяція гармонізованих стійких процесів за спостереженнями з шумом

Київський національний університет імені Тараса Шевченка, 01033, Київ, вул. Володимирська, 64. e-mail: mmp@univ.kiev.ua

M.P. Moklyachuk, *Doctor of Sciences, Professor*
V.I. Ostapenko, *PhD Student*

Minimax extrapolation problem for harmonizable stable processes with noise observations

Taras Shevchenko National University of Kyiv, 01033, Kyiv, 64 Volodymyrska st. e-mail: mmp@univ.kiev.ua

Досліджується задача оптимального оцінювання лінійного функціонала від невідомих значень гармонізованого α-стійкого випадкового процесу $\xi(t), t > 0$ за спостереженнями процесу $\xi(t) + \eta(t)$ у точках $t < 0$, де $\xi(t), t \in \mathbb{R}$ та $\eta(t), t \in \mathbb{R}$ – взаємно незалежні гармонізовані α-стійкі випадкові процеси. Отримані формули для обчислення величини похибки та визначення спектральної характеристики оптимальної оцінки функціонала за умови, що спектральні щільності процесів відомі. У тому випадку, коли спектральні щільності невідомі, але задані множини допустимих спектральних щільностей, застосовано мінімаксний метод оцінювання.

Ключові слова: гармонізований процес, оптимальна лінійна оцінка, робастна мінімаксна оцінка, найменшсприятлива спектральна щільність, мінімаксна спектральна характеристика.

The problem of optimal linear estimation of the linear functional which depends on the unknown values of a harmonizable α-stable stochastic processes $\xi(t), t > 0$, from observations of the process $\xi(t) + \eta(t), t < 0$, where $\xi(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$ are mutually independent harmonizable α-stable stochastic processes is considered. The problem is considered in the case of spectral certainty where spectral densities are exactly known as well as in the case of spectral uncertainty where spectral densities are not exactly unknown while classes of admissible spectral densities are specified. Formulas for calculating the error and the spectral characteristic of the optimal linear estimate of the functional are proposed in the case of spectral certainty. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristics are proposed in the case of spectral uncertainty for some classes of admissible spectral densities.

Key Words: harmonizable process, optimal linear estimate, robust estimate, least favorable spectral density, minimax spectral characteristic.

Communicated by Prof. Kozachenko Yu.V.

1 Introduction

The interpolation, extrapolation and filtering problems for stationary stochastic processes were investigated by A. Kolmogorov, N. Wiener, A. Yaglom (see [2] for details). The extrapolation problem for harmonizable stochastic process was investigated by S. Rajput and K. Sandberg [8]. The proposed methods of solution the estimation problems are based on the assumption that we known spectral densities of the considered processes. However, usually we do not know exactly spectral densities of the processes while we often know a set of admissible spectral densities. In this case we can apply the minimax-robust approach to estimate the functional $A^{extr}\xi$. This method let us find an estimate that minimizes the maximum of the errors for all spectral densities from the given set of admissible spectral densities simultaneously (see book [5] for more details). The paper by U. Grenander [3] is the first one where the minimax-robust approach to the problem of extrapolation of stationary stochastic processes was formulated and investigated. In the book by M. Moklyachuk [5] results of investigations of the problem of optimal linear estimation of functionals which depend on unknown values of stationary and related sequences and processes are presented. In the papers by M. P. Moklyachuk and V. I. Ostapenko [6], [7] results of investigations of the interpolation problem for harmonizable stable sequences are proposed.

In this paper we deal with the problem of optimal linear estimation of the functional $A^{extr}\xi = \int_0^\infty a(t)\xi(t)dt$ which depends on the unknown values of a harmonizable α-stable stochastic processes $\xi(t), t > 0$, from observations of the process $\xi(t) + \eta(t), t < 0$, where $\xi(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$ are mutually independent harmonizable α-stable stochastic processes. Formulas for calculating the error and the spectral characteristic of the optimal linear estimate of the functional are proposed in the case of spectral certainty where spectral densities of the processes are exactly known. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristics are proposed in the case of spectral uncertainty where spectral densities are not exactly unknown while a set of admissible densities is given.

2 Harmonizable symmetric α-stable stochastic process

Definition 2.1 *(symmetric α-stable random variable).* A real random variable ξ is said to be symmetric α-stable, $S\alpha S$, if its characteristic function has the form $Eexp(it\xi) = exp(-c|t|^\alpha)$ for some $c \geqslant 0$ and $0 < \alpha \leqslant 2$.

Definition 2.2 *(symmetric α-stable stochastic process).* A stochastic process $\{\xi = \xi(t), t \in \mathbb{R}\}$ is called symmetric α-stable, $S\alpha S$, stochastic process, if all finite dimensional distributions $(\xi(t_1), \xi(t_2), \ldots, \xi(t_d)), t_1, t_2, \ldots, t_d \in \mathbb{R}, d \geqslant 1$ are symmetric α-stable.

For jointly $S\alpha S$ random variables $\xi = \xi_1 + i\xi_2$ and $\eta = \eta_1 + i\eta_2$ the covariation of ξ with η is defined as [1]

$$[\xi, \eta]_\alpha = \int_{S_4} (x_1 + ix_2)\times \times (y_1 + iy_2)^{<\alpha-1>} d\Gamma_{\xi_1,\xi_2,\eta_1,\eta_2}(x_1, x_2, y_1, y_2), \tag{1}$$

where $z^{<\beta>} = |z|^{\beta-1}\bar{z}$ for a complex number z and $\beta > 0$. The covariation is linear on the first argument but in general is not symmetric and linear on the second argument [1]. For ξ, ξ_1, ξ_2, η jointly $S\alpha S$ we have $[\xi_1 + \xi_2, \eta]_\alpha = [\xi_1, \eta]_\alpha + [\xi_2, \eta]_\alpha$,

$$|[\xi, \eta]_\alpha| \leqslant ||\xi||_\alpha ||\eta||_\alpha^{\alpha-1} \tag{2}$$

and $||\xi||_\alpha = [\xi, \xi]_\alpha^{1/\alpha}$ is a norm in the linear space of $S\alpha S$ random variables which is equivalent to convergence in probability. The norm $||\cdot||_\alpha$ is not necessarily the standard L^α norm.

Let $Z = \{Z(t) : -\infty < t < \infty\}$ be a complex $S\alpha S$ process with independent increments. The spectral measure of the process Z is defined as $\mu\{(s, t]\} = \|Z(t) - Z(s)\|_\alpha^\alpha$. The integrals $\int f(t)dZ(t)$ can be defined for all $f \in L^\alpha(\mu)$ with properties [1]:

$$\left\|\int f(t)dZ(t)\right\|_\alpha^\alpha = \int |f(t)|^\alpha d\mu,$$

$$\left[\int f(t)dZ(t), \int g(t)dZ(t)\right]_\alpha = = \int f(t)(g(t))^{<\alpha-1>} d\mu. \tag{3}$$

Definition 2.3 *(Harmonizable symmetric α-stable stochastic process).* A $S\alpha S$ stochastic process $\{\xi(t), t \in \mathbb{R}\}$ is said to be harmonizable, $HS\alpha S$, if there exists a $S\alpha S$ process $Z = \{Z(\theta); \theta \in [-\infty, \infty]\}$ with independent increments and finite spectral measure μ such that process $\xi(t)$ has the spectral representation $\xi(t) = \int_{-\infty}^\infty e^{in\theta} dZ(\theta)$, $t \in \mathbb{R}$, and the covariation has the representation $[\xi(t), \xi(s)]_\alpha = \int_{-\infty}^\infty e^{i(t-s)\theta} d\mu(\theta)$, $t, s \in \mathbb{R}$.

Note that a $HS\alpha S$ stochastic process is not always stationary, but for $\alpha = 2$ the $HS\alpha S$ processes are stationary with Gaussian distribution. We consider the case where $1 < \alpha \leqslant 2$.

Denote by $H(\xi)$ the time domain of the $HS\alpha S$ process $\{\xi(t), t \in \mathbb{R}\}$, which is a closed in the norm $\|\cdot\|_\alpha$ linear manifold generated by all values of the $HS\alpha S$ process $\{\xi(t), t \in \mathbb{R}\}$. It follows from the spectral representation of the $HS\alpha S$ process $\{\xi(t), t \in \mathbb{R}\}$ that the mapping $\xi(t) \leftrightarrow e^{it\theta}, t \in \mathbb{R}$, extents to an isomorphism between the spaces $H(\xi)$ and $L^\alpha(\mu)$. Under this isomorphism to each $\eta \in H(\xi)$ corresponds a unique $f \in L^\alpha(\mu)$ such that $\eta = \int_{-\infty}^\infty f(\theta)dZ(\theta)$.

For a closed linear subspace $M \subseteq L^\alpha(\mu)$ and $f \in L^\alpha(\mu)$, there exists a unique element from M which minimizes the distance to f. This element is called projection of f onto M or the best approximation of f in M. This projection is denoted by $P_M f$ and is uniquely determined by the condition

$$\int_{-\infty}^\infty g\,(f - P_M f)^{<\alpha-1>}\, d\mu = 0, \quad g \in M. \tag{4}$$

Similarly, for $HS\alpha S$ stochastic process $\{\xi(t), t \in \mathbb{Z}\}$ and a closed linear subspace $H^-(\xi)$ of the space $H(\xi)$ there is a uniquely determined element $\hat{\xi}_n \in H^-(\xi)$ which minimizes the distance to ξ_n and is determined from the condition

$$\left[\eta, \xi(t) - \hat{\xi}(t)\right]_\alpha = 0, \quad \eta \in H^-(\xi). \tag{5}$$

3 Extrapolation problem. Projection approach

Consider the problem of the optimal estimation of the linear functional

$$A^{extr}\xi = \int_0^\infty a(t)\xi(t)dt = \int_{-\infty}^\infty A^{extr}(\theta)dZ^\xi(\theta),$$

$$A^{extr}(\theta) = \int_0^\infty a(t)e^{it\theta}dt,$$

that depends on the unknown values of the $HS\alpha S$-process $\xi(t), t > 0$, from observations of the process $\{\xi(t) + \eta(t), t < 0\}$, where $\xi(t)$ and $\eta(t)$ are harmonizable symmetric α-stable stochastic processes.

We consider the problem for mutually independent harmonizable symmetric α-stable stochastic processes $\{\xi(t), t \in \mathbb{R}\}$ and $\{\eta(t), t \in \mathbb{R}\}$ which have absolutely continuous spectral measures and the spectral densities $f(\theta) > 0$ and $g(\theta) > 0$ satisfying the minimality condition

$$\int_{-\infty}^\infty \frac{|\gamma(\theta)|^{\frac{\alpha}{\alpha-1}}}{(f(\theta)+g(\theta))^{\frac{1}{\alpha-1}}}d\theta < \infty. \tag{6}$$

for a non-zero function of the exponential type $\gamma(\theta) = \int_0^\infty \alpha(t)e^{it\theta}dt$, $\int_{-\infty}^\infty |\gamma(\theta)|^{\frac{\alpha}{\alpha-1}}d\theta < \infty$.

Denote by $H^-(\xi+\eta)$ closed in the $||\cdot||_\alpha$ norm linear manifold generated by values of the harmonizable symmetric α-stable stochastic process $\xi(t) + \eta(t), t < 0$, in the space $H(\xi+\eta)$ generated by all values of the process $\{\xi(t) + \eta(t), t \in \mathbb{R}\}$.

It follows from the relation between spaces $H(\xi)$ and $L_\alpha(f)$ that the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is of the form

$$\hat{A}^{extr}\xi = \int_{-\infty}^\infty h(\theta)\left(dZ^\xi(\theta) + dZ^\eta(\theta)\right). \tag{7}$$

This estimate is determined by the spectral characteristic $h(\theta)$ from $L_\alpha^-(f+g)$ which is a subspace of $L_\alpha(f+g)$ generated by $e^{it\theta}$ for $t < 0$.

The optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is a projection of $A^{extr}\xi$ on the subspace $H^-(\xi+\eta)$ which is determined by relations

$$[\zeta, A^{extr}\xi - \hat{A}^{extr}\xi]_\alpha = 0, \quad \forall\zeta \in H^-(\xi+\eta),$$

or, equivalently, by relations

$$[\xi(t) + \eta(t), A^{extr}\xi - \hat{A}^{extr}\xi]_\alpha = 0, \quad \forall t < 0. \tag{8}$$

It follows from these equations that the spectral characteristic $h(\theta)$ of the estimate is determined by the relation

$$\left(A^{extr}(\theta) - h(\theta)\right)^{<\alpha-1>} f(\theta) - (h(\theta))^{<\alpha-1>} g(\theta) = \tag{9}$$

$$= \overline{C^{extr}(\theta)}, \quad C^{extr}(\theta) = \int_0^\infty c(t)e^{it\theta}dt$$

and satisfies condition

$$\int_{-\infty}^\infty e^{-i\theta t}h(\theta)d\theta = 0, t < 0. \tag{10}$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\begin{aligned} &\left\|\hat{A}^{extr}\xi - A^{extr}\xi\right\|_\alpha^\alpha = \\ &= \int_{-\infty}^\infty \left|A^{extr}(\theta) - h(\theta)\right|^\alpha f(\theta)d\theta + \\ &+ \int_{-\infty}^\infty |h(\theta)|^\alpha g(\theta)d\theta. \end{aligned} \tag{11}$$

We can conclude that the following theorem holds true.

Theorem 3.1. *Let $\{\xi(t), t \in \mathbb{R}\}$ and $\{\eta(t), t \in \mathbb{R}\}$ be mutually independent harmonizable symmetric α-stable stochastic processes which have absolutely continuous spectral measures and the spectral densities $f(\theta) > 0$ and $g(\theta) > 0$ satisfying the minimality condition* (6). *The optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi = \int_0^\infty a(t)\xi(t)dt$ that depends on the unknown values $\xi(t), t > 0$, of the process $\xi(t), t \in \mathbb{R}$ from observations of the process $\{\xi(t)+\eta(t), t < 0\}$ is calculated by the formula* (7). *The spectral characteristic $h(\theta)$ of the estimate is determined by equations* (9) *and* (10). *The variance of the optimal estimate of the functional is calculated by the formula* (11).

3.1 Extrapolation problem. Observations without noise

Consider the problem of optimal linear estimation of the functional $A^{extr}\xi$ that depends on the unknown values of a $HS\alpha S-$ process $\xi(t), t>0$, from observations of the process $\{\xi(t), t<0\}$.

Let $\{\xi(t), t \in \mathbb{R}\}$ be a harmonizable symmetric α-stable stochastic process which has absolutely continuous spectral measure and the spectral density $f(\theta)>0$ satisfying the minimality condition

$$\int_{-\infty}^{\infty} \frac{|\gamma(\theta)|^{\frac{\alpha}{\alpha-1}}}{(f(\theta))^{\frac{1}{\alpha-1}}} d\theta < \infty, \tag{12}$$

for a non-zero function of the exponential type $\gamma(\theta)=\int_0^\infty \alpha(t)e^{it\theta}dt$, $\int_{-\infty}^{\infty}|\gamma(\theta)|^{\frac{\alpha}{\alpha-1}}d\theta<\infty$.

Denote by $H^-(\xi)$ the closed in the $||\cdot||_\alpha$ norm linear manifold generated by values of the $HS\alpha S$ process $\xi(t), t<0$, in the space $H(\xi)$ generated by all values of the $HS\alpha S$ process $\{\xi(t), t\in\mathbb{R}\}$.

From isomorphism between spaces $H(\xi)$ and $L_\alpha(f)$ it follows that the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is of the form

$$\hat{A}^{extr}\xi = \int_{-\infty}^{\infty} h(\theta) dZ^{\xi}(\theta). \tag{13}$$

The spectral characteristic $h(\theta)$ of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional is determined by the formula $h(\theta)=$

$$= A^{extr}(\theta) - \left(\overline{C^{extr}(\theta)}\right)^{<\frac{1}{\alpha-1}>} (f(\theta))^{\frac{-1}{\alpha-1}}, \tag{14}$$

where $C^{extr}(\theta)$ is determined by condition

$$\int_{-\infty}^{\infty} e^{-i\theta t} h(\theta) d\theta = 0, t<0. \tag{15}$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\|\hat{A}^{extr}\xi - A^{extr}\xi\right\|_\alpha^\alpha = \\ = \int_{-\infty}^{\infty} \left| \left(\overline{C^{extr}(\theta)}\right)^{<\frac{1}{\alpha-1}>} (f(\theta))^{\frac{-1}{\alpha-1}} \right|^\alpha f(\theta) d\theta. \tag{16}$$

From the theorem 3.1 the following corollary holds true.

Corollary 1. *Let $\{\xi(t), t\in\mathbb{R}\}$ be a harmonizable symmetric α-stable stochastic process which has absolutely continuous spectral measure and the spectral density $f(\theta)>0$ satisfying the minimality condition* (12)*. The optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$, that depends on the unknown values $\xi(t), t>0$, of the process $\{\xi(t), t\in\mathbb{R}\}$, from observations of the process $\{\xi(t), t<0\}$ is of the form* (13)*. The spectral characteristic $h(\theta)$ of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional is calculated by formula* (14)*, where $C^{extr}(\theta)$ is determined by condition* (15)*. The variance of the optimal estimate of the functional is calculated by formula* (16)*.*

3.2 Extrapolation problem. Stationary processes

Consider the problem of the optimal estimation of the functional $A^{extr}\xi$ that depends on the unknown values $HS\alpha S-$ process $\xi(t), t>0$, from observations of the process $\{\xi(t)+\eta(t), t<0\}$, where $\xi(t)$ and $\eta(t)$ are mutually independent harmonizable symmetric α-stable stochastic processes in the case $\alpha=2$.

We will suppose that stationary process $\{\xi(t), t\in\mathbb{R}\}$ and $\{\eta(t), t\in\mathbb{R}\}$ have spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition (6) with $\alpha=2$.

The spectral characteristic $h(\theta)$ of the optimal estimate is of one of the forms

$$h(\theta) = \frac{A^{extr}(\theta)f(\theta) - C^{extr}(\theta)}{f(\theta)+g(\theta)}, \tag{17}$$

$$h(\theta) = A^{extr}(\theta) - \frac{A^{extr}(\theta)g(\theta) + C^{extr}(\theta)}{f(\theta)+g(\theta)}. \tag{18}$$

The variance of estimate has the form

$$\left\|\hat{A}^{extr}\xi - A^{extr}\xi\right\|_2^2 = \\ = \int_{-\infty}^{\infty} \left| \frac{A^{extr}(\theta)g(\theta) + C^{extr}(\theta)}{f(\theta)+g(\theta)} \right|^2 f(\theta) d\theta \\ + \int_{-\infty}^{\infty} \left| \frac{A^{extr}(\theta)f(\theta) - C^{extr}(\theta)}{f(\theta)+g(\theta)} \right|^2 g(\theta) d\theta = \\ = \langle \mathbf{Bc}, \mathbf{c} \rangle + \langle \mathbf{Ra}, \mathbf{a} \rangle, \tag{19}$$

$$\text{where } c(t) = \left(\mathbf{B}^{-1}\mathbf{Da}\right)(t), t \geqslant 0, \tag{20}$$

$$(\mathbf{Bc})(t) = \int_0^\infty c(u) \int_{-\infty}^{\infty} e^{i(u-t)\theta} \frac{1}{f(\theta)+g(\theta)} d\theta du;$$

$$(\mathbf{Dc})(t) = \int_0^\infty c(u) \int_{-\infty}^{\infty} e^{i(u-t)\theta} \frac{f(\theta)}{f(\theta)+g(\theta)} d\theta du;$$

$$(\mathbf{Rc})(t)=\int_0^\infty c(u)\int\limits_{-\infty}^{\infty} e^{i(u-t)\theta}\frac{f(\theta)g(\theta)}{f(\theta)+g(\theta)}d\theta du.$$

So, the following theorem holds true.

Theorem 3.2. *Let $\{\xi(t),t\in\mathbb{R}\}$ and $\{\eta(t),t\in\mathbb{R}\}$ be mutually independent stationary stochastic processes which have absolutely continuous spectral measures and the spectral densities $f(\theta)>0$ and $g(\theta)>0$ satisfying the minimality condition* (6) *with $\alpha=2$. The optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$, that depends on the unknown values $\xi(t),t>0$ of the process $\{\xi(t),t\in\mathbb{R}\}$, from observations of the process $\{\xi(t)+\eta(t),t<0\}$, is calculated by the formula* (7)*. The spectral characteristic $h(\theta)$ of the estimate is calculated by the formula* (17) *or* (18)*. The variance of the optimal estimate of the functional is calculated by the formula* (19)*.*

3.3 Extrapolation problem. Stationary processes. Observations without noise

Consider the problem of optimal linear estimation of the functional $A^{extr}\xi$ that depends on the unknown values $\xi(t),t>0$, of a stationary stochastic process $\{\xi(t),t\in\mathbb{R}\}$ from observations of the process $\xi(t),t<0$. Suppose that the stationary stochastic process $\{\xi(t),t\in\mathbb{R}\}$ has the spectral density $f(\theta)>0$ satisfying the minimality condition (12) with $\alpha=2$.

The spectral characteristic $h(\theta)$ of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is calculated by the formula

$$h(\theta)=A^{extr}(\theta)-C^{extr}(\theta)\left(f(\theta)\right)^{-1}, \qquad (21)$$

The variance of the optimal estimate of the functional is calculated by the formula

$$\left\|\hat{A}^{extr}\xi-A^{extr}\xi\right\|_2^2= \\ =\int_{-\infty}^{\infty}\left|C^{extr}(\theta)\left(f(\theta)\right)^{-1}\right|^2 f(\theta)d\theta. \qquad (22)$$

where

$$C^{extr}(\theta)=\int_0^\infty c(t)e^{i\theta}d\theta,$$
$$c(t)=\left(\mathbf{B}^{-1}\mathbf{a}\right)(t),t\geqslant 0, \qquad (23)$$
$$(\mathbf{Bc})(t)=\frac{1}{2\pi}\int_0^\infty c(u)\int_{-\infty}^{\infty}\left(f(t)\right)^{-1}dtdu.$$

So, the following theorem holds true.

Theorem 3.3. *Let $\{\xi(t),t\in\mathbb{R}\}$ be a stationary stochastic process which has absolutely continuous spectral measure and the spectral density $f(\theta)>0$ satisfying the minimality condition* (12) *with $\alpha=2$. The optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ that depends on the unknown values $\xi(t),t>0$, of the process $\{\xi(t),t\in\mathbb{R}\}$ from observations of the process $\{\xi(t),t<0\}$ is of the form* (13)*, where the spectral characteristic $h(\theta)$ of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional is calculated by the formula* (21)*. The variance of the optimal estimate of the functional can be calculated by formula* (22)*.*

4 Extrapolation problem. Minimax approach

The derived formulas for calculation the error

$$\Delta\left(h(f,g);f,g\right):=\left\|\hat{A}^{extr}\xi-A^{extr}\xi\right\|_\alpha^\alpha$$

and the spectral characteristic $h(f,g):=h(\theta)$ of the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ can be applied only in the case where we know spectral densities of the processes. However, usually we do not have exact values of the spectral densities of the processes while we often know a set $D=D_f\times D_g$ of admissible spectral densities. In this case we can apply the minimax-robust approach to estimate the functional $A^{extr}\xi$. This method let us find an estimate that minimizes the maximum of the errors for all spectral densities from the given set $D=D_f\times D_g$ of admissible spectral densities simultaneously (see book [5] for more details).

Definition 4.1. For a given class of spectral densities $D=D_f\times D_g$ the spectral densities $f_0(\theta)\in D_f$, $g_0(\theta)\in D_g$ are called the least favorable in $D=D_f\times D_g$ for the optimal linear estimation of the functional $A^{extr}\xi$ if the following relation holds true

$$\Delta\left(f_0,g_0\right)=\Delta\left(h\left(f_0,g_0\right);f_0,g_0\right)= \\ \max_{(f,g)\in D_f\times D_g}\Delta\left(h\left(f,g\right);f,g\right). \qquad (24)$$

Definition 4.2. For a given class of spectral densities $D=D_f\times D_g$ the spectral characteristic $h^0=h(f_0,g_0)$ of the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is called minimax (robust) for the optimal linear estimation of the functional $A^{extr}\xi$ if the following relations hold true

$$h^0 \in H_D = \bigcap_{(f,g)\in D_f\times D_g} L^\alpha(f+g),$$

$$\min_{h\in H_D}\max_{(f,g)\in D}\Delta(h;f,g) = \max_{(f,g)\in D}\Delta\left(h^0;f,g\right).$$

The least favorable spectral densities $f_0(\theta)$, $g_0(\theta)$ and the minimax spectral characteristic $h^0 = h(f_0,g_0)$ form a saddle point of the function $\Delta(h;f,g)$ on the set $H_D\times D$. The saddle point inequalities

$$\Delta(h;f_0,g_0) \geqslant \Delta\left(h^0;f_0,g_0\right) \geqslant \Delta\left(h^0;f,g\right)$$

$$\forall h\in H_D, \forall f\in D_f, \forall g\in D_g$$

hold true if $h^0 = h(f_0,g_0)$ and $h(f_0,g_0)\in H_D$, where (f_0,g_0) is a solution to the constrained optimization problem

$$\max_{(f,g)\in D_f\times D_g}\Delta(h(f_0,g_0);f,g) = \Delta(h(f_0,g_0);f_0,g_0), \tag{25}$$

$$\begin{aligned}\Delta(h(f_0,g_0);f,g) &= \left\|A^{extr}\xi - \hat{A}^{extr}\xi\right\|_\alpha^\alpha \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|A^{extr}(\theta)-h^0(\theta)\right|^\alpha f(\theta)d\theta \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h^0(\theta)\right|^\alpha g(\theta)d\theta.\end{aligned} \tag{26}$$

The constrained optimization problem (25) is equivalent to the unconstrained optimization problem

$$\begin{aligned}\Delta_D(f,g) = -\Delta(h(f_0,g_0);f,g) + \\ +\delta(f,g\,|D_f\times D_g) \to \inf,\end{aligned} \tag{27}$$

where $\delta(f,g\,|D_f\times D_g)$ is the indicator function of the set $D = D_f\times D_g$. Solution (f_0,g_0) to the problem (27) is characterized by the condition $0\in\partial\Delta_D(f_0,g_0)$, where $\partial\Delta_D(f_0,g_0)$ is the subdifferential of the functional $\Delta_D(f,g)$ at point (f_0,g_0). This condition makes it possible to find the least favorable spectral densities in some special classes of spectral densities $D = D_f\times D_g$ ([5]).

Note, that the form of the functional $\Delta(h(f_0,g_0);f,g)$ is convenient for application the method of Lagrange multipliers for finding solution to the problem (25). In the following part of the paper we propose relations that determine the least favourable spectral densities in some special classes of spectral densities.

Basing on the presented definitions and the derived formulas we come to conclusion that the following lemmas hold true.

Lemma 1. *Let $\{\xi(t),t\in\mathbb{R}\}$ and $\{\eta(t),t\in\mathbb{R}\}$ be mutually independent harmonizable symmetric α-stable stochastic processes which have absolutely continuous spectral measures and the spectral densities $f_0(\theta)>0$ and $g_0(\theta)>0$ satisfying the minimality condition* (6)*. Let the spectral densities $(f_0,g_0)\in D_f\times D_g$ gives a solution to the constrained optimization problem* (25)*. The spectral densities (f_0,g_0) are the least favorable spectral densities in $D_f\times D_g$ and $h^0 = h(f_0,g_0)$ is the minimax spectral characteristic of the optimal linear estimation $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ that depends on the unknown values $\xi(t),t>0$, of the process $\{\xi(t),t\in\mathbb{R}\}$ from observations of the process $\{\xi(t)+\eta(t),t<0\}$ if $h^0 = h(f_0,g_0)\in H_D$.*

Lemma 2. *Let $\{\xi(t),t\in\mathbb{R}\}$ be a harmonizable symmetric α-stable stochastic process which has absolutely continuous spectral measure and the spectral density $f_0(\theta)>0$ satisfying the minimality condition* (12)*. Let the spectral density $f_0\in D_f$ gives a solution to the constrained optimization problem*

$$\max_{f\in D_f}\Delta(h(f_0);f) = \Delta(h(f_0);f_0), \tag{28}$$

$$\begin{aligned}\Delta(h(f_0);f) &= \left\|A^{extr}\xi - \hat{A}^{extr}\xi\right\|_\alpha^\alpha = \\ &\int_{-\infty}^{\infty}\left|\left(C^{extr0}(\theta)\right)^{<\frac{1}{\alpha-1}>}(f_0(\theta))^{\frac{-1}{\alpha-1}}\right|^\alpha f(\theta)d\theta.\end{aligned} \tag{29}$$

The spectral density f_0 is the least favorable spectral density in D_f and $h^0 = h(f_0)$ is the minimax spectral characteristic of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$, that depends on the unknown values $\xi(t),t>0$, of the process $\{\xi(t),t\in\mathbb{R}\}$, from observations of the process $\{\xi(t),t<0\}$ if $h^0 = h(f_0)\in H_D$.

Lemma 3. *Let $\{\xi(t),t\in\mathbb{R}\}$ and $\{\eta(t),t\in\mathbb{R}\}$ be mutually independent stationary stochastic processes which have absolutely continuous spectral measures and the spectral densities $f_0(\theta)>0$ and $g_0(\theta)>0$ satisfying the minimality condition* (6) *with $\alpha=2$. Let spectral densities $(f_0,g_0)\in D_f\times D_g$ gives a solution to the constrained optimization problem*

$$\max_{(f,g)\in D_f\times D_g}\Delta(h(f_0,g_0);f,g) = \Delta(h(f_0,g_0);f_0,g_0), \tag{30}$$

$$\Delta\left(h(f_0,g_0);f,g\right)=$$
$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\left|A^{extr}(\theta)g_0(\theta)+\int\limits_0^{\infty}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\theta}dt\right|^2}{(f_0(\theta)+g_0(\theta))^2}\times$$
$$\times f(\theta)d\theta+\frac{1}{2\pi}\times$$
$$\times\int\limits_{-\infty}^{\infty}\frac{\left|A^{extr}(\theta)f_0(\theta)-\int\limits_0^{\infty}((\mathbf{B}^0)^{-1}\mathbf{R}^0\mathbf{a})(t)e^{it\theta}dt\right|^2}{(f_0(\theta)+g_0(\theta))^2}\times$$
$$\times g(\theta)d\theta. \tag{31}$$

The spectral densities (f_0,g_0) are the least favorable spectral densities in $D_f\times D_g$ and $h^0=h(f_0,g_0)$ is the minimax spectral characteristic of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$, that depends on the unknown values $\xi(t),t>0$, of the sequence $\{\xi(t),t\in\mathbb{R}\}$ from observations of the process $\{\xi(t)+\eta(t),t<0\}$ if $h^0=h(f_0,g_0)\in H_D$.

Lemma 4. *Let $\{\xi(t),t\in\mathbb{R}\}$ be a stationary stochastic process which has absolutely continuous spectral measure and the spectral density $f_0(\theta)>0$ satisfying the minimality condition (12) with $\alpha=2$. Let the spectral density $f_0\in D_f$ gives a solution to the constrained optimization problem*

$$\max_{f\in D_f}\Delta\left(h(f_0);f\right)=\Delta\left(h(f_0);f_0\right), \tag{32}$$

$$\Delta\left(h(f_0);f\right)=$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int\limits_0^{\infty}((\mathbf{B}^0)^{-1}\mathbf{a})(t)e^{it\theta}dt\right|^2 f_0^{-2}(\theta)f(\theta)d\theta. \tag{33}$$

The spectral density f_0 is the least favorable spectral density in D_f and $h^0=h(f_0)$ is the minimax spectral characteristic of the optimal linear estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ that depends on the unknown values $\xi(t),t>0$, of the process $\{\xi(t),t\in\mathbb{R}\}$ from observations of the process $\{\xi(t),t<0\}$ if $h^0=h(f_0)\in H_D$.

4.1 Least favorable spectral densities in the class $D_f^0\times D_g^0$

Consider the problem for the class $D_f^0\times D_g^0$, where

$$D_f^0=\{f(\theta)|\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\theta)d\theta\leqslant P_1\},$$

$$D_g^0=\{g(\theta)|\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\theta)d\theta\leqslant P_2\}.$$

Let $f_0(\theta)\in D_f^0, g_0(\theta)\in D_g^0$ and let the functions

$$h_f(f_0,g_0)=\left|\int_0^{\infty}a(t)e^{it\theta}dt-h^0(\theta)\right|^{\alpha}, \tag{34}$$

$$h_g(f_0,g_0)=\left|h^0(\theta)\right|^{\alpha} \tag{35}$$

be bounded.

Under these conditions it follows that the functional

$$\tilde{\triangle}(f,g)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_0^{\infty}a(t)e^{it\theta}dt-h^0(\theta)\right|^{\alpha}f(\theta)d\theta$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h^0(\theta)\right|^{\alpha}g(\theta)d\theta. \tag{36}$$

is bounded in the space $L_1\times L_1$.

From the condition $0\in\partial\triangle_D(f_0,g_0), D=D_f^0\times D_g^0$ it follows that the least favorable densities satisfy equations

$$\left|\int_0^{\infty}a(t)e^{it\theta}dt-h^0(\theta)\right|^{\alpha}=\alpha_1(f_0(\theta)+g(\theta)) \tag{37}$$

$$\left|h^0(\theta)\right|^{\alpha}=\alpha_2(f_0(\theta)+g(\theta)), \tag{38}$$

where $\alpha_1\geqslant 0,\alpha_2\geqslant 0$ are constant, $\alpha_1\neq 0$, if $\int_{-\infty}^{\infty}f_0(\theta)d\theta=P_1$; $\alpha_2\neq 0$, if $\int_{-\infty}^{\infty}g_0(\theta)d\theta=P_2$.

We have the following statement holds true.

Theorem 4.1. *Let the spectral densities $f_0(\theta)\in D_f^0, g_0(\theta)\in D_g^0$ satisfy the minimality condition (6). Let functions h_f,h_g determined by (34), (35) be bounded. The spectral densities $f_0(\theta),g_0(\theta)$ are the least favorable in the class $D=D_f^0\times D_g^0$ for the optimal extrapolation of the functional $A^{extr}\xi$ if $f_0(\theta),g_0(\theta)$ give a solution to the system of equations (37), (38) and define solution of optimization problem (25). The spectral characteristic $h_(f_0,g_0)$ of optimal linear estimation $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is defined by equations (9), (10).*

4.2 Least favorable spectral densities for class D_1. Observations without noise

Consider the problem for the class of spectral densities

$$D_1=\left\{f(\theta):\int_{-\infty}^{\infty}f(\theta)d\theta=\gamma\right\}.$$

Making use of the Lagrange multipliers we get

$$f_0(\theta) = \lambda \left| \overline{C^{extr^0}(\theta)} \right|. \qquad (39)$$

$$h(f_0) = A^{extr}(\theta) + \left(\overline{C^{extr^0}(\theta)} \right)^{<\frac{1}{\alpha-1}>} (f_0(\theta))^{-\frac{1}{\alpha-1}}. \qquad (40)$$

Thus, the following theorem holds true.

Theorem 4.2. *Let the spectral density $f_0 \in D_1$ satisfy minimality condition* (12) *with $\alpha = 2$. The least favorable spectral density in the class D_1 for the optimal estimate of functional $A^{extr}\xi$ is density* (39) *which determine solution of optimization problem* (28). *The minimax spectral characteristic of the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is defined from* (40) *and* (15).

4.3 Least favorable spectral densities for class $D_{2\epsilon_1} \times D_{1\epsilon_2}$. Stationary processes

Consider problem for the class $D_{2\epsilon_1} \times D_{1\epsilon_2}$, where

$$D_{2\epsilon_1} = \{f(\theta) | \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda) - f_1(\lambda)|^2 d\lambda \leqslant \epsilon_1\},$$

$$D_{1\epsilon_2} = \{g(\theta) | \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)| d\lambda \leqslant \epsilon_2\}.$$

Let the functions $h_f(f_0, g_0) =$

$$= \frac{\left| A^{extr}(\theta) g_0(\theta) + \int_0^{\infty} \left((\mathbf{B}^0)^{-1} \mathbf{D}^0 \mathbf{a} \right)(t) e^{it\theta} dt \right|^2}{(f_0(\theta) + g_0(\theta))^2}, \qquad (41)$$

$$h_g(f_0, g_0) = \frac{\left| A^{extr}(\theta) f_0(\theta) - \int_0^{\infty} \left((\mathbf{B}^0)^{-1} \mathbf{D}^0 \mathbf{a} \right)(t) e^{it\theta} dt \right|^2}{(f_0(\theta) + g_0(\theta))^2}, \qquad (42)$$

be bounded. From condition $0 \in \partial \triangle_D(f_0, g_0), D = D_f^0 \times D_g^0$, we have that the least favorable spectral densities are defined by equation

$$\left| A^{extr}(\theta) g_0(\theta) + \int_0^{\infty} \left((\mathbf{B}^0)^{-1} \mathbf{D}^0 \mathbf{a} \right)(t) e^{it\theta} dt \right|^2 = \alpha_1 (f_0(\theta) + g_0(\theta))^2 (f_0(\theta) - f_1(\theta)) \qquad (43)$$

$$\left| A^{extr}(\theta) f_0(\theta) - \int_0^{\infty} \left((\mathbf{B}^0)^{-1} \mathbf{D}^0 \mathbf{a} \right)(t) e^{it\theta} dt \right|^2 = \alpha_2 (f_0(\theta) + g_0(\theta)) \Psi(\theta), \qquad (44)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda) - f_1(\lambda)|^2 d\lambda = \epsilon_1 \qquad (45)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)| d\lambda = \epsilon_2 \qquad (46)$$

$|\Psi(\theta)| < 1, \Psi(\theta) = sign(g_0(\theta) - g_1(\theta))$, if $g_0(\theta) \neq g_1(\theta)$, α_1, α_2 are constants.

Thus, the following statement holds true.

Theorem 4.3. *Let $f_0(\theta) \in D_{2\epsilon_1}, g_0(\theta) \in D_{1\epsilon_2}$ satisfy minimality condition* (6). *Let function h_f, h_g determined by* (41), (42) *be bounded. The spectral densities $f_0(\theta), g_0(\theta)$ are the least favorable in class $D = D_{2\epsilon_1} \times D_{1\epsilon_2}$ for optimal extrapolation of functional $A^{extr}\xi = \int_0^{\infty} a(t)\xi(t)dt$, if $f_0(\theta), g_0(\theta)$ are solution of equations* (43), (44), (45), (46) *and determine solution of optimization problem* (30). *The spectral characteristic $h(f_0, g_0)$ of the optimal estimate $\hat{A}^{extr}\xi$ of the functional $A^{extr}\xi$ is determined by equations* (18).

4.4 Least favorable spectral densities D_f^{β}. Stationary processes. Observations without noise

Consider the problem for class D_f^{β}, where

$$D_f^{\beta} = \{f(\theta) | \int_{-\infty}^{\infty} (f(\lambda))^{\beta} d\lambda = P_1\}.$$

It follows from condition $0 \in \partial \triangle_D(f_0, g_0), D = D_f^{\beta}$ that the least favorable spectral density satisfies equation

$$\left| \int_0^{\infty} \left((\mathbf{B}^0)^{-1} \mathbf{a} \right)(t) e^{i\theta} d\theta \right|^2 f_0^{-2}(\theta) = \gamma_1 (f_0(\theta))^{\beta-1}, \qquad (47)$$

where γ_1 is determined by condition

$$\int_{-\infty}^{\infty} f_0(\theta)^{\beta} d\theta = P_1. \qquad (48)$$

Thus, the following statement holds true.

Theorem 4.4. *Let spectral density $f_0 \in D_f^{\beta}$ satisfy minimality condition* (12) *with $\alpha = 2$. The least favorable spectral density D_f^{β} for the optimal estimate of the functional $A^{extr}\xi$ is determined by equation* (47) *and the optimization problem* (32). *The minimax spectral characterisation $h(f_0)$ is determined by equation* (21).

5 Conclusion

In this article we propose methods of solution of the problem of optimal linear estimation of the functional $A^{extr}\xi = \int_0^\infty a(t)\xi(t)dt$ which depends on the unknown values of a harmonizable α-stable process $\xi(t), t > 0$ from observations of the process $\xi(t) + \eta(t), t \in \mathbb{R}$ at points $t < 0$. The problem is investigated in the case of certainty where spectral densities of the processes are exactly known as well as in the case of uncertainty where spectral densities of the processes are not exactly known while sets of admissible densities are described. In the case of certainty we derive formulas for calculation the error and spectral characteristic of the optimal estimate of the functional. The minimax approach is applied in the case of spectral uncertainty. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristics are proposed for some classes of admissible spectral densities.

Список використаних джерел

1. *Cambanis S.* Complex stable variables and processes / S. Cambanis // Contributions to Statistics, North-Holland, New York. – 1983. – P. 63 – 79.
2. *Gikhman I.I.* The Theory of Stochastic Processes. I. Classics in Mathematics/ I.I. Gikhman, A.V. Skorokhod // Berlin: Springer-Verlag, 2004. – 574 p.
3. *Grenander U.* A prediction problem in game theory / U. Grenander // Ark. Mat. 3.– 1957. – 371-379.
4. *Kassam S.A.* Robust techniques for signal processing: A survey / S.A. Kassam, H.V. Poor // Proceedings of the IEEE. – 1985. – 73, №3. – P. 433 – 481.
5. *Моклячук М. П.* Робастні оцінки функціоналів від стохастичних процесів./ М. П. Моклячук // К.:ВПЦ "Київський університет". – 2008. – 320 с.
6. *Moklyachuk M. P.* Minimax interpolation problem for harmonizable stable sequences with noise observations / M. P. Moklyachuk, V. I. Ostapenko // J. Appl. Math. Stat. – 2015. – Vol. 2. – No. 1. –P. – 21 – 42
7. *Moklyachuk M. P.* Minimax interpolation of harmonizable sequences / M. P. Moklyachuk, V. I. Ostapenko // Theor. Probab. Math. Stat. – 2016. – No. 92. – P. – 135 -–146.
8. *Rajput S.* On some extremal problems in H^p and the prediction of L^p-harmonizable stochastic processes / S. Rajput // Probab. Theory Relat. Fields. – 1994. – 99, No. 2. – P. – 197 –210.

References

1. CAMBANIS S. (1983), “Complex stable variables and processes” *Contributions to Statistics, North-Holland,* New York.– pp. 63-79.
2. GIKHMAN I.I. and SKOROKHOD A.V. (2004), *The theory of stochastic processes. I. Classics in Mathematics*, Berlin: Springer-Verlag.
3. GRENANDER U. (1957), “A prediction problem in game theory” *Ark. Mat.* № 3. – pp. 371-379.
4. KASSAM S.A. and POOR H.V. (1985), “Robust techniques for signal processing: A survey”, *Proceedings of the IEEE*, 73(3), pp. 433 – 481.
5. MOKLYACHUK M.P. (2008), “Robust estimates of functional of stochastic processes.” *Kyiv.– Kyiv University.*
6. MOKLYACHUK M.P., OSTAPENKO V. I. (2015) “Minimax interpolation problem for harmonizable stable sequences with noise observations” *J. Appl. Math. Stat.* Vol. 2, No. 1, pp. 21 – 42
7. MOKLYACHUK M.P., OSTAPENKO V. I. (2016) “Minimax interpolation of harmonizable sequences” *Theor. Probab. Math. Stat.,* No. 92, pp. 135-146.
8. RAJPUT, S., SUNDBERG, C. (1994), “On some extremal problems in H^p and the prediction of L^p-harmonizable stochastic processes.” *Probab. Theory Relat. Fields.* Vol. 99, Iss. 2.– pp. 197-210.

Received: 02.10.2015