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Approximation of autonomous affine control systems in the sense of time optimality and algebraic approximation

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In the paper conditions are given under which, for autonomous affine control systems, approximation in the sense of time optimality implies the algebraic approximation.
Keywords: nonlinear affine control system, problem of time optimality, algebraic approximation.

Ігнатович С. Ю. **Апроксимація автономних афінних керованих систем у сенсі швидкодії та алгебраїчна апроксимація.** У роботі даються умови, за яких для автономних афінних систем з апроксимації у сенсі швидкодії випливає алгебраїчна апроксимація.
Ключові слова: нелінійні керовані системи, задача швидкодії, алгебраїчна апроксимація.

Игнатович С. Ю. **Аппроксимация автономных аффинных управляемых систем в смысле быстродействия и алгебраическая аппроксимация.** В работе даются условия, при которых для автономных аффинных управляемых систем из аппроксимации в смысле быстродействия вытекает алгебраическая аппроксимация.
Ключевые слова: нелинейные управляемые системы, задача быстродействия, алгебраическая аппроксимация.
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1. Background and statement of the problem

In this paper we deal with the time-optimal control problem for autonomous nonlinear affine systems of the form

$$\dot{x} = a(x) + ub(x), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}, \ a(0) = 0, \tag{1}$$

$$|u(t)| \le 1, \ x(0) = x^0, \ x(\theta) = 0, \ \theta \to \min, \tag{2}$$

where $a(x)$ and $b(x)$ are real analytic vector fields in a neighborhood of the origin. The requirement $a(0) = 0$ means that the origin is a rest point for this system. For brevity, we denote the system (1) by $\{a, b\}$.

Now we briefly recall some results obtained in [1, 2]. Below, $S_{a,b} = S_{a,b}(\theta, u)$ denotes the map taking a pair (θ, u) to the initial point x^0 which is steered to the origin by the control $u = u(t)$ in the time θ. This map can be expressed as a series

$$x^0 = S_{a,b}(\theta, u) = \sum_{m=1}^{\infty} \sum_{m_1+\cdots+m_k+k=m} v_{i_1\ldots i_k} \xi_{m_1\ldots m_k}(\theta, u),$$

where $\xi_{i_1\ldots i_k}(\theta, u)$ are *nonlinear power moments* of the form

$$\xi_{m_1\ldots m_k}(\theta, u) = \int_0^\theta \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} \prod_{j=1}^{k} \tau_j^{m_j} u(\tau_j) d\tau_k \cdots d\tau_1,$$

and $v_{m_1\ldots m_k}$ are constant vector coefficients which can be found by the formula

$$v_{m_1\ldots m_k} = \frac{(-1)^k}{m_1! \ldots m_k!} \mathrm{ad}_{R_a}^{m_1} R_b \circ \cdots \circ \mathrm{ad}_{R_a}^{m_k} R_b E(x)|_{x=0}, \tag{3}$$

where the operators R_a and R_b are defined as $R_a\phi(x) = \phi_x(x)a(x)$ and $R_b\phi(x) = \phi_x(x)b(x)$, operator brackets $\mathrm{ad}_{R_a}^m R_b$ are defined as $\mathrm{ad}_{R_a}^0 R_b = R_b$, $\mathrm{ad}_{R_a}^{m+1} R_b = [R_a, \mathrm{ad}_{R_a}^m R_b]$, $m \geq 0$ ($[\cdot, \cdot]$ means the operator commutator), and $E(x) \equiv x$. Since $a(x)$ and $b(x)$ are real analytic, there exist $C_1, C_2 > 0$ such that $\|v_{m_1\ldots m_k}\| \leq k! C_1 C_2^{m_1+\cdots+m_k+k}$ for all $k \geq 1$ and $m_1, \ldots, m_k \geq 0$ [3].

For any fixed $\theta > 0$, let us consider nonlinear power moments as functionals defined on the unit ball of the space $L_\infty[0, \theta]$, i.e., on the set $B^\theta = \{u \in L_\infty[0, \theta] : \|u(t)\| \leq 1\}$. The linear span (over $\mathbb{R}$) of all such functionals form an associative algebra $\mathcal{A}^\theta$ with the concatenation product

$$\xi_{m_1\ldots m_k}(\theta, \cdot) \vee \xi_{j_1\ldots j_q}(\theta, \cdot) = \xi_{m_1\ldots m_k j_1\ldots j_q}(\theta, \cdot).$$

One can show that the algebra $\mathcal{A}^\theta$ is *free* for any $\theta > 0$. On the other hand, since $\xi_{m_1\ldots m_k}(\theta, u) = \theta^{m_1+\cdots+m_k+k} \xi_{m_1\ldots m_k}(1, \widehat{u})$ where $\widehat{u}(t) = u(t\theta)$, $t \in [0, 1]$, we can regard the number $\mathrm{ord}(\xi_{m_1\ldots m_k}) = m_1 + \cdots + m_k + k$ as the *order* of the functional $\xi_{m_1\ldots m_k}(\theta, \cdot)$. This concept allows us to introduce a graded structure in $\mathcal{A}^\theta$.

Notice that algebras $\mathcal{A}^\theta$ with different $\theta > 0$ are isomorphic to each other. Therefore, it is convenient to deal with more abstract object. Namely, let us consider the set of abstract free elements (letters) ξ_m, $m \geq 0$. Strings of letters (words) $\xi_{m_1} \cdots \xi_{m_k}$ are denoted by $\xi_{m_1\ldots m_k}$. In the set of words, the concatenation is defined: $\xi_{m_1\ldots m_k} \vee \xi_{j_1\ldots j_q} = \xi_{m_1\ldots m_k j_1\ldots j_q}$. All finite linear combinations of words (over $\mathbb{R}$) form a graded free associative algebra $\mathcal{A} = \sum_{m=1}^\infty \mathcal{A}^m$, where homogeneous subspaces $\mathcal{A}^m$ are defined as follows,

$$\mathcal{A}^m = \mathrm{Lin}\{\xi_{m_1\ldots m_k} : m_1 + \cdots + m_k + k = m\}, \ m \geq 1.$$

This algebra is isomorphic to $\mathcal{A}^\theta$ for any $\theta > 0$; we call it *the algebra of nonlinear power moments*. Below we identify $\mathcal{A}^\theta$ and $\mathcal{A}$.

We say that an element $z \in \mathcal{A}^m$ is homogeneous and the number $\mathrm{ord}(z) = m$ is its order. It is convenient to supplement $\mathcal{A}$ with the unity element 1 (which can be thought of as the empty word) and consider the algebra $\mathcal{A}^e = \mathcal{A} + \mathbb{R}$. Throughout the paper we assume $\xi_{m_p \ldots m_q} = 1$ if $p > q$. We also use the notation $\ell^{\vee q} = \ell \vee \cdots \vee \ell$ (q times).

In $\mathcal{A}$ we consider the free graded Lie algebra $\mathcal{L} = \sum_{m=1}^\infty \mathcal{L}^m$ generated by the letters ξ_m, $m \geq 0$, with the Lie brackets $[\ell_1, \ell_2] = \ell_1 \vee \ell_2 - \ell_2 \vee \ell_1$; then $\mathcal{A}$ is its universal enveloping algebra. We also use *the shuffle product* operation in $\mathcal{A}$ defined by the following recurrent formula

$$\xi_{i_1 \ldots i_k} ш\, \xi_{j_1 \ldots j_q} = \xi_{i_1} \vee (\xi_{i_2 \ldots i_k} ш\, \xi_{j_1 \ldots j_q}) + \xi_{j_1} \vee (\xi_{i_1 \ldots i_k} ш\, \xi_{j_2 \ldots j_q}),$$

and such that $1 ш z = z ш 1 = z$ for any $z \in \mathcal{A}^e$. Below we also use the notation $z^{ш q} = z ш \cdots ш z$ (q times). We say that $P(z_1, \ldots, z_k)$ is *a homogeneous shuffle polynomial* of order m if $P(z_1, \ldots, z_k) = \sum \alpha_{q_1 \ldots q_k} z_1^{ш q_1} ш \cdots ш z_k^{ш q_k}$ where $\alpha_{q_1 \ldots q_k} \in \mathbb{R}$ and the sum is taken over all $q_1, \ldots, q_n$ such that $\sum_{i=1}^k q_i \mathrm{ord}(z_i) = m$.

Finally, we introduce the inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{A}$ so that the basis $\xi_{i_1 \ldots i_k}$ becomes orthonormal.

Let us now consider the set of vector coefficients (3). They generate the linear map $v : \mathcal{A} \to \mathbb{R}^n$ defined as $v(\xi_{m_1 \ldots m_k}) = v_{m_1 \ldots m_k}$. The important role is played by the restriction of this map to the Lie algebra $\mathcal{L} \subset \mathcal{A}$. Namely, let us suppose that the Rashevsky-Chow condition holds,

$$v(\mathcal{L}) = \mathbb{R}^n \tag{4}$$

and consider the following subspaces

$$\mathcal{P}^1 = \{\ell \in \mathcal{L}^1 : v(\ell) = 0\}, \quad \mathcal{P}^k = \{\ell \in \mathcal{L}^k : v(\ell) \in v(\mathcal{L}^1 + \cdots + \mathcal{L}^{k-1})\}, \ k \geq 2.$$

We say that

$$\mathcal{L}_{a,b} = \sum_{k=1}^\infty \mathcal{P}^k$$

is *a core Lie subalgebra* corresponding to the system $\{a, b\}$. We say that

$$\mathcal{J}_{a,b} = \mathrm{Lin}\{\ell \vee z : \ell \in \mathcal{L}_{a,b}, z \in \mathcal{A}\}$$

is *a right ideal* corresponding to the system $\{a, b\}$. Due to properties of the map v, if $z \in \mathcal{J}_{a,b} \cap \mathcal{A}^m$ then $v(z) \in v(\mathcal{A}^1 + \cdots + \mathcal{A}^{m-1})$. One can show that $\mathcal{L}_{a,b} = \mathcal{J}_{a,b} \cap \mathcal{L}$, hence, $\mathcal{L}_{a,b}$ and $\mathcal{J}_{a,b}$ define each other.

We notice that the Rashevsky-Chow condition (4) implies the *attainability* for the system $\{a, b\}$. This means that the set of all initial vectors x^0 which can be steered to the origin has nonempty interior and the origin belongs to the closure of this interior.

Suppose $\ell_1, \ldots, \ell_n$ are homogeneous Lie elements such that

$$\mathcal{L} = \mathrm{Lin}\{\ell_1, \ldots, \ell_n\} + \mathcal{L}_{a,b},$$

and $\{\ell_j\}_{j=n+1}^{\infty}$ is a homogeneous basis of $\mathcal{L}_{a,b}$. As is well known [4], the set

$$\{\ell_{i_1}^{\vee q_1} \vee \cdots \vee \ell_{i_k}^{\vee q_k} : i_1 < \ldots < i_k, k \geq 1\}$$

forms a basis of the algebra $\mathcal{A}$; we call it a Poincaré-Birkhoff-Witt basis.

Suppose $\{d_{i_1 \ldots i_k}^{q_1 \cdots q_k}\}$ is a dual basis, that is,

$$\langle d_{i_1 \ldots i_k}^{q_1 \cdots q_k}, \ell_{j_1}^{\vee r_1} \vee \cdots \vee \ell_{j_s}^{\vee r_s} \rangle = 1 \text{ iff } \ s = k, \ i_m = j_m, \ q_m = r_m.$$

Then [5] it can be expressed as

$$d_{i_1 \ldots i_k}^{q_1 \cdots q_k} = \frac{1}{q_1! \cdots q_k!} d_{i_1}^{ш q_1} ш \cdots ш d_{i_k}^{ш q_k},$$

where $d_i = d_i^1$. In other words, the sequence $\{d_j\}_{j=1}^{\infty}$ defines all other elements of the dual basis. Hence, the map $S_{a,b}$ can be expressed as a series w.r.t. the dual basis,

$$S_{a,b}(\theta, u) = \sum_{k \geq 1, \ i_1 < \cdots < i_k, \ q_j \geq 1} \frac{1}{q_1! \cdots q_k!} v(\ell_{i_1}^{\vee q_1} \vee \cdots \vee \ell_{i_k}^{\vee q_k}) d_{i_1}^{ш q_1} ш \cdots ш d_{i_k}^{ш q_k}.$$

Moreover, if $i_1 \geq n+1$ then $\ell_{i_1} \in \mathcal{L}_{a,b}$ and therefore $\ell_{i_1}^{\vee q_1} \vee \cdots \vee \ell_{i_k}^{\vee q_k} \in \mathcal{J}_{a,b}$. This representation justifies the result which was obtained in [2]: for any system $\{a, b\}$ satisfying condition (4) there exists (polynomial) nonsingular change of variables $y = \Phi(x)$ $(\Phi(0) = 0)$ such that

$$y_k^0 = (\Phi(x^0))_k = d_k(\theta, u) + \rho_k(\theta, u), \ k = 1, \ldots, n,$$

where $\rho_k \in \sum_{i=\mathrm{ord}(d_k)+1}^{\infty} \mathcal{A}_i$. It turns out that there exists a (autonomous) system $\{a^*, b^*\}$ such that

$$(S_{a^*,b^*})_k = d_k(\theta, u), \ k = 1, \ldots, n.$$

Let us notice that the components of the series of this system are homogeneous as elements of $\mathcal{A}$. In such a case we say that the system $\{a^*, b^*\}$ is *homogeneous*. It can be shown that if $v(\mathcal{J}_{a^*,b^*}) = 0$ then there exists such a change of coordinates that $(F(S_{a^*,b^*}))_k = d_k^*(\theta, u)$, $k = 1, \ldots, n$, where d_k^* are homogeneous elements (of dual basis). In other words, the algebraic representation becomes homogeneous after a change of coordinates $y = F(x)$. Then we also say that the system is homogeneous and the coordinates y are *privileged* for the system $\{a^*, b^*\}$.

Definition 1 *Suppose a homogeneous system $\{a^*, b^*\}$ is such that $\mathcal{J}_{a^*,b^*} = \mathcal{J}_{a,b}$ (or, what is the same, $\mathcal{L}_{a^*,b^*} = \mathcal{L}_{a,b}$). Then we say that $\{a^*, b^*\}$ is an algebraic approximation of $\{a, b\}$.*

It can be shown that if $\{a, b\}$ is autonomous then its algebraic approximation $\{a^*, b^*\}$ can be chosen as autonomous.

In [2] we propose the connection of such approximation with time optimality. Let us adopt the following definition of equivalence in the sense of time optimality. Consider two time-optimal control problems of the form (1), (2) for systems $\{a^*, b^*\}$ and $\{a, b\}$. Suppose there exists an open domain $\Omega \subset \mathbb{R}^n \backslash \{0\}$, $0 \in \overline{\Omega}$, such that the time-optimal control problem for the system $\{a^*, b^*\}$ has a unique solution $(\theta^*_{x^0}, u^*_{x^0})$ for any $x^0 \in \Omega$. Denote by $U_{x^0}(\theta) \subset B^\theta$ the set of all controls which steer the point x^0 to the origin by virtue of the system $\{a, b\}$ in the time θ, then the optimal time for this system equals $\theta_{x^0} = \min\{\theta : U_{x^0}(\theta) \neq \varnothing\}$.

Definition 2 *We say that the system $\{a^*, b^*\}$ approximates the system $\{a, b\}$ in the sense of time optimality in the domain Ω if there exists a (real analytic) nonsingular map $\Phi(x)$ of the neighborhood of the origin ($\Phi(0) = 0$) and a set of pairs $(\widetilde{\theta}_{x^0}, \widetilde{u}_{x^0})$, $x^0 \in \Omega$, such that $\widetilde{u}_{x^0} \in U_{\Phi(x^0)}(\widetilde{\theta}_{x^0})$ and*

$$\frac{\theta_{\Phi(x^0)}}{\theta^*_{x^0}} \to 1, \quad \frac{\widetilde{\theta}_{x^0}}{\theta^*_{x^0}} \to 1, \quad \frac{1}{\theta}\int_0^\theta \left|u^*_{x^0}(t) - \widetilde{u}_{x^0}(t)\right| dt \to 0 \quad \text{as } x^0 \to 0,\ x^0 \in \Omega,$$

where $\theta = \min\{\theta^*_{x^0}, \widetilde{\theta}_{x^0}\}$.

Controls $\widetilde{u}_{x^0}(t)$ can be regarded as "almost optimal" controls for the system $\{a, b\}$ which steer the point $\Phi(x^0)$ to the origin in the "almost optimal" time $\widetilde{\theta}_{x^0}$.

In [2] the following result was obtained. Suppose the system $\{a^*, b^*\}$ is an algebraic approximation of the system $\{a, b\}$. Suppose also that there exists an open domain $\Omega \subset \mathbb{R}^n \backslash \{0\}$, $0 \in \overline{\Omega}$, such that

(i) the time-optimal control problem for the system $\{a^*, b^*\}$ has a unique solution $(\theta^*_{x^0}, u^*_{x^0})$ for any $x^0 \in \Omega$;

(ii) the function $\theta^*_{x^0}$ is continuous w.r.t. $x^0 \in \Omega$;

(iii) for the set $K = \{u^*_{x^0}(t\theta^*_{x^0}) : x^0 \in \Omega\} \subset L_2[0, 1]$, the weak convergence implies the strong convergence.

Then there exists a set $\{\Omega(\delta)\}_{\delta > 0}$ of domains, $\Omega(\delta_1) \subset \Omega(\delta_2)$ if $\delta_1 > \delta_2$, $\bigcup_{\delta > 0} \Omega(\delta) = \Omega$, such that $\{a^*, b^*\}$ approximates $\{a, b\}$ in the sense of time optimality in each domain $\Omega(\delta)$.

In other words, if the system $\{a^*, b^*\}$ approximates $\{a, b\}$ in the algebraic sense then, under some conditions, it approximates $\{a, b\}$ in the sense of time optimality.

In [1] we considered a subclass of systems $\{a, b\}$ whose approximation $\{a^*, b^*\}$ is linear. In this case we proved also the converse implication. Roughly speaking, the result is as follows: if the system $\{a, b\}$ is approximated by a linear system in the sense of time optimality then its algebraic approximation is linear, i.e., $d_i = \xi_{m_i}$, $i = 1, \ldots, n$. The proof used essentially the fact that optimal controls for linear systems are piecewise constant and, for a set of initial points of nonzero measure, have $n - 1$ switchings.

The question remains whether this statement can be proved for more general class of approximating systems. In [6] we partially answered this question. The main idea was to consider those systems $\{a^*, b^*\}$ whose optimal controls are piecewise constant with $n-1$ switchings for a set of initial points with nonempty interior.

In the present paper we develop the idea proposed in [6] and prove analogous statement for autonomous systems under much weaker assumptions concerning optimal controls. Preliminary lemmas are given in Section 2. The main result (Theorem 1) is proved in Section 3.

2. Preliminary results

Notation. (a) Denote by $\varphi : \mathcal{A}+\mathbb{R} \to \mathcal{A}$ and $\varphi' : \mathcal{A} \to \mathcal{A}+\mathbb{R}$ differentiations in $\mathcal{A}$ defined by

$$\varphi(\xi_m) = (m+1)\xi_{m+1}, \quad \varphi(1) = 0,$$
$$\varphi'(\xi_0) = 0, \quad \varphi'(\xi_m) = m\xi_{m-1}, \ m \geq 1,$$

then

$$\varphi(\xi_{m_1\dots m_k}) = \sum_{i=1}^{k}(m_i+1)\xi_{m_1\dots(m_i+1)\dots m_k}, \quad \varphi'(\xi_{m_1\dots m_k}) = \sum_{i=1}^{k} m_i\xi_{m_1\dots(m_i-1)\dots m_k}.$$

(b) Denote by $\psi_0 : \mathcal{A}+\mathbb{R} \to \mathcal{A}$ and $\psi_0' : \mathcal{A} \to \mathcal{A}+\mathbb{R}$ linear mappings defined by

$$\psi_0(\xi_{m_1\dots m_k}) = \xi_{m_1\dots m_k} \vee \xi_0, \quad \psi_0(1) = \xi_0,$$
$$\psi_0'(\xi_0) = 1, \quad \psi_0'(\xi_{m_1\dots m_k}) = \begin{cases} 0, & m_k \neq 0, \\ \xi_{m_1\dots m_{k-1}}, & m_k = 0. \end{cases}$$

Lemma 1 *(a) Mappings φ and φ' are transpose to each other, i.e., for any $y_1 \in \mathcal{A}+\mathbb{R}$ and any $y_2 \in \mathcal{A}$*

$$\langle \varphi(y_1), y_2\rangle = \langle y_1, \varphi'(y_2)\rangle.$$

(b) Mappings ψ_0 and ψ_0' are transpose to each other, i.e., for any $y_1 \in \mathcal{A}+\mathbb{R}$ and any $y_2 \in \mathcal{A}$

$$\langle \psi_0(y_1), y_2\rangle = \langle y_1, \psi_0'(y_2)\rangle.$$

Proof. (a) Notice that $\langle \varphi(\xi_{i_1\dots i_s}), \xi_{m_1\dots m_k}\rangle = 0$ and $\langle \xi_{i_1\dots i_s}, \varphi'(\xi_{m_1\dots m_k})\rangle = 0$ if $s \neq k$. Hence, suppose $s = k$. For any $q = 1, \dots, k$

$$\langle \xi_{i_1\dots(i_q+1)\dots i_k}, \xi_{m_1\dots m_q\dots m_k}\rangle = \langle \xi_{i_1\dots i_q\dots i_k}, \xi_{m_1\dots(m_q-1)\dots m_k}\rangle,$$
$$\langle \xi_{i_1\dots i_q\dots i_k}, \xi_{m_1\dots(m_q-1)\dots m_k}\rangle = 0 \ \text{ if } \ i_q+1 \neq m_q.$$

Hence, for any $\xi_{m_1\dots m_k} \in \mathcal{A}$ and any $\xi_{i_1\dots i_k} \in \mathcal{A}+\mathbb{R}$

$$\langle \varphi(\xi_{i_1\dots i_k}), \xi_{m_1\dots m_k}\rangle = \sum_{q=1}^{k}(i_q+1)\langle \xi_{i_1\dots(i_q+1)\dots i_k}, \xi_{m_1\dots m_q\dots m_k}\rangle =$$
$$= \sum_{q=1}^{k} m_q\langle \xi_{i_1\dots i_q\dots i_k}, \xi_{m_1\dots(m_q-1)\dots m_k}\rangle = \langle \xi_{i_1\dots i_k}, \varphi'(\xi_{m_1\dots m_k})\rangle.$$

(b) For any $\xi_{m_1\ldots m_k}\in\mathcal{A}$ and any $\xi_{i_1\ldots i_s}\in\mathcal{A}+\mathbb{R}$

$$\langle\psi_0(\xi_{i_1\ldots i_s}),\xi_{m_1\ldots m_k}\rangle=\langle\xi_{i_1\ldots i_s}\vee\xi_0,\xi_{m_1\ldots m_k}\rangle=\begin{cases}\langle\xi_{i_1\ldots i_s},\xi_{m_1\ldots m_{k-1}}\rangle & \text{if } m_k=0,\\ 0 & \text{otherwise,}\end{cases}$$

which, obviously, equals $\langle\xi_{i_1\ldots i_s},\psi_0'(\xi_{m_1\ldots m_k})\rangle$.

Lemma 2 *(a)* $\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0)=\mathcal{A}$; *(b)* $\ker(\varphi')\cap\ker(\psi_0')=\{0\}$.

Proof. (a) First, let us show that any $\xi_{m_1\ldots m_k}\in\mathcal{A}$ belongs to $\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0)$. We use the induction w.r.t. m_k.

If $m_k=0$ then $\xi_{m_1\ldots m_k}=\psi_0(\xi_{m_1\ldots m_{k-1}})\in\mathrm{Im}(\psi_0)$ for any $m_1,\ldots,m_{k-1}$.

Suppose $p\geq0$ and $\xi_{m_1\ldots m_{k-1}p}\in\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0)$ for any $m_1,\ldots,m_{k-1}$. Then

$$\varphi(\xi_{m_1\ldots m_{k-1}p})=\varphi(\xi_{m_1\ldots m_{k-1}}\vee\xi_p)=(p+1)\xi_{m_1\ldots m_{k-1}(p+1)}+\varphi(\xi_{m_1\ldots m_{k-1}})\vee\xi_p.$$

By the induction supposition, $\varphi(\xi_{m_1\ldots m_{k-1}})\vee\xi_p\in\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0)$. Hence,

$$\xi_{m_1\ldots m_{k-1}(p+1)}=\tfrac{1}{p+1}\left(\varphi(\xi_{m_1\ldots m_{k-1}p})-\varphi(\xi_{m_1\ldots m_{k-1}})\vee\xi_p\right)\in\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0).$$

The induction arguments complete the proof.

(b) Now, let $y_1\in\ker(\varphi')\cap\ker(\psi_0')$. Then Lemma 1 implies that for any $y_2\in\mathcal{A}+\mathbb{R}$

$$\langle\varphi(y_2),y_1\rangle=\langle y_2,\varphi'(y_1)\rangle=0,\quad\langle\psi_0(y_2),y_1\rangle=\langle y_2,\psi_0'(y_1)\rangle=0.$$

Hence, y_1 is orthogonal to $\mathrm{Im}(\varphi)+\mathrm{Im}(\psi_0)=\mathcal{A}$, therefore, $y_1=0$.

Remark. It follows from [3] that if $\mathcal{J}_{a,b}$ is a right ideal corresponding to the system $\{a,b\}$ then φ and ψ_0 are $\mathcal{J}_{a,b}$-invariant, i.e.,

$$\varphi(\mathcal{J}_{a,b})\subset\mathcal{J}_{a,b},\qquad\psi_0(\mathcal{J}_{a,b})\subset\mathcal{J}_{a,b}.\tag{5}$$

Relation (5) is necessary and sufficient for the ideal $\mathcal{J}_{a,b}$ to be a right ideal of an *autonomous* control system.

Corollary 1 *Suppose $\mathcal{J}_{a,b}$ is a right ideal corresponding to the system $\{a,b\}$. Then φ' and ψ_0' are $\mathcal{J}_{a,b}^\perp$-invariant, i.e., $\varphi'(\mathcal{J}_{a,b}^\perp)\subset\mathcal{J}_{a,b}^\perp$ and $\psi_0'(\mathcal{J}_{a,b}^\perp)\subset\mathcal{J}_{a,b}^\perp$.*

Remark. Formally, Corollary 1 requires the system $\{a,b\}$ to be autonomous. However, one can weaken this condition by assuming that *the algebraic approximation* of $\{a,b\}$ is autonomous. On this way, Theorem 1 (which uses Corollary 1) can be slightly generalized.

Lemma 3 *Let us fix $\theta>0$ and consider $u(t)$, $t\in[0,\theta]$, such that there exists $u(0)=\lim_{t\to+0}u(t)$. Let us consider $\theta_\delta=\theta-\delta$ and $u_\delta(t)=u(t+\delta)$, $t\in[0,\theta_\delta]$ for $0<\delta<\delta_0<\theta$. Then for any $z\in\mathcal{A}$*

$$\frac{d}{d\delta}z(\theta_\delta,u_\delta)|_{\delta=+0}=-\varphi'(z)(\theta,u)-u(0)\psi_0'(z)(\theta,u).$$

Proof. It suffices to consider $z = \xi_{m_1\dots m_k}$. We have

$$\xi_{m_1\dots m_k}(\theta_\delta, u_\delta) = \int_0^{\theta-\delta}\int_{\tau_k}^{\theta-\delta}\cdots\int_{\tau_2}^{\theta-\delta}\prod_{j=1}^{k}\tau_j^{m_j}u(\tau_j+\delta)d\tau_1\cdots d\tau_k =$$

$$= \int_\delta^{\theta}\int_{\tau_k}^{\theta}\cdots\int_{\tau_2}^{\theta}\prod_{j=1}^{k}(\tau_j-\delta)^{m_j}u(\tau_j)d\tau_1\cdots d\tau_k.$$

Then

$$\frac{d}{d\delta}\xi_{m_1\dots m_k}(\theta_\delta, u_\delta) =$$

$$= -(\tau_k-\delta)^{m_k}u(\tau_k)\int_{\tau_k}^{\theta}\int_{\tau_{k-1}}^{\theta}\cdots\int_{\tau_2}^{\theta}\prod_{j=1}^{k-1}(\tau_j-\delta)^{m_j}u(\tau_j)d\tau_1\cdots d\tau_{k-1}|_{\tau_k=\delta}-$$

$$-\sum_{i=1}^{k}m_i\int_\delta^{\theta}\int_{\tau_k}^{\theta}\cdots\int_{\tau_2}^{\theta}\prod_{j\neq i}(\tau_j-\delta)^{m_j}(\tau_i-\delta)^{m_i-1}\prod_{j=1}^{k}u(\tau_j)d\tau_1\cdots d\tau_k.$$

Hence, when $\delta\to+0$ we get

$$\frac{d}{d\delta}\xi_{m_1\dots m_k}(\theta_\delta, u_\delta)|_{\delta=+0} = \begin{cases} -\sum\limits_{i=1}^{k} m_i\xi_{m_1\dots(m_i-1)\dots m_k}(\theta,u) - u(0)\xi_{m_1\dots m_{k-1}}(\theta,u) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } m_k=0, \\ -\sum\limits_{i=1}^{k} m_i\xi_{m_1\dots(m_i-1)\dots m_k}(\theta,u) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } m_k\neq 0 \end{cases}$$

which completes the proof.

3. Equivalence of autonomous homogeneous systems

In this section, a system $\{a^*, b^*\}$ is supposed to be homogeneous. Then in privileged coordinates we get $(S_{a^*,b^*})_k = d_k^*$, where $\operatorname{ord}(d_k^*) = w_k^*$, $k=1,\dots,n$. For such a system we introduce a dilation $H_\varepsilon(x)$ acting as $(H_\varepsilon(x))_k = \varepsilon^{w_k^*}x_k$, $k=1,\dots,n$. Notice that

$$\theta^*_{H_\varepsilon(x^0)} = \varepsilon\theta^*_{x^0} \quad\text{and}\quad u^*_{H_\varepsilon(x^0)}(t) = u^*_{x^0}(\tfrac{t}{\varepsilon}),\ t\in[0,\varepsilon\theta^*_{x^0}]. \qquad (6)$$

Let us suppose that an open domain $\Omega\subset\mathbb{R}^n\backslash\{0\}$, $0\in\overline{\Omega}$, is such that the time-optimal control problem for the (homogeneous) system $\{a^*, b^*\}$ has a unique solution $(\theta^*_{x^0}, u^*_{x^0})$ for any $x^0\in\Omega$. We assume that in Ω optimal controls are continuous from the right at $t=0$, i.e., there exists $u^*_{x^0}(0) = \lim_{t\to+0}u^*_{x^0}(t)$ for any $x^0\in\Omega$. Without loss of generality we may assume that the domain Ω is *pseudo-conic* w.r.t. $\{a^*, b^*\}$, i.e., if $x\in\Omega$ then $H_\varepsilon(x)\in\Omega$ for any $0<\varepsilon<\varepsilon_0$.

Now let us denote by $x^*(t)$, $t\in[0,\theta^*_{x^0}]$ the optimal trajectory corresponding to an optimal control $u^*_{x^0}(t)$ (here $x^*(0)=x^0$ and $x^*(\theta^*_{x^0})=0$). Then obviously

$$\theta^*_{x^*(\delta)} = \theta^*_{x^0}-\delta \quad\text{and}\quad u^*_{x^*(\delta)}(t) = u^*_{x^0}(t+\delta),\ t\in[0,\theta^*_{x^*(\delta)}],\quad \text{for } \delta\in(0,\theta^*_{x^0}). \quad (7)$$

We assume that Ω is open, hence, for any $x^0 \in \Omega$ some segment of the optimal trajectory starting at x^0 belongs to Ω, i.e., there exists $\delta_0 > 0$ such that $x^*(\delta) \in \Omega$ for $0 < \delta < \delta_0 < \theta^*_{x^0}$.

Finally, we call the set $L = \{x \in \mathbb{R}^n : x_1, \ldots, x_{n-1} \text{ are fixed}, \ x_n \in \mathbb{R}\}$ *a vertical line.*

Theorem 1 *Suppose a homogeneous autonomous system $\{a^*, b^*\}$ approximates the autonomous system $\{a, b\}$ in the sense of time optimality in any of (pseudo-conic) domains Ω_i, $i \in I$, with the same map $\Phi(x)$ (where I may be finite or infinite set of indices) and for all $x^0 \in \bigcup_{i\in I} \Omega_i$ the time-optimal control problem for $\{a^*, b^*\}$ has a unique solution $(\theta^*_{x^0}, u^*_{x^0})$ such that $u^*_{x^0}(t)$ is continuous from the right at $t = 0$. Suppose there exists an open subset $\Omega' \subset \bigcup_{i\in I} \Omega_i$ which satisfies the following condition in privileged coordinates for the system $\{a^*, b^*\}$:*

For any vertical line L, if the intersection $M = \Omega' \cap L$ is nonempty then the function $f(x) = u^*_x(0)$, $x \in M$, is not constant. (L)

Then $\{a^*, b^*\}$ is an algebraic approximation of $\{a, b\}$.

Proof. Let $\mathcal{L}_{a^*,b^*}$ and $\mathcal{L}_{a,b}$ be core Lie subalgebras corresponding to the systems $\{a^*, b^*\}$, $\{a, b\}$ and let $\{\ell^*_k\}_{k=1}^n$, $\{\ell_k\}_{k=1}^n$ be homogeneous Lie elements such that

$$\mathcal{L} = \mathrm{Lin}\{\ell^*_1, \ldots, \ell^*_n\} + \mathcal{L}_{a^*,b^*} = \mathrm{Lin}\{\ell_1, \ldots, \ell_n\} + \mathcal{L}_{a,b}.$$

Suppose $\{d^*_k\}_{k=1}^n$ and $\{d_k\}_{k=1}^n$ are the corresponding elements of dual basis and $w_k = \mathrm{ord}(d_k)$, $w^*_k = \mathrm{ord}(d^*_k)$, where $w^*_1 \le \cdots \le w^*_n$ and $w_1 \le \cdots \le w_n$. We notice that for autonomous systems without loss of generality we may assume $\ell_1 = \ell^*_1 = d_1 = d^*_1 = \xi_0$.

We suppose the coordinates are privileged for the system $\{a^*, b^*\}$, then

$$(S_{a^*,b^*})_k = d^*_k(\theta, u), \quad k = 1, \ldots, n,$$

and

$$x^0_k = d^*_k(\theta^*_{x^0}, u^*_{x^0}), \quad k = 1, \ldots, n. \tag{8}$$

Also, without loss of generality we assume

$$(S_{a,b})_k = d_k(\theta, u) + \rho_k(\theta, u), \quad k = 1, \ldots, n.$$

and $\rho_k \in \sum_{m=w_k+1}^{\infty} \mathcal{A}^m$. By the supposition, the system $\{a^*, b^*\}$ approximates $\{a, b\}$ in the sense of time optimality. Taking into account Definition 2 we suppose $\widetilde{u}_{x^0} \in U_{\Phi(x^0)}(\widetilde{\theta}_{x^0})$, then

$$(\Phi(x^0))_k = d_k(\widetilde{\theta}_{x^0}, \widetilde{u}_{x^0}) + \rho_k(\widetilde{\theta}_{x^0}, \widetilde{u}_{x^0}), \quad k = 1, \ldots, n.$$

Therefore,

$$\begin{aligned} d_k(\widetilde{\theta}_{x^0}, \widetilde{u}_{x^0}) + \rho_k(\widetilde{\theta}_{x^0}, \widetilde{u}_{x^0}) = (\Phi(d^*_1(\theta^*_{x^0}, u^*_{x^0}), \ldots, d^*_n(\theta^*_{x^0}, u^*_{x^0})))_k = \\ = \sum_{i=1}^{n} \alpha_{ik} d^*_i(\theta^*_{x^0}, u^*_{x^0}) + \sum_{m=1}^{w_k} p_{mk}(d^*_1, \ldots, d^*_n)(\theta^*_{x^0}, u^*_{x^0}) + R_k(\theta^*_{x^0}, u^*_{x^0}), \end{aligned} \tag{9}$$

where the matrix $\{\alpha_{ik}\}$ is nonsingular (it equals $\Phi'(0)$), p_{mk} are shuffle polynomial without linear terms, $\mathrm{ord}(p_{mk}(d_1^*,\ldots,d_n^*)) = m$, and $R_k \in \sum_{m=w_k+1}^{\infty}\mathcal{A}^m$. Without loss of generality we assume that the elements $\{\ell_k^*\}_{k=1}^n$ are chosen so that $\Phi'(0)$ equals the identical matrix.

Due to Definition 2 for any $z\in\mathcal{A}^m$ and any $i\in I$ we have

$$z(\widetilde{\theta}_{x^0},\widetilde{u}_{x^0}) = z(\theta^*_{x^0},u^*_{x^0}) + \bar{o}((\theta^*_{x^0})^m) \ \text{ as } \ x^0\to 0,\ x^0\in\Omega_i.$$

Then (9) implies for any $x^0\in\Omega_i$

$$d_k(\theta^*_{x^0},u^*_{x^0}) = d_k^*(\theta^*_{x^0},u^*_{x^0}) + \sum_{m=1}^{w_k} p_{mk}(d_1^*,\ldots,d_n^*)(\theta^*_{x^0},u^*_{x^0}) + \bar{o}((\theta^*_{x^0})^{w_k}), \quad (10)$$

$k=1,\ldots,n$. Let us denote

$$P_{mk}(d_1^*,\ldots,d_n^*) = \begin{cases} d_k^* + p_{mk}(d_1^*,\ldots,d_n^*) & \text{if } \ m = w_k^*, \\ p_{mk}(d_1^*,\ldots,d_n^*) & \text{otherwise.}\end{cases} \quad (11)$$

Considering (10) for $x^0_\varepsilon = H_\varepsilon(x^0)\in\Omega_i$, $0<\varepsilon<\varepsilon_0$, instead of x^0, we get

$$\varepsilon^{w_k} d_k(\theta^*_{x^0},u^*_{x^0}) = \sum_{m=1}^{w_k}\varepsilon^m P_{mk}(d_1^*,\ldots,d_n^*)(\theta^*_{x^0},u^*_{x^0}) + \bar{o}(\varepsilon^{w_k})$$

as $\varepsilon\to 0$, which implies

$$P_{mk}(d_1^*,\ldots,d_n^*)(\theta^*_{x^0},u^*_{x^0}) = 0,\ m\le w_k-1, \quad (12)$$

$$d_k(\theta^*_{x^0},u^*_{x^0}) = P_{w_k k}(d_1^*,\ldots,d_n^*)(\theta^*_{x^0},u^*_{x^0}), \quad k=1,\ldots,n, \quad (13)$$

for any $x^0\in\Omega_i$. Using (8) we get from (12)

$$P_{mk}(x_1^0,\ldots,x_n^0) = 0,\ m\le w_k-1,$$

for any $x^0\in\bigcup_{i\in I}\Omega_i$, which implies that polynomials P_{mk} are zero, $P_{mk}\equiv 0$. In particular, (11) gives $w_k^*\ge w_k$.

Now we consider (13) and use the induction arguments. Assume

$$\begin{aligned} &w_j = \cdots = w_{j+q} = c,\\ &w_s < c \ \text{ if } \ s\le j-1 \ \text{ and } w_s > c \ \text{ if } \ s\ge j+q+1.\end{aligned}$$

Suppose $j=1$ or

$$d_k^* = d_k, \quad k=1,\ldots,j-1. \quad (14)$$

As is shown above, $w_j^*\ge w_j$. Hence, if $j\ge 2$ then, due to the induction supposition,

$$\mathcal{J}^{\perp}_{a,b}\cap\mathcal{A}^m = \mathcal{J}^{\perp}_{a^*,b^*}\cap\mathcal{A}^m, \quad m=1,\ldots,c-1. \quad (15)$$

Since $\mathrm{ord}(P_{c(j+r)}(d_1^*,\ldots,d_n^*)) = c$ and $w_r^*\ge w_s > c$ for $s > j+q$, we get $P_{c(j+r)}(d_1^*,\ldots,d_n^*) = P_{c(j+r)}(d_1^*,\ldots,d_{j+q}^*)$. For brevity, we temporarily denote $f_{j+r} = P_{c(j+r)}(d_1^*,\ldots,d_{j+q}^*)$.

Since Ω_i is open then $x^*(\delta) \in \Omega_i$ for $0 < \delta < \delta_0$. Therefore, considering (13) for $x^*(\delta)$ instead of x^0, we get

$$d_{j+r}(\theta_\delta, u_\delta) = f_{j+r}(\theta_\delta, u_\delta), \ r = 0, \dots, q, \ 0 < \delta < \delta_0,$$

where $\theta_\delta = \theta^*_{x^0} - \delta$, $u_\delta(t) = u^*_{x^*(\delta)}(t) = u^*_{x^0}(t+\delta)$, $t \in [0, \theta_\delta]$. Hence, Lemma 3 gives

$$\varphi'(d_{j+r})(\theta, u) + u(0)\psi'(d_{j+r})(\theta, u) = \varphi'(f_{j+r})(\theta, u) + u(0)\psi'(f_{j+r})(\theta, u), \quad (16)$$

where $\theta = \theta^*_{x^0}$, $u = u^*_{x^0}$. By construction, $d_{j+r} \in \mathcal{J}^\perp_{a,b}$ and $f_{j+r} \in \mathcal{J}^\perp_{a^*,b^*}$, hence, applying Corollary 1 and using (15) we have

$$\varphi'(d_{j+r}), \psi'_0(d_{j+r}) \in \mathcal{J}^\perp_{a,b} \cap \mathcal{A}^{c-1} = \mathcal{J}^\perp_{a^*,b^*} \cap \mathcal{A}^{c-1}, \ \varphi'(f_{j+r}), \psi'_0(f_{j+r}) \in \mathcal{J}^\perp_{a^*,b^*} \cap \mathcal{A}^{c-1},$$

therefore, for any $r = 0, \dots, q$

$$\varphi'(d_{j+r} - f_{j+r}) \in \mathcal{J}^\perp_{a^*,b^*} \cap \mathcal{A}^{c-1}, \quad \psi'_0(d_{j+r} - f_{j+r}) \in \mathcal{J}^\perp_{a^*,b^*} \cap \mathcal{A}^{c-1}.$$

However, a basis of $\mathcal{J}^\perp_{a^*,b^*}$ is formed by polynomials of $\{d^*_1, \dots, d^*_n\}$. Let us take into account that $\mathrm{ord}(d_{j+r}) = \mathrm{ord}(f_{j+r}) = c \le w^*_j$. Hence, for some polynomials P_{1r} and P_{2r}

$$\varphi'(d_{j+r} - f_{j+r}) = P_{1r}(d^*_1, \dots, d^*_{j-1}), \quad \psi'_0(d_{j+r} - f_{j+r}) = P_{2r}(d^*_1, \dots, d^*_{j-1}). \ (17)$$

Hence, (16) implies

$$P_{1r}(d^*_1, \dots, d^*_{j-1})(\theta, u) + u(0)P_{2r}(d^*_1, \dots, d^*_{j-1})(\theta, u) = 0$$

where $\theta = \theta^*_{x^0}$, $u = u^*_{x^0}$. Now recalling (8) we get

$$P_{1r}(x^0_1, \dots, x^0_{j-1}) + u(0)P_{2r}(x^0_1, \dots, x^0_{j-1}) = 0 \quad (18)$$

for any $x^0 \in \bigcup_{i \in I} \Omega_i$, where $u(0) = u^*_{x^0}(0)$.

Suppose the polynomial P_{2r} is not identically zero. Let us apply condition (L). Namely, let us consider the set $\Omega'' = \{x \in \Omega' : P_{2r}(x_1, \dots, x_{j-1}) \neq 0\}$ which is nonempty since the nonempty set Ω' is open. For any $x \in \Omega''$ the optimal control equals $u(x) = -\frac{P_{1r}(x_1,\dots,x_{j-1})}{P_{2r}(x_1,\dots,x_{j-1})}$, hence, it depends only on the first $j-1$ coordinates of the point x (where $j - 1 \le n - 1$). Hence, the optimal control is constant on the intersection of Ω'' with any vertical line, what contradicts condition (L).

Hence, the polynomial P_{2r} is zero, therefore, P_{1r} also is zero. Then (17) implies $d_{j+r} - f_{j+r} \in \ker(\varphi') \cap \ker(\psi'_0)$. Now, Lemma 2 gives $d_{j+r} = f_{j+r}$. Thus,

$$d_{j+r} = P_{c(j+r)}(d^*_1, \dots, d^*_{j+q}). \quad (19)$$

If $w^*_{j+r} > w_{j+r} = c$ then, by (11), $P_{c(j+r)}(d^*_1, \dots, d^*_{j+q}) = p_{c(j+r)}(d^*_1, \dots, d^*_{j-1})$ is a shuffle polynomial without linear term, hence, $P_{c(j+r)}(d^*_1, \dots, d^*_{j+q}) \in \mathcal{L}^\perp$. However, $d_{j+r} \notin \mathcal{L}^\perp$, therefore, (19) leads to contradiction.

Hence, $w^*_{j+r} = w_{j+r} = c$ for all $r = 0, \ldots, q$. Then (11) and (19) give

$$d_{j+r} = d^*_{j+r} + p_{c(j+r)}(d^*_1, \ldots, d^*_{j-1}), \quad r = 0, \ldots, q. \tag{20}$$

We recall that monomials of $p_{c(j+r)}$ are elements of the dual basis. So, if $p_{c(j+r)}$ contains the monomial $(d^*_1)^{ш q_1} ш \cdots ш (d^*_{j-1})^{ш q_{j-1}}$ with nonzero coefficient then $p_{c(j+r)}(d^*_1, \ldots, d^*_{j-1})$ is not orthogonal to the element $(\ell^*_1)^{\vee q_1} \vee \cdots \vee (\ell^*_{j-1})^{\vee q_{j-1}}$. However, the induction supposition (14) implies $\ell^*_k = \ell_k$, $k = 1, \ldots, j-1$, hence, both d_{j+r} and d^*_{j+r} are orthogonal to this element. Then (20) implies that the polynomial $p_{c(j+r)}$ is zero, $p_{c(j+r)}(d^*_1, \ldots, d^*_{j-1}) \equiv 0$, and therefore,

$$d_{j+r} = d^*_{j+r}, \quad r = 0, \ldots, q.$$

Using the induction arguments we get that $d_k = d^*_k$ for $k = 1, \ldots, n$. Therefore, $\mathcal{J}^{\perp}_{a,b} = \mathcal{J}^{\perp}_{a^*,b^*}$, which implies $\mathcal{J}_{a,b} = \mathcal{J}_{a^*,b^*}$. The theorem is proved.

Remark. In Theorem 1, the controls $u^*_{x^0}$ are time-optimal. However, the optimality itself is not used in the proof. Instead, the following two properties of controls $u^*_{x^0}$ are applied: the requirement (6) connected with the homogeneity, and the property (7) which is justified by the autonomy of the system. One can generalize the theorem assuming that for any point $x^0 \in \Omega$ a control $u^*_{x^0}$ is chosen which steers the point x^0 to the origin by virtue of the system $\{a^*, b^*\}$ and steers the point $\Phi(x^0)$ to the origin by virtue of the system $\{a, b\}$ and, in addition, satisfies (6), (7), and condition (L). Then equality (10) holds and, as one can obtain by repeating the rest of the proof of the theorem, the systems $\{a^*, b^*\}$ and $\{a, b\}$ have the same right ideals.

Remark. Let us also notice that condition (L), which is used in the proof in order to conclude the identities $P_{1r} = P_{2r} = 0$ from equality (18), can be replaced by some other condition. For example, one can require the existence of $\alpha \in \mathbb{R}$ and two open sets $M_1, M_2 \subset \bigcup_{i \in I} \Omega_i$ such that $\widehat{u}^*_{x^0}(0) = \alpha$ for any $x^0 \in M_1$ and $\widehat{u}^*_{x^0}(0) \neq \alpha$ for any $x^0 \in M_2$.

Example. As $\{a^*, b^*\}$, let us consider the nonlinear homogeneous system of the form

$$\dot{x}_1 = u, \ \dot{x}_2 = x_1, \ \dot{x}_3 = x_1^3.$$

As was shown in [7], for any x^0 the optimal control $u^*_{x^0}(t)$ equals ± 1 or 0 and has finite number of switchings. In [8] domains where the time-optimal control problem for this system has a unique solution were described. In particular, it turns out that conditions (i)–(iii) mentioned in Section 1 are satisfied in several open domains. Hence, if $\{a^*, b^*\}$ is an algebraic approximation of $\{a, b\}$ then $\{a^*, b^*\}$ approximates $\{a, b\}$ in the sense of time optimality in these domains. Moreover, there exists a domain Ω' satisfying the conditions of Theorem 1; for example, one can choose $\Omega' = \{x : x_1 > 1, -x_1^2 < x_2 < 0\}$. Therefore, if $\{a^*, b^*\}$ approximates $\{a, b\}$ in the sense of time optimality in some domains Ω_i such that $\Omega' \subset \bigcup_{i \in I} \Omega_i$ then $\{a^*, b^*\}$ is an algebraic approximation of $\{a, b\}$.

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