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PERIODIC WORDS CONNECTED WITH THE TRIBONACCI WORDS

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We introduce two families of periodic words (TLP-words of type 1 and TLP-words of type 2) that are connected with the Tribonacci numbers and Tribonacci words, respectively.

Key words: Tribonacci numbers, Tribonacci words.

1. Introduction.

The Tribonacci numbers T_n are defined by the recurrence relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, for all integer $n > 2$, and with initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ (see, e.g., [1, 2, 3]). Many properties of the Tribonacci numbers require the full ring structure of the integers. However, generalizations to the ring $\mathbb{Z}_m$ have been considered (see, e.g., [4]). The sequence $T_n \pmod m$ is periodic and repeats by returning to its starting values because there are only a finite number m^3 of triples of terms possible, and the recurrence of a triple results in recurrence of all the following terms.

In analogy to the definition of Tribonacci numbers, one defines the Tribonacci finite words as the contatenation of the three previous terms $t_n = t_{n-1}t_{n-2}t_{n-3}$, $n > 2$, with initial values $t_0 = 0$, $t_1 = 01$ and $t_2 = 0102$ and defines the infinite Tribonacci word t, $t = \lim t_n$ (see [5]). It is the archetype of an Arnoux-Rauzy word (see, e.g., [6, 7]).

Using Tribonacci words, in the present article we shall introduce some new kinds of the infinite words, mainly TLP words, and investigate some of their properties.

For any notations not explicitly defined in this article we refer to [1, 6].

2. Tribonacci sequence modulo m.

The letter p is reserved to designate a prime and m may be arbitrary integer, $m > 1$.

Let $T^*_n(m)$, $0 \leqslant T^*_n(m) < m$, denote the n-th member of the sequence of integers $T_n \equiv T_{n-1} + T_{n-2} + T_{n-3} \pmod m$, for all integer $n > 2$, and with initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$ ($T^*_0(m) = 0$, $T^*_1(m) = 0$ and $T^*_2(m) = 1$). We reduce T_n modulo m taking the least nonnegative residues, and let $k(m)$ denote the length of the period of the repeating sequence $T_n^*(m)$. A few properties of the function $k(m)$ are in the following theorem [4, 8]. Let p be an odd prime, $p \neq 11$, and $\left(\frac{p}{11}\right)$ denotes the Legendre symbol.

Theorem 1. *In $\mathbb{Z}_m$ the following hold:*

1) Any Tribonacci sequence modulo m is periodic.
2) If $\left(\frac{p}{11}\right) = 1$, then $k(p)|(p^2 + p + 1)$ if $x^3 - x^2 - x - 1$ is irreducible $\bmod p$, otherwise $k(p)|(p - 1)$.
3) If $\left(\frac{p}{11}\right) = -1$, then $k(p)|(p^2 - 1)$.

The results in Theorem 8 give upper bounds for $k(p)$ but there are primes for which $k(p)$ is less than the given upper bound.

3. Tribonacci words.

Let $t_0 = 0$, $t_1 = 01$ and $t_2 = 0102$. Now $t_n = t_{n-1}t_{n-2}t_{n-3}$, $n > 2$, the contatenation of the three previous terms. The successive initial finite Tribonacci words are:

$$t_0 = 0,\ t_1 = 01,\ t_2 = 0102,\ t_3 = 0102010,\ t_4 = 0102010010201, \ldots \tag{1}$$

The infinite Tribonacci word t is the limit $t = \lim_{n\to\infty} t_n$. It is referenced A080843 in the On-line Encyclopedia of Integer Sequences [9]. The properties of the Tribonacci infinite words are of great interest in some fields of mathematics and its application such as number theory, fractal geometry, formal language etc. See [5, 10, 11, 12, 13].

We denote as usual by $|t_n|$ the length (the number of symbols) of t_n. The following proposition summarizes basic properties of Tribonacci words [5].

Theorem 2. *The infinite Tribonacci word and the finite Tribonacci words satisfy the following properties:*

1) The words 11, 22 and 000 are not subwords of the infinite Tribonacci word.
2) For all $n \geqslant 0$ let a_n be the last symbol of t_n and $k \equiv n \pmod 3$, $k \in \{0, 1, 2\}$. Then we have $a_n = k$.
3) For all $n \geqslant 0$ $|t_n| = T_{n+3}$.

4. Periodic TLP words.

Let us start with the classical definition of periodicity of words over arbitrary alphabet $\{a_0, a_1, a_2, \ldots\}$ (see [14]).

Definition 1. *Let $w = a_0a_1a_2\ldots$ be an infinite word. We say that w is*

1) a periodic word if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geqslant 0$. The smallest t satisfying the previous condition is called the period of w;
2) an eventually periodic word if there exist two positive integers t, s such that $a_i = a_{i+t}$, for all $i \geqslant s$;
3) an aperiodic word if it is not eventually periodic.

Theorem 3. *The infinite Tribonacci word is aperiodic.*

This statement is proved in [10].

We consider the finite Tribonacci words t_n (3) as numbers written in the ternary numeral system and denote them by b_n. Denote by d_n the value of the number b_n in usual decimal numeration system. We write $d_n = b_n$ meaning that d_n and b_n are writing of the same number in different numeral systems.

Example 1.

$$t_0 = 0,\ t_1 = 01,\ t_2 = 0102,\ t_3 = 0102010,\ t_4 = 0102010010201, \dots,$$
$$b_0 = 0,\ b_1 = 1,\ b_2 = 102,\ b_3 = 102010,\ b_4 = 102010010201, \dots,$$
$$d_0 = 0,\ d_1 = 1,\ d_2 = 11,\ d_3 = 300,\ d_4 = 218800,\ d_5 = 38759787911, \dots.$$

Theorem 4. *For any finite Tribonacci word $t_n$we have*

$$d_0 = 0, \quad d_1 = 1, \quad d_2 = 11, \quad d_n = d_{n-1}3^{T_{n+1}+T_n} + d_{n-2}3^{T_n} + d_{n-3}\,(n > 2). \qquad (2)$$

Proof. One can easily verify (4) for the first few $n : d_3 = b_3 = 102010 = 102000+10+0 = b_2 3^3 + b_1 3^1 + b_0 = d_2 3^3 + d_1 3^1 + d_0$, $d_4 = b_4 = 102010010201 = 102010000000+10200+1 = d_3 3^6 + d_2 3^2 + d_1$. Equality (4) follows from Theorem 9 (statement 3)) and the equality $d_n = b_n = b_{n-1}\underbrace{0\dots0}_{T_{n+1}+T_n} + b_{n-2}\underbrace{0\dots0}_{T_n} + b_{n-3} = d_{n-1}3^{T_{n+1}+T_n} + d_{n-2}3^{T_n} + d_{n-3}$.

Let $w_0(m) = 0$ and for arbitrary integer n, $n \geqslant 1$, $d_n(m) = d_n \pmod m$, $0 \leqslant d_n(m) < m$, $b_n(m)$ is $d_n(m)$ in the ternary numeration system, $w_n(m) = w_{n-1}(m)b_n(m)$. Denote by $w(m)$ the limit $w(m) = \lim_{n\to\infty} w_n(m)$.

Definition 2. *We say that*
1) $w_n(m)$ is a finite TLP word type 1 modulo m;
2) $w(m)$ is an infinite TLP word type 1 modulo m.

Theorem 5. *The infinite TLP word of type 1 $w(p)$ is a periodic word.*

Proof. The statement follows from (4) because there are only a finite number of $d_n(p)$ and $3^{T_n} \pmod p$ possible, and the recurrence of the first few terms sequence $d_n(p)$ and $3^{T_n} \pmod p$ gives recurrence of all subsequent terms.

Using Tribonacci words (3) we define a periodic TLP word $w^*(m)$ (infinite TLP word type 2 by modulo m). As usual, we denote by ϵ the empty word. Let $v_n^*(m)$ be the last $T_{n+3}^*(m)$ symbols of the word t_n if $T_{n+3}^*(m) \neq 0$, otherwise $v_n^*(m) = \epsilon$.

Theorem 6. *The word length $|v_n^*(m)|$ coincides with $T_{n+3}^*(m)$.*

Proof. This is clear by construction of $v_n^*(m)$.

Since $T_n^*(m)$ is a periodic sequence with period $k(m)$, the sequence $|v_n^*(m)|$ is periodic with the same period. Let $w_0^*(m) = 0$ and for arbitrary integer n, $n \geqslant 1$, $w_n^*(m) = w_{n-1}^*(m)v_n^*(m)$. Denote by $w^*(m)$ the limit $w^*(m) = \lim_{n\to\infty} w_n^*(m)$.

Definition 3. *We say that*
1) $w_n^*(m)$ is a finite TLP word of type 2 modulo m;
2) $w^*(m)$ is an infinite TLP word of type 2 modulo m.

Theorem 7. *The infinite TLP word of type 2 $w^*(m)$ is a periodic word.*

The Theorem 14 is a direct corollary of Theorem 9 (statement 2)) and Theorem 13.

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ПЕРІОДИЧНІ СЛОВА, ПОВ'ЯЗАНІ ЗІ СЛОВАМИ ТРІБОНАЧЧІ

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Означено два види періодичних слів (TLP-слова типу 1 та TLP-слова типу 2), які пов'язані з числами Трібоначчі та словами Трібоначчі.

Ключові слова: числа Трібоначчі, слова Трібоначчі.