*Наведено математичні моделі зміни курсу судна при повороті без урахування часу перекладання пера керма. Всього розглянуто три моделі різного ступеня адекватності реальному процесу повороту і виявлено відповідність математичних моделей експериментальним натурним спостереженням конкретних суден. Отримано аналітичні вирази для розрахунку тривалості обох фаз повороту способом простих ітерацій*

*Ключові слова: забезпечення безпеки судноводіння, управління рухом, поворот судна, динамічні моделі повороткості*

*Приведены математические модели изменения курса судна при повороте без учета времени перекладки пера руля. Всего рассмотрено три модели разной степени адекватности реальному процессу поворота и выявлено соответствие математических моделей экспериментальным натурным наблюдениям конкретных судов. Получены аналитические выражения для расчета продолжительности обоих фаз поворота способом простых итераций*

*Ключевые слова: обеспечение безопасности судовождения, управление движением, поворот судна, динамические модели поворотливости*

UDC 656.61.052.484
DOI: 10.15587/1729-4061.2016.85839

# ANALYSIS OF MATHEMATICAL MODELS OF CHANGING THE VESSEL'S COURSE WHEN TURNING

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## 1. Introduction

The accuracy of implementation of curvilinear sections of the program trajectory of a vessel's motion depends on the degree of adequacy of the model of the vessel's turning ability to the actual process of its turning and correctness of selection of maneuver parameters. In the situations of different constraint, the models of the vessel's turning ability of different degree of adequacy are required; moreover, with an increase in the level of adequacy, procedure for the calculation of maneuver parameters becomes more complicated.

Therefore, an analysis of mathematical models of the process of a vessel's turn with a different degree of calculation of its inertness and corresponding procedures of calculation of maneuver parameters is a relevant direction of scientific studies on providing the navigation safety.

## 2. Literature review and problem statement

Papers [1–3] are devoted to the problems of studying the curvilinear motion of a vessel at turning. Formation of the transfer trajectory of a vessel's turn, taking into account experimental data of the vessel's turning ability, is examined in article [1], and paper [2] gives the results of experimental research into the models of the vessel's turning ability. Article [4] gives the characteristics of turning ability of the container carrier "Oxford" and explores the imitation simulation of its turning.

Paper [6] examines the problems of considering the vessel's dynamics in the calculation of parameters of divergence strategy in case of a dangerous rapprochement, and article [7] proposes the model of rotary motion of a vessel considering the duration of putting the rudder blade over. The estimation of roughness of the mathematical model, accepted for the study, by means of exploring the influence of additional nonlinear terms on the motion modes of a vessel was carried out in paper [8]. The ineffectiveness of standard automatic steering device in case of wind was established. An introduction of the intelligent component into the algorithm of automatic steering device made it possible not only to substantially increase its effectiveness, but also to overcome a certain degree of uncontrollability. The concept of coefficients of parameters influence of mathematical model of a vessel on its maneuverability characteristics is introduced in article [9]. As an example, influence coefficients for a number of the maneuverability characteristics were calculated: a radius of the steady circulation of a vessel, advance in the evolutionary circulation period, initial vessel's turning ability and all characteristics of checking helm. However, the problem of calculation of maneuver parameters of a vessel's turn for the implementation of program curvilinear trajectory was not examined.

In paper [10], it is indicated that for controlling a vessel under extreme conditions, it is necessary to calculate the influence of all external forces, which contribute or complicate control under difficult conditions. The work presents the basic principles of the estimation of external forces in controlling a vessel and possibility to use these forces under extreme conditions, taking into account the existence or absence of tugs. The proposed approach does not make it possible to reveal the dependence of the maneuver parameters of a vessel on its inertial characteristics. Article [11] is devoted to the problems of developing automatic steering systems of controlling the course of a vessel using the principles of fuzzy logic. Problems of the system's adaptation are considered, taking into account the complex dynamics and nonlinearity of mathematical model of a vessel as an object of control under randomly changing conditions of the system's operation.

In article [12], it is noted that in the majority of contemporary systems of controlling complex dynamic objects, the model of a controlled object is used for forming controlling influences. The use of neural networks is one of the approaches to obtaining the model. Marine mobile objects and, in particular, sea vessels belong to the class of the so-called indeterminate objects. A special feature of such objects is that, as a rule, their mathematical model at the stage of developing a control system is known with accuracy to some unknown characteristics.

In paper [16], it is noted that an increase in the load capacity of contemporary vessels causes the need for applying improved computer systems for their safe navigation. Such systems use the forecast tools of the vessels' motion, which have been successfully applied for a long time. However, simplification of the existing forecasts restricts their use for immediate mapping of the vessel's motion with a change in the rudder position and in engine revolutions. The required accuracy of implementation of the curvilinear trajectory of the vessel's motion can be provided with the more improved forecast models of the vessel's motion.

At present, as it is indicated in article [17], an information system of imitative simulation of the vessels' motion with complex dynamic models is being developed, depending on the rudder angle and engine revolutions. This system will make it possible to provide a new type of planning the vessel's maneuvers and control of implementation of the assigned maneuver. The assigned maneuvering is supposed to be mapped in the process of maneuvering simultaneously with the actual vessel's motion and with indication of the forecasted trajectory, which is determined by real input data from the vessel's sensors. It should be noted that this system solves a direct problem of mapping of the forecasted trajectory by the assigned maneuver parameters, although relevant is the inverse problem of determining the forthcoming maneuver parameters along with the assigned program of the motion trajectory. In other words, it is necessary to calculate the starting moment of turning and its duration for the assigned rudder blade angle, after providing the vessel's entrance to the assigned point.

Paper [18] is devoted to the problems of identification of the ship models maneuvering, which are the key to studying the vessel's maneuverability, designing control systems of the vessel's motion and development of systems of controlling ship trainers. In this work, the nonlinear model of the vessel's maneuvering is formed on the basis of an analysis of vessel's hydrodynamics. The theory of the systems identification is used for evaluating the parameters of the model, which can be calculated with the proposed algorithm, based on the extended filter theory of Kalman. For obtaining input and output data of the systems, which are necessary for identification of parameters of experiment, the circulation and the zigzag maneuver were used, which are carried out on the imitator of the vessel control. The errors, introduced during the process of measurement, are removed with the help of this algorithm.

In work [19], author examines the intelligent system of prediction of the vessel's motion, which imitates the process of training the autonomous control unit, created with the help of the artificial neural network. A control unit observes input signals and calculates the values of the required maneuvering parameters of the vessel in confined waters. The basic task of the system is continuous inspection of navigation parameters of a vessel and the prediction of their values after a particular time interval. The result of prognostication can be used as a warning to a navigator about the appearing threat.

In the process of vessel's turning, the characteristics of its turning ability are considered approximately, which decreases the accuracy of vessel's entering the next section of the program motion and leads to an increase in probability of navigation emergency. That is why there is a necessity for the development of the procedure of calculation of maneuvering parameters of the vessel's turn, which include the starting time of turning and its duration, depending considerably on the selected model of a vessel's turning ability that takes into account its inertia. Despite its relevance, this problem was not examined in the above analyzed works. Moreover, at present there are navigation information systems on the vessels, in which it is expedient to develop the function of calculation of parameters of a vessel's turn for the assigned initial data. This circumstance requires comparison and analysis of different models of the vessel's rotary motion and selection of the base model of a vessel's turning ability, taking into account its inertial characteristics in connection with predominant navigating conditions. The indicated task requires its solution, and the examination of some of its aspects is proposed in this work.

## 3. The aim and tasks of the study

The purpose of the study is the examination of mathematical models of the vessel's turning ability, the estimation of the degree of their adequacy to experimental field observations with the purpose of selecting the most acceptable for describing the rotary motion of a vessel

To achieve the set goal, the following tasks were to be solved:

– to examine the simplest kinematic model of the rotary motion of a vessel with a constant angular velocity;

– to analyze dynamic models of the vessel's turn taking into account its inertia;

– to present procedures for the calculation of the duration of a vessel's turn by numerical methods for each model;

– to substantiate the selection of the most acceptable model of the vessel's rotary motion on the basis of the field observations data.

## 4. Kinematic model of a change in the vessel's course when turning

The turn of a vessel consists of two phases. At the first phase of a turn, at the initial moment of time $t_n$ the rudder is put over to the angle $t_n$ and the rudder is retained in this position during the time interval $\Delta t_k$. Then, at the second phase of a turn, the rudder is put over to the opposite hard to the same value and the inertia of vessel's turn is dissipated during the time interval $\Delta t$. When it is over, the vessel sets the predetermined course, the angular velocity of a turn becomes zero, the rudder blade is put into the center-line plane of a vessel, and a vessel from the initial course $K_o$ sets course $K_y$ (Fig. 1)

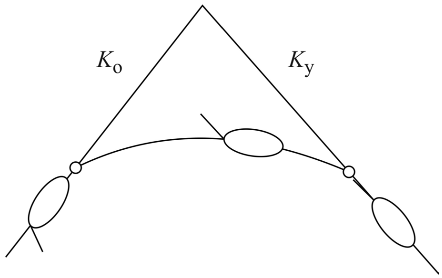


Fig. 1. Vessel's turn

To describe a vessel's turn, we will consider the following models of rotary motion [5–7]: the first model is simplified, which describes a vessel's turn at unchanged angular velocity, the second and the third models consider existence of time variables, characterizing the vessel's dynamics. We will note that the first model serves as an initial rapprochement in irrational solution of expressions, obtained by the more complex models of a vessel's turn.

The first kinematic model of the vessel's motion, which characterizes a change in the course of vessel K under the influence of the rudder blade, implies vessel's turning with a constant angular velocity and is described by the differential equation, which takes the following form:

$$\dot{K} = k_\omega \beta_k,$$

where $k_\omega$ is the coefficient of rudder effectiveness; $\beta_k$ is the angle of putting the rudder blade over.

Thus $\omega = \dot{K}$, initial equation may be written down in the form:

$$\omega = k_\omega, \beta_k = a_\omega,$$

by integrating which, we will obtain the expression for the current course of vessel K, as the time function, that is,

$$K = K_o + \int_0^t \omega d\tau = K_o + \int_0^t a_\omega d\tau.$$

Thus, in a general form we obtain expression for the course of a vessel in the time function t:

$$K = K_o + a_\omega t.$$

In this model the vessel's turn is made without checking helm, i. e. a turn has only one phase and therefore $\Delta t_k = \tau$. Consequently, $\Delta K = a_\omega \tau$ or $\tau = \Delta K / a_\omega$.

Increment of coordinates of an operating vessel within the period of maneuvering $\tau$ is determined by the following expressions:

$$\Delta x_o = \int_0^\tau V_o \sin\left[K_o + K(t)\right] dt,$$

$$\Delta y_o = \int_0^\tau V_o \cos\left[K_o + K(t)\right] dt.$$

Substituting the expression for K(t), it is possible to write down:

$$\Delta x_o = \int_0^\tau V_o \sin\left(K_o + a_\omega t\right) dt,$$

$$\Delta y_o = \int_0^\tau V_o \cos\left(K_o + a_\omega t\right) dt.$$

Let us consider the expression for $\Delta x_o$:

$$\Delta x_o = \int_0^\tau V_o \sin\left(K_o + a_\omega t\right) dt =$$
$$= V_o \sin K_o \int_0^\tau \cos\left(a_\omega t\right) dt + V_o \cos K_o \int_0^\tau \sin\left(a_\omega t\right) dt.$$

We will designate

$$\Im s = \int_0^\tau \sin\left(a_\omega t\right) dt \quad \text{and} \quad \Im c = \Im s = \int_0^\tau \cos\left(a_\omega t\right) dt.$$

Then the last expression for $\Delta x_o$ takes the following form:

$$\Delta x_o = V_o \left(\sin K_o \Im c + \cos K_o \Im s\right).$$

We will find expressions for $\Im s$ and $\Im c$:

$$\Im s = \int_0^\tau \sin\left(a_\omega t\right) dt = -\frac{1}{a_\omega} \cos\left(a_\omega t\right)\Big|_0^\tau = -\frac{1}{a_\omega}\left[\cos\left(a_\omega t\right) - 1\right].$$

$$\Im c = \int_0^\tau \cos\left(a_\omega t\right) dt = \frac{1}{a_\omega} \sin\left(a_\omega t\right)\Big|_0^\tau = \frac{1}{a_\omega} \sin\left(a_\omega \tau\right).$$

Taking into account the obtained expressions for the unknown integrals, we will write down increment in coordinate $\Delta x_o$ in the following form:

$$\Delta x_o = \frac{V_o}{a_\omega}\left\{\sin K_o \sin\left(a_\omega \tau\right) - \cos K_o \left[\cos\left(a_\omega \tau\right) - 1\right]\right\}.$$

We consider that $\Delta K = a_\omega \tau$, which is why the expression for $\Delta x_o$ takes the form:

$$\Delta x_o = \frac{V_o}{a_\omega}\left\{\sin K_o \sin \Delta K - \cos K_o \left[\cos \Delta K - 1\right]\right\} =$$
$$= \frac{V_o}{a_\omega}\left[\cos K_o - \cos\left(K_o + \Delta K\right)\right].$$

We note that the relationship $K_o + \Delta K = K_y$, where $K_y$ is the course of the second section of a vessel's program trajectory, is correct; therefore it is possible to finally write down expression for $\Delta x_o$:

$$\Delta x_o = \frac{V_o}{a_\omega}\left(\cos K_o - \cos K_y\right).$$

Now we will find the expression for increment for the second coordinate $\Delta y_o$:

$$\Delta y_o = \int_0^\tau V_o \cos\left[K_o + K(t)\right] dt =$$
$$= V_o \cos K_o \int_0^\tau \cos\left(a_\omega t\right) dt - V_o \sin K_o \int_0^\tau \sin\left(a_\omega t\right) dt$$

or

$$\Delta y_o = V_o \left(\cos K_o \Im c - \sin K_o \Im s\right).$$

We substitute the values of integrals $\Im c$ and $\Im s$, obtained earlier:

$$\Delta y_o = V_o \left(\cos K_o \Im c - \sin K_o \Im s\right) =$$
$$= \frac{V_o}{a_\omega}\left\{\cos K_o \sin\left(a_\omega \tau\right) + \sin K_o \left[\cos\left(a_\omega \tau\right) - 1\right]\right\},$$

similarly to the previous case for $\Delta y_o$, we obtain:

$$\Delta y_o = \frac{V_o}{a_\omega}\left[\cos K_o \sin\Delta K + \sin K_o\left(\sin\Delta K - 1\right)\right]$$

or

$$\Delta y_o = \frac{V_o}{a_\omega}\left(\sin K_y - \sin K_o\right).$$

Thus, summing up the obtained results, it is possible to note that the calculation of increments in coordinates $\Delta x_o$ and $\Delta y_o$ is performed according to the following formulas:

$$\Delta x_o = \frac{V_o}{a_\omega}\left(\cos K_o - \cos K_y\right),$$

$$\Delta y_o = \frac{V_o}{a_\omega}\left(\sin K_y - \sin K_o\right).$$

The obtained expressions are used for initial assessment of the magnitude of correction for vessel's inertia, as well as the initial approximation of solution by the method of simple iterations.

## 5. Dynamic models of a vessel's turn, which consider its inertia

The second dynamic model of the vessel's motion, which characterizes a change in the course of vessel K under the influence of rudder blade, is described by the non-homogeneous linear differential equation with constant coefficients, which takes the following form [5]

$$T_1\ddot{K} + \dot{K} = k_\omega \beta_k,$$

where $T_1$ is the constant of time, characterizing inertia properties of a vessel; $k_\omega$ is the coefficient of rudder effectiveness.

Let us find the solution of the reduced differential equation, for which we will write it down relative to angular velocity $\omega$, taking into account that $\omega=\dot{K}$:

$$T_1\dot{\omega} + \omega = k_\omega \beta_k.$$

The equation in question is non-uniform; therefore, its general solution is the sum of the solution of corresponding homogeneous equation and particular solution [13], that is:

$$\omega = \omega_{od} + \omega_r, \tag{1}$$

where $\omega_{od}$ and $\omega_r$ are, respectively, the solution of similar homogeneous equation and the particular solution.

Let us find solution for $\omega_{od}$, writing down the appropriate homogeneous differential equation:

$$T_1\dot{\omega}_{od} + \omega_{od} = 0,$$

to which the following characteristic equation corresponds:

$$T_1 k + 1 = 0,$$

with root $k=-1/T_1$. Therefore [13], the solution of homogeneous differential equation is:

$$\omega_{od} = C_1 \exp\left(-t/T_1\right).$$

That is why expression (1) takes the following form:

$$\omega = C_1 \exp\left(-t/T_1\right) + \omega_r. \tag{2}$$

We will find the integration constant $C_1$ from the initial conditions at t=0. We will designate the initial value of angular velocity with $\omega_o$ and, substituting t=0 in (2), we will obtain expression for integration constant $C_1$:

$$C_1 = \omega_o - \omega_r. \tag{3}$$

Then, substituting (3) in (2), for w we will finally obtain the following expression:

$$\omega = \left(\omega_o - \omega_r\right)\exp\left(-t/T_1\right) + \omega_r. \tag{4}$$

Integrating the last equation, we will obtain the expression for current course of vessel K, that is:

$$K = K_o + \int_0^t \left[\left(\omega_o - \omega_r\right)\exp\left(-\tau/T_1\right) + \omega_r\right]d\tau,$$

We will introduce the following designation for definite integral in the previous interval:

$$J_o = \int_0^t \left[\left(\omega_o - \omega_r\right)\exp\left(-\tau/T_1\right) + \omega_r\right]d\tau,$$

and find the expression for it. It is evident that:

$$J_o = \int_0^t \left[\left(\omega_o - \omega_r\right)\exp\left(-\tau/T_1\right) + \omega_r\right]d\tau =$$

$$= \int_0^t \left[\left(\omega_o - \omega_r\right)\exp\left(-\tau/T_1\right)\right]d\tau + \int_0^t \omega_r d\tau,$$

or

$$J_o = \left(\omega_o - \omega_r\right)\int_0^t \exp\left(-\tau/T_1\right)d\tau + \omega_r t.$$

Finally, we obtain the following expression for integral $J_o$:

$$J_o = \omega_r t - T_1\left(\omega_r - \omega_o\right)\left[1 - \exp\left(-t/T_1\right)\right].$$

That is why expression for a vessel's course has the following form:

$$K = K_o + \omega_r t - T_1\left(\omega_r - \omega_o\right)\left[1 - \exp\left(-t/T_1\right)\right]. \tag{5}$$

Let us find expressions for the current value of the vessel's course at the first and second phases of its turn, which are different by the position of the rudder blade relative to the center-line plane of a vessel.

At the first phase of a turn, the duration of which makes the time interval $\Delta t_k$, the initial $\omega_o$ and steady-state $\omega_r$ value of angular velocity are expressed as follows:

$$\omega_o = 0 \text{ and } \omega_r = k_\omega \beta_k = a_\omega.$$

In this case, expression (5) takes the following form:

$$K = K_o + a_\omega t - T_1 a_\omega\left[1 - \exp\left(-\tau/T_1\right)\right]$$

or

$$K = K_o + a_\omega\left\{t - T_1\left[1 - \exp\left(-t/T_1\right)\right]\right\}. \tag{6}$$

At the second phase of a turn, in the moment of time $t_n+\Delta t_k$, the rudder is put over to the opposite hard to angle $-\beta_k$ and, within time interval $\Delta t$, checking helm takes place. In this case, the initial value of angular velocity in the moment of time $\Delta t_k$ is determined by the expression

$$\omega_o = a_\omega\left[1-\exp\left(-\Delta t_k/T_1\right)\right],$$

and steady-state value of angular velocity $\omega_r=-a_\omega$. We substitute the obtained values in (5):

$$\tilde{K} = K - a_\omega t - -T_1\left\{-a_\omega - a_\omega\left[1-\exp\left(-\Delta t_k/T_1\right)\right]\right\}\left[1-\exp\left(-t/T_1\right)\right].$$

That is why the value of the current course is described by dependence:

$$\tilde{K} = K - a_\omega\left\{T_1\left[2-\exp\left(-\Delta t_k/T_1\right)\right]\left[1-\exp\left(-t/T_1\right)\right]-t\right\}. \quad (7)$$

To calculate time intervals $\Delta t_k$ and $\Delta t$, it is necessary to make the system of equations, which in general case formalizes the requirements of the turn for the assigned increment in course $\Delta K$, as well as the loss of angular velocity by the moment of setting a new course, and takes the following form:

$$\begin{cases}\Delta K = K\left(\Delta t_k\right)+\tilde{K}\left(\Delta t\right),\\ \omega(\Delta t_k,\Delta t)=0.\end{cases} \quad (8)$$

We will write down the first equation of system (8) in explicit form, using expressions (6) and (7) for the components $K(\Delta t_k)$ and $\tilde{K}$ $(\Delta t)$:

$$\Delta K = a_\omega\left\{\Delta t_k - T_1\left[1-\exp\left(-\Delta t_k/T_1\right)\right]\right\}+ +a_\omega\left\{T_1\left[2-\exp\left(-\Delta t_k/T_1\right)\right]\left[1-\exp\left(-t/T_1\right)\right]-\Delta t\right\}.$$

Let us divide both parts of the obtained equation by value $a_\omega$ and obtain:

$$\frac{\Delta K}{a_\omega} = \left\{\Delta t_k - T_1\left[1-\exp\left(-\Delta t_k/T_1\right)\right]\right\}+ +a_\omega\left\{T_1\left[2-\exp\left(-\Delta t_k/T_1\right)\right]\left[1-\exp\left(-t/T_1\right)\right]-\Delta t\right\},$$

hence, we find the expression for calculating $\Delta t_k$ by the method of simple iterations [14 ]:

$$\Delta t_k = T_1\left[1-\exp\left(-\Delta t_k/T_1\right)\right]+\Delta t- -T_1\left[2-\exp\left(-\Delta t_k/T_1\right)\right]\left[1-\exp\left(-t/T_1\right)\right]+\frac{\Delta K}{a_\omega}$$

with initial approximation

$$\Delta t_k = \frac{\Delta K}{a_\omega}.$$

In the reduced expression for simple iterations it is also necessary to find connection between variables $\Delta t_k$ и $\Delta t$, that is, value $\Delta t$ must be expressed through $\Delta t_k$. For this, we will use the second equation of system (8) and the initial values of angular velocity at the second phase of a turn

$$\omega_o = a_\omega\left[1-\exp\left(-\Delta t_k/T_1\right)\right],\ \omega_r = -a_\omega.$$

Substituting these values in (4), we will obtain:

$$\omega(\Delta t_k,\Delta t)= =\left\{a_\omega\left[1-\exp\left(-\Delta t_k/T_1\right)\right]+a_\omega\right\}\exp\left(-\Delta t/T_1\right)-a_\omega=0,$$

$$a_\omega\left[1-\exp\left(-\Delta t_k/T_1\right)+1\right]\exp\left(-\Delta t/T_1\right)-a_\omega=0,$$

$$a_\omega\left\{\left[2-\exp\left(-\Delta t_k/T_1\right)\right]\exp\left(-\Delta t/T_1\right)-1\right\}=0,$$

from this expression, it is possible to write down:

$$\left[2-\exp\left(-t_k/T_1\right)\right]\exp\left(-\Delta t/T_1\right)=1,$$

$$\left[2-\exp\left(-\Delta t_k/T_1\right)\right]^{-1}=\exp\left(-\Delta t/T_1\right),$$

taking the logarithms of both parts of the last equation, we obtain:

$$-\Delta t/T_1 = -\ln\left[2-\exp\left(-\Delta t_k/T_1\right)\right],$$

$$\Delta t = T_1\ln\left[2-\exp\left(-\Delta t_k/T_1\right)\right].$$

The latter obtained equation makes it possible to connect variables $\Delta t_k$ and $\Delta t$, which provides for iterative calculation of the durations of each phase of the vessel's turn, as well as turn duration $\tau$ from one assigned vessel's course to another one.

To calculate corrections, which consider the inertia of a vessel, by the moments of its turn, it is necessary to calculate increment in coordinates $\Delta x_o$ and $\Delta y_o$ of the operating vessel over the time of turn $\tau$. It is obvious that:

$$\Delta x_o = \int_0^\tau V_o\sin\left[K_o+K\left(t\right)\right]dt,$$

$$\Delta y_o = \int_0^\tau V_o\cos\left[K_o+K\left(t\right)\right]dt.$$

If we take into account that a vessel's turn consists of two phases, and in addition, the expression for the current course at the first and second phases takes different forms, the previous integrals for $\Delta x_o$ and $\Delta y_o$ take the following form:

$$\Delta x_o = \int_0^\tau V_o\sin\left[K_o+\left(K+\tilde{K}\right)\right]dt,$$

$$\Delta y_o = \int_0^\tau V_o\cos\left[K_o+\left(K+\tilde{K}\right)\right]dt.$$

Each of the reduced definite integrals is the sum of two others, describing increment in coordinates at the first and second phases of a vessel's turn, that is:

$$\Delta x_o = \int_0^{\Delta t_k} V_o\sin\left[K_o+K\right]dt+\int_0^{\Delta t} V_o\sin\left[K_o+K\left(\Delta t_k\right)+\tilde{K}\right]dt,$$

$$\Delta y_o = \int_0^{\Delta t_k} V_o\cos\left[K_o+K\right]dt+\int_0^{\Delta t} V_o\cos\left[K_o+K\left(\Delta t_k\right)+\tilde{K}\right]dt.$$

Substituting expressions (6) and (7) in previous equations, first we will obtain expression for increment $\Delta x_o$:

$$\Delta x_o = \int_0^{\Delta t_k} V_o\sin\left[K_o+a_\omega\left\{t-T_1\left[1-\exp\left(-t/T_1\right)\right]\right\}\right]dt+ +\int_0^{\Delta t} V_o\sin\left[K_o+K\left(\Delta t_k\right)+a_\omega\left\{T_1\left[2-\exp\left(-t_k/T_1\right)\right]\left[1-\exp\left(-t/T_1\right)\right]-t\right\}\right]dt,$$

or remove constant values from the integral sign:

$$\Delta x_o = V_o \sin K_o \int_0^{\Delta t_k} \cos\left[a_\omega\left\{t - T_1\left[1-\exp(-t/T_1)\right]\right\}\right]dt +$$
$$+V_o \sin K_o \int_0^{\Delta t_k} \sin\left[a_\omega\left\{t - T_1\left[1-\exp(-t/T_1)\right]\right\}\right]dt +$$
$$+V_o \sin\left[K_o + K(\Delta t_k)\right]\int_0^{\Delta t} \cos\left[a_\omega\left\{T_1\left[2-\exp(-\Delta t_k/T_1)\right]\left[1-\exp(-t/T_1)\right]-t\right\}\right]dt +$$
$$+V_o \cos\left[K_o + K(\Delta t_k)\right]\int_0^{\Delta t} \sin\left[a_\omega\left\{T_1\left[2-\exp(-\Delta t_k/T_1)\right]\left[1-\exp(-t/T_1)\right]-t\right\}\right]dt,$$

where

$$K(\Delta t_k) = a_\omega\left\{\Delta t_k - T_1\left[1-\exp(-\Delta t_k/T_1)\right]\right\}.$$

We will find expression for increment in coordinate $\Delta y_o$:

$$\Delta y_o = \int_0^{\Delta t_k} V_o \cos\left[K_o + a_\omega\left\{t - T_1\left[1-\exp(-t/T_1)\right]\right\}\right]dt +$$
$$+\int_0^{\Delta t} V_o \cos\left[K_o + K(\Delta t_k) + a_\omega\left\{T_1\left[2-\exp(-t_k/T_1)\right]\left[1-\exp(-t/T_1)\right]-t\right\}\right]dt.$$

We remove constant values from the integral sign and obtain:

$$\Delta y_o = V_o \cos K_o \int_0^{\Delta t_k} \cos\left[a_\omega\left\{t - T_1\left[1-\exp(-t/T_1)\right]\right\}\right]dt -$$
$$-V_o \sin K_o \int_0^{\Delta t_k} \sin\left[a_\omega\left\{t - T_1\left[1-\exp(-t/T_1)\right]\right\}\right]dt +$$
$$+V_o \cos\left[K_o + K(\Delta t_k)\right]\int_0^{\Delta t} \cos\left[a_\omega\left\{T_1\left[2-\exp(-\Delta t_k/T_1)\right]\left[1-\exp(-t/T_1)\right]-t\right\}\right]dt -$$
$$-V_o \sin\left[K_o + K(\Delta t_k)\right]\int_0^{\Delta t} \sin\left[a_\omega\left\{T_1\left[2-\exp(-\Delta t_k/T_1)\right]\left[1-\exp(-t/T_1)\right]-t\right\}\right]dt.$$

In this way the calculations of the necessary values of magnitudes $\Delta t$, $\Delta t_k$, $\Delta x_o$ and $\Delta y_o$ are performed in case when the second dynamic model of the rotary motion of vessel is used.

We note the circumstance that definite integrals, which are included in the expressions for $\Delta x_o$ and $\Delta y_o$, are not expressed in elementary functions and their values are found by numerical methods, for example, we use the method of trapezoids or the Simpson method, which gives more accurate results.

The third dynamic model of a change in the course of vessel K with its turn is described by the non-homogeneous linear differential equation of the third order with constant coefficients, which takes the following form [5]:

$$T_1T_2\dddot{K} + (T_1+T_2)\ddot{K} + \dot{K} = K_\omega \beta_k,$$

where $T_1$ and $T_2$ are the time constants, characterizing inertia properties of a vessel

Let us find the solution for this differential equation, for which we will write it down depending on angular velocity $\omega$, taking into account that $\omega=\dot{K}$:

$$T_1T_2\ddot{\omega} + (T_1+T_2)\dot{\omega} + \omega = K_\omega \beta_k.$$

A general solution of this non-uniform equation of the second order is the sum of the solution of the corresponding uniform equation $\omega_{od}$ and particular solution $\omega_r$, that is:

$$\omega = \omega_{od} + \omega_r.$$

First, we will find solution $\omega_{od}$, for which we write down the corresponding uniform equation:

$$T_1T_2\ddot{\omega} + (T_1+T_2)\dot{\omega} + \omega = 0, \quad (9)$$

which has the following characteristic equation:

$$T_1T_2k^2 + (T_1+T_2)k + 1 = 0, \quad (10)$$

the roots of which are necessary to determine.

From (10), we obtain:

$$k^2 + \frac{T_1+T_2}{T_1T_2}k + \frac{1}{T_1T_2} = 0,$$

hence, we find expressions for the equation roots:

$$k_{1,2} = -(T_1+T_2)/2T_1T_2 \pm$$
$$\pm\left[(T_1+T_2)^2/4T_1^2T_2^2 - 1/T_1T_2\right]^{1/2} =$$
$$= -(T_1+T_2)/2T_1T_2 \pm$$
$$\pm\left[(T_1^2+T_2^2+2T_1T_2-4T_1T_2)/4T_1^2T_2^2\right]^{1/2} =$$
$$= -(T_1+T_2)/2T_1T_2 \pm$$
$$\pm\left[(T_1^2+T_2^2-2T_1T_2)/4T_1^2T_2^2\right]^{1/2} =$$
$$= -(T_1+T_2)/(2T_1T_2) \pm$$
$$\pm(T_1-T_2)/(2T_1T_2).$$

From the last equation we find the following expressions for the required roots of characteristic equation:

$$k_1 = -(T_1+T_2)/(2T_1T_2) + (T_1-T_2)/(2T_1T_2),\ k_1 = -1/T_1,$$
$$k_2 = -(T_1+T_2)/(2T_1T_2) - (T_1-T_2)/(2T_1T_2),\ k_2 = -1/T_2.$$

According to the obtained values of roots $k_1=-1/T_1$ and $k_1=-1/T_2$, it is possible to write down a solution for the initial differential equation (10) for angular velocity $\omega$ [15]:

$$\omega = C_1\exp(-t/T_1) + C_2\exp(-t/T_2) + \omega_r. \quad (11)$$

Integration constants $C_1$ and $C_2$ are found from the initial conditions at t=0. Initial value of angular velocity will be designated through $\omega_o$, and angular acceleration $\dot{\omega}_o=0$. Substituting initial values at t=0 in (11), we will obtain the following expression:

$$\omega_o = C_1 + C_2 + \omega_r.$$

For the second expression, it is necessary to write down analytical form of first-order derivative $\dot{\omega}$. For this purpose, it is necessary to differentiate (11):

$$\dot{\omega} = -C_1/T_1\exp(-t/T_1) - C_2/T_2\exp(-t/T_2),$$

at t=0, we will obtain the second expression for defining $C_1$ and $C_2$:

$$\dot{\omega}_o = -C_1/T_1 - C_2/T_2,$$

$$C_1T_2 + C_2T_1 = 0.$$

From the obtained equations, we write down the following expression:

$$C_1 + C_2 = \omega_o - \omega_r,$$

$$C_1T_2 + C_2T_1 = 0.$$

From the second equation, we obtain:

$$C_1 = -C_2T_1/T_2,$$

which, when substituting in the first expression, will give:

$$-C_2T_1/T_2 + C_2 = \omega_o - \omega_r,$$

hence:

$$C_2 = -T_2(\omega_o - \omega_r)/(T_1 - T_2),$$

$$C_1 = -T_1(\omega_o - \omega_r)/(T_1 - T_2).$$

That is why the expression for angular velocity ω takes the form:

$$\omega = \omega_r + (\omega_o - \omega_r)\times$$
$$\times\left[T_1\exp(-t/T_1) - T_2\exp(-t/T_2)\right]/(T_1 - T_2). \quad (12)$$

Integrating the obtained expression, we find analytical dependence for the current value of the vessel's course in general form:

$$K = \int_0^t \omega d\tau = \frac{(\omega_o - \omega_r)}{(T_1 - T_2)}\times$$
$$\times\int_0^t\left[T_1\exp(-\tau/T_1) - T_2\exp(-\tau/T_2)\right]d\tau + \int_0^t \omega_r d\tau;$$

$$K = K_o + \frac{(\omega_o - \omega_r)}{(T_1 - T_2)}\times$$
$$\times\left[-T_1^2\exp(-\tau/T_1) + T_2^2\exp(-\tau/T_2)\right]\Big|_0^\tau + \omega_r t.$$

Substituting integration limits, we obtain the following expression for the current course of vessel K:

$$K = K_o + \omega_r t + (\omega_o - \omega_r)\times$$
$$\times\left\{T_1^2\left[1 - \exp(-t/T_1)\right] + T_2^2\left[1 - \exp(-t/T_2)\right]\right\}/(T_1 - T_2).$$

Let us find expression for K, substituting the values of the initial and steady-state angular velocity of the vessel's turn at the first phase in the obtained equation, in this case $\omega_o$=0 and $\omega_r$=$a_\omega$:

$$K = K_o + a_\omega t -$$
$$-a_\omega\left\{T_1^2\left[1 - \exp(-t/T_1)\right] - T_2^2\left[1 - \exp(-t/T_2)\right]\right\}/(T_1 - T_2),$$

$$K = K_o +$$
$$+a_\omega\left\{t - \left\{T_1^2\left[1 - \exp(-t/T_1)\right] + T_2^2\left[1 - \exp(-t/T_2)\right]\right\}/(T_1 - T_2)\right\}. \quad (13)$$

We obtain the expression for the current vessel's course at the second phase of turn $\tilde{K}$, substituting in the initial expression of formula for $\omega_o$=$\omega_o(\Delta t_k)$ and $\omega_r$=$-a_\omega$. First we will find expression for $\omega_o(\Delta t_k)$, taking into account that the moment of time $\Delta t_k$ is the finishing moment of the first phase of the vessel's turn. Therefore, we substitute $\Delta t_k$ in (12), taking into account initial data for the first phase of the turn:

$$\omega(\Delta t_k) = a_\omega -$$
$$-a_\omega\left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2),$$

$$\omega_o =$$
$$= a_\omega\left\{1 - \left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2)\right\}.$$

Substituting the obtained values $\omega_o$ and $\omega_r$ in the initial expression for the current course of vessel K, we will obtain:

$$\tilde{K} = K - a_\omega t +$$
$$+a_\omega\left\{1 - \left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2) + 1\right\}\times$$
$$\times\left\{T_1^2\left[1 - \exp(-t/T_1)\right] - T_2^2\left[1 - \exp(-t/T_2)\right]\right\}/(T_1 - T_2),$$

$$\tilde{K} = K - a_\omega t +$$
$$+a_\omega\left\{2 - \left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2)\right\}\times$$
$$\times\left\{T_1^2\left[1 - \exp(-t/T_1)\right] - T_2^2\left[1 - \exp(-t/T_2)\right]\right\}/(T_1 - T_2). \quad (14)$$

Expressions (13) and (14) make it possible to write down the first equation for the assigned change in the course ΔK at the turn of an operating vessel. However, first it is necessary to write down expressions for an increment in the course at the first phase of the turn $K(\Delta t_k)$ and at the second phase of the turn $\tilde{K}$ (Δt).

We substitute value $\Delta t_k$ in expression (13) and obtain:

$$K(\Delta t_k) = a_\omega\left\{\Delta t_k - \left\{T_1^2\left[1 - \exp(-\Delta t_k/T_1)\right] - T_2^2\left[1 - \exp(-\Delta t_k/T_2)\right]\right\}/(T_1 - T_2)\right\}.$$

Similarly, substitution of Δt in (14) allows us to obtain the expression for an increment in the course at the second phase of the turn $\tilde{K}$ (Δt):

$$\tilde{K}(\Delta t) = -a_\omega\Delta t +$$
$$+a_\omega\left\{2 - \left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2)\right\}\times$$
$$\times\left\{T_1^2\left[1 - \exp(-\Delta t/T_1)\right] - T_2^2\left[1 - \exp(-\Delta t/T_2)\right]\right\}/(T_1 - T_2).$$

We substitute the obtained expressions in the formula of increment in the course

$$\Delta K = K(\Delta t_k) + \tilde{K}(\Delta t)$$

and obtain the following analytical expression:

$$\Delta K = a_\omega\left\{\Delta t_k - \left\{T_1^2\left[1 - \exp(-\Delta t_k/T_1)\right] -\right.\right.$$
$$\left.\left.-T_2^2\left[1 - \exp(-\Delta t_k/T_2)\right]\right\}/T_1 - T_2\right\} -$$
$$-a_\omega\Delta t + a_\omega\left\{2 - \left[T_1\exp(-\Delta t_k/T_1) - T_2\exp(-\Delta t_k/T_2)\right]/(T_1 - T_2)\right\}\times$$
$$\times\left\{T_1^2\left[1 - \exp(-\Delta t/T_1)\right] - T_2^2\left[1 - \exp(-\Delta t/T_2)\right]\right\}/(T_1 - T_2).$$

We will divide both parts of the equation by magnitude $a_\omega$:

$$\Delta t_k - \left\{T_1^2\left[1-\exp\left(-\Delta t_k/T_1\right)\right]-T_2^2\left[1-\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}-$$
$$-\Delta t+\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}\times$$
$$\times\left\{T_1^2\left[1-\exp\left(-\Delta t/T_1\right)\right]-T_2^2\left[1-\exp\left(-\Delta t/T_2\right)\right]\right\}\big/\left(T_1-T_2\right)=\Delta K/a_\omega.$$

From the last expression, we write down dependence $\Delta t_k$ on $\Delta t$:

$$\Delta t_k=\Delta t+\left\{T_1^2\left[1-\exp\left(-\Delta t_k/T_1\right)\right]-T_2^2\left[1-\exp\left(-\Delta t_k/T_2\right)\right]\right\}\big/\left(T_1-T_2\right)-$$
$$-\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}\times$$
$$\times\left\{T_1^2\left[1-\exp\left(-\Delta t/T_1\right)\right]-T_2^2\left[1-\exp\left(-\Delta t/T_2\right)\right]\right\}\big/\left(T_1-T_2\right)+\Delta K/a_\omega. \quad (15)$$

We compile the second equation, substituting in (12) the values of the initial and steady-state angular velocity at the second phase of a turn and equaling the obtained expression to zero, that is:

$$\omega\left(\Delta t_k,\Delta t\right)=$$
$$=a_\omega\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}\times$$
$$\times\left[T_1\exp\left(-\Delta t/T_1\right)-T_2\exp\left(-\Delta t/T_2\right)\right]\big/\left(T_1-T_2\right)-a_\omega=0.$$

Reducing both parts by $a_\omega$, we obtain the following equation:

$$\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}\times$$
$$\times\left[T_1\exp\left(-\Delta t/T_1\right)-T_2\exp\left(-\Delta t/T_2\right)\right]\big/\left(T_1-T_2\right)-1=0,$$

$$\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}\times$$
$$\times\left[T_1\exp\left(-\Delta t/T_1\right)-T_2\exp\left(-\Delta t/T_2\right)\right]\big/\left(T_1-T_2\right)=1.$$

We perform elementary transformation in the last equation:

$$\left[T_1\exp\left(-\Delta t/T_1\right)-T_2\exp\left(-\Delta t/T_2\right)\right]=\left(T_1-T_2\right)\times$$
$$\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}^{-1},$$

which allows us to write down the obtained equation in the form, convenient for calculation by the method of simple iterations. For this purpose, we will write down:

$$\exp\left(-\Delta t/T_1\right)=\left(T_2/T_1\right)\exp\left(-\Delta t/T_2\right)+\left[\left(T_1-T_2/T_1\right)\right]\times$$
$$\times\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}^{-1}.$$

We take logarithms of both parts of the last expression and obtain:

$$-\Delta t/T_1=\ln\left\{\left(T_2/T_1\right)\exp\left(-\Delta t/T_2\right)+\left[\left(T_1-T_2/T_1\right)\right]\times\right.$$
$$\left.\times\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}^{-1}\right\},$$

hence, it is possible to write down the following expression:

$$\Delta t=-T_1\ln\left\{\left(T_2/T_1\right)\exp\left(-\Delta t/T_2\right)+\left[\left(T_1-T_2/T_1\right)\right]\times\right.$$
$$\left.\times\left\{2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right)\right\}^{-1}\right\}.$$

Designating

$$L=2-\left[T_1\exp\left(-\Delta t_k/T_1\right)-T_2\exp\left(-\Delta t_k/T_2\right)\right]\big/\left(T_1-T_2\right),$$

finally, we receive the expression, connecting $\Delta t_k$ with $\Delta t$:

$$\Delta t=-T_1\ln\left\{\left(T_2/T_1\right)\exp\left(-\Delta t/T_2\right)+\right.$$
$$\left.+\left[\left(T_1-T_2/T_1\right)\right]L^{-1}\right\}. \quad (16)$$

Thus, for the calculation of values $\Delta t_k$ and $\Delta t$ by the method of simple iterations, assigned by the previous value $\Delta t_k$, value $\Delta t$ is calculated with the help of expression (16), which is then substituted in expression (15) for calculating the subsequent value $\Delta t_k$.

Let us find expressions for calculating the increments in coordinates of the operating vessel $\Delta x_o$ and $\Delta y_o$ within the period of turn $\tau$. As it was shown earlier, the required increments in coordinates are expressed as follows with the help of definite integrals:

$$\Delta x_o=\int_0^\tau V_o\sin\left[K_o+K(t)\right]dt,$$

$$\Delta y_o=\int_0^\tau V_o\cos\left[K_o+K(t)\right]dt.$$

We also consider that each of the reduced integrals is the sum of two integrals, corresponding to increments of coordinates at the first and second phase of a turn, that is:

$$\Delta x_o=\int_0^{\Delta t_k} V_o\sin\left[K_o+K\right]dt+$$
$$+\int_0^{\Delta t} V_o\sin\left[K_o+K\left(\Delta t_k\right)+\tilde{K}\right]dt,$$

$$\Delta y_o=\int_0^{\Delta t_k} V_o\cos\left[K_o+K\right]dt+$$
$$+\int_0^{\Delta t} V_o\cos\left[K_o+K\left(\Delta t_k\right)+\tilde{K}\right]dt.$$

First, we will find expression for increment $\Delta x$, removing the constants from the integral sign:

$$\Delta x_o=V_o\sin K_o\int_0^{\Delta t_k}\cos(K)dt+V_o\cos K_o\int_0^{\Delta t}\sin(K)dt+$$
$$+V_o\sin\left[K_o+K\left(\Delta t_k\right)\right]\int_0^{\Delta t}\cos\left(\tilde{K}\right)dt+$$
$$+V_o\cos\left[K_o+K\left(\Delta t_k\right)\right]\int_0^{\Delta t}\sin\left(\tilde{K}\right)dt,$$

then, substituting formulas (13) and (14) in the last expression, we obtain the following analytical dependence:

$$\Delta x_o = V_o \sin K_o \int_0^{\Delta tk} \cos\left\{a_\omega \left\{t - \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\}\right\} dt +$$
$$+V_o \cos K_o \int_0^{\Delta tk} \sin\left\{a_\omega \left\{t - \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\}\right\} dt +$$
$$+V_o \sin\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \cos\left\{-a_\omega t + a_\omega \left\{2 - \left[T_1 \exp\left(-\Delta t_k/T_1\right) - T_2 \exp\left(-\Delta t_k/T_2\right)\right] / \left(T_1 - T_2\right)\right\} \times\right.$$
$$\left. \times \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\} dt +$$
$$+V_o \cos\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \sin\left\{-a_\omega t + a_\omega \left\{2 - \left[T_1 \exp\left(-\Delta t_k/T_1\right) - T_2 \exp\left(-\Delta t_k/T_2\right)\right] / \left(T_1 - T_2\right)\right\} \times\right.$$
$$\left. \times \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\} dt.$$

Similarly, we find expression for $\Delta y_o$:

$$\Delta y_o = V_o \cos K_o \int_0^{\Delta t_k} \cos\left(K\right) dt - V_o \sin K_o \int_0^{\Delta t} \sin\left(K\right) dt +$$
$$+V_o \cos\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \cos\left(\tilde{K}\right) dt - V_o \sin\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \sin\left(\tilde{K}\right) dt,$$

either substituting expressions for K and $\tilde{K}$, we will obtain:

$$\Delta x_o = V_o \cos K_o \int_0^{\Delta tk} \cos\left\{a_\omega \left\{t - \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\}\right\} dt -$$
$$-V_o \sin K_o \int_0^{\Delta tk} \sin\left\{a_\omega \left\{t - \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\}\right\} dt +$$
$$+V_o \cos\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \cos\left\{-a_\omega t + a_\omega \left\{2 - \left[T_1 \exp\left(-\Delta t_k/T_1\right) - T_2 \exp\left(-\Delta t_k/T_2\right)\right] / \left(T_1 - T_2\right)\right\} \times\right.$$
$$\left. \times \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\} dt -$$
$$-V_o \sin\left[K_o + K\left(\Delta t_k\right)\right] \int_0^{\Delta t} \sin\left\{-a_\omega t + a_\omega \left\{2 - \left[T_1 \exp\left(-\Delta t_k/T_1\right) - T_2 \exp\left(-\Delta t_k/T_2\right)\right] / \left(T_1 - T_2\right)\right\} \times\right.$$
$$\left. \times \left\{T_1^2\left[1-\exp\left(-t/T_1\right)\right] - T_2^2\left[1-\exp\left(-t/T_2\right)\right]\right\} / \left(T_1 - T_2\right)\right\} dt.$$

Thus, we perform the calculation of increment in the coordinates of the operating vessel during its turn, taking into account the third dynamic model of motion by the yaw angle.

## 6. Discussion of results. Selection of the most acceptable model of the rotary motion of a vessel

For the selection of an adequate model of the rotary motion of a vessel, which can be used for solving the problem of determining parameters of the turn, the imitation simulation of trajectories of the turn with the help of experimental material was carried out.

In the process of operating the container carrier "Oxford", experimental materials were obtained as a result of field observations of the vessel turning ability, which were used for determining the type of dynamic model and the calculation of numerical values of its parameters. The vessel was made in 1998, its length is 216 meters, its width is 26,8 meters, and the height of its side is 21,8 meters. Displacement in ballast is 17130 tons, full load displacement – 23660 tons with the average draught of 9,40 meters. The power plant consists of two diesels of MAN|B&W 9L 58|64 type, with capacity of 12510 kW each. The propeller is 7000 mm in diameter and weighs 45000 kg. Effective area of the rudder is 25,6 sq. m. The maximum speed in full load with both engines in operation is 27 knots.

The experimental material was obtained as follows. At the vessel's motion by unchanged course and velocity, the rudder was put over to the assigned angle and the chosen hard. From the starting moment of putting the rudder over, in the equal intervals of approximately 5 seconds, the moments of time $t_i$ and the correspondent values of an increment in the vessel's course ΔKi were registered. Under operating conditions it was possible to obtain experimental data for the vessel's turn to the right and to the left at the angles of putting the rudder over to 5, 10 and 15 degrees. In this case, each of six indicated maneuvers was made several times under the maximally similar conditions: at the sea state of 2–4 points, wind of 2–3 points and in still waters. Table 1 displays data on the conditions for field observations.

The materials of all series of one maneuver were averaged, obtaining the dependence of an increment in the course on the time. The averaged results of each of the maneuvers are given in Table 2.

Table 1

Data on the conditions for field observations

| Number of maneuver | Side of turn | Angle of putting rudder over β | Vessel's velocity | Number of series | Load state |
|---|---|---|---|---|---|
| Maneuver 1 | right | 5 | 24 | 5 | in ballast |
| Maneuver 2 | left | 5 | 24 | 5 | in ballast |
| Maneuver 3 | right | 10 | 21 | 5 | in load |
| Maneuver 4 | left | 10 | 21 | 5 | in load |
| Maneuver 5 | right | 15 | 27 | 4 | in load |
| Maneuver 6 | left | 15 | 27 | 4 | in load |

For the approximation of initial experimental material by the dynamic model of the third order and the calculation of the corresponding values of parameters $T_1$ and $T_2$, we used the method of least squares under the assumption of normal law of distribution of measurement errors. Results of the calculations are given in Table 3.

Table 2

Averaged data on maneuvers

| Maneuvers | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 2 | | 3 | | 4 | | 5 | | 6 | |
| $t_i$ | $\Delta K_i$ | $t_i$ | $\Delta K_i$ | $t_i$ | $\Delta K_i$ | $t_i$ | $\Delta K_i$ | $t_i$ | $\Delta K_i$ | $t_i$ | $\Delta K_i$ |
| 5 | 0,5 | 5 | 0,4 | 5 | 0,9 | 5 | 1,1 | 5 | 0 | 5 | 0 |
| 10 | 2,0 | 10 | 2,1 | 10 | 3,0 | 10 | 2,8 | 14 | 0 | 11 | 0 |
| 15 | 4,2 | 15 | 4,4 | 15 | 6,2 | 15 | 6,3 | 18 | 1,5 | 15 | 2,0 |
| 20 | 7,5 | 20 | 7,3 | 20 | 11,3 | 20 | 11,1 | 21 | 5,1 | 19 | 6,1 |
| 25 | 11,2 | 25 | 11,4 | 25 | 17,2 | 25 | 17,1 | 26 | 12,0 | 27 | 21,9 |
| 30 | 15,5 | 30 | 15,2 | 30 | 23,0 | 30 | 23,2 | 31 | 22,1 | 33 | 35,1 |
| 35 | 19,8 | 35 | 20,0 | 35 | 29,0 | 35 | 29,1 | 36 | 35,2 | 39 | 50,2 |
| 40 | 23,8 | 40 | 23,6 | 40 | 35,5 | 40 | 35,3 | 40 | 44,8 | 43 | 60,1 |
| 44 | 28,0 | 44 | 28,1 | | | | | 45 | 58,0 | | |

Table 3

Parameters of dynamic models of container carrier "Oxford"

| Rudder | Model 1 | Model 2 | Model 3 |
|---|---|---|---|
| $5^\circ$ | $a_\omega = 0,92$ | $a_\omega = 0,92$; $T_1 = 14,23$ | $a_\omega = 0,88$; $T_1 = 9,61$; $T_2 = 1,69$; |
| $10^\circ$ | $a_\omega = 1,38$ | $a_\omega = 1,38$; $T_1 = 13,22$ | $a_\omega = 1,28$; $T_1 = 8,89$; $T_2 = 1,54$; |
| $15^\circ$ | $a_\omega = 2,70$ | $a_\omega = 2,70$; $T_1 = 10,23$ | $a_\omega = 2,65$; $T_1 = 9,62$; $T_2 = 1,23$; |

Correctness of the calculated parameters of the vessel's turning ability for the dynamic models of the second and third orders was verified with the help of calculation of the curve of dependence $\Delta K$ on t and comparison of results of the calculations with experimental data.

The following analytical dependence was used for the dynamic model of the vessel's turning ability of the second order:

$$\Delta K = 0, \text{ if } t < t_Z,$$

$$\Delta K = a_\omega \left\{ (t - t_Z) - T_1 \left[ 1 - \exp\left( -(t - t_Z)/T_1 \right) \right] \right\}, \text{ if } t \ge t_Z,$$

and for the dynamic model of rotary motion of a vessel of the third order we used analytical dependence :

$$\Delta K = 0, \text{ if } t < t_Z,$$

$$\Delta K = a_\omega \left\{ (t - t_Z) - \left\{ T_1^2 \left[ 1 - \exp\left( -(t - t_Z)/T_1 \right) \right] - T_2^2 \left[ 1 - \exp\left( -(t - t_Z)/T_2 \right) \right] \right\} \Big/ (T_1 - T_2) \right\},$$

if $t \ge t_Z$.

As an example, Fig. 2 represents results of the calculations of the maneuver with the rudder angle of 15° for the dynamic model of the second order.

Results of the calculations on maneuver with the rudder angle equal to 15 degrees for the dynamic model of the third order are represented in Fig. 3.

The conducted analysis of correspondence of experimental data and calculated results revealed that the method of least squares, applied for the model of rotary motion of the third order, provides for a good agreement between experimental data and those calculated.

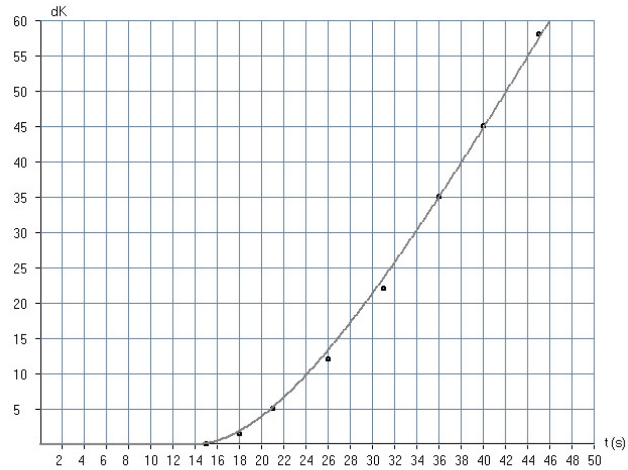


Fig. 2. Calculated dependence $\Delta K$ on t for maneuver to 15°, obtained with the aid of the second dynamic model

For the bulk carrier "Sheila Ann" (built in China in 1999), materials on its turning ability were obtained under actual operating conditions; according to these materials, the best correspondence to the experimental trajectory of the vessel's turn is reached for the third type of the model.

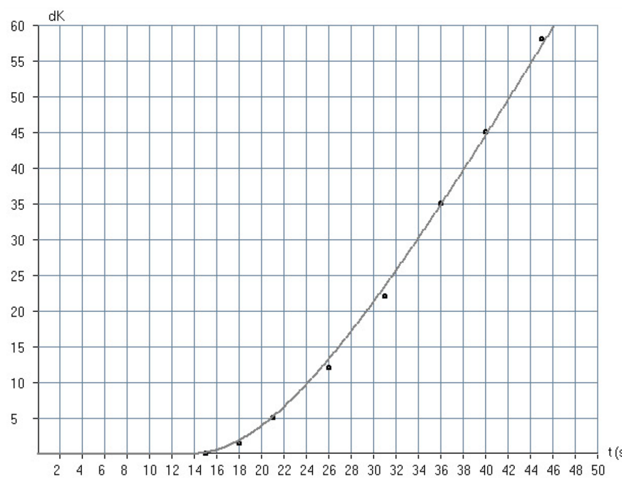


Fig. 3. Calculated dependence $\Delta K$ on t for the maneuver to 15°, obtained with the help of the third dynamic model

Paper [4] also examined imitation simulation of the turn of container carrier "Oxford", whose characteristics of turning ability were given above, with the calculation of magnitude of trajectory error by the moment of the maneuver completion (Fig. 4).

As a result of the imitation simulation of the turn to 90°, it was established that with the use for the prediction of the curvilinear section of the first type of mathematical model, the trajectory error was 150÷200 m, for the second type of the model this magnitude was 35÷40 m, and for the third type – 25÷30 m.

Therefore, mathematical model of the vessel's turning ability of the third type is the most acceptable, since, at sufficient simplicity, it possesses the required accuracy (maximum divergence of experimental and model trajectories is 25÷30 for both experiments). Therefore, it is expedient to use a mathematical model of the vessel's turning ability of the third type as the model of the vessel's rotary motion.

At the National University "Odessa Marine Academy" (Ukraine), a navigation information system of providing for vessel maneuvering is under development, which contains the module of calculation of parameters of a vessel's turn when using a model of the vessel's turning ability of the third type. By the assigned sections of program trajectory and increment in the course between them, the system calculates the starting moment of a turn and its trajectory taking into account the rudder blade angle and characteristics of the vessel's inertia, predicting the trajectory of the vessel's motion, as shown in Fig. 5.

Thus, present work presents an analysis of three models of a vessel's rotary motion, which to varying degrees correspond to the real process of turning and can be used when calculating the parameters of a vessel's turn in the situations of different constraint. A drawback in the examined models is that they do not consider the time of putting the rudder blade over.

It is expedient to use results of the study in the navigation information systems for developing the function of calculation of parameters of a vessel's turn for the assigned initial data, which will substantially increase the accuracy of performing a turn by the vessel and entering a new phase of the program trajectory. In future, it is necessary to analyze the models of the vessel's rotary motion, which consider the duration of putting the rudder blade over, and perform their comparative characteristic with the models, proposed in this article by the parameter of accuracy.

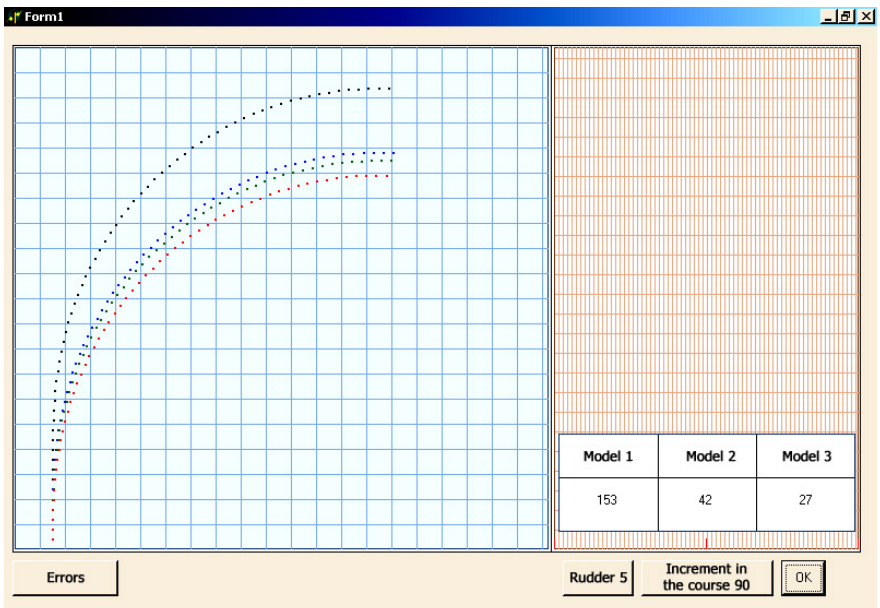


Fig. 4. Trajectories of the vessel's turn when using different models

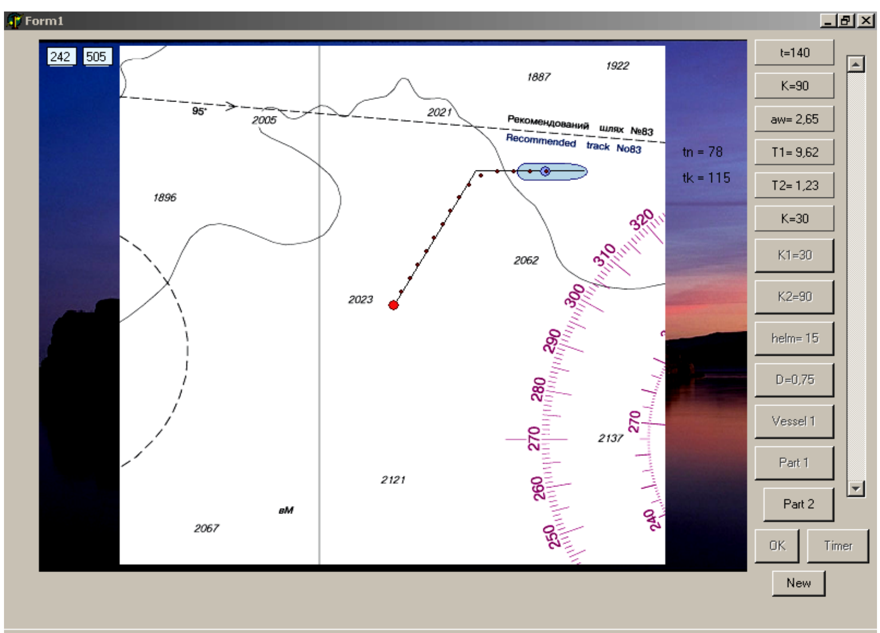


Fig. 5. Results of predicting a vessel's turn

## 7. Conclusions

1. On the basis of an analysis of dynamic models of the vessel's turn, which consider its inertia, the analytical expressions for calculating the duration of both phases of a turn were obtained.

2. With the help of experimental data, obtained during the field observations, the substantiation of selecting the most acceptable model of the vessel's rotary motion was performed, which minimizes a trajectory error in vessel entering the predetermined trajectory.

3. It was shown that the results of the study are used in the developed navigation information system of providing vessels maneuvering, which contains the module of calculation of parameters of a vessel's turn with the help of the model of the vessel's turning ability of the third type.

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