

Розглядається задача ідентифікації нестационарних параметрів лінійного об'єкта, які можна описати Марківською моделлю першого порядку, за допомогою найбільш простих в обчислювальному відношенні однокрокових адаптивних алгоритмів ідентифікації – модифікованих алгоритмів Качмажа і Нагумо-Ноди. Ці алгоритми не потребують знання інформації щодо ступеня нестационарності об'єкту, що досліджується, вони використовують при побудові моделі інформацію тільки про один такт (крок) вимірювань. Наявність в цих алгоритмах в знаменнику значень вхідних сигналів свідчить про необхідність введення в них деяких обмежень на ці сигнали. Модифікація полягає в використанні в алгоритмах регуляризуючого додатку, з метою покращення їх обчислювальних властивостей та уникнення поділу на нуль. Застосування Марковської моделі є досить ефективним, бо дає можливість отримати аналітичні оцінки властивостей алгоритмів.

Показано, що використання регуляризуючого додатку в алгоритмах ідентифікації, покращуючи стійкість алгоритмів, призводить до деякого уповільнення процесу побудови моделі. Визначено умови збіжності регуляризованих алгоритмів Качмажа і Нагумо-Ноди при оцінюванні нестационарних параметрів в середньому та середньоквадратичному і наявності завад вимірів.

Отримані оцінки відрізняються від існуючих більшою точністю. Незважаючи на це, вони є досить загальними і залежать як від ступеня нестационарності об'єкту, так і від статистичних характеристик завад. Крім того, визначено вирази для оптимальних значень параметрів алгоритмів, що забезпечують їх максимальну швидкість збіжності в умовах не стаціонарності та присутності гаусовських завад. Отримані аналітичні вирази містять ряд невідомих параметрів (помилка оцінювання, ступінь нестационарності об'єкту, статистичні характеристики завад). Для їх практичного застосування слід скористатися будь-якою рекурентною процедурою оцінювання цих невідомих параметрів і використовувати одержувані оцінки для уточнення тих параметрів, що входять в алгоритми

**Ключові слова:** Марківська модель, адаптивний алгоритм Качмажа, Нагумо-Ноди, регуляризація, рекурентна процедура, оптимальний параметр

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# ANALYSIS OF CONVERGENCE OF ADAPTIVE SINGLE-STEP ALGORITHMS FOR THE IDENTIFICATION OF NON-STATIONARY OBJECTS

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## 1. Introduction

Many problems of control, prediction, and pattern recognition, etc. relate to the construction of a model of the following type:

$$y_{n+1} = \theta^{*T} x_{n+1} + \xi_{n+1}, \quad (1)$$

where  $y_{n+1}$  is the observed output signal;

$$x_{n+1} = (x_{1,n+1}, x_{2,n+1}, \dots, x_{N,n+1})^T$$

is the vector of input signals  $N \times 1$ ;  $\theta^* = (\theta_1^*, \theta_2^*, \dots, \theta_N^*)^T$  is the vector of sought-for parameters  $N \times 1$ ;  $\xi_{n+1}$  is the interfer-

ence and it is reduced to minimization of a certain functional of quality, chosen in advance (identification criterion). The most widely used in practice quadratic functional leads to different identification algorithms, which make it possible to obtain the estimates of sought-for vector  $\theta^*$  at normal interference distributions, that is  $\xi_{n+1} \sim N(0, \sigma_\xi^2)$ .

It should be noted that the problem of constructing a model (1) (identification problem) is of interest not only in itself, but as a part of a general optimization problem.

It should be noted that the effectiveness of application of this or that algorithm depends on availability of information about a drift. At minimal or no information, it is necessary to apply adaptive algorithms that make it possible to refine the estimates as soon as new information becomes

available. These algorithms include gradient algorithms, the algorithm by Kaczmarz and Nagumo-Noda. Unlike the gradient algorithms, these algorithms, though they use only current information about  $x$  and  $y$ , are more responsive. This demonstrates the feasibility of their application for identification of non-stationary parameters. Existing estimates, characterizing the properties of these algorithms are quite rough. Apparently, this is caused by the difficulty of the theoretical studies of the properties of these algorithms under non-stationary conditions. That is why the analytical study on convergence and getting more accurate estimates at existence of measurement interferences is relevant.

However, the identification problem becomes essentially more complicated if parameters  $\theta$  change (drift) over time, that is  $\theta^*(k) = \text{var}$ .

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## 2. Literature review and problem statement

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The effectiveness of application of this or that algorithm for estimating the drifting parameters essentially depends on the volume of a priori information about the nature of a drift. In accordance with this, adaptive algorithms of identification of non-stationary parameters can be divided into two classes:

- 1) algorithms for the estimation of parameters at known law of their drift;
- 2) algorithms for the estimation of parameters at unknown law of drift.

The first direction originates in [1]. In this paper, the generalization of the stochastic approximation was proposed in case the drift of the regression equation root is close to linear. Most further papers [2–4] deal with the case where a drift is either on average close to linear, or fades. Paper [2] considered the case where the drift was parameterized and the identification problem was reduced to the estimation of the unknown parameter  $\theta^*(k)$  using the multidimensional method of stochastic approximation.

Lack of information about the nature of a drift requires the development of identification algorithms, using the minimum amount of information  $\theta^*(k)$  and maintaining performance in a wide range of variation  $\theta^*(k)$ .

The algorithms of the second direction, first of all, include the recurrent least-squares-method with the exponential weighting of information [3, 4]. It should be noted that the problem of choosing this parameter has not yet been resolved. The only compensation for the lack of a priori information about the nature of the parameters drift is the lag in the evaluation of a change of parameters and, consequently, lower observation accuracy due to the inertia of the algorithms.

Among the simplest in computing single-step identification algorithms, the most effective are algorithms by Kaczmarz and Nagumo-Noda [5, 6]. The Kaczmarz algorithm was proposed in paper [5] for the solution of the systems of linear algebraic equations, and subsequently was applied successfully to solve the identification problem when constructing the model of type (1).

The estimates of convergence rate of this algorithm at the identification of stationary objects were first obtained in [7–10]. While papers [7–9] considered a regular case, the estimates that take into consideration the statistical properties of signals and interferences were obtained in paper [10].

To improve computational sustainability of the Kaczmarz algorithm, V. M. Chadeev [7, 8] proposed its modification – a regularized algorithm.

Algorithm by Nagumo-Noda, considered in [6], contained non-linearity, representing the operation of using the sign of an input signal and possessed the lower convergence than that of the Kaczmarz algorithm. In this work, the rate of the algorithm was proved, and the convergence rate was determined in [10]. The algorithm by Nagumo-Noda is known in problems of filtering as a signed-regressor NLMS (SR-NMLS) [11–14]. In paper [15], non-asymptotic and asymptotic estimates of convergence rate, in particular, of the regularized algorithms by Kaczmarz and Nagumo–Noda were obtained, which showed that the introduction of the regularizing addition, improving stability of algorithms, leads to slow down of their convergence rate.

However, the performance of these algorithms often turns out to be insufficient when assessing non-stationary parameters. Knowledge (or approximation) of the law of drift makes it possible to receive effective algorithms of tracking non-stationary parameters. Paper [16] considered the algorithm called dynamic algorithm by Kaczmarz, which uses some a priori model of a drift. It should be noted, however, that errors in assigning the law of changes in parameters may cause the loss of properties of algorithm convergence.

The possibility of accelerating the Kaczmarz algorithm by using not one but a series of measurements was considered in [17–19]. In these articles, the modifications of the given algorithm at the identification of stationary objects were considered. A slightly different modification of algorithm by Kaczmarz, which turned out to be quite effective in evaluating non-stationary parameters, was proposed and studied in [20].

Articles [17, 18] have become the impetus for the creation of multi-step projection algorithms [19, 21, 22]. In [19], a recurrent form of these algorithms was proposed, and in [21, 22], the properties of random pseudo-inverse matrices and projection matrices were established. This enabled determining the rate of convergence of these algorithms. It was concluded that accounting of information about  $L$  previous steps in these algorithms ( $L$ ) is equivalent in terms of convergence to reducing the dimensionality of the original space  $N$  by  $L$ . Thus, application of multi-step projection algorithms makes it possible to accelerate significantly the identification process and is quite effective when assessing non-stationary parameters.

The algorithm by Kaczmarz, better known as NLMS – normalized least-mean-square algorithm, is widely used not only in the systems of identification of stationary [23] and non-stationary [24–26] system. In [27–29], its application to solving problems of filtration was described. It should be noted that in papers [19, 24–26], to describe the non-stationary parameters, the first-order Markovian model was used, while papers [30, 31] used the modified first-order Markovian model (this model received fairly wide use in training artificial neural networks [31]). It should be noted that the use of such a model is very convenient, because it allows receiving the analytical estimates of the dynamic properties of specific algorithms quite easily. This makes it possible to determine the suitability of these algorithms for solving the problem of identification of non-stationary objects.

In paper [32], a comparative analysis of the operation of the NLMS algorithm and the affine projection algorithm was performed. In studies [33–35], different variants of

affine projection algorithms with respect to the problem of signals processing were described.

A rather large body of research into the algorithms by Kaczmarz and Nagumo-Noda was predetermined by their wide applicability due to computational simplicity and efficiency. Despite this, the study of the properties of these algorithms under non-stationary conditions in the presence of measurement interferences is a very important problem. Although some estimates of the properties of these algorithms were obtained in the above papers, they are quite rough due to the use of all kinds of assumptions about the statistical properties of signals and interferences. The recent research from authors of [15] presents new theoretical results, enabling obtaining more accurate estimates compared to the known estimates during solving the problem of identification of a stationary object. In this regard, it is of interest to apply the results of this research to the problem of identification of a non-stationary object, that is, to generalize the results obtained in [15] for a non-stationary case.

### 3. The aim and objectives of the study

The aim of this research is to study the problems of convergence of single-step algorithms for the identification of non-stationary parameters described by the Markovian first-order model in the presence of measurement interferences and to determine parameters for the algorithms that ensure their maximum convergence rate.

To accomplish the aim, the following tasks have been set:

- to obtain more accurate analytical estimates of convergence in mean and root-mean-square of the algorithm by Kaczmarz in comparison with the existing ones;
- determine the optimal values of the algorithm relaxation parameter ensuring its maximum convergence rate under considered conditions;
- to obtain more accurate analytic estimates of convergence in mean and root-mean-square of the algorithm by Nagumo-Noda in comparison with the existing ones;
- to determine the optimal values of the algorithm relaxation parameter ensuring its maximum convergence rate under considered conditions.

### 4. Studying the convergence of the Kaczmarz regularized algorithm

A regularized Kaczmarz algorithm takes the form

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \gamma \frac{e_{n+1} x_{n+1}}{\|x_{n+1}\|^2 + \delta}, \tag{2}$$

where

$$e(k+1) = y(k+1) - \hat{y}(k+1) = y(k+1) - \hat{\theta}^T(k)x(k+1),$$

$\hat{y}(k+1)$  is the output signal of the model;

$$\hat{\theta}(k) = (\hat{\theta}_1(k), \hat{\theta}_2(k), \dots, \hat{\theta}_N(k))^T$$

is the vector of the estimated parameters  $N \times 1$ ;  $\gamma$  is a certain parameter (relaxation parameter), influencing the convergence rate of the algorithm;  $\delta > 0$  is the regularization

parameter;  $\|\bullet\|$  is the Euclidean norm, one of the most applicable in both solving the identification problem, and in the problems of signals processing (interference compensation, echo removal, etc.).

To obtain analytical estimates in the non-stationary case, it was supposed, like in papers [19, 24–26] that non-stationary parameters of an object can be represented by a Markovian first-order model

$$\theta_{n+1} = \theta_n + S_{n+1}, \tag{3}$$

where

$$S_{n+1} = (S_{n+1,1}, S_{n+1,2}, \dots, S_{n+1,N})^T$$

is the vector of random sequence  $N \times 1$ ;  $S_i \sim N(0, \sigma_s^2)$ .

Let us introduce into consideration the estimation error

$$\tilde{\theta}_{n+1} = \theta_{n+1} - \hat{\theta}_{n+1}. \tag{4}$$

Taking into consideration (3), expression for  $e_{n+1}$  can be written down as follows:

$$e_{n+1} = \tilde{\theta}_n^T x_{n+1} + S_{n+1}^T x_{n+1} + \xi_{n+1}. \tag{5}$$

As it is assumed that  $\xi(k) \sim N(0, \sigma_\xi^2)$ , we obtain

$$M\{e_{n+1}^2\} = \sigma_\xi^2 + \sigma_x^2 M\{\|\tilde{\theta}_{n+1}\|^2\}, \tag{6}$$

where  $M$  is the symbol of mathematic expectation.

Taking into consideration the accepted kind of non-stationarity (3), the Kaczmarz algorithm regarding identification errors can be written as follows:

$$\begin{aligned} \tilde{\theta}_{n+1} &= \tilde{\theta}_n + S_{n+1} - \gamma \frac{x_{n+1} x_{n+1}^T}{\|x_{n+1}\| + \delta} \tilde{\theta}_n - \\ &- \gamma \frac{x_{n+1} x_{n+1}^T}{\|x_{n+1}\| + \delta} S_{n+1} - \frac{x_{n+1}}{\|x_{n+1}\|^2 + \delta} \xi_{n+1} = \\ &= \left(1 - \gamma \frac{x_{n+1} x_{n+1}^T}{\|x_{n+1}\|^2 + \delta}\right) (\tilde{\theta}_n + S_{n+1}) - \frac{x_{n+1}}{\|x_{n+1}\|^2 + \delta} \xi_{n+1}. \end{aligned} \tag{7}$$

It is assumed that the components of the vector of estimation error  $\tilde{\theta}_n$  obey the normal law of distribution with  $\tilde{\theta}_{i,n} \triangleq M\{\tilde{\theta}_{i,n}\}$  and dispersion  $\sigma_{i,n}^2$  [36], that is, all components of estimation vector  $\hat{\theta}_{i,n}$  are distributed according to the normal law  $\hat{\theta}_{i,n} \sim N(m_{i,n}, \sigma_{i,n}^2)$ , with a probability density function

$$f(\hat{\theta}_{i,n}) = \frac{1}{\sqrt{2\pi\sigma_{i,n}^2}} \left[ e^{-\frac{(\hat{\theta}_{i,n}) - m_{i,n}}{2\sigma_{i,n}^2}} + e^{-\frac{(\hat{\theta}_{i,n}) + m_{i,n}}{2\sigma_{i,n}^2}} \right] U(\hat{\theta}_{i,n}),$$

where

$$m_{i,n} = \theta_{i,n} - \tilde{\theta}_{i,n};$$

$$\sigma_{i,n}^2 \triangleq M\{\tilde{\theta}_{i,n}^2\} - M^2\{\tilde{\theta}_{i,n}\};$$

$$U(\hat{\theta}_{i,n}) = \begin{cases} 0, & \hat{\theta}_{i,n} < 0, \\ 1, & \hat{\theta}_{i,n} \geq 0. \end{cases}$$

The mean of this distribution is determined from formula

$$M\{\hat{\theta}_n\} = \int_{-\infty}^{\infty} |\hat{\theta}_n| f(|\hat{\theta}_n|) d|\hat{\theta}_n| =$$

$$= m_{i,n} \operatorname{erf}\left(\frac{m_{i,n}}{\sqrt{2\sigma_{i,n}^2}}\right) + \sqrt{\frac{2}{\pi}} \sigma_{i,n} e^{-\frac{m_{i,n}^2}{2\sigma_{i,n}^2}},$$

where

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the interval of Gaussian probability

#### 4. 1. Analysis of convergence in the mean and root-mean-square

Let us calculate  $M\{\tilde{\theta}_{n+1}\}$ .

It should be noted that in all the studies that address the Kaczmarz algorithm it is assumed that

$$M\left\{\frac{x_{n+1}x_{n+1}^T}{\|x_{n+1}\|^2 + \delta}\right\} = \frac{M\{x_{n+1}x_{n+1}^T\}}{M\{\|x_{n+1}\|^2 + \delta\}}, \quad (8)$$

$$M\left\{\frac{1}{\|x_{n+1}\|^2 + \delta}\right\} = \frac{1}{N\sigma_x^2 + \delta}. \quad (9)$$

In papers [10, 15], other expressions for these ratios were received. Taking into consideration statistical independence of  $x_{n+1}$  and  $\xi_{n+1}$  and that [10, 15]

$$M\left\{\frac{x_{n+1}x_{n+1}^T}{\|x_{n+1}\|^2 + \delta}\right\} = \frac{1}{N}I - M\left\{\frac{\delta}{\|x_{n+1}\|^2 + \delta}\right\}I; \quad (10)$$

$$M\left\{\frac{1}{\|x_{n+1}\|^2 + \delta}\right\} = \frac{1}{(N-2)\sigma_x^2 + \delta} \left(1 - \frac{2\delta\sigma_x^2}{((N-2)\sigma_x^2 + \delta)^2}\right) \quad (11)$$

after regular transformations, we obtain

$$M\{\tilde{\theta}_{n+1}\} = 1 - \gamma \frac{1}{N} \left( \begin{aligned} & \left(1 - \frac{\delta N}{(N-2)\sigma_x^2 + \delta} \times \right. \\ & \left. \times \left(1 - \frac{2\delta\sigma_x^2}{((N-2)\sigma_x^2 + \delta)^2}\right) \right) M\{\tilde{\theta}_n\}. \quad (12) \end{aligned} \right)$$

Here,  $I$  is the identity matrix  $N \times N$ .

It follows from the obtained expression (12) that for the algorithm convergence (2) in mean, it is necessary to meet the inequality

$$\left| 1 - \gamma \frac{1}{N} \left(1 - \frac{\delta N}{(N-2)\sigma_x^2 + \delta} \left(1 - \frac{2\delta\sigma_x^2}{((N-2)\sigma_x^2 + \delta)^2}\right)\right) \right| < 1, \quad (13)$$

hence, it follows

$$0 < \gamma <$$

$$< \frac{2N((N-2)\sigma_x^2 + \delta)^3}{((N-2)\sigma_x^2 + \delta)^3 - \delta N((N-2)\sigma_x^2 + \delta)^2 + 2N\delta\sigma_x^2}. \quad (14)$$

Therefore, by satisfying (14), the algorithm (2) converges in the mean, that is

$$\lim_{n \rightarrow \infty} M\{\tilde{\theta}_{n+1}\} = 0, \quad (15)$$

that is, the obtained estimate is non-shifted.

To investigate the convergence in the root-mean-square, we multiply (7) on the left by  $\tilde{\theta}_{n+1}^T$

$$\|\tilde{\theta}_{n+1}\|^2 = \|\tilde{\theta}_n\|^2 + \|S_{n+1}\|^2 - 2\gamma \frac{\tilde{\theta}_n^T x_{n+1}}{\|x_{n+1}\|^2 + \delta} [\xi_{n+1} + \tilde{\theta}_n^T x_{n+1}] -$$

$$- 2\gamma \frac{x_{n+1}^T S_{n+1}}{\|x_{n+1}\|^2 + \delta} [\xi_{n+1} + \tilde{\theta}_n^T x_{n+1}] + \gamma^2 \frac{\|x_{n+1}\|^2}{(\|x_{n+1}\|^2 + \delta)^2} [\xi_{n+1} + \tilde{\theta}_n^T x_{n+1}]^2.$$

Compute  $M\{\|\tilde{\theta}_{n+1}\|^2\}$

$$M\{\|\tilde{\theta}_{n+1}\|^2\} = M\{\|\tilde{\theta}_n\|^2\} + M\{\|S_{n+1}\|^2\} -$$

$$- 2\gamma M\left\{\frac{\tilde{\theta}_n^T x_{n+1} x_{n+1}^T \tilde{\theta}_n}{\|x_{n+1}\|^2 + \delta}\right\} -$$

$$- 2\gamma M\left\{\frac{S_{n+1}^T x_{n+1} x_{n+1}^T S_{n+1}}{(\|x_{n+1}\|^2 + \delta)^2} \|x_{n+1}\|^2\right\} +$$

$$+ \gamma^2 M\left\{\frac{\xi_{n+1}^2 \|x_{n+1}\|^2}{(\|x_{n+1}\|^2 + \delta)^2}\right\}.$$

In known papers, in the calculation of summands, included in the right part  $M\{\|\tilde{\theta}_{n+1}\|^2\}$ , the simplifications that are similar to those in (8), (9) were used.

Taking into consideration (10), (11) and using the results from paper [15], in which it was proved that

$$M\left\{\frac{(\tilde{\theta}_n^T x_{n+1})^2}{\|x_{n+1}\|^2 + \delta}\right\} =$$

$$= \left[ \frac{1}{N}I - \frac{\delta}{(N-2)\sigma_x^2 + \delta} \left(1 - \frac{2\sigma_x^2\delta}{((N-2)\sigma_x^2 + \delta)^2}\right) \right] M\{\|\tilde{\theta}_n\|^2\};$$

$$M\left\{\frac{\|x_{n+1}\|^2}{(\|x_{n+1}\|^2 + \delta)^2} \xi_{n+1}^2\right\} = \frac{(N-2)\sigma_x^2\sigma_\xi^2}{((N-2)\sigma_x^2 + \delta)^2};$$

$$M\left\{\frac{\|x_{n+1}\|^4}{(\|x_{n+1}\|^2 + \delta)^2}\right\} =$$

$$= -2\delta \left( \frac{1}{(N-2)\sigma_x^2 + \delta} \left(1 - \frac{2\sigma_x^2\delta}{((N-2)\sigma_x^2 + \delta)^2}\right) - \frac{\delta^2}{((N-2)\sigma_x^2 + \delta)^2} \right) =$$

$$= \frac{\delta^2((N-2)\sigma_x^2 + \delta) - 2\delta((N-2)\sigma_x^2 + \delta)^2 + 4\delta^2\sigma_x^2}{((N-2)\sigma_x^2 + \delta)^3},$$

After simple transformations, we obtain

$$M\{\|\tilde{\theta}_{n+1}\|^2\} = \left[ 1 - \frac{\gamma}{N} \left[ \frac{(2-\gamma) - \frac{2(\gamma-1)(\delta(N-2)\sigma_x^2 + \delta)^2 + 2\sigma_x^2\delta}{((N-2)\sigma_x^2 + \delta)^3}}{\frac{\gamma^2\delta^2}{((N-2)\sigma_x^2 + \delta)^2}} \right] \right] \times M\{\|\tilde{\theta}_n\|^2 + \|S_{n+1}\|^2\} + \gamma^2 \frac{(N-2)\sigma_x^2\sigma_\xi^2}{((N-2)\sigma_x^2 + \delta)^2},$$

hence, it follows that in order to transform algorithm (2) into the root-mean-square, it is necessary to meet the condition

$$\left| 1 - \frac{\gamma}{N} \left[ \frac{(2-\gamma) - \frac{2(\gamma-1)(\delta(N-2)\sigma_x^2 + \delta)^2 + 2\sigma_x^2\delta}{((N-2)\sigma_x^2 + \delta)^3}}{\frac{\gamma^2\delta^2}{((N-2)\sigma_x^2 + \delta)^2}} \right] \right| < 1. \quad (16)$$

**4.2. Determining the optimal value for relaxation parameter  $\gamma$**

It should be noted that in papers [7, 8], which proposed the regularization of the algorithm by Kaczmarz, the case  $\gamma=1$  was studied and the problem of selecting the optimal value of parameter  $\gamma$  was not stated. In articles [9, 19], it was shown that this parameter that is called the relaxation parameter must be selected from the condition  $\gamma \in (0,2)$ . At  $\gamma=1$ , we obtain the algorithm of complete relaxation, while at  $\gamma \neq 1$  – of incomplete relaxation. It should be stressed that these papers examined the problem of stationary object identification.

Consider the problem of selecting the optimal value of the relaxation parameter  $\gamma$ , ensuring a maximum rate of algorithm convergence in the case of a non-stationary object. Because convergence is characterized by magnitude  $M\{\|\tilde{\theta}_{n+1}\|^2\}$ , the expression for optimal value  $\gamma$  can be obtained by solving equation

$$\frac{\partial M\{\|\tilde{\theta}_{n+1}\|^2\}}{\partial \gamma} = 0. \quad (17)$$

Omitting simple transformations, we will obtain

$$\gamma^{OPT} = \frac{\left[ ((N-2)\sigma_x^2 + \delta)^3 + \delta((N-2)\sigma_x^2 + \delta)^2 + \delta\sigma_x^2 \right] M\{\|\tilde{\theta}_n\|^2 + \|S_{n+1}\|\}}{\left[ ((N-2)\sigma_x^2 + \delta)^3 + 2\delta((N-2)\sigma_x^2 + \delta)^2 + \delta^2((N-2)\sigma_x^2 + \delta) \right] M\{\|\tilde{\theta}_n\|^2 + \|S_{n+1}\|\} + N(N-2)\sigma_x^2\sigma_\xi^2((N-2)\sigma_x^2 + \delta)}. \quad (18)$$

Expression (18) makes it possible to determine the optimal value for parameter  $\gamma$  that ensures a maximum rate of convergence of the algorithm by Kaczmarz when evaluating the non-stationary parameters described by the Markovian model. As it can be seen from (18), in order to use this value, the information about the statistical properties of signals and interferences, degree of non-stationarity of an object and error estimations are needed. That is why this result is rather of theoretical interest. In addressing the practical problems and in the absence of such information, it can only be concluded that the chosen relaxation parameter should be smaller than unity.

It should be noted that the obtained ratios generalize the known results. For the classic algorithm by Kaczmarz with  $\delta=0$  from (14), known condition for convergence in the mean follows [7–10]

$$0 < \gamma < 2N. \quad (19)$$

Conditions for its convergence in root-mean-square for a non-stationary case are derived from (16) at  $\delta=0$

$$\gamma^{opt} = \frac{((N-2)^3\sigma_x^6(M\{\|\tilde{\theta}_n\|^2\} + \sigma_s^2))}{((N-2)\sigma_x^6M\{\|\theta_n\|^2\} + \sigma_s^2) + N(N-2)^2\sigma_x^4\sigma_\xi^2} = 1 - \frac{N\sigma_\xi^2}{(N-2)\sigma_x^2M\{\|\tilde{\theta}_n\|^2 + \sigma_s^2\}}, \quad (20)$$

and for a stationary case and absence of interference

$$\gamma^{OPT} = 1. \quad (21)$$

In the latter case, we have an algorithm for complete relaxation, in which estimation error  $\|\tilde{\theta}_n\|^2$  is projected on direction  $x_{n+1}$  (this ensures a maximum algorithm convergence rate).

Result (21) is known, starting from the work by Kaczmarz [5]. Expression (20) makes it possible to re-evaluate the choice of the optimum value of relaxation parameter  $\gamma$  in the identification of non-stationary objects.

Similarly to analyzing ratio (18), one can notice that the expression  $\gamma_n^{OPT}$  contains a number of unknown magnitudes. That is why in the practical application of the Kaczmarz algorithm, it is possible to use the same recommendation as above, that is to select the relaxation parameter that is smaller than unity. This will ensure the algorithm convergence in mean and root-mean-square.

**5. Studying the convergence of the Nagumo-Noda regularized algorithm**

A regularized algorithm by Nagumo-Noda, known in problems of filtering as signed-regressor NLMS (SR-NMLS) [11–14] takes the form (here the same designations are accepted as in (2))

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \gamma \frac{e_{n+1} \text{sign} x_{n+1}}{|x_{n+1}| + \delta}, \quad (22)$$

where  $\gamma$  is a certain parameter (relaxation parameter) influencing the rate of algorithm convergence.

$$|x_{n+1}| = x_{n+1}^T \text{sign} x_{n+1};$$

$$\text{sign} x_{n+1} = \begin{cases} 1, & \text{at } x_{n+1} > 0, \\ 0, & \text{at } x_{n+1} = 0, \\ -1, & \text{at } x_{n+1} < 0. \end{cases}$$

When evaluating non-stationary parameters described by model (3), and existence of interference, the algorithm can be written down regarding identification errors (4) as follows:

$$\tilde{\theta}_{n+1} = \left( I - \gamma \frac{\text{sign} x_{n+1} x_{n+1}^T}{|x_{n+1}| + \delta} \right) (\tilde{\theta}_n + S_{n+1}) - \frac{\text{sign} x_{n+1} x_{n+1}^T}{|x_{n+1}| + \delta} \xi_{n+1}. \quad (23)$$



It should be noted that the statistical properties of estimates obtained using the algorithm (22) have not been studied. In papers [11–14, 27], to approximate the expressions containing modules, the Price's theorem was used, which made it possible to obtain quite rough estimates.

When studying the problems on the convergence of this algorithm, we will use the results from paper [15].

### 5.1. Analysis of convergence in the mean and root-mean-square

To analyze convergence in the mean, we calculate  $M\{\tilde{\theta}_{n+1}\}$ .

Taking into consideration statistical independence of signals and interferences, as well as formulas [10, 15]

$$M\left\{\frac{1}{|x_{n+1}|+\delta}\right\}\approx\frac{\sqrt{\pi\omega}}{N+\delta\sqrt{\pi\omega}}; \quad (24)$$

$$\begin{aligned} M\left\{\frac{\text{sign}x_{n+1}x_{n+1}^T}{|x_{n+1}|+\delta}\right\} &= \\ &= \left(\frac{1}{N}-\delta M\left\{\frac{1}{|x_{n+1}|+\delta}\right\}\right)I\approx\frac{1}{N}-\delta\frac{\sqrt{\pi\omega}}{N+\delta\sqrt{\pi\omega}}, \end{aligned} \quad (25)$$

where  $\omega=(2\sigma_x^2)^{-1}$ , we will obtain the ratios, from which it follows that algorithm (22) will converge in mean when meeting the condition

$$\left|1-\gamma\frac{N+\sqrt{\pi\omega}-N\delta\sqrt{\pi\omega}}{N(N+\sqrt{\pi\omega})}\right|<1$$

or

$$0<\gamma<\frac{2N(N+\sqrt{\pi\omega})}{N+\sqrt{\pi\omega}-N\delta\sqrt{\pi\omega}}, \quad (26)$$

that is, if this condition is met, the assessment obtained by this algorithm will be non-shifted (15).

It should be noted that the expression (26) does not include the characteristic of non-stationarity  $S_{n+1}$ . It is explained by the choice of the model of non-stationarity (3) and its properties ( $S_i\sim N(0,\sigma_s^2)$ ).

It should be noted that from (26), there follows the condition of convergence in mean for the algorithm by Nagumo-Noda  $0<\gamma<2N$ , that coincides with known from [9, 15].

To establish convergence in root-mean-square, we will act by analogy with the above. After multiplying both sides of (23) on the left by  $\tilde{\theta}_{n+1}^T$  and the computation of mathematical expectation, given that [15]

$$M\left\{\frac{\|x_{n+1}\|^2}{(|x_{n+1}|+\delta)^2}\right\}=\frac{\pi N}{2(N+\delta\sqrt{\pi\omega})^2}+O\left(\frac{1}{N^2}\right), \quad (27)$$

we will obtain

$$\begin{aligned} M\{\|\tilde{\theta}_{n+1}\|^2\} &= \left(1-\frac{\gamma}{N+\delta\sqrt{\pi\omega}}\left(2-\gamma\frac{\pi N}{2(N+\delta\sqrt{\pi\omega})}\right)\right)\times \\ &\times M\{\|\tilde{\theta}_n\|^2\}+\sigma_s^2+\gamma^2\frac{\pi N\omega\sigma_\xi^2}{(N+\delta\sqrt{\pi\omega})^2}. \end{aligned}$$

It follows from these expressions that algorithm (22) will converge in root-mean square when meeting the condition

$$\left|1-\frac{\gamma}{N+\delta\sqrt{\pi\omega}}\left(2-\gamma\frac{\pi N}{2(N+\delta\sqrt{\pi\omega})}\right)\right|<1. \quad (28)$$

When satisfying (28), the region of convergence of the algorithm (22) (considering that  $\omega=(2\sigma_x^2)^{-1}$ ) is determined by the following expression

$$\lim_{n\rightarrow\infty}M\{\|\tilde{\theta}_{n+1}\|^2\}=\frac{\gamma\pi N\sigma_\xi^2}{\left(4(N+\delta\sqrt{\pi\omega})^2-\gamma\pi N\right)\sigma_x^2}.$$

The known ratios for the Nagumo-Noda algorithm follow from the obtained expressions at  $\delta=0$ .

### 5.2. Determining the optimal value for relaxation parameter $\gamma$

Approach to determining the optimal value of parameter  $\gamma$  for the algorithm by Nagumo-Noda remains unchanged: found from condition (17).

Omitting simple transformations, we will obtain the following expression for  $\gamma$ :

$$\gamma^{OPT}=\frac{2(N+\delta\sqrt{\pi\omega})(M\{\|\tilde{\theta}_n\|^2\}+\sigma_s^2)}{\pi N\left[M\{\|\tilde{\theta}_n\|^2\}+\sigma_s^2+2\omega\sigma_\xi^2\right]}. \quad (29)$$

Analysis of this expression is similar to the above analysis of selection of relaxation parameter for the Kaczmarz algorithm. It should be noted that expression (29) was derived for the first time and it generalizes the known results. So, for the classic algorithm by Nagumo-Noda for a non-stationary case ( $\sigma_s^2\neq 0$  and  $\delta=0$ ), it follows from (29) that

$$\gamma_n^{OPT}=\frac{2N\left[M\{\|\tilde{\theta}_n\|^2\}+\sigma_s^2\right]}{\pi N\left[M\{\|\tilde{\theta}_n\|^2\}+\sigma_s^2+2\omega\sigma_\xi^2\right]}. \quad (30)$$

It should be noted that this result is derived for the first time.

From this expression (29), it follows that for stationary case  $\sigma_s^2=0$ , absence of interference  $\sigma_\xi^2=0$  and  $\delta=0$

$$\gamma^{OPT}=\frac{2}{\pi}, \quad (31)$$

which coincides with known from [15].

## 6. Modelling

We solved the problem of identification of a non-stationary object described by equation (1) with the following parameters  $N=10$ , and 4 out of 10 parameters were non-stationary, represented by the Markovian model (3) with  $\sigma_s^2=0,5$ . The other six parameters were accepted as equal to

$$\theta_5=0,8; \theta_6=0,4; \theta_7=-0,2;$$

$$\theta_8=0,55; \theta_9=-0,67; \theta_{10}=0,15.$$

The sequences of normally distributed magnitudes  $x_{n+1}\sim N(0;1)$ ,  $\xi_{n+1}\sim N(0;3)$ , were selected as input  $x_{n+1}$  and additive interference  $\xi_{n+1}$ .

As the criterion for comparison of the operation of algorithms, we used magnitude

$$MSE = M \left\{ \left\| \tilde{\theta}_{n+1} \right\|^2 \right\}.$$

Fig. 1, *a, b* shows the diagrams of changing magnitude *MSE* for the regularized algorithms by Kaczmarz (2) and Nagumo-Noda (22), respectively, at different selection of relaxation parameter  $\gamma$ . Since value  $\left\| \tilde{\theta}_{n+1} \right\|^2$  is known at the simulation modeling, this makes it possible to compare the limit (optimal) capabilities of the algorithms. That is why the curves without markers in the figures correspond to the theoretically optimal choice of parameters  $\gamma^{OPT}$  in accordance with (20) and (29), and the other correspond to the practical selection of these parameters. Thus, curves with circles correspond to the selection (task) of  $\gamma = 0,7$  and  $\delta = 0$ , and curved triangles correspond to the selection of  $\gamma = 0,7$  and  $\delta = 0,01$ , curves with squares – to selection of  $\gamma = 0,7$  and  $\delta = 0,02$ , and curves with rhombs – to the selection of  $\gamma = 0,6$  and  $\delta = 0,02$ .

Fig. 1 shows the results compared with the analysis of operation of the studied algorithms in the selection of appropriate  $\gamma^{OPT}$  (curves without any markers) and task  $\gamma = 0,7$  and  $\delta = 0,01$  (curves with triangles).

Analyzing the simulation results, it can be concluded that the introduction of regularization parameter  $\delta \neq 0$  leads to a slowdown in the rate of convergence of the algorithms. However, there arises no problem of dividing by zero, that is, the stability of algorithms increases.

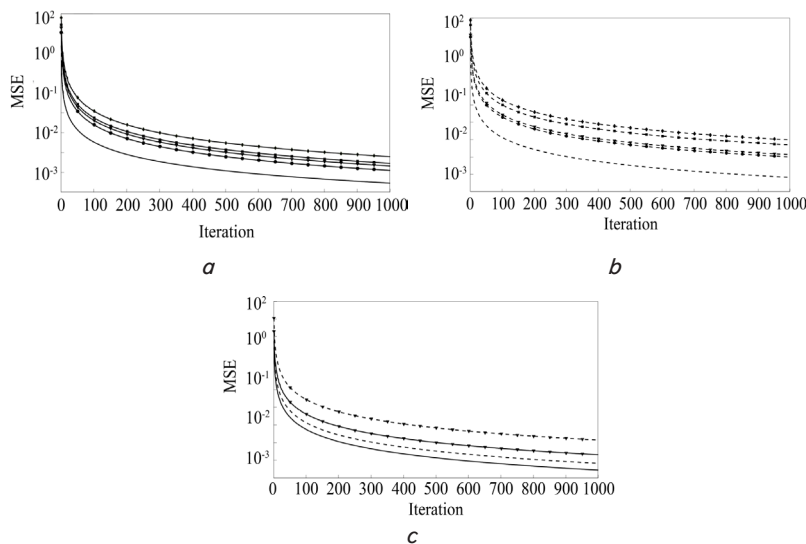


Fig. 1. Modeling results

### 7. Discussion of results of studying the convergence of adaptive single-step algorithms for the identification of non-stationary objects

Studies carried out in this work are the continuation and development of earlier studies, described in [15]. The results, obtained when identifying stationary objects,

strictly proved in [15], were used to identify non-stationary objects.

As shown by the results of the research, the use of regularizing addition in identification algorithms while improving the stability of algorithms leads to a slowdown in the process of model construction. The conditions of convergence of the regularized algorithms by Kaczmarz and Nagumo-Noda when evaluating non-stationary parameters and existence of measurement interference were determined.

The resulting estimates are fairly general and depend both on the degree of non-stationarity of an object, and on the statistical characteristics of useful signals and interferences. In addition, the expressions for optimal values of parameters of algorithms relaxation, ensuring their maximal convergence rate, were determined. Because these expressions contain a series of unknown parameters (estimation error  $\left\| \theta_n \right\|^2$ , degree of non-stationarity of object  $\sigma_s^2$ ), the estimates of these parameters should be used for their practical application. Thus, during identification in the on-line mode, it is possible to apply any recurring evaluation procedure and to use the resulting estimates to clarify the parameters within the algorithms. At identification under an off-line mode, it is necessary to adjust the result derived after all computations.

It should be noted that the estimates obtained in this paper are more accurate than the existing ones. Therefore, in addressing practical problems, a researcher can preliminarily assess with great certainty the capabilities of this or that algorithm and the effectiveness of its application.

### 8. Conclusions

1. More accurate, compared to known, conditions for the convergence of the regularized algorithm by Kaczmarz in the mean and root-mean-square during the assessment of non-stationary parameters and existence of measurement interferences were obtained.

2. The expressions for the optimal values of the relaxation parameter of the Kaczmarz algorithm, ensuring its maximum rate of convergence in the identification of non-stationary objects and measurement interferences were determined.

3. More accurate, compared to the known, conditions of convergence of the regularized algorithm by Nagumo-Noda in mean and root-mean-square during the assessment of non-stationary parameters were obtained.

4. The expressions for the optimal values of the relaxation parameter of the algorithm by Nagumo-Noda, ensuring its maximum convergence rate in the identification of non-stationary objects and existence of measurement interferences were determined.

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Розвинуто застосування алгоритмів методу базисних матриць, які оснащені технологією довгої арифметики для покращення точності виконання основних операцій при дослідженні погано обумовлених лінійних систем, зокрема систем лінійних алгебраїчних рівнянь (СЛАР). Встановлення факту поганої обумовленості системи є досить трудомісткою обчислювальною процедурою. Закладено проведення контролю входження обчислень в стан некоректності та унеможливлення накопичення похибок обчислень, що є бажаною властивістю методів та алгоритмів розв'язання практичних задач.

В сучасних ЕОМ, як правило, використовуються стандартні типи цілих чисел, розмір яких не перевищує 64 байта. Було подолано це апаратне обмеження програмним шляхом, а саме, розробкою власного типу даних у вигляді спеціальної бібліотеки Longint мовою C++ з використанням стандартної бібліотеки шаблонів STL (Standard Template Library). Програмна реалізація була розвинута на проведення обчислень за методами базисних матриць (МБМ) та Гауса, тобто використано довгу арифметику для моделей з раціональними елементами. Запропоновано алгоритми та комп'ютерну реалізацію методів типу Гауса та штучних базисних матриць (варіант методу базисних матриць) в середовищах Matlab та Visual C++ з використанням технології точних обчислень елементів методів, в першу чергу, для погано обумовлених систем різної розмірності. Розроблено бібліотеку Longint з типами довгих цілих чисел (longint3) та раціональних чисел (longrat3) із чисельником та знаменником типу longint3. Арифметичні операції над довгими цілими числами реалізовано на основі сучасних методів: зокрема, методу Штрассена множення. Наведено результати обчислювального експерименту за згаданими методами, в якому тестові моделі систем генерувались, зокрема, на основі матриць Гільберта різної розмірності, які характеризуються як "незручні"

**Ключові слова:** метод базисних матриць, точні обчислення, погано обумовлена система лінійних рівнянь

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# ADVANCEMENT OF A LONG ARITHMETIC TECHNOLOGY IN THE CONSTRUCTION OF ALGORITHMS FOR STUDYING LINEAR SYSTEMS

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## 1. Introduction

Mathematical modeling of processes of different nature is known to lead to the need to explore non-linear equations and systems of varying complexity (mathematical models).

In many cases, they are solved by the introduction of certain simplifications in statements, transition, in particular, to difference analogues (discrete variant) and eventually to the systems of linear algebraic equations (SLAE) of different dimension, often with a square matrix of restrictions. The